STABLE LAWS FOR THE DOUBLING MAP

SÉBASTIEN GOUËZEL

Abstract. We prove stable limit theorems for the functions $f_\alpha(x) = x^{-\alpha}$, $\alpha \geq 1/2$, under the iteration of the doubling map $T : x \to 2x \mod 1$. The limiting distributions are smaller (resp. larger) than the sum of corresponding i.i.d. random variables when $\alpha > 1$ (resp. < 1).

Let $T : [0, 1] \to [0, 1]$ be the doubling map $T(x) = 2x \mod 1$, preserving Lebesgue measure Leb. Our goal in this expository article is to study the limit theorems satisfied by the Birkhoff sums of the functions $f_\alpha(x) = x^{-\alpha}$. Our main concern is not the result in itself (it could very easily be extended to general uniformly expanding maps of the interval using similar arguments, or to more general observables), but rather the techniques: they were developed primarily to study intermittent dynamics, and our goal here is to present them in a simple different setting, where they also prove quite efficient.

When $\alpha < 1/2$, the function f_α is in $L^2(\mathrm{Leb})$. In this case, numerous criteria apply to show that the Birkhoff sums of f_α satisfy a central limit theorem. While most criteria are formulated in terms of the L^2modulus of continuity of f_α, we can for instance use the following one, due to Dedecker, which has the great advantage of avoiding completely computations:

Theorem 0.1. *If $f \in L^2(\mathrm{Leb})$ is piecewise monotonic (with a finite number of branches), then the Birkhoff sums of f satisfy the central limit theorem: there exists $\sigma^2 \geq 0$ such that $S_n(f - \int f)/\sqrt{n}$ converges in distribution to a Gaussian distribution $\mathcal{N}(0, \sigma^2)$.*

This theorem is proved using classical martingale techniques, and a clever covariance inequality given in [Ded04].

Our main focus will be on the case $\alpha \geq 1/2$, where martingale techniques do not apply.

1. The independent case

To see what is likely to happen, let us first consider the (easier) i.i.d. case. So, let $X_0, X_1, \dots$ be distributed like f_α, we will show that this sequence satisfies a limit theorem. Let us first estimate the characteristic function of X_j, i.e. $\phi(t) = E(e^{itX_j})$. Since it satisfies $\phi(-t) = \overline{\phi(t)}$, we can without loss of generality consider only $t \geq 0$

Proposition 1.1. *Let $\alpha \geq 1/2$. For $t \geq 0$,*

- *If $\alpha > 1$, then $E(e^{itX_0}) = 1 - \Gamma\left(1 - \frac{1}{\alpha}\right) \cos\left(\frac{\pi}{2\alpha}\right) t^{1/\alpha} \left(1 - i \tan\left(\frac{\pi}{2\alpha}\right)\right) + o(t^{1/\alpha})$.*
- *If $1/2 < \alpha < 1$, then $E(e^{itX_0}) = 1 + \frac{i}{1-\alpha} t - \Gamma\left(1 - \frac{1}{\alpha}\right) \cos\left(\frac{\pi}{2\alpha}\right) t^{1/\alpha} \left(1 - i \tan\left(\frac{\pi}{2\alpha}\right)\right) + o(t^{1/\alpha})$.*
- *If $\alpha = 1$, then $E(e^{itX_0}) = 1 - it \ln t + ((1 - \gamma)i - \pi/2)t + o(t)$.*
- *If $\alpha = 1/2$, then $E(e^{itX_0}) = 1 + 2it + t^2 \ln t + o(t^2 \ln t)$.*

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In the $\alpha = 1$ case, γ is the Euler constant, i.e., the limit of $\sum_{k=1}^n 1/k - \ln n$. When $\alpha < 1$, the leading term $\frac{i}{1-\alpha} t$ is simply $itE(X_0)$ (this is not surprising, since $e^{ix} = 1 + ix + o(x)$, hence $E(e^{itX_0}) = 1 + itE(X_0) + o(t)$ whenever X_0 is integrable).

Proof. Since the values of the different constants will not be important to us, we will leave their computation to the interested reader (or to Maple): we will obtain them as (explicit) integrals.

We have

$$\begin{aligned} E(e^{itX_0}) &= \int_{x=0}^1 e^{itf_\alpha(x)} \,\mathrm{dLeb}(x) = \int_{x=0}^1 e^{itx^{-\alpha}} \,\mathrm{d}x = \frac{1}{\alpha} \int_{y=1}^\infty e^{ity} \frac{\mathrm{d}y}{y^{1+1/\alpha}} \\ &= \frac{t^{1/\alpha}}{\alpha} \int_{z=t}^\infty e^{iz} \frac{\mathrm{d}z}{z^{1+1/\alpha}}. \end{aligned}$$

If $\alpha > 1$, we have $e^{iz} = 1 + O(z)$. Moreover, $\int_0^t z^{-1/\alpha} \,\mathrm{d}z = O(t^{1-1/\alpha})$. Hence,

$$E(e^{itX_0}) = \frac{t^{1/\alpha}}{\alpha} \int_{z=t}^\infty (e^{iz} - 1) \frac{\mathrm{d}z}{z^{1+1/\alpha}} + \frac{t^{1/\alpha}}{\alpha} \int_{z=t}^\infty \frac{\mathrm{d}z}{z^{1+1/\alpha}} = \frac{t^{1/\alpha}}{\alpha} (C_\alpha + O(t^{1-1/\alpha})) + 1,$$

where $C_\alpha = \int_0^\infty (e^{iz} - 1) \frac{\mathrm{d}z}{z^{1+1/\alpha}}$. This proves the first case of the proposition.

Assume now that $1/2 < \alpha < 1$. The previous computation breaks down since $(e^{iz} - 1)/z^{1+1/\alpha}$ is not any more integrable at 0. The trick is to go one step further in the Taylor expansion of e^{iz} to recover integrability:

$$\begin{aligned} E(e^{itX_0}) &= \frac{t^{1/\alpha}}{\alpha} \int_{z=t}^\infty (e^{iz} - 1 - iz) \frac{\mathrm{d}z}{z^{1+1/\alpha}} + \frac{t^{1/\alpha}}{\alpha} \int_{z=t}^\infty (1 + iz) \frac{\mathrm{d}z}{z^{1+1/\alpha}} \\ &= \frac{t^{1/\alpha}}{\alpha} (C_\alpha + O(t^{2-1/\alpha})) + 1 + \frac{it^{1/\alpha}}{\alpha} \frac{t^{1-1/\alpha}}{1/\alpha - 1}, \end{aligned}$$

since $\int_0^t z^{1-1/\alpha} = O(t^{2-1/\alpha})$. This again proves the proposition.

When $\alpha = 1$, we have a problem at 0 if we consider $e^{iz} - 1$ (as in the case $\alpha > 1$), and a problem at infinity if we consider $e^{iz} - 1 - iz$ (as in the case $\alpha < 1$). The idea is to consider $e^{iz} - 1 - i\phi(z)$ where ϕ is a function behaving like z at 0 and bounded at infinity, to avoid both problems. The choice of ϕ is essentially arbitrary, common choices in the literature are $\phi(z) = z/(1 + z^2)$ or $\phi(z) = \sin(z)$, we will use $\phi(z) = z1_{z\leq 1}$ to simplify the computations. We get

$$\begin{aligned} E(e^{itX_0}) &= t \int_{z=t}^\infty (e^{iz} - 1 - iz1_{z\leq 1}) \frac{\mathrm{d}z}{z^2} + t \int_{z=t}^\infty (1 + iz1_{z\leq 1}) \frac{\mathrm{d}z}{z^2} \\ &= t(C_1 + O(t)) + 1 - it \ln t, \end{aligned}$$

where $C_1 = \int_0^\infty (e^{iz} - 1 - iz1_{z\leq 1}) \frac{\mathrm{d}z}{z^2}$. This proves the third case of the proposition.

The case $\alpha = 1/2$ is similar, but we need to write $e^{iz} = 1 + iz - z^2/2 + O(z^3)$ to conclude. □

Corollary 1.2. *For $\alpha \geq 1/2$, define sequences $A_n = A_n(\alpha)$ and $B_n = B_n(\alpha)$ by*

(1) *if $\alpha > 1$, then $A_n = 0$ and $B_n = n^\alpha$.*
(2) *if $1/2 < \alpha < 1$, then $A_n = nE(X_0) = n/(1 - \alpha)$ and $B_n = n^\alpha$.*
(3) *if $\alpha = 1$, then $A_n = n \ln n$ and $B_n = n$.*
(4) *if $\alpha = 1/2$, then $A_n = nE(X_0) = n/(1 - \alpha)$ and $B_n = \sqrt{n \ln n}$.*

There exists a nondegenerate random variable W_α such that $(\sum_{k=0}^{n-1} X_k - A_n)/B_n \to W_\alpha$.

By nondegenerate, we mean that W_α is not almost surely constant. The characteristic function of W_α is explicit, and given in the proof.

Proof. Assume first that $\alpha > 1$. By Proposition 1.1, for $t \geq 0$, the characteristic function ϕ of X_0 is equal to $1 + C_\alpha t^{1/\alpha}(1 + o(1))$. The characteristic function of $(\sum_{k=0}^{n-1} X_k)/n^\alpha$ is equal to $\phi(t/n^\alpha)^n$, i.e. $(1+C_\alpha t^{1/\alpha}/n+o(1/n))^n$. This quantity converges to $\exp(C_\alpha t^{1/\alpha})$ when n tends to infinity. Since convergence of random variables is equivalent to the pointwise convergence of characteristic functions, this concludes the proof of the convergence of $(\sum_{k=0}^{n-1} X_k)/n^\alpha$. Moreover, the characteristic function of the limit law W_α is given, for $t \geq 0$, by $\exp(C_\alpha t^{1/\alpha})$.

Assume now $1/2 < \alpha < 1$. The characteristic function of $X_0 - E(X_0)$ is equal to $1 + C_\alpha t^{1/\alpha}(1 + o(1))$, again by Proposition 1.1. The result follows as in the previous case.

Let now $\alpha = 1$. In this case, $\phi(t) = 1 - it \ln t + ct + o(t) = \exp(-it \ln t + ct + o(t))$. Hence, the characteristic function of $(\sum_{k=0}^{n-1} X_k - n \ln n)/n$ is equal to the exponential of

$$\begin{aligned} n\left[-i\frac{t}{n}\ln\left(\frac{t}{n}\right) + c\frac{t}{n} + o(1/n)\right] - it\ln n &= [-it\ln t + it\ln n + ct + o(1)] - it\ln n \\ &= ct - it\ln t + o(1). \end{aligned}$$

Since this converges when $n \to \infty$, this concludes the proof.

Finally, if $\alpha = 1/2$, the characteristic function of $X_0 - E(X_0)$ is equal to $\exp(t^2 \ln t + o(t^2 \ln t))$, hence the characteristic function of $(\sum_{k=0}^{n-1} X_k - nE(X_0))/\sqrt{n \ln n}$ is the exponential of

$$n\frac{t^2}{n\ln n}\ln\left(\frac{t}{\sqrt{n\ln n}}\right)(1 + o(1)) = (t^2\ln t/\ln n - t^2\ln\sqrt{n\ln n}/\ln n)(1 + o(1)). \tag{1}$$

This quantity converges to $-t^2/2$, as desired. □

Proposition 1.1 and Corollary 1.2 are special cases of the well-known description of the *domain of attraction of stable laws* (see e.g. [GK68, Fel66]), i.e., the random variables X for which there exist A_n and B_n such that $(\sum_{k=0}^{n-1} X_k - A_n)/B_n$ converges in distribution to a non-constant random variables, when X_k are i.i.d. and distributed like X. Essentially, there should exist $p < 2$, a slowly varying function L (as defined below in Definition 4.2) and $c_1, c_2 > 0$ with $c_1 + c_2 = 1$ such that $P(X > x) = (c_1 + o(1))x^{-p}L(x)$ and $P(X < -x) = (c_2 + o(1))x^{-p}L(x)$ when $x \to \infty$ (or a slightly different condition in the $p = 2$ case). Of course, the random variables considered in Proposition 1.1 and Corollary 1.2 satisfy this condition, for some constant function L.

2. The strategy

We now turn back to the dynamical situation: T is the doubling map, and $f_\alpha(x) = x^{-\alpha}$. If $f_\alpha(x)$ is large, then x is close to 0, so that $T(x)$ is also close to 0, hence $f_\alpha(Tx)$ is also large. This argument indicates that f_α and $f_\alpha \circ T$ are not independent, and the asymptotic behavior may be different from the independent case.

In fact, two opposite phenomena coexist:

- If $f_\alpha(x)$ is large, then $f_\alpha(Tx)$ is also large, so the Birkhoff sums $S_n f(x)$, when they are large, should be larger than the sums of the corresponding i.i.d. random variables.
- If $f_\alpha(x)$ is large, then $f_\alpha(Tx) \leq f_\alpha(x)$: one can not add a much larger term in the end. This phenomenon tends to make the Birkhoff sums smaller than the sums of the corresponding i.i.d. random variables.

These two phenomena compete. It will turn out that, for $\alpha < 1$, the first one is prominent, while for $\alpha > 1$ the second one is more important. For $\alpha = 1$, the two phenomena balance almost exactly, and the asymptotic behavior of $S_n f_1$ is very close to the behavior of i.i.d. random variables.

Our main theorem is the following. For $\alpha \geq 1/2$, consider the sequences $A_n = A_n(\alpha)$ and $B_n = B_n(\alpha)$ given by Corollary 1.2, ensuring the converge of $(\sum_{k=0}^{n-1} X_k - A_n)/B_n$ to a nondegenerate limit law W_α, when X_k is an i.i.d. sequence distributed like f_α.

Theorem 2.1. *Let $\alpha \geq 1/2$. If $\alpha \neq 1$, the sequence $(2^\alpha - 1)(\sum_{k=0}^{n-1} f_\alpha \circ T^k - A_n)/B_n$ converges in distribution to W_α. If $\alpha = 1$, the sequence $(\sum_{k=0}^{n-1} f_\alpha \circ T^k - A_n + 2n \ln 2)/B_n$ converges in distribution to W_α.*

Since $2^\alpha - 1$ is > 1 (resp. < 1) when $\alpha > 1$ (resp. $\alpha < 1$), this justifies the above claims: the Birkhoff sums are smaller than the sum of corresponding i.i.d. random variables when $\alpha > 1$, and larger when $\alpha < 1$, while the asymptotic behavior is the same for $\alpha = 1$, up to a centering term.

To study the behavior of the sums, one would like to use a Markov partition. The most natural one is the partition into $[0, 1/2)$ and $[1/2, 1)$, but this partition is too small: knowing that x belongs to $[0, 1/2)$ gives almost no information on the value of $f_\alpha(x)$. One is therefore led to consider the partition into the intervals $I_n = [1/2^{n+1}, 1/2^n)$: on I_n, $f_\alpha(x)$ is of the order of magnitude $2^{\alpha n}$. However, the lack of independence becomes apparent: while $T(I_0) = \bigcup_{n\in\mathbb{N}} I_n$, we have $T(I_n) = I_{n-1}$ for $n > 0$.

To regain independence, it is natural to induce on the interval $Y = I_0$. For $x \in Y$, let $\phi(x) = \inf\{n > 0 \mid T^n(x) \in Y\}$ be its first return time, and define the induced map $T_Y : Y \to Y$ by $T_Y(x) = T^{\phi(x)}(x)$. This map is Markov for the partition of the interval $(1/2, 1]$ into the intervals $J_n = [1/2 + 1/2^{n+1}, 1/2 + 1/2^n]$. More precisely, T_Y is affine on J_n with slope 2^n, corresponds there to n iterates of T, and $T_Y(J_n) = I_Y$ for all n. The return time ϕ is equal to n on J_n.

Define the induced function g_α by $g_\alpha(x) = \sum_{k=0}^{\phi(x)-1} f_\alpha(x)$. In this way, the Birkhoff sums of g_α for T_Y form a subsequence of the Birkhoff sums of f_α for T. It is therefore reasonable to hope to study the latter by understanding the former.

We can compute g_α: for $x \in J_n$,

$$g_\alpha(x) = x^{-\alpha} + \sum_{k=0}^{n-2} (2^k(2x-1))^{-\alpha} = x^{-\alpha} + \frac{1}{(2x-1)^\alpha} \frac{1 - 2^{-(n-1)\alpha}}{1 - 2^{-\alpha}}. \tag{2}$$

In particular, since $2x - 1 \geq 2^{-n}$, we get

$$g_\alpha(x) = \frac{1}{(1 - 2^{-\alpha})(2x-1)^\alpha} + O(1) = \frac{1}{(2^\alpha - 1)(x - 1/2)^\alpha} + O(1). \tag{3}$$

Let $a_\alpha(x) = \frac{1}{(2^\alpha-1)(x-1/2)^\alpha}$ and $b_\alpha = g_\alpha - a_\alpha$. Since b_α is bounded, it will satisfy the central limit theorem, and will therefore play no important role in the limit theorem. Moreover, in this induced setting, there is enough independence gained at each iteration of T_Y so that the Birkhoff sums of a_α satisfy the same limit behavior as the sum of corresponding i.i.d. random variables (which are distributed like a rescaled version of f_α, hence Corollary 1.2 applies to give their asymptotic behavior). In this way, we will show in Theorem 3.9 that the sums $\sum_{k=0}^{n-1} g_\alpha \circ T_Y^k$ satisfy an explicit limit theorem (and the norming constants are rescaled versions of the constants appearing in the i.i.d. case for f_α).

Since the Birkhoff sums $\sum_{k=0}^{n-1} g_\alpha \circ T_Y^k$ form a subsequence of the Birkhoff sums $\sum_{j=0}^{N-1} f_\alpha \circ T^j$, it is then possible to obtain a limit theorem for f_α using the previously mentioned limit theorem for g_α: this is a very general mechanism, that we describe in details in Section 4.

3. The induced map

The main result of this paragraph is Theorem 3.9, in which we describe the limit theorems satisfied by the Birkhoff sums $\sum_{k=0}^{n-1} g_\alpha \circ T_Y^k$. It will be proved using the decomposition $g_\alpha = a_\alpha + b_\alpha$ where $a_\alpha(x) = \frac{1}{(2^\alpha - 1)(x-1/2)^\alpha}$ and, for $x \in J_n$,

$$b_\alpha(x) = x^{-\alpha} - \frac{2^{-(n-1)\alpha}}{1-2^{-\alpha}} \frac{1}{(2x-1)^\alpha}. \tag{4}$$

Since b_α is bounded, its Birkhoff sums will be quite easy to understand. Our main interest it therefore to describe the Birkhoff sum $S_n^Y a_\alpha = \sum_{k=0}^{n-1} a_\alpha \circ T_Y^k$, or rather its characteristic function. We will do so by using the classical Nagaev argument on perturbations of Perron-Frobenius operators.

Let $\mathcal{L}$ be the Perron-Frobenius operator associated to T_Y, defined by duality by $\int u \cdot v \circ T_Y \,\mathrm{dLeb}_Y = \int \mathcal{L}u \cdot v \,\mathrm{dLeb}_Y$, where Leb_Y denotes the Lebesgue measure on Y normalized to be of mass 1. This operator is explicitly given by

$$\mathcal{L}u(x) = \sum_{T_Y(y)=x} \frac{u(y)}{T_Y'(y)} = \sum_{n \geq 1} 2^{-n} u(v_n x), \tag{5}$$

where $v_n : Y \to J_n$ is the inverse branch of T_Y, given by $v_n(x) = 2^{-n}x + 1/2$ (the space of functions it acts on will be specified later, let us say for now that if $u \in L^1$ then $\mathcal{L}u$ is well defined and also belongs to L^1). Let us also define, for $t \in \mathbb{R}$, an operator $\mathcal{L}_t$ by $\mathcal{L}_t u = \mathcal{L}(e^{ita_\alpha} u)$. The interest of this definition is that

$$\int e^{itS_n^Y a_\alpha} \,\mathrm{dLeb}_Y = \int \mathcal{L}_t^n(1) \,\mathrm{dLeb}_Y . \tag{6}$$

Hence, understanding the iterates of $\mathcal{L}_t$ will give a good description of the characteristic function of $S_n^Y a_\alpha$, and enable us to prove limit theorems for these Birkhoff sums.

The strategy is to find a good functional space $\mathcal{B}$ (contained in L^1) on which the operator $\mathcal{L}$ has a simple eigenvalue at 1 and a spectral gap, and such that $\mathcal{L}_t$ is a small perturbation of $\mathcal{L}$: in this way, $\mathcal{L}_t^n(1)$ will essentially be described by $\lambda(t)^n$ where $\lambda(t)$ is the eigenvalue close to 1 of $\mathcal{L}_t$.

For $0 < \gamma \leq 1$, let $\mathcal{B}_\gamma$ be the Banach space of functions on Y which are γ-Hölder continuous. Its norm is given by

$$\|u\|_{\mathcal{B}_\gamma} = \sup |u| + \inf\{C \mid \forall x, y,\ |u(x) - u(y)| \leq C|x-y|^\gamma\}. \tag{7}$$

Let us choose once and for all some $\gamma \in (0,1]$ with $1/(2\alpha) < \gamma < 1/\alpha$ if $\alpha > 1/2$, and $\gamma = 1$ if $\alpha = 1/2$. We will work on $\mathcal{B} = \mathcal{B}_\gamma$.

Lemma 3.1. *We have* $\|\mathcal{L}^n u\|_{\mathcal{B}} \leq 2^{1-\gamma n} \|u\|_{\mathcal{B}} + \|u\|_{L^1}$.

Proof. Let $u \in \mathcal{B}$. Denote by $\mathrm{Höl}(u)$ its best Hölder constant.

We have $\mathcal{L}^n u(x) = \sum_{i_1,\dots,i_n} 2^{-(i_1+\dots+i_n)} u(v_{i_1}\cdots v_{i_n} x)$. Since $|v_{i_1}\cdots v_{i_n}x - v_{i_1}\cdots v_{i_n}y| \leq 2^{-n}|x-y|$ for any choice of $i_1,\dots,i_n$, we get

$$|\mathcal{L}^n u(x) - \mathcal{L}^n u(y)| \leq \sum_{i_1,\dots,i_n} 2^{-(i_1+\dots+i_n)} \operatorname{Höl}(u)(2^{-n}|x-y|)^\gamma = 2^{-\gamma n}\operatorname{Höl}(u)|x-y|^\gamma. \tag{8}$$

Hence, $\operatorname{Höl}(\mathcal{L}^n u) \leq 2^{-\gamma n}\operatorname{Höl}(u)$.

For any x,y, we obtain $|\mathcal{L}^n u(x)| \leq |\mathcal{L}^n u(y)| + \operatorname{Höl}(\mathcal{L}^n u)$. Integrating over y and using the fact that $\int \mathcal{L}^n|u| = \int |u|$, we obtain $|\mathcal{L}^n u(x)| \leq \|u\|_{L^1} + \operatorname{Höl}(\mathcal{L}^n u)$. This yields the desired control on the supremum of $|\mathcal{L}^n u|$, and concludes the proof of the lemma. □

By an abstract functional-analytic argument that we shall explain later, this lemma implies that $\mathcal{L}$ acting on $\mathcal{B}$ has a finite number of eigenvalues of modulus 1. To understand the corresponding eigenfunctions, the next lemma will prove useful.

Lemma 3.2. *For any $u \in L^1$, the function $\mathcal{L}^n u$ converges in L^1 to $\int u \,\mathrm{dLeb}_Y$ when $n\to\infty$.*

Proof. If u is constant on each interval J_n, then $\mathcal{L}u$ is constant equal to $\int u$. More generally, if u is constant on each interval of the partition $\bigvee_{i=0}^{k-1} T_Y^{-i}\{J_n\}$, then $\mathcal{L}^k u$ is constant equal to $\int u$. Hence, the statement of the lemma holds for these functions. Since they are dense in L^1, the lemma follows. □

The proof of this lemma is quite specific to the situation of T_Y, whose derivative is constant on each branch. This implies that, for any function u constant on each interval of the partition $\bigvee_{i=0}^{k-1} T_Y^{-i}\{J_n\}$, and any function v, then u and $v\circ T_Y^k$ are independent. The proof is a reformulation of this fact. The lemma would still hold for more general piecewise expanding maps, but the proof would require additional (very classical) dynamical arguments.

We will now study the perturbations $\mathcal{L}_t$ of $\mathcal{L}$.

Lemma 3.3. *There exists $C>0$ such that, for any $t\in\mathbb{R}$ and any $u\in\mathcal{B}$,*

$$\|(\mathcal{L}_t - \mathcal{L})u\|_{\mathcal{B}} \leq C|t|^\gamma \|u\|_{\mathcal{B}}. \tag{9}$$

Proof. Let us first estimate the Hölder constant of $(\mathcal{L}_t-\mathcal{L})u$. For $x,y\in Y$, we write

$$\begin{aligned}
&(\mathcal{L}_t-\mathcal{L})u(x) - (\mathcal{L}_t-\mathcal{L})u(y)\\
&= \sum 2^{-n}(e^{ita_\alpha(v_n x)}-1)u(v_n x) - \sum 2^{-n}(e^{ita_\alpha(v_n y)}-1)u(v_n y)\\
&= \sum 2^{-n}(e^{ita_\alpha(v_n y)}-1)(u(v_n y)-u(v_n x)) + \sum 2^{-n}(e^{ita_\alpha(v_n x)} - e^{ita_\alpha(v_n y)})u(v_n x).
\end{aligned} \tag{10}$$

Let us bound the first term. Let C be such that $|e^{ia}-1|\leq C|a|^\gamma$ for any $a\in\mathbb{R}$. Then

$$|e^{ita_\alpha(v_n y)}-1| \leq C|ta_\alpha(v_n y)|^\gamma \leq C|t|^\gamma 2^{\alpha\gamma n}. \tag{11}$$

Since $|u(v_n x)-u(v_n y)|\leq \operatorname{Höl}(u)|x-y|^\gamma$, we get

$$\left|\sum 2^{-n}(e^{ita_\alpha(v_n y)}-1)(u(v_n y)-u(v_n x))\right| \leq C|t|^\gamma \sum 2^{-n}2^{\alpha\gamma n}|x-y|^\gamma \operatorname{Höl}(u). \tag{12}$$

Since $\alpha\gamma<1$ by construction, the sum is finite.

We now turn to the second term of (10). By (2) and the formula for v_n,

$$|a_\alpha(v_n x) - a_\alpha(v_n y)| \leq C\left|\frac{1}{(2v_n(x)-1)^\alpha} - \frac{1}{(2v_n(y)-1)^\alpha}\right| \leq C2^{\alpha n}|x-y|. \tag{13}$$

Hence,

$$\begin{aligned}\left|\sum 2^{-n}(e^{ita_\alpha(v_n x)} - e^{ita_\alpha(v_n y)})u(v_n x)\right| &\leq \sum 2^{-n}|e^{it(a_\alpha(v_n x)-a_\alpha(v_n y))} - 1| \sup|u| \\ &\leq \sum 2^{-n}C|t|^\gamma |a_\alpha(v_n x) - a_\alpha(v_n y)|^\gamma \sup|u| \\ &\leq C\left\|u\right\|_{\mathcal{B}} |t|^\gamma \sum 2^{-n}2^{\gamma\alpha n}|x-y|^\gamma.\end{aligned}$$

Since $\alpha\gamma < 1$, the sum is finite. We have proved that

$$\text{Höl}((\mathcal{L}_t - \mathcal{L})u) \leq C|t|^\gamma \left\|u\right\|_{\mathcal{B}}. \tag{14}$$

As in the end of the proof of Lemma 3.1, we have $\sup|(\mathcal{L}_t - \mathcal{L})u| \leq \left\|(\mathcal{L}_t - \mathcal{L})u\right\|_{L^1} + \text{Höl}((\mathcal{L}_t - \mathcal{L})u)$. Moreover,

$$\left\|(\mathcal{L}_t - \mathcal{L})u\right\|_{L^1} = \left\|\mathcal{L}((e^{ita_\alpha} - 1)u)\right\|_{L^1} \leq \left\|(e^{ita_\alpha} - 1)u\right\|_{L^1} \leq \left\|e^{ita_\alpha} - 1\right\|_{L^1} \sup|u|. \tag{15}$$

The conclusion of the proof therefore follow from the inequality

$$\int |e^{ita_\alpha} - 1| \leq C\int |ta_\alpha|^\gamma \leq C|t|^\gamma \int_{x=1/2}^{1} \frac{\mathrm{d}x}{(2x-1)^{\alpha\gamma}}, \tag{16}$$

the last integral being finite since $\alpha\gamma < 1$. $\square$

Putting together the previous lemmas gives the following spectral description of the operators $\mathcal{L}_t$.

Theorem 3.4. *There exist $C > 0$, $\sigma < 1$ and $\epsilon > 0$ such that, for $|t| \leq \epsilon$, there exists a decomposition $\mathcal{B} = E_t \oplus F_t$, where E_t is one-dimensional and F_t is a closed hyperplane in $\mathcal{B}$, with the following properties:*

1. *The subspaces E_t and F_t are invariant under $\mathcal{L}_t$.*
2. *The restriction of $\mathcal{L}_t$ to E_t is the multiplication by a number $\lambda(t)$, with $|\lambda(t) - 1| \leq C|t|^\gamma$.*
3. *The restriction of $\mathcal{L}_t$ to F_t satisfies $\left\|(\mathcal{L}_t)^n_{|F_t}\right\| \leq C\sigma^n$.*
4. *The projection P_t on E_t with kernel F_t satisfies $\|P_t - P_0\| \leq C|t|^\gamma$.*

Proof. The inclusion of L^1 into $\mathcal{B}$ is compact. Moreover, the operator $\mathcal{L}$ satisfies a Doeblin-Fortet inequality $\|\mathcal{L}^n u\| \leq C2^{-\gamma n}\|u\| + C\|u\|_{L^1}$, by Lemma 3.1. Heuristically, this inequality can be interpreted as follows: $\mathcal{L}$ should be the sum of an operator of spectral radius at most $2^{-\gamma}$ and of a compact operator. The first operator would have no spectrum outside of the disk of radius $2^{-\gamma}$, while the second operator would only have eigenvalues of finite multiplicity there. Hopefully, the operator $\mathcal{L}$ should share the same kind of property.

This intuition is made precise by Hennion's theorem [Hen93]: using only the Doeblin-Fortet inequality and compactness, it shows that $\mathcal{L}$ has finitely many eigenvalues of modulus 1, of finite multiplicity, and that the rest of its spectrum is contained in a disk of radius < 1.

If u is an eigenfunction for an eigenvalue λ of modulus 1, then $\mathcal{L}^n u = \lambda^n u$. However, $\mathcal{L}^n u$ converges in L^1 to $\int u$ by Lemma 3.2, hence u is constant. This shows that 1 is the only eigenvalue of modulus 1 of $\mathcal{L}$, and that the corresponding eigenspace is one-dimensional. This proves the statement of the theorem for $t = 0$.

For t small, the operator $\mathcal{L}_t$ is close to $\mathcal{L}$, by Lemma 3.3. Hence, their spectra are close, as well as the corresponding spectral projections, by classical perturbation theory (it easily follows from the integral expression of the spectral projections, see [Kat66]). This yields the statement of the theorem for small t. $\square$

Proposition 3.5. *The eigenvalue $\lambda(t)$ given in the previous theorem satisfies*

$$\lambda(t) = \phi_\alpha\left(\frac{2^\alpha}{2^\alpha - 1}t\right) + O(|t|^{2\gamma}), \tag{17}$$

where ϕ_α is the characteristic function of f_α, studied in Proposition 1.1.

Proof. Let $\xi_t = P_t(1)/\int P_t(1)\,\mathrm{dLeb}_Y$: it satisfies $\mathcal{L}_t\xi_t = \lambda(t)\xi_t$, moreover $\int \xi_t = 1$ and $\|\xi_t - \xi_0\| \leq C|t|^\gamma$. Therefore,

$$\lambda(t) = \int \lambda(t)\xi_t = \int \mathcal{L}_t\xi_t = \int (\mathcal{L}_t - \mathcal{L}_0)(\xi_t - \xi_0) + \int \mathcal{L}_t\xi_0. \tag{18}$$

The first term is $O(|t|^{2\gamma})$ since $\|\mathcal{L}_t - \mathcal{L}_0\| = O(|t|^\gamma)$ and $\|\xi_t - \xi_0\| = O(|t|^\gamma)$. Since $\xi_0 = 1$, the second term is equal to $\int e^{ita_\alpha}$. Moreover, $a_\alpha(x) = \frac{1}{(2^\alpha-1)(x-1/2)^\alpha}$, hence

$$\begin{aligned}\int_{1/2}^1 e^{ita_\alpha(x)}\,\mathrm{dLeb}_Y(x) &= 2\int_0^{1/2} \exp(itx^{-\alpha}/(2^\alpha - 1))\,\mathrm{d}x \\ &= \int_0^1 \exp(it(y/2)^{-\alpha}/(2^\alpha-1))\,\mathrm{d}y = \phi_\alpha(t2^\alpha/(2^\alpha - 1)).\end{aligned}$$

□

Corollary 3.6. *Let W_α and $A_n = A_n(\alpha)$, $B_n = B_n(\alpha)$ be defined in Corollary 1.2. Then $((1-2^{-\alpha})S_n^Y a_\alpha - A_n)/B_n \to W_\alpha$.*

Proof. For $A_n \in \mathbb{R}$ and $B_n \to \infty$, we have

$$\begin{aligned}\int e^{it(S_n^Y a_\alpha - A_n)/B_n} &= e^{-itA_n/B_n}\int \mathcal{L}_{t/B_n}(1) = e^{-itA_n/B_n}\left[\lambda(t/B_n)^n \int P_{t/B_n}(1) + O(\sigma^n)\right] \\ &= e^{-itA_n/B_n}\lambda(t/B_n)^n(1+o(1)) + O(\sigma^n).\end{aligned}$$

If A_n and B_n are such that $e^{-itA_n/B_n}\lambda(t/B_n)^n$ converges to the characteristic function of a random variable W, then the convergence of $(S_n^Y a_\alpha - A_n)/B_n$ to W follows.

Due to the previous proposition and the inequality $2\gamma > 1/\alpha$ (or $2\gamma = 1/\alpha$ if $\alpha = 1/2$), we get that $\lambda((1-2^{-\alpha})t)$ satisfies the same asymptotics as $\phi_\alpha(t)$ described in Proposition 1.1. This is enough to apply the proof of Corollary 1.2 in our setting, and obtain the statement of the corollary. □

We now turn to the study of the Birkhoff sums of b_α. Since this function is bounded, these sums will satisfy the central limit theorem. However, we will only prove the following weaker estimate, which is sufficient for our purposes.

Lemma 3.7. *For $\alpha \geq 1/2$, the sequence $(S_n^Y b_\alpha - n\int b_\alpha\,\mathrm{dLeb}_Y)/\sqrt{n\ln n}$ converges to 0 in probability.*

Proof. Let us define a new perturbed transfer operator $\tilde{\mathcal{L}}_t$ acting on $\mathcal{B}_1$ the space of Lipschitz functions, by $\tilde{\mathcal{L}}_t u = \mathcal{L}(e^{itb_\alpha}u)$. It satisfies the same kind of properties as the operators defined using a_α. In particular, we can check that $\left\|\tilde{\mathcal{L}}_t - \mathcal{L}\right\| \leq C|t|$ (the estimate is better for $\tilde{\mathcal{L}}_t$ than $\mathcal{L}_t$ because we can work on $\mathcal{B}_1$ instead of $\mathcal{B}_\gamma$, the function b_α being smooth enough). we can check as in the proof of Proposition 3.5 that $\tilde{\lambda}(t) = \int e^{itb_\alpha} + O(t^2)$. Since b_α is bounded, this gives $\tilde{\lambda}(t) = 1 + itE(b_\alpha) + O(t^2)$. Then

$$E(e^{it(S_n^Y b_\alpha - nE(b_\alpha))/\sqrt{n\ln n}}) = e^{-itnE(b_\alpha)/\sqrt{n\ln n}}\tilde{\lambda}(t/\sqrt{n\ln n})^n(1+o(1)) + O(\sigma^n) \tag{19}$$

converges to 1, as desired. □

To proceed, it is necessary to compute $E(b_\alpha)$.

Lemma 3.8. *We have* $\int b_\alpha \,\mathrm{dLeb}_Y = \frac{1}{1-\alpha}\frac{2^\alpha-2}{2^\alpha-1}$ *if* $\alpha \neq 1$, *and* $\int b_1 \,\mathrm{dLeb}_Y = -2\ln 2$.

Proof. On $J_n = [1/2^{n+1}, 1/2^n]$, we have $b_\alpha(x) = x^{-\alpha} - \frac{2^{-(n-1)\alpha}}{2^\alpha-1}(x-1/2)^{-\alpha}$. Hence,

$$\int b_\alpha \,\mathrm{dLeb}_Y = 2\left[\int_{1/2}^1 x^{-\alpha}\,\mathrm{d}x - \sum_{n=1}^\infty \frac{2^{-(n-1)\alpha}}{2^\alpha-1}\int_{1/2^{n+1}}^{1/2^n} x^{-\alpha}\,\mathrm{d}x\right]. \tag{20}$$

If $\alpha \neq 1$, we obtain

$$\begin{aligned}\int b_\alpha \,\mathrm{dLeb}_Y &= 2\left[\frac{1-(1/2)^{1-\alpha}}{1-\alpha} - \sum_{n=1}^\infty \frac{2^{-(n-1)\alpha}}{2^\alpha-1}\frac{2^{-n(1-\alpha)}-2^{-(n+1)(1-\alpha)}}{1-\alpha}\right]\\ &= \frac{2}{1-\alpha}\left[1-2^{\alpha-1} - \frac{2^\alpha}{2^\alpha-1}(1-2^{\alpha-1})\right] = \frac{2}{1-\alpha}\frac{2^{\alpha-1}-1}{2^\alpha-1}.\end{aligned}$$

If $\alpha = 1$, we can compute in the same way (or use the computation for $\alpha \neq 1$ and let α tend to 1). □

Theorem 3.9. *The Birkhoff sums of g_α satisfy the following limit theorem. The limiting distribution W_α has been defined in Corollary 1.2.*

- *If* $\alpha > 1$, *then* $(1-2^{-\alpha})S_n^Y g_\alpha / n^\alpha \to W_\alpha$.
- *If* $1/2 < \alpha < 1$, *then* $(1-2^{-\alpha})(S_n^Y g_\alpha - 2n/(1-\alpha))/n^\alpha \to W_\alpha$.
- *If* $\alpha = 1$, *then* $(2^{-1}S_n^Y g_1 - n\ln(n/2))/n \to W_1$.
- *If* $\alpha = 1/2$, *then* $(1-2^{-\alpha})(S_n^Y g_\alpha - 2n/(1-\alpha))/\sqrt{n\ln n} \to W_\alpha$.

Proof. We write $S_n^Y g_\alpha = S_n^Y a_\alpha + S_n^Y(b_\alpha - E(b_\alpha)) + nE(b_\alpha)$. The contribution of $S_n^Y(b_\alpha - E(b_\alpha))$ is negligible by Lemma 3.7.

Assume first $\alpha > 1$. Then the contribution of $nE(b_\alpha)/n^\alpha$ also tends to 0, hence $S_n^Y g_\alpha$ satisfies the same limit theorem as $S_n^Y a_\alpha$. The conclusion follows by Corollary 3.6.

Assume now $1/2 \leq \alpha < 1$. Corollary 3.6 gives the convergence

$$(1-2^{-\alpha})\left(S_n^Y a_\alpha - \frac{n}{(1-\alpha)(1-2^{-\alpha})}\right)/B_n \to W_\alpha, \tag{21}$$

where $B_n = n^\alpha$ (resp. $\sqrt{n\ln n}$) if $\alpha > 1/2$ (resp. $\alpha = 1/2$). Therefore, the theorem follows if

$$\frac{1}{(1-\alpha)(1-2^{-\alpha})} + E(b_\alpha) = \frac{2}{1-\alpha}. \tag{22}$$

This equality follows from Lemma 3.8.

Finally, for $\alpha = 1$, the statement is a consequence of the limit behavior of $S_n^Y a_1$ given by Corollary 3.6 and the integral of b_1 given in Lemma 3.8. □

In the $\alpha < 1$ case, the centering factor $2/(1-\alpha)$ is the integral of g_α, as should be expected from Birkhoff theorem.

4. Inducing limit theorems

In this section, we deduce Theorem 2.1 from Theorem 3.9. We will rely on a general argument ensuring that a limit theorem for an induced map implies a limit theorem for the original map (see [ADU93, Zwe03, MT04, Gou07]).

4.1. **Limit theorems do not depend on the reference measure.** The following theorem has been proved by Eagleson [Eag76] and popularized by Zweimüller (in much more general contexts, [Zwe07]).

Theorem 4.1. *Let $T : X \to X$ be an ergodic probability preserving map. Let $f : X \to \mathbb{R}$ be measurable, let $A_n \in \mathbb{R}$, let B_n tend to ∞ and let m' be an absolutely continuous probability measure. Then $(S_n f - A_n)/B_n$ converges in distribution to a random variable W with respect to m if and only if it satisfies the same convergence with respect to m'.*

Proof. For the proof, let us write $M(n, g, \phi) = \int g((S_n f - A_n)/B_n)\phi \,\mathrm{d}m$, where g is a bounded Lipschitz function and ϕ is an integrable function.

We claim that

$$M(n, g, \phi) - M(n, g, \phi \circ T) \to 0 \text{ when } n \to \infty. \tag{23}$$

Let us first prove this assuming that ϕ is bounded. Then

$$\begin{aligned}
|M(n, g, \phi) - M(n, g, \phi \circ T)| &= \left| \int \left(g\left(\frac{S_n f(x) - A_n}{B_n}\right) - g\left(\frac{S_n f(Tx) - A_n}{B_n}\right)\right) \phi(Tx) \,\mathrm{d}m(x) \right| \\
&\leq C \int \min(1, |S_n f(x) - S_n f(Tx)|/B_n) \,\mathrm{d}m \\
&= C \int \min(1, |f(x) - f(T^n x)|/B_n) \,\mathrm{d}m \\
&\leq C \int (\min(1, |f|/B_n) + \min(1, |f| \circ T^n/B_n)) \,\mathrm{d}m \\
&\leq C \int \min(1, |f|/B_n) \,\mathrm{d}m.
\end{aligned}$$

This quantity converges to 0 when $n \to \infty$, since $B_n \to \infty$. This proves (23) for bounded ϕ.

In general, we have

$$|M(n, g, \psi)| \leq \|g\|_\infty \|\psi\|_{L^1}. \tag{24}$$

Hence, the general case of (23) follows by writing $\phi = \phi_1 + \phi_2$ with ϕ_1 bounded and $\|\phi_2\|_{L^1} \leq \epsilon$: we obtain $\limsup |M(n, g, \phi) - M(n, g, \phi \circ T)| \leq 2\epsilon$.

Assume now that $(S_n f - A_n)/B_n$ converges in distribution with respect to m' towards W. Write $\mathrm{d}m' = \phi \mathrm{d}m$ with ϕ integrable (and of integral 1). Let g be a bounded Lipschitz function. Then $M(n, g, \phi) \to E(g(W))$, hence $M(n, g, \phi \circ T^k) \to E(g(W))$ by (23). Hence, $M(n, g, S_k\phi/k) \to E(g(W))$. Let $\epsilon > 0$, and choose k large enough so that $\|S_k\phi/k - 1\|_{L^1} \leq \epsilon$. Then

$$\begin{aligned}
&\limsup |M(n, g, 1) - E(g(W))| \\
&\qquad \leq \limsup |M(n, g, 1) - M(n, g, S_k\phi/k)| + \limsup |M(n, g, S_k\phi/k) - E(g(W))|.
\end{aligned}$$

The first term is at most ϵ by (24), while the second one is 0. Hence, $M(n, g, 1)$ converges to $E(g(W))$. This proves the convergence of $(S_n f - A_n)/B_n$ to W with respect to m.

Conversely, if $(S_n f - A_n)/B_n$ converges to W with respect to m, the convergence with respect to m' follows in the same way:

$$\begin{aligned}
\limsup |M(n, g, \phi) - E(g(W))| &\leq \limsup |M(n, g, \phi) - M(n, g, S_k\phi/k)| \\
&\quad + \limsup |M(n, g, S_k\phi/k) - M(n, g, 1)| + \limsup |M(n, g, 1) - E(g(W))|.
\end{aligned}$$

The third term tends to 0 by assumption, the first one tends to 0 by (23), and the second one is at most $C\left\|S_k\phi/k-1\right\|_{L^1}$, which can be made arbitrarily small by choosing k large enough. □

4.2. **Limit theorem do not depend on random indices.**

Definition 4.2. *A continuous function $L:\mathbb{R}_+^*\to\mathbb{R}_+^*$ is slowly varying if, for any $\lambda>0$, $L(\lambda x)/L(x)\to 1$ when $x\to\infty$. A function f is regularly varying with index d if it can be written as $x^dL(x)$ where L is slowly varying. A sequence a_n is regularly varying with index d is there exists a function f, regularly varying with index d, such that $a_n=f(n)$.*

Theorem 4.3. *Let $T:X\to X$ be a probability preserving map, and let $\alpha(n)$ and B_n be two sequences of integers which are regularly varying with positive indexes. Let also $A_n\in\mathbb{R}$. Let $f:X\to\mathbb{R}$ measurable be such that $(S_nf-A_n)/B_n$ converges in distribution to a random variable W. Let also $t_1,t_2,\dots$ be a sequence of integer valued measurable functions on X, and let $c>0$. Assume that either*

$$\frac{t_n-cn}{\alpha(n)} \text{ tends in probability to } 0 \text{ and } \max_{0\le k\le\alpha(n)}|S_kf|/B_n \text{ is tight} \tag{25}$$

or

$$\frac{t_n-cn}{\alpha(n)} \text{ is tight and } \max_{0\le k\le\alpha(n)}|S_kf|/B_n \text{ tends in probability to } 0. \tag{26}$$

Then the sequence $(S_{t_n}f-A_{\lfloor cn\rfloor})/B_{\lfloor cn\rfloor}$ converges in distribution to W.

Proof. We will show that, under (25) or (26), there exists a sequence $\beta(n)$ of integers such that

$$|t_n-cn|/\beta(n) \text{ and } \max_{0\le k\le 2\beta(n)}|S_kf|/B_n \text{ tend in probability to } 0. \tag{27}$$

Let us show how it implies the theorem. It is sufficient to prove that

$$m\left\{x \mid \left|\frac{S_{t_n(x)}f-S_{\lfloor cn\rfloor}f}{B_{\lfloor cn\rfloor}}\right|\ge\epsilon\right\}\to 0. \tag{28}$$

Abusing notations, we will omit the integer parts. The measure of the set in the last equation is bounded by $m\{|t_n-cn|\ge\beta(n)\}+m\{\exists i\in[\gamma(n),\beta(n)],\ |S_{cn+i}f-S_{cn}f|\ge\epsilon B_{cn}\}$, where $\gamma(n)=-\min(cn,\beta(n))$. The measure of the first set tends to 0 by (27). If x belongs to the second set, then either $|S_{cn}f-S_{cn+\gamma(n)}f|\ge\epsilon B_{cn}/2$ or $|S_{cn+i}f-S_{cn+\gamma(n)}f|\ge\epsilon B_{cn}/2$. In both cases, $\max_{0\le k\le 2\beta(n)}|S_kf|(T^{cn+\gamma(n)}x)\ge\epsilon B_{cn}/2$. Since B_n is regularly varying, there exists C such that $B_{cn}/2\ge CB_n$. Hence, the measure of the second set is bounded by $m\{\max_{0\le k\le 2\beta(n)}|S_kf|\ge C\epsilon B_n\}$, which also tends to 0 by (27).

To conclude the proof, it is therefore sufficient to construct $\beta(n)$ satisfying (27).

Lemma 4.4. *Let Y_n be a sequence of real random variables tending in probability to 0. There exists a non-decreasing sequence $A(n)\to\infty$ such that $A(n)Y_n$ still tends in probability to 0.*

Proof. For $k>0$, let $N(k)$ be such that, for $n\ge N(k)$, $P(|Y_n|>1/k^2)\le 1/k$. We can also assume that $N(k+1)>N(k)$. Define A by $A(n)=k$ when $N(k)\le n<N(k+1)$, this sequence tends to infinity. Consider $k\in\mathbb{N}$, and $n\ge N(k)$. Let $l\ge k$ be such that $N(l)\le n<N(l+1)$. Then

$$P(A(n)|Y_n|>1/k)\le P(A(n)|Y_n|>1/l)=P(|Y_n|>1/l^2)\le 1/l=1/A(n). \tag{29}$$

Hence, $P(A(n)|Y_n|>1/k)$ tends to 0 for any k. □

Lemma 4.5. *Let B_n be a regularly varying sequence with positive index, and let Y_n be a sequence of real random variables such that Y_n/B_n converges in probability to 0. Then there exists a non-decreasing sequence $\phi(n) = o(n)$ such that $Y_n/B_{\phi(n)}$ still converges in probability to 0. We can also ensure $\phi(n+1) \leq 2\phi(n)$ for any n, and $\phi(n) \to \infty$.*

Proof. Applying the previous lemma to Y_n/B_n, we obtain a non-decreasing sequence $A(n)$ tending to infinity such that $A(n)Y_n/B_n$ converges in probability to 0. Replacing $A(n)$ with $\min(A(n), \log n)$ if necessary, we can assume $A(n) = O(\log n)$. Write $B_n = n^d L(n)$ where L is slowly varying. Let $\phi(n)$ be the integer part of $n/A(n)^{1/(2d)}$, it satisfies the equation $\phi(n+1) \leq 2\phi(n)$ since A is non-decreasing, tends to infinity since $A(n) = O(\log n)$, and

$$\frac{Y_n}{B_{\phi(n)}} = \frac{A(n)Y_n}{B_n} \cdot \frac{B_n}{A(n)B_{\phi(n)}}. \tag{30}$$

The first factor tends to 0 in probability, while the second one is equivalent to

$$\frac{n^d L(n)}{A(n)(n^d/A(n)^{1/2})L(n/A(n)^{1/(2d)})}. \tag{31}$$

By Potter's bounds [BGT87, Theorem 1.5.6], for any $\epsilon > 0$, there exists $C > 0$ such that $L(n)/L(n/A(n)^{1/(2d)}) \leq CA(n)^{\epsilon}$. Taking $\epsilon = 1/4$, we obtain that the last equation is bounded by $C/A(n)^{1/4}$, and therefore tends to 0. □

We can now prove (27).

Assume first (25). Applying the last lemma to $Y_n = (t_n - cn)$, we obtain a non-decreasing sequence $\phi(n) = o(n)$ such that $(t_n - cn)/\alpha(\phi(n)) \to 0$. Let $\beta(n) = \alpha(\phi(n))/2$, then

$$\max_{0 \leq k \leq 2\beta(n)} |S_k f|/B_n = \frac{B_{\phi(n)}}{B_n} \max_{0 \leq k \leq \alpha(\phi(n))} |S_k f|/B_{\phi(n)}. \tag{32}$$

The factor $B_{\phi(n)}/B_n$ tends to 0 since $\phi(n) = o(n)$ and B_n is regularly varying with positive index. The second factor is tight by assumption. Hence, (32) tends in probability to 0, as desired.

Assume now (26). Applying the last lemma to $Y_n = \max_{0 \leq k \leq \alpha(n)} |S_k f|$, we obtain a non-decreasing sequence $\phi(n) = o(n)$ such that

$$\max_{0 \leq k \leq \alpha(n)} |S_k f|/B_{\phi(n)} \text{ tends in probability to } 0. \tag{33}$$

Let $\psi(n)$ be the smallest integer p such that $\phi(p) \geq n/2$. Then $\phi(\psi(n)) \geq n/2$. Moreover, $\phi(\psi(n) - 1) < n/2$, hence $\phi(\psi(n)) < n$ by the inequality $\phi(k+1) \leq 2\phi(k)$. Therefore, $B_n \geq C^{-1} B_{\phi(\psi(n))}$ since the sequence B_n, being regularly varying with positive index, is increasing up to a constant multiplicative factor.

Let $\beta(n) = \alpha(\psi(n))/2$, we get

$$\max_{0 \leq k \leq 2\beta(n)} |S_k f|/B_n = \max_{0 \leq k \leq \alpha(\psi(n))} |S_k f|/B_n \leq C \max_{0 \leq k \leq \alpha(\psi(n))} |S_k f|/B_{\phi(\psi(n))}. \tag{34}$$

This converges to 0 in probability by (33).

Since $\phi(\psi(n)) \geq n/2$ and $\phi(k) = o(k)$, we have $n = o(\psi(n))$. Since α is regularly varying with positive index, this yields $\alpha(n) = o(\beta(n))$. In particular, the tightness of $(t_n - cn)/\alpha(n)$ implies the convergence to 0 of $(t_n - cn)/\beta(n)$. We have proved (27) as desired. □

4.3. **Inducing limit theorems.**

Theorem 4.6. *Let $T : X \to X$ be an ergodic probability preserving map, let α_n and B_n be two sequences of integers which are regularly varying with positive indexes, let $A_n \in \mathbb{R}$, and let $Y \subset X$ be a subset with positive measure. We will denote by $m_Y := m_{|Y}/m(Y)$ the induced probability measure.*

Let $\phi : Y \to \mathbb{N}^*$ be the return time of T to Y, and $T_Y = T^\phi : Y \to Y$ the induced map. Consider a measurable function $f : X \to \mathbb{R}$ and define $f_Y : Y \to \mathbb{R}$ by $f_Y = \sum_{k=0}^{\phi-1} f \circ T^k$. Let us define the sequence of Birkhoff sums $S_n^Y f_Y = \sum_{k=0}^{n-1} f_Y \circ T_Y$. Assume that $(S_n^Y f_Y - A_n)/B_n$ converges in distribution (with respect to m_Y) to a random variable W. Additionally, assume either that

$$\frac{S_n^Y \phi - n/m(Y)}{\alpha(n)} \text{ tends in probability to } 0 \text{ and } \max_{0\leq k\leq \alpha(n)} |S_k^Y f_Y|/B_n \text{ is tight} \tag{35}$$

or

$$\frac{S_n^Y \phi - n/m(Y)}{\alpha(n)} \text{ is tight and } \max_{0\leq k\leq \alpha(n)} |S_k^Y f_Y|/B_n \text{ tends in probability to } 0. \tag{36}$$

Then $(S_n f - A_{\lfloor nm(Y)\rfloor})/B_{\lfloor nm(Y)\rfloor}$ converges in distribution (with respect to m) to W.

Proof. Going to the natural extension, we can without loss of generality assume that T is invertible. Abusing notations, we will write $B_{nm(Y)}$ instead of $B_{\lfloor nm(Y)\rfloor}$. We will prove that $(S_n f - A_{nm(Y)})/B_{nm(Y)}$ converges to W in distribution with respect to m_Y: this will imply the desired result by Theorem 4.1, since m_Y is absolutely continuous with respect to m.

For $x \in Y$ and $N \in \mathbb{N}$, let $n(x, N) = \mathrm{Card}\{1 \leq i < N, T^i x \in Y\}$ denote the number of visits of x to Y. By construction, it satisfies

$$n(x, N) \geq k \iff S_k^Y \phi(x) < N. \tag{37}$$

Define also a function H on X by $H(x) = \sum_{k=1}^{\psi(x)} f(T^{-k}x)$, where $\psi(x) = \inf\{n \geq 1 \mid T^{-n}x \in Y\}$. By construction, for $x \in Y$,

$$S_N f(x) = S_{n(x,N)}^Y f_Y(x) + H(T^N x). \tag{38}$$

Moreover, $H \circ T^N / B_{Nm(Y)}$ converges to 0 in distribution on X (since the measure is invariant and B_n tends to infinity), and therefore on Y. To prove the theorem, it is therefore sufficient to show that

$$\frac{S_{n(x,N)}^Y f_Y - A_{Nm(Y)}}{B_{Nm(Y)}} \to W. \tag{39}$$

This will follow from Theorem 4.3 if we can check its assumptions for $t_N(x) = n(x, N)$ and $c = m(Y)$ (the assumptions for the Birkhoff sums of f_Y are already part of the assumptions of Theorem 4.6).

Birkhoff's theorem ensures that $n(x, N) = Nm(Y) + o(N)$ for almost every x. Therefore, along any subsequence N_k for which $\alpha(N_k) \geq \delta N_k$ with $\delta > 0$, we get that $m\{|n(x, N_k) - N_k m(Y)|/\alpha(N_k)\}$ converges in probability to 0, and there is nothing left to prove. Thus, it is sufficient to consider only values of N along which $\alpha(N)/N \to 0$.

For any $a > 0$, we have by (37)

$$m\left\{\frac{n(x,N)-Nm(Y)}{\alpha(N)} \geq a\right\} = m\left\{S^Y_{Nm(Y)+\alpha(N)a}\phi < N\right\}$$
$$= m\left\{\frac{S^Y_{Nm(Y)+\alpha(N)a}\phi - (Nm(Y)+\alpha(N)a)/m(Y)}{\alpha(Nm(Y)+\alpha(N)a)} < -\frac{\alpha(N)}{\alpha(Nm(Y)+\alpha(N)a)}\frac{a}{m(Y)}\right\}.$$

Since we are considering values of N for which $\alpha(N) = o(N)$, we have $Nm(Y) + \alpha(N)a \leq 2Nm(Y)$ if N is large enough. Since α is regularly varying with positive index, this yields $\frac{\alpha(N)}{\alpha(Nm(Y)+\alpha(N)a)} \geq C > 0$. Hence,

$$m\left\{\frac{n(x,N)-Nm(Y)}{\alpha(N)} \geq a\right\} \leq m\left\{\frac{S^Y_{p(N)}\phi - p(N)/m(Y)}{\alpha(p(N))} < -Ca\right\} \tag{40}$$

for some integer $p(N)$ which tends to infinity with N.

Let us now study $m\left\{\frac{n(x,N)-Nm(Y)}{\alpha(N)} < -a\right\}$. Using again $\alpha(N) = o(N)$, we get $Nm(Y) - \alpha(N)a \geq Nm(Y)/2 > 0$ for large enough N. Hence,

$$m\left\{\frac{n(x,N)-Nm(Y)}{\alpha(N)} < -a\right\} = m\left\{S^Y_{Nm(Y)-\alpha(N)a}\phi \geq N\right\}$$
$$= m\left\{\frac{S^Y_{Nm(Y)-\alpha(N)a}\phi - (Nm(Y)-\alpha(N)a)/m(Y)}{\alpha(Nm(Y)-\alpha(N)a)} \geq \frac{\alpha(N)}{\alpha(Nm(Y)-\alpha(N)a)}\frac{a}{m(Y)}\right\}.$$

Since $\frac{\alpha(N)}{\alpha(Nm(Y)-\alpha(N)a)} \geq C > 0$, we obtain

$$m\left\{\frac{n(x,N)-Nm(Y)}{\alpha(N)} < -a\right\} \leq m\left\{\frac{S^Y_{q(N)}\phi - q(N)/m(Y)}{\alpha(q(N))} \geq Ca\right\}, \tag{41}$$

for some $q(N)$ tending to infinity with N.

The equations (40) and (41) together show that the tightness (resp. the convergence in probability to 0) of $(S^Y_n\phi - n/m(Y))/\alpha(n)$ implies the tightness (resp. the convergence in probability to 0) of $(n(x,N) - Nm(Y))/\alpha(N)$. We can therefore apply Theorem 4.3, to conclude the proof. □

Example 1. The setting of [MT04].

Assume that $S^Y_n\phi$ and $S^Y_n f_Y$ satisfy the central limit theorem (so that $B_n = \sqrt{n}$), and that f_Y is integrable with $\int f_Y = 0$. By Birkhoff's theorem, $S^Y_n f_Y = o(n)$ almost surely. Hence, $\max_{0\leq k\leq n}|S_k f| = o(n)$ almost surely. The assumption (36) is therefore satisfied for $\alpha(n) = B_n = \sqrt{n}$.

Example 2. The setting of [CG06]: convergence with tight maxima.

By Birkhoff's theorem, $(S^Y_n\phi - n/m(Y))/n$ converges in probability to 0. We obtain that, if $S^Y_n f_Y/B_n$ converges in distribution and $\max_{0\leq k\leq n}|S^Y_k f_Y|/B_n$ is tight, the assumption (35) is satisfied for $\alpha(n) = n$.

4.4. **Application to the doubling map.**

Proof of Theorem 2.1. Let $\alpha \geq 1/2$. Theorem 3.9 gives two explicit regularly varying sequences A'_n and B'_n such that $(S_n^Y g_\alpha - A'_n)/B'_n$ converges in distribution to W_α. We claim that $(S_n f_\alpha - A'_{n/2})/B'_{n/2}$ also converges in distribution to W_α – this is the desired conclusion.

To prove it, we will apply Theorem 4.6 (and more precisely (36)) with $\alpha(n) = \sqrt{n}$, to $Y = [1/2, 1]$. The return time function ϕ is square-integrable and locally constant, hence it satisfies the central limit theorem (in this specific case, the successive iterates $\phi \circ T_Y^k$ are independent, hence this is the usual central limit theorem for i.i.d. random variables). The tightness of $(S_n^Y \phi - 2n)/\sqrt{n}$ therefore holds.

We have to prove that $\max_{0 \leq k \leq \sqrt{n}} |S_k^Y g_\alpha|/B'_n$ tends in probability to 0. Since g_α is positive, the maximum is attained for $k = \sqrt{n}$. Moreover, the distributional convergence of Theorem 3.9 shows that

- For $\alpha > 1$, $S_k^Y g_\alpha / k^\alpha$ is tight.
- For $\alpha = 1$, $S_k^Y g_\alpha / (k \ln k)$ is tight.
- For $\alpha < 1$, $S_k^Y g_\alpha / k$ is tight.

In the three cases, this shows that $S_{\sqrt{n}}^Y g_\alpha / B'_n$ tends in distribution to 0. This concludes the proof. □

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IRMAR, Université de Rennes 1, 35042 Rennes, France

E-mail address: sebastien.gouezel@univ-rennes1.fr