Theorem 3.1. In Theorem 3.1, it is possible to remove the assumptions on the return time. This is proved in "On almost-sure versions of classical limit theorems for dynamical systems" by J.-R. Chazottes and myself, Theorem 2.19. The idea of the argument is that Gibbs-Markov maps satisfy strong limit theorems, not only distributional ones. By using a "limit theorem with tight maxima", we are able to induce the limit theorem without any assumption on the return time.

Theorem B.3. Theorem B.3 is not optimal. Indeed, it is possible to remove the factor $(\ln n)^{\frac{p-1}{p}}$ in the conclusion of the theorem. That is, one gets the same bound for $S_n f$ and for $M_n f$. This is a result of Serfling, in "Moment inequalities for the maximum cumulative sum", Ann. Math. Statist. 41 1970 1227–1234.

We give here a quick proof. For any $a_1, \dots, a_n$ and $\alpha_1, \dots, \alpha_n$, for any $p \geq 1$, we have

$$(a_1+\cdots+a_n)^p = (\alpha_1+\cdots+\alpha_n)^p \left(\frac{\sum \alpha_i(a_i/\alpha_i)}{\sum \alpha_i}\right)^p \leq (\alpha_1+\cdots+\alpha_n)^p \frac{\sum \alpha_i(a_i/\alpha_i)^p}{\sum \alpha_i},$$

since $x \mapsto x^p$ is convex. Therefore,

$$(a_1+\cdots+a_n)^p \leq (\alpha_1+\cdots+\alpha_n)^{p-1}\left[\frac{a_1^p}{\alpha_1^{p-1}}+\cdots+\frac{a_n^p}{\alpha_n^{p-1}}\right]. \tag{1}$$

Plugging this formula in the proof of Theorem 3.1 (with the dyadic decomposition), we get

$$\int |M_{2^n-1}f|^p \leq (\alpha_1+\cdots+\alpha_n)^{p-1}\sum_{j=0}^{n-1}\frac{1}{\alpha_j^{p-1}}2^{n-j-1}2^{pj/2}. \tag{2}$$

We have to choose the α_j. In the proof in the article, I use $\alpha_j = 1$ for all j. However, a small computation with Lagrange multipliers shows that, when minimizing $\sum b_j/\alpha_j^{p-1}$, it is a good idea to choose α_j proportional to $b_j^{1/p}$. So, we take $\alpha_j = 2^{(p/2-1)j/p}$. We get

$$\int |M_{2^n-1}f|^p \leq C2^{(p-1)(p/2-1)n/p}2^n 2^{(p/2-1)n/p} = C2^{(p/2-1)n}2^n = C2^{pn/2}. \tag{3}$$

Hence,

$$\left(\int |M_{2^n-1}f|^p\right)^{1/p} \leq C2^{n/2}. \tag{4}$$