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Asymptotic combinatorics of Artin–Tits monoids and of some other monoids

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We introduce methods to study the combinatorics of the normal form of large random elements in Artin–Tits monoids. These methods also apply in an axiomatic framework that encompasses other monoids such as dual braid monoids.

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1. Introduction

Braid monoids are finitely presented monoids with a rich combinatorics, mainly based on the existence of a normal form for their elements. Consider the braid monoid B_n^+ on n strands: it has $n-1$ *Artin generators* which represent the elementary twists of two neighbor strands. The *Artin length* of an element $x \in B_n^+$ is the number m such that x can be written as a product of m Artin generators. The decomposition is not unique, yet m is well defined. The Garside normal form, also called the greedy normal form, arises from the following fact: there exists a finite subset $\mathcal{S}$ of B_n^+, that not only generates B_n^+, but also such that any $x \in B_n^+$ writes uniquely as a product $x_1 \cdot \ldots \cdot x_k$ of elements of $\mathcal{S}$, provided that the sequence $(x_1, \ldots, x_k)$ satisfies a condition of the form $x_1 \to x_2 \to \cdots \to x_k$, where $x \to y$ is some binary relation on $\mathcal{S}$.

Fix an integer j and pick an element x in B_n^+ at random among elements of Artin length m, with m large. Then consider the j first elements $(x_1, \ldots, x_j)$ in the normal form of x. This is a complex random object: the sequence $(x_1, \ldots, x_j)$ has no Markovian structure in general, and the law of $(x_1, \ldots, x_j)$ depends on m. However, some regularity appears when considering the limit with $m \to \infty$. Indeed, the limit law of the sequence $(x_1, \ldots, x_j)$ is *Markovian*, which already brings qualitative information on the typical behavior of "large random elements" in a braid monoid. First motivated by this kind of question, we have introduced in a previous work [1] the notion of uniform measure at infinity for braid monoids. The marginals of the uniform measure give back the limit form of the law of $(x_1, \ldots, x_j)$ when $m \to \infty$, hence yielding an appropriate notion of uniform measure for "infinite braid elements". The notion of uniform measure at infinity allowed us to settle open questions regarding the behavior of the normal form of large braids, asked by Gebhardt and Tawn in [2].

The following questions were left open in [1]: how much of these techniques extend to more general monoids? Furthermore, on top of the qualitative information brought by the Markovian properties of the uniform measure at infinity, can we also obtain asymptotics on some combinatorial statistics? Here is a typical statistics for which precise asymptotics might be expected. Beside the Artin length of elements in B_n^+, the *height* of $x \in B_n^+$ is the length k of the normal form $(x_1, \ldots, x_k)$ of x. What is the typical behavior of the ratio m/k when k is the height of a random element in B_n^+ of Artin length m large? Assume that this ratio is close to some constant γ. Is there a limit law for the quantity $\sqrt{k}(m/k-\gamma)$ when $m \to \infty$? Hence, we were left with the two following natural extensions to carry over: 1) extension to other monoids than braid monoids; and 2) asymptotics for combinatorial statistics.

A natural candidate for the generalization of braid monoids is the class of Artin–Tits monoids. This class encompasses both braid monoids and trace monoids—the later are also called heap monoids [3] and free partially commutative monoids [4] in the literature. Contrasting with braid monoids, a trace monoid is not a lattice for the left divisibility order; in particular, it has no Garside element. Actually, trace monoids and braid monoids can be seen as two typical representatives of Artin–Tits monoids of type FC, which is a subclass of Artin–Tits monoids already large enough to provide a good view on Artin–Tits monoids.

In this paper, we develop the notion of uniform measure at infinity for Artin–Tits monoids. More generally, we consider the class of *multiplicative measures at infinity* for an Artin–Tits monoid, among which is the uniform measure at infinity. We first define the *boundary at infinity* $\partial\mathbf{A}$ of an Artin–Tits monoid $\mathbf{A}$ as the topological space of "$\mathbf{A}$-infinite elements". It has the property that $\mathbf{A} \cup \partial\mathbf{A}$ is a compactification of $\mathbf{A}$. Multiplicative measures at infinity are probability measures on $\partial\mathbf{A}$ with a purely algebraic definition. The combinatorics of the monoid is reflected within these measures. For instance, the explicit expression for the uniform measure at infinity involves the root of smallest modulus of the Möbius polynomial of the monoid $\mathbf{A}$. The Möbius polynomial encodes some information on the combinatorics of the monoid. Its root of smallest modulus is the inverse of the growth rate of the monoid. By analogy, multiplicative measures at infinity play in an Artin–Tits monoid a role which corresponds to the role of the standard Bernoulli measures in a free monoid, as illustrated below:

$$\begin{array}{ccccc}
\begin{array}{c}\Sigma^*\text{: monoid}\\ \text{of }\Sigma\text{-words}\end{array} & \longrightarrow & \begin{array}{c}\Sigma^\infty\text{: space of}\\ \text{infinite }\Sigma\text{-sequences}\end{array} & \longrightarrow & \begin{array}{c}m\text{: Bernoulli}\\ \text{measure on }\Sigma^\infty\end{array}\\[2ex]
\begin{array}{c}\mathbf{A}\text{: Artin–Tits}\\ \text{monoid}\end{array} & \longrightarrow & \begin{array}{c}\partial\mathbf{A}\text{: boundary}\\ \text{at infinity of }\mathbf{A}\end{array} & \longrightarrow & \begin{array}{c}m\text{: multiplicative}\\ \text{measure on }\partial\mathbf{A}\end{array}
\end{array}$$

The uniform measure at infinity describes the qualitative behavior of a large random element of the monoid. As for braid monoids, the first j elements in the normal form of a random element $x \in \mathbf{A}$ of large size converge in law toward a Markov chain, for which the initial distribution and the transition matrix are explicitly described. This Markov chain corresponds to the marginal of the uniform measure at infinity. Multiplicative measures play an analogous role when large elements, instead of being picked uniformly at random, are chosen at random with multiplicative weights associated to the elementary generators of the monoid.

A second device that we introduce is the conditioned weighted graph (CWG) associated with an Artin–Tits monoid. This is essentially a non-negative matrix encoding the combinatorics of the monoid, together with weights attributed to the generators of the monoid. CWG are reminiscent of several tools and techniques found elsewhere in the literature and can almost be considered as folklore; we show in particular the relationship with the classical notion of survival process. Our contribution consists in applying spectral methods to derive asymptotics for CWG, including a concentration result with a convergence in law and a Central Limit Theorem.

We have thus two devices for studying the asymptotic combinatorics of Artin–Tits monoids. On the one hand, the uniform measure at infinity has a purely algebraic definition, and naturally encodes some information on the combinatorics of the monoid. On the other hand, the CWG associated with the monoid entirely encodes the combinatorics of the monoid and analytical methods can be used to obtain information on its asymptotic. Our program consists in using the analytical results for CWG in order to 1) derive asymptotic results for large random elements in the monoid; and 2) derive additional information regarding multiplicative measures at infinity. In particular, the set of finite probabilistic parameters that entirely describe multiplicative measures at infinity is shown to be homeomorphic to a standard simplex, and by this way we prove the existence of the uniform measure at infinity in all cases, which is a non-trivial task.

The paper brings original contributions in the area of pure combinatorics of Artin–Tits monoids. In particular, the following results are of interest *per se*: 1) If an Artin–Tits monoid is irreducible, its graph of proper simple elements is strongly connected—this graph contains as vertices all the simple elements of the monoid, excepted the unit element and the Garside element if it exists. 2) We introduce a generalized Möbius transform and we give an explicit form of its inverse for general Artin–Tits monoids. 3) We give a simple characterization of monoids of type FC among Artin–Tits monoids.

Finally, we observe that our methods also apply outside the framework of Artin–Tits monoids. We isolate a working framework where the chain of arguments that we use can be repeated *mutatis mutandis*. We illustrate this working framework by giving examples of monoids fitting into it, yet living outside the class of Artin–Tits monoids. Of particular interest is the class of dual braid monoids and their generalizations for Artin–Tits monoids of spherical type. We show that they fit indeed into our general framework.

Outline In Section 2, we first recall the basic definitions and some results regarding the combinatorics of Artin–Tits monoids. The notion of normal form of elements of an Artin–Tits monoid leads us to the definition of the boundary at infinity of the monoid. We describe the relationship between measures at infinity and Möbius transform on the monoid. We particularize then our study to the class of multiplicative measures at infinity, among which is the uniform measure at infinity. We characterize multiplicative measures at infinity by a finite number of probabilistic parameters with suitable normalization conditions, and we state a uniqueness result for the uniform measure.

In Section 3 we introduce a technical device: the notion of conditioned weighted graph (CWG). Based on the Perron–Frobenius theory for primitive matrices, we observe that a natural notion of weighted measure at infinity can be attached to any CWG.

In Section 4, we show how the theory of CWG applies in particular to Artin–Tits monoids. The two notions of uniform measure at infinity introduced above are related to each other. We obtain in particular a parametrization of all multiplicative measures at infinity for an Artin–Tits monoid, and the existence of the uniform measure at infinity.

Section 5 is devoted to the asymptotic study of combinatorial statistics, first in the general case of CWG, and then applied to the case of Artin–Tits monoids. Using spectral methods, we state a concentration result and a Central Limit Theorem in the framework of CWG, relatively to weak convergences. We state the corresponding convergence results for the asymptotic combinatorics of Artin–Tits monoids, yielding very general answers to the questions raised in the first part of this introduction, for braids.

Finally, in Section 6, we extract some minimal properties that were needed to carry our analysis. Provided that these minimal properties hold indeed, one states the corresponding results for more general monoids. We also give examples of such monoids outside the class of Artin–Tits monoids, including the class of dual monoids of Artin–Tits monoids of spherical type.

2. Probability measures on the boundary of Artin–Tits monoids

In this section we review some background on the combinatorics of Artin–Tits monoids. It turns out that the notion of Garside family is of great interest, both from a theoretical and from a computational viewpoint. Therefore we focus on Garside subsets for Artin–Tits monoids and related notions: simple elements and the associated normal form in particular. We shall see later that the probabilistic viewpoint also takes great advantage of these notions, which were originally devised mainly to explore combinatorial aspects of Artin–Tits monoids.

2.1. Definitions and examples of Artin–Tits monoids

2.1.1. Definition of Artin–Tits monoids

Let a finite, non-empty alphabet Σ be equipped with a symmetric function $\ell : \Sigma\times\Sigma \mapsto \{2,3,4,\ldots\} \cup \{\infty\}$, *i.e.*, such that $\ell(a,b) = \ell(b,a)$ for all $(a,b) \in \Sigma\times\Sigma$. Associate to the pair (Σ,ℓ) the binary relation I on Σ, and the monoid $\mathbf{A} = \mathbf{A}(\Sigma,\ell)$ defined by the following presentation [5]:

$$\mathbf{A} = \big\langle \Sigma \;\big|\; ababa\ldots = babab\ldots \text{ for } (a,b) \in I \big\rangle^+ ,$$
$$\text{where } ababa\ldots \text{ and } babab\ldots \text{ both have length } \ell(a,b),$$
$$\text{and } I = \big\{(a,b) \in \Sigma\times\Sigma \;\big|\; a \neq b \text{ and } \ell(a,b) < \infty\big\}.$$

Note that the values of ℓ on the diagonal of $\Sigma\times\Sigma$ are irrelevant. Such a monoid is called an *Artin–Tits monoid.*

2.1.2. Examples of Artin–Tits monoids

The free monoid generated by Σ is isomorphic to $\mathbf{A}(\Sigma,\ell)$ with $\ell(\cdot,\cdot) = \infty$. The free Abelian monoid generated by Σ is isomorphic to $\mathbf{A}(\Sigma,\ell)$ with $\ell(\cdot,\cdot) = 2$. More generally, considering $\mathbf{A}(\Sigma,\ell)$ with ℓ ranging only over $\{2,\infty\}$, but assuming possibly both values, yields the class of so-called *heap monoids* on Σ, also called *trace monoids* on Σ. They are the monoids analogous to the so-called right-angled Artin groups [6].

Braid monoids are also specific instances of Artin–Tits monoids. Indeed, for every integer $n \geq 3$, the braid monoid on n strands is the monoid B_n^+ generated by $n-1$

elements $\sigma_1, \ldots, \sigma_{n-1}$ with the following relations [7]: the braid relations $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $1 \le i \le n-2$ and $j = i+1$, and the commutativity relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $1 \le i, j \le n-1$ and $|i-j| > 1$. Hence B_n^+ is isomorphic to $\mathbf{A}(\Sigma, \ell)$ by choosing $\Sigma = \{\sigma_1, \ldots, \sigma_{n-1}\}$ and $\ell(\sigma_i, \sigma_j) = 3$ for $|i-j| = 1$ and $\ell(\sigma_i, \sigma_j) = 2$ for $|i-j| > 1$.

2.1.3. Length and orders

Let $\mathbf{A} = \mathbf{A}(\Sigma, \ell)$ be an Artin–Tits monoid. The *length* of $x \in \mathbf{A}$, denoted by $|x|$, is the length of any word in the equivalence class x, with respect to the congruence defining $\mathbf{A}$ (it does not depend on the choice of the word as the relations do not modify the length).

In particular, elements of Σ are characterized as those $x \in \mathbf{A}$ such that $|x| = 1$. It follows that, when considering an Artin–Tits monoid $\mathbf{A}$, we may refer to the pair (Σ, ℓ) such that $\mathbf{A} = \mathbf{A}(\Sigma, \ell)$. In particular, elements of Σ are called *generators* of $\mathbf{A}$.

The monoid $\mathbf{A}$ is equipped with the *left* and with the *right divisibility* relations, denoted respectively by $\le_{\mathrm{l}}$ and by $\le_{\mathrm{r}}$, which are both partial orders on $\mathbf{A}$, defined by:

$$x \le_{\mathrm{l}} y \iff \exists z \in \mathbf{A} \quad y = x \cdot z\,,$$
$$x \le_{\mathrm{r}} y \iff \exists z \in \mathbf{A} \quad y = z \cdot x\,.$$

We also denote by $<_{\mathrm{l}}$ and $<_{\mathrm{r}}$ the associated strict orders.

The results stated in this paragraph and the next one are proved in [8, Ch. IX, Prop. 1.26]. Every Artin–Tits monoid $\mathbf{A}$ is both left and right cancellative, meaning:

$$\forall x, y, z \in \mathbf{A} \quad (z \cdot x = z \cdot y \implies x = y) \wedge (x \cdot z = y \cdot z \implies x = y).$$

The partially ordered set $(\mathbf{A}, \le_{\mathrm{l}})$ is a lower semilattice, *i.e.*, any finite set has a greatest lower bound. Furthermore, any $\le_{\mathrm{l}}$-bounded set has a $\le_{\mathrm{l}}$-least upper bound.

We denote by $\bigwedge_{\mathrm{l}} X$ and by $\bigvee_{\mathrm{l}} X$ the greatest $\le_{\mathrm{l}}$-lower bound and the least upper bound of a subset $X \subseteq \mathbf{A}$, if they exist. We use the standard notations $a \wedge_{\mathrm{l}} b = \bigwedge_{\mathrm{l}}\{a, b\}$ for $a, b \in \mathbf{A}$. We also write $a \vee_{\mathrm{l}} b = \bigvee_{\mathrm{l}}\{a, b\}$ when defined. The analogous notations $\bigvee_{\mathrm{r}}$, *etc.*, are defined with respect to the right divisibility relation.

Unlike braid monoids, $(\mathbf{A}, \le_{\mathrm{l}})$ is not necessarily a lattice. For instance, distinct generators a and b of a free monoid have no common multiple and thus $\{a, b\}$ has no $\le_{\mathrm{l}}$-upper bound.

If a and b are two generators of $\mathbf{A}$, then $a \vee_{\mathrm{l}} b$ exists if and only if $\ell(a, b) < \infty$, if and only if $a \vee_{\mathrm{r}} b$ exists, and then $a \vee_{\mathrm{l}} b = a \vee_{\mathrm{r}} b = ababa\ldots$, where the rightmost member has length $\ell(a, b)$. More generally, let $S \subseteq \Sigma$ be a set of generators of $\mathbf{A}$. Then it follows from [8, Ch. IX, Proposition 1.29] that the following conditions are equivalent: (i) S is $\le_{\mathrm{l}}$-bounded; (ii) $\bigvee_{\mathrm{l}} S$ exists; (iii) S is $\le_{\mathrm{r}}$-bounded; (iv) $\bigvee_{\mathrm{r}} S$ exists. If these conditions are fulfilled, the element $\Delta_S = \bigvee_{\mathrm{l}} S$ has the following property:

$$\forall x \in \mathbf{A} \quad x \le_{\mathrm{l}} \Delta_S \iff x \le_{\mathrm{r}} \Delta_S\,. \tag{2.1}$$

It implies in particular:

$$\Delta_S = \bigvee_{\mathrm{r}} S, \qquad \text{and} \qquad \forall x \in S \quad x \leq_{\mathrm{r}} \Delta_S\,. \tag{2.2}$$

Furthermore, it follows from [8, Ch. IX, Cor. 1.12] that the sub-monoid of $\mathbf{A}$ generated by S is isomorphic to the Artin–Tits monoid $\mathbf{A}(S, \ell\big|_{S\times S})$. In particular:

$$\Delta_S \in \langle S\rangle, \qquad \text{the sub-monoid of } \mathbf{A} \text{ generated by } S. \tag{2.3}$$

If X is a subset of $\mathbf{A}$, we say that X is *closed under existing* $\vee_{\mathrm{l}}$ to mean that, if a and b are elements of X such that $a \vee_{\mathrm{l}} b$ exists in $\mathbf{A}$, then $a \vee_{\mathrm{l}} b \in X$.

2.1.4. Artin–Tits monoids of spherical type

The monoid $\mathbf{A} = \mathbf{A}(\Sigma, \ell)$ is said to have *spherical type* if $\bigvee_{\mathrm{l}} \Sigma$ exists. In this case, the partially ordered set $(\mathbf{A}, \leq_{\mathrm{l}})$ is a lattice. The element $\bigvee_{\mathrm{l}} \Sigma$ is denoted by Δ and is called the *Garside element of* $\mathbf{A}$.

An equivalent definition, often found in the literature, is the following. Let $\mathbf{C} = \mathbf{C}(\Sigma, \ell)$ be the presented group with Σ as set of generators, and with the same relations given in Section 2.1.1 in the definition of $\mathbf{A}(\Sigma, \ell)$, together with all the relations $s^2 = 1$ for s ranging over Σ. The group $\mathbf{C}$ is the Coxeter group [9] associated with $\mathbf{A}$. Then $\mathbf{A}$ has spherical type if and only if $\mathbf{C}$ is finite, see [5, Th. 5.6].

For instance, it is well known that the braid monoid B_n^+ defined as in Section 2.1.2 has spherical type, with Garside element given by $\Delta = (\sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_{n-1}) \cdot (\sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_{n-2}) \cdot \ldots \cdot (\sigma_1 \cdot \sigma_2) \cdot \sigma_1$. A heap monoid is of spherical type if and only if it is a free Abelian monoid.

2.1.5. Irreducibility of Artin–Tits monoids and Coxeter graph

Definition 2.1. An Artin–Tits monoid $\mathbf{A}$ is called *irreducible* if it is not isomorphic to the direct product of two Artin–Tits monoids.

For instance, braid monoids and free monoids are all irreducible. A free Abelian monoid is irreducible if and only if it has only one generator. For a heap monoid $\mathcal{M} = \mathbf{A}(\Sigma, \ell)$ with $\ell(\cdot,\cdot) \in \{2, \infty\}$, define $D = \{(a, b) \in \Sigma \times \Sigma \bigm| a = b \text{ or } \ell(a, b) = \infty\}$. Then $\mathcal{M}$ is irreducible if and only if the undirected graph (Σ, D) is connected.

More generally, the irreducibility is related to the Coxeter graph of the monoid, defined as follows.

Definition 2.2. The *Coxeter graph* of an Artin–Tits monoid $\mathbf{A}(\Sigma, \ell)$ is the undirected graph $\mathbf{G} = (\Sigma, E)$, with $E = \{(s, t) \in \Sigma \times \Sigma \bigm| s = t \text{ or } \ell(s, t) \geq 3 \text{ or } \ell(s, t) = \infty\}$.

As observed in [5, § 7.1], we have the following result.

Proposition 2.3. *An Artin–Tits monoid is irreducible if and only if its Coxeter graph is connected.*

2.2. Normal sequences and normal form of elements

We fix an Artin–Tits monoid $\mathbf{A} = \mathbf{A}(\Sigma, \ell)$.

2.2.1. Garside subsets

A subset $\mathcal{G}$ of $\mathbf{A}$ is a *Garside subset* of $\mathbf{A}$ if it contains Σ and if it is closed under existing $\vee_l$ and downward closed under $\leq_r$, the latter meaning:

$$\forall x \in \mathcal{G} \quad \forall y \in \mathbf{A} \quad y \leq_r x \implies y \in \mathcal{G}.$$

The following result is proved in [10].

Proposition 2.4. *Any Artin–Tits monoid admits a* finite *Garside subset.*

The class of Garside subsets of $\mathbf{A}$ is obviously closed by intersection, hence $\mathbf{A}$ admits a *smallest* Garside subset, which we denote throughout the paper by $\mathcal{S}$. The subset $\mathcal{S}$ is the closure of Σ under $\leq_r$ and existing $\vee_l$. Proposition 2.4 tells us that the set $\mathcal{S}$ thus constructed is finite. By construction, $\mathcal{S}$ contains $\Sigma \cup \{\boldsymbol{e}\}$, where $\boldsymbol{e}$ is the unit element of the monoid.

Definition 2.5. The elements of the smallest Garside subset of an Artin–Tits monoid are called its *simple elements.*

Assume that $\mathbf{A}$ is of spherical type. Then, according to [8, Ch. IX, Prop. 1.29], the set $\mathcal{S}$ coincides with the set of left divisors of Δ, which is also the set of right divisors of Δ. Hence: $\Delta = \bigvee_l \Sigma = \bigvee_r \Sigma = \bigvee_l \mathcal{S} = \bigvee_r \mathcal{S}$, and Δ is the maximum of the sub-lattice $(\mathcal{S}, \leq_l)$.

2.2.2. Artin–Tits monoids of type FC

If the smallest Garside subset of an Artin–Tits monoid is closed under *left* divisibility, then the combinatorics of the monoid is a bit more simple to study.

Definition 2.6. An Artin–Tits monoid $\mathbf{A}$, with smallest Garside subset $\mathcal{S}$, is of *type FC* if $\mathcal{S}$ is closed under left divisibility.

It is proved in [11, Th. 2.85] that this definition is indeed equivalent to the one found in the literature [12,8]. In particular, heap monoids and braid monoids are monoids of type FC.

Not all Artin–Tits monoids are of type FC: see an example in Section 2.3.4 below.

2.2.3. Normal sequences

Most of what is known on the combinatorics of Artin–Tits monoids is based on the notion of normal sequence.

Definition 2.7. Let $\mathbf{A}$ be an Artin–Tits monoid with smallest Garside subset $\mathcal{S}$. A sequence $(x_1, \ldots, x_k)$, with $k \geq 1$, of elements of $\mathbf{A}$ is *normal* if it satisfies:

1. $x_i \in \mathcal{S}$ for all $i = 1, \ldots, k$.
2. $x_i = \bigvee_1 \{\zeta \in \mathcal{S} \mid \zeta \leq_1 x_i \cdot \ldots \cdot x_k\}$ for all $i = 1, \ldots, k$.

A fundamental property of normal sequences is the following.

Lemma 2.8. *Let $\mathbf{A}$ be an Artin–Tits monoid.*

1. *A sequence $(x_1, \ldots, x_k)$, with $k \geq 1$, of simple elements of $\mathbf{A}$ is normal if and only if it satisfies:*

$$x_i = \bigvee\nolimits_1 \{\zeta \in \mathcal{S} \mid \zeta \leq_1 x_i \cdot x_{i+1}\} \quad \textit{for all } i = 1, \ldots, k-1.$$

2. *A sequence $(x_1, \ldots, x_k)$, with $k \geq 1$, of simple elements of $\mathbf{A}$ is normal if and only if all sequences (x_i, x_{i+1}) are normal, for $i \in \{1, \ldots, k-1\}$.*

Proof. Point 1 is proved in [10], and point 2 follows at once from point 1. □

Let $x \to y$ denote the relation $x = \bigvee_1 \{\zeta \in \mathcal{S} \mid \zeta \leq x \cdot y\}$, defined for $(x, y) \in \mathcal{S} \times \mathcal{S}$. Point 2 of Lemma 2.8 reduces the study of normality of sequences to the study of the binary relation $\to$ on $\mathcal{S}$.

The unit element $\boldsymbol{e}$ of $\mathbf{A}$ satisfies:

$$\forall x \in \mathcal{S} \quad x \to \boldsymbol{e}, \qquad \forall x \in \mathcal{S} \quad \boldsymbol{e} \to x \iff x = \boldsymbol{e}. \tag{2.4}$$

Hence $\boldsymbol{e}$ can only occur at the end of normal sequences, and $\boldsymbol{e}$ is the only element of $\mathcal{S}$ satisfying the two properties in (2.4).

Dually, if $\mathbf{A}$ is of spherical type, the Garside element $\Delta = \bigvee_1 \Sigma$ satisfies:

$$\forall x \in \mathcal{S} \quad \Delta \to x, \qquad \forall x \in \mathcal{S} \quad x \to \Delta \iff x = \Delta. \tag{2.5}$$

Hence Δ can only occur at the beginning of normal sequences. Obviously, Δ is the only simple element satisfying the two properties in (2.5).

2.2.4. Charney graph

We define the Charney graph $(\mathscr{C}, \to)$ of an Artin–Tits monoid $\mathbf{A}$ as follows. If $\mathbf{A}$ is of spherical type, we put $\mathscr{C} = \mathcal{S} \setminus \{\Delta, \boldsymbol{e}\}$, where Δ is the Garside element of $\mathbf{A}$. If not, we put $\mathscr{C} = \mathcal{S} \setminus \{\boldsymbol{e}\}$. In all cases, the edge relation $\to$ is the restriction to $\mathscr{C} \times \mathscr{C}$ of the relation $\to$ defined above on $\mathcal{S} \times \mathcal{S}$.

The relevance of this definition will appear in Section 2.2.7 below.

2.2.5. Normal form of elements and height

Let $\mathbf{A}$ be an Artin–Tits monoid. Then, for every element $x \in \mathbf{A}$ with $x \neq \boldsymbol{e}$, there exists a unique integer $k \geq 1$ and a unique normal sequence $(x_1, \ldots, x_k)$ of *non-unit simple elements* such that $x = x_1 \cdot \ldots \cdot x_k$. This sequence is called the *normal form* of x. By convention, we decide that $(\boldsymbol{e})$ is the normal form of the unit element of $\mathbf{A}$.

The integer k is called the *height* of x, and we denote it:

$$k = \tau(x).$$

The following result is standard [8, Chap. III, Prop. 1.36]. It shows that comparing elements can be done after 'cutting' at the right height.

Lemma 2.9. *Let $\mathbf{A}$ be an Artin–Tits monoid, and let $\mathcal{S}$ be the smallest Garside subset of $\mathbf{A}$. Then:*

1. *If $(x_1, \ldots, x_k)$ is the normal form of and element $x \in \mathbf{A}$, then $x_1 \cdot \ldots \cdot x_j = \bigvee_{\mathrm{l}}\{z \in \mathcal{S}^j \bigm| z \leq_{\mathrm{l}} x\}$ for all $j \in \{1, \ldots, k\}$, where $\mathcal{S}^j = \{u_1 \cdot \ldots \cdot u_j \bigm| (u_1, \ldots, u_j) \in \mathcal{S} \times \cdots \times \mathcal{S}\}$.*
2. *The height $\tau(x)$ of an element $x \in \mathbf{A}$ is the smallest integer $j \geq 1$ such that $x \in \mathcal{S}^j$.*

The normal form of elements does not behave 'nicely' with respect to the monoid multiplication. For instance, the multiplication of an element $x \in \mathbf{A}$, of normal form $(x_1, \ldots, x_k)$, by an element $y \in \mathbf{A}$, yields in general an element z of normal form $(z_1, \ldots, z_{k'})$ with no simple relation between x_j and z_j.

It is even possible that the multiplication $x \cdot \sigma$ of $x \in \mathbf{A}$ with a generator $\sigma \in \Sigma$, satisfies $\tau(x \cdot \sigma) < \tau(x)$ (see an example in Section 2.3.4). This contrasts with monoids of type FC, where $\tau(x \cdot \sigma) \geq \tau(x)$ always holds.

2.2.6. Conditions for normality of sequences

Aiming at studying the connectedness of the Charney graph in the next section, one needs theoretical tools to construct normal sequences. Such tools exist in the literature. They include the *letter set*, the *left set* and the *right set* of an element $x \in \mathbf{A}$, which are respectively defined as the following subsets of Σ:

$$\begin{aligned}
\mathcal{L}(x) &= \{\sigma \in \Sigma \bigm| \exists y, z \in \mathbf{A} \quad x = y \cdot \sigma \cdot z\}, \\
L(x) &= \{\sigma \in \Sigma \bigm| \sigma \leq_{\mathrm{l}} x\}, \\
R(x) &= \{\sigma \in \Sigma \bigm| \sigma \leq_{\mathrm{r}} x \quad \text{or} \quad (\exists \eta \in \mathcal{L}(x) \quad \sigma \neq \eta \text{ and } \ell(\sigma, \eta) = \infty)\}.
\end{aligned}$$

The letter set $\mathcal{L}(x)$ of x is the set of letters that appear in some word representing x or, equivalently, in any word representing x, as relations in Artin–Tits monoids use the same letters on both sides of the relation.

Lemma 2.10. *Let $\mathbf{A} = \mathbf{A}(\Sigma, \ell)$ be an Artin–Tits monoid. The subsets*

$$\mathcal{G} = \{x \in \mathbf{A} \bigm| \forall\sigma \in \Sigma \quad \forall y, z \in \mathbf{A} \quad x \neq y \cdot \sigma^2 \cdot z\},$$
$$\mathcal{I} = \{x \in \mathbf{A} \bigm| \forall\sigma, \eta \in \mathcal{L}(x) \quad \sigma \neq \eta \implies \ell(\sigma, \eta) < \infty\},$$

are Garside subsets of $\mathbf{A}$.

Proof. It is proved in [8, Theorem 6.27] that $\mathcal{G}$ is a Garside subset. Since $\mathcal{I}$ is clearly downward closed under $\leq_{\mathrm{r}}$ and contains Σ, we focus on proving that $\mathcal{I}$ is closed under $\vee_{\mathrm{l}}$. In passing, we also note that $\mathcal{I}$ is downward closed under $\leq_{\mathrm{l}}$.

Seeking a contradiction, assume the existence of $x, y \in \mathcal{I}$ such that $z = x \vee_{\mathrm{l}} y$ exists in $\mathbf{A}$ but $z \notin \mathcal{I}$. Without loss of generality, we assume that z is such an element of minimal length and that the element $w = x \wedge_{\mathrm{l}} y$ is maximal among all the elements of the set $\{u \wedge_{\mathrm{l}} v \bigm| u, v \in \mathcal{I} \text{ and } u \vee_{\mathrm{l}} v = z\}$. Consequently, a contradiction, and therefore a proof of the lemma, is obtained by proving the following claim:

(†) *There exist elements x' and y' in $\mathcal{I}$ such that $x' \vee_{\mathrm{l}} y' = z$ and $w <_{\mathrm{l}} (x' \wedge_{\mathrm{l}} y')$.*

Since $z \notin \{x, y\}$, observe that neither $x \leq_{\mathrm{l}} y$ nor $y \leq_{\mathrm{l}} x$ hold, whence $w <_{\mathrm{l}} x$ and $w <_{\mathrm{l}} y$. Thus we pick $\sigma, \eta \in \Sigma$ such that $w \cdot \sigma \leq_{\mathrm{l}} x$ and $w \cdot \eta \leq_{\mathrm{l}} y$. Then $w \cdot \sigma \leq_{\mathrm{l}} z$ and $w \cdot \eta \leq_{\mathrm{l}} z$, which implies that σ and η have a common $\leq_{\mathrm{l}}$-upper bound, and thus $\sigma \vee_{\mathrm{l}} \eta$ exists by the remarks made in Section 2.1.3, and it is equal to $\sigma\eta\sigma\cdots$.

Since $\mathcal{I}$ is $\leq_{\mathrm{l}}$-downward closed, and since x and y belong to $\mathcal{I}$, the elements $w \cdot \sigma$ and $w \cdot \eta$ both belong to $\mathcal{I}$. It follows that $\ell(a, b) < \infty$ for all pairs (a, b) with $a \neq b$ and $a, b \in \mathcal{L}(w) \cup \{\sigma, \eta\}$. Therefore, putting $t = w \cdot (\sigma \vee_{\mathrm{l}} \eta)$, we have that $t \in \mathcal{I}$. Since t also writes as $t = (w \cdot \sigma) \vee_{\mathrm{l}} (w \cdot \eta)$, it is clear that $t \leq_{\mathrm{l}} z$.

Hence, the element $u = t \vee_{\mathrm{l}} x$ exists, and $u \leq_{\mathrm{l}} z$. If $u = z$, then setting $x' = x$ and $y' = t$ gives us $w <_{\mathrm{l}} (w \cdot \sigma) \leq_{\mathrm{l}} (x' \wedge_{\mathrm{l}} y')$ and $z = x' \vee_{\mathrm{l}} y'$. If $u <_{\mathrm{l}} z$, then by minimality of z we have $u \in \mathcal{I}$, and therefore setting $x' = u$ and $y' = y$ gives us $w <_{\mathrm{l}} (w \cdot \eta) \leq_{\mathrm{l}} (t \wedge_{\mathrm{l}} y) \leq_{\mathrm{l}} (x' \wedge_{\mathrm{l}} y')$, $z = (x \vee_{\mathrm{l}} y) \leq_{\mathrm{l}} (x' \vee_{\mathrm{l}} y')$ and $(x' \vee_{\mathrm{l}} y') \leq_{\mathrm{l}} z$, whence $z = x' \vee_{\mathrm{l}} y'$. This completes the proof of the claim and of the lemma. □

We obtain the following *sufficient* criterion for detecting normal sequences.

Corollary 2.11. *Let* $\mathbf{A}$ *be an Artin–Tits monoid.*

1. *If x and y are two non-unit simple elements of $\mathbf{A}$ satisfying $L(y) \subseteq R(x)$, then (x, y) is normal.*

2. *If $(x_1, \ldots, x_k)$, with $k \geq 1$, is a sequence of non-unit simple elements of $\mathbf{A}$ satisfying $L(x_{i+1}) \subseteq R(x_i)$ for all $i \in \{1, \ldots, k-1\}$, then it is normal.*

Proof. Point 2 follows from point 1, in view of Lemma 2.8, point 2. To prove point 1, let x and y be as in the statement and let $u = \bigvee_{\mathrm{l}} \{z \in \mathcal{S} \mid z \leq_{\mathrm{l}} x \cdot y\}$; we prove that $x = u$.

Clearly, $x \leq_{\mathrm{l}} u$. Seeking a contradiction, assume that $u \neq x$. Then there exists $z \in \mathcal{S}$ such that $x <_{\mathrm{l}} z \leq_{\mathrm{l}} x \cdot y$. Let $\sigma \in \Sigma$ be such that $x \cdot \sigma \leq_{\mathrm{l}} z$. Then $x \cdot \sigma \leq_{\mathrm{l}} x \cdot y$ and thus $\sigma \leq_{\mathrm{l}} y$ since $\mathbf{A}$ is left cancellative, hence $\sigma \in L(y) \subseteq R(x)$.

Let $\mathcal{G}$ and $\mathcal{I}$ be the Garside subsets of $\mathbf{A}$ introduced in Lemma 2.10. Discussing the property $\sigma \in R(x)$, one has: (1) if $\sigma \leq_{\mathrm{r}} x$ then $x \cdot \sigma \notin \mathcal{G}$, and (2) if $\ell(\sigma, \eta) = \infty$ for some $\eta \in \mathcal{L}(x)$ with $\eta \neq \sigma$, then $x \cdot \sigma \notin \mathcal{I}$. In both cases, we have thus $x \cdot \sigma \notin \mathcal{G} \cap \mathcal{I}$. Since $\mathcal{G}$ and $\mathcal{I}$ are both closed under $\leq_{\mathrm{l}}$, so is $\mathcal{G} \cap \mathcal{I}$, and thus $z \notin \mathcal{G} \cap \mathcal{I}$ since $x \cdot \sigma \leq_{\mathrm{l}} z$. But $\mathcal{S} \subseteq \mathcal{G} \cap \mathcal{I}$ according to Lemma 2.10, which contradicts that $z \in \mathcal{S}$ and completes the proof. □

2.2.7. Connectedness of the Charney graph

Recall that we have defined the Charney graph of an Artin–Tits monoid in Section 2.2.4. We aim to prove the following result.

Theorem 2.12. *The Charney graph of an irreducible Artin–Tits monoid is strongly connected.*

We postpone the proof of this theorem to the end of the section, and prove two intermediate results first.

Lemma 2.13. *Let $\mathbf{A} = \mathbf{A}(\Sigma, \ell)$ be an Artin–Tits monoid, let S be a subset of Σ such that the element $\Delta_S = \bigvee_{\mathrm{l}} S$ exists, and let $\sigma \in \Sigma \setminus S$.*

1. *Let $\mathcal{L}^*(\sigma, S) = \{\eta \in S \mid \ell(\sigma, \eta) = 2\}$. Then:*

$$L(\sigma \cdot \Delta_S) = \{\sigma\} \cup \mathcal{L}^*(\sigma, S), \qquad R(\sigma \cdot \Delta_S) \supseteq S \cup \{\eta \in \Sigma \mid \ell(\sigma, \eta) = \infty\}.$$

2. *If $\ell(\sigma, \eta) < \infty$ for all $\eta \in S$, then $\sigma \cdot \Delta_S$ is simple.*

Proof. 1. The rightmost inclusion follows from the following two observations: the relation $R(\sigma \cdot \Delta_S) \supseteq \{\eta \in \Sigma \mid \ell(\sigma, \eta) = \infty\}$ is immediate from the definitions stated at the beginning of Section 2.2.6, and the relation $R(\sigma \cdot \Delta_S) \supseteq S$ derives from the property (2.2), Section 2.1.3.

For the leftmost equality, we first observe that the inclusion $\{\sigma\} \cup \mathcal{L}^*(\sigma, S) \subseteq L(\sigma \cdot \Delta_S)$ is obvious. For the converse inclusion, the relation $L(\sigma \cdot \Delta_S) \subseteq \mathcal{L}(\sigma \cdot \Delta_S)$ is immediate, and $\mathcal{L}(\Delta_S) = S$ follows from (2.3), hence we obtain $L(\sigma \cdot \Delta_S) \subseteq \{\sigma\} \cup S$. Thus, it is enough to prove that $\eta \notin L(\sigma \cdot \Delta_S)$ for every $\eta \in S$ such that $\ell(\sigma, \eta) > 2$.

Suppose, for the sake of contradiction, that $\eta \leq_{\mathrm{l}} \sigma \cdot \Delta_S$ for such an element η. Then σ and η have the common $\leq_{\mathrm{l}}$-upper bound $\sigma \cdot \Delta_S$. This implies that $\ell(\sigma, \eta) < \infty$ according to the remarks of Section 2.1.3, and that $\sigma \vee_{\mathrm{l}} \eta \leq_{\mathrm{l}} \sigma \cdot \Delta_S$. Furthermore, $\sigma \cdot \eta \cdot \sigma \leq_{\mathrm{l}} (\sigma \vee_{\mathrm{l}} \eta)$ since $\ell(\sigma, \eta) > 2$, whence $\sigma \cdot \eta \cdot \sigma \leq_{\mathrm{l}} \sigma \cdot \Delta_S$, and thus $\eta \cdot \sigma \leq_{\mathrm{l}} \Delta_S$. Since $\sigma \notin S$, the latter relation is impossible.

2. Let $\mathcal{S}$ denote the set of simple elements of $\mathbf{A}$. Then for all $\eta \in S$, it follows from the relation $\ell(\sigma, \eta) < \infty$ that $\sigma \vee_{\mathrm{l}} \eta \in \mathcal{S}$, and from the relation $\sigma \cdot \eta \leq_{\mathrm{r}} (\sigma \vee_{\mathrm{l}} \eta)$ that $\sigma \cdot \eta \in \mathcal{S}$. Since $\sigma \cdot \Delta_S = \bigvee_{\mathrm{l}} \{\sigma \cdot \eta \bigm| \eta \in S\}$, it implies that $\sigma \cdot \Delta_S \in \mathcal{S}$. $\square$

For the next result, we follow the same lines as in the proof of [13, Proposition 4.9].

Proposition 2.14. *Let $\mathbf{A}$ be an irreducible Artin–Tits monoid, and let $\mathcal{S}$ be the smallest Garside subset of $\mathbf{A}$. Let a and b be non-unit elements of $\mathcal{S}$. If either $\mathbf{A}$ does not have spherical type, or $a = \Delta$, or $b \neq \Delta$, then there exists some normal sequence $(x_1, \ldots, x_k)$ such that $a = x_1$ and $b = x_k$.*

Proof. Let $\mathbf{G}$ be the Coxeter graph of $\mathbf{A}$ (see Section 2.1.5), and let $d_{\mathbf{G}}(\cdot, \cdot)$ denote the graph metric in $\mathbf{G}$. For each set $S \subseteq \Sigma$, we denote by $c(S)$ the number of connected components of S in the graph $\mathbf{G}$, and by $d_{\mathbf{G}}(\cdot, S)$ the sum $\sum_{s \in S} d_{\mathbf{G}}(\cdot, s)$.

First, observe that if $L(b) = \Sigma$, then b is a $\leq_{\mathrm{l}}$-upper bound of Σ, hence $\mathbf{A}$ has spherical type and $\Delta = b$ since $\Delta = \bigvee_{\mathrm{l}} \Sigma$. According to our assumptions, this only occurs with $a = \Delta$, and then the normal sequence (Δ) has first and last letters a and b. Hence, we assume that $L(b) \neq \Sigma$.

Next, since $a \neq \boldsymbol{e}$, the set $R(a)$ is non-empty. Hence, let $\rho \in R(a)$ be some node of $\mathbf{G}$. Since (a, ρ) is a normal sequence according to Corollary 2.11, point 1, we assume without loss of generality that a is an element of Σ.

Now, we put $S = L(b)$, and we discuss different cases. We take into account that $\mathbf{G}$ is connected, which follows from the assumption that $\mathbf{A}$ is irreducible *via* Proposition 2.3. We also note that $\bigvee_{\mathrm{l}} S$ exists, since b is a $\leq_{\mathrm{l}}$-upper bound of S.

1. If $d_{\mathbf{G}}(a, S) = 0$, then $S = \{a\}$, hence $L(b) \subseteq R(a)$ and therefore the sequence (a, b) is normal by Corollary 2.11, point 1.
2. If $c(S) = |S|$ and $d_{\mathbf{G}}(a, S) > 0$. Then the set S is $\mathbf{G}$-independent and contains some vertex $\sigma \neq a$.
   Let $n = d_{\mathbf{G}}(\sigma, a) \geq 1$, and let $s_0, s_1, \ldots, s_n$ be a path in $\mathbf{G}$ such that $s_0 = a$, $s_n = \sigma$. Since S is $\mathbf{G}$-independent, the vertex s_{n-1} does not belong to S. Consider the sets:

$$Q = \{s_{n-1}\} \cup \{t \in S \bigm| d_{\mathbf{G}}(t, s_{n-1}) \geq 2\}, \qquad T = \{t \in S \bigm| \ell(s_{n-1}, t) < \infty\}.$$

   We note that $\Delta_T = \bigvee_{\mathrm{l}} T$ exists since $\bigvee_{\mathrm{l}} S$ exists. Lemma 2.13, point 1, proves:

$$L(s_{n-1} \cdot \Delta_T) \subseteq Q, \qquad R(s_{n-1} \cdot \Delta_T) \supseteq T \cup \{t \in S : \ell(s_{n-1}, t) = \infty\} = S,$$

and Lemma 2.13, point 2, proves that $s_{n-1}\cdot\Delta_T\in\mathcal{S}$. By construction, the set Q is $\mathbf{G}$-independent, whence $c(Q)=|Q|$. Finally, we observe:

$$\begin{aligned} d_{\mathbf{G}}(a,S)-d_{\mathbf{G}}(a,Q) &= \sum_{t\in S}\mathbf{1}\big(d_{\mathbf{G}}(t,s_{n-1})=1\big)d_{\mathbf{G}}(a,t)-d_{\mathbf{G}}(a,s_{n-1}) \\ &\geq d_{\mathbf{G}}(a,\sigma)-d_{\mathbf{G}}(a,s_{n-1})=1, \end{aligned}$$

and thus $d_{\mathbf{G}}(a,Q)<d_{\mathbf{G}}(a,S)$.
Since we have observed that $L(s_{n-1}\cdot\Delta_T)\subseteq Q$, we may assume as an induction assumption on $d_{\mathbf{G}}(a,S)$ the existence of a normal sequence $(w_1,\ldots,w_j)$ such that $w_1=a$ and $w_j=s_{n-1}\cdot\Delta_T$. Since we have also observed that $R(w_j)\supseteq S=L(b)$, it follows from Corollary 2.11, point 1, that (w_j,b) is a normal sequence, and thus the sequence $(w_1,\ldots,w_j,b)$ is also normal according to Lemma 2.8, point 2.

3. If $c(S)<|S|$, then, for every subset T of Σ, let $p(T)$ be the maximal cardinal of a connected component of T in $\mathbf{G}$, and let $q(T)$ be the number of connected components of T with this maximal cardinal.
Consider some connected component S' of S in $\mathbf{G}$ of maximal cardinal, and let $\eta_0\notin S$ be some neighbor of S' in $\mathbf{G}$. Such a vertex η_0 exists since $S\neq\Sigma$. In addition, consider the sets:

$$Q=\{\eta_0\}\cup\{s\in S\bigm| d_{\mathbf{G}}(\eta_0,s)\geq 2\},\qquad T=\{t\in S\bigm|\ell(\eta_0,t)<\infty\}.$$

As in case 2 above, we note that $\Delta_T=\bigvee_1 T$ exists since $\bigvee_1 S$ exists, and we apply Lemma 2.13 to obtain:

$$L(\eta_0\cdot\Delta_T)\subseteq Q,\qquad R(\eta_0\cdot\Delta_T)\supseteq S,\qquad \eta_0\cdot\Delta_T\in\mathcal{S}.$$

It is obvious that $p(Q)\leq p(S)$. Moreover, since η_0 is a neighbor of S', $S'\cap Q\subsetneq S'$. Therefore, since S' has been chosen of maximal cardinal among the connected components of S on the one hand, and since $|S'|\geq 2$ by the assumption $c(S)<|S|$ on the other hand, at least one of the inequalities $p(Q)<p(S)$ and $q(Q)<q(S)$ holds. It implies that $\big(p(Q),q(Q)\big)<\big(p(S),q(S)\big)$ holds in the lexicographical order on $\mathbb{N}\times\mathbb{N}$. We may thus assume as an induction hypothesis (using Case 2 if $p(Q)=1$) the existence of a normal sequence $(w_1,\ldots,w_j)$ such that $w_1=a$ and $w_j=\eta_0\cdot\Delta_T$. As in case 2 above, we use that $L(b)=S\subseteq R(w_j)$ to conclude that (w_j,b), and thus $(w_1,\ldots,w_j,b)$ are normal sequences.

The proof is complete. $\square$

Theorem 2.12 follows at once from Proposition 2.14.

2.3. Finite measures on the completion of Artin–Tits monoids

2.3.1. Boundary at infinity of an Artin–Tits monoid

Let $\mathbf{A}$ be an Artin–Tits monoid, with smallest Garside subset $\mathcal{S}$. Elements of $\mathbf{A} \setminus \{\boldsymbol{e}\}$ are in bijection with normal sequences according to the results recalled in Section 2.2.5. Hence, they identify with finite paths in the graph $(\mathcal{S} \setminus \{\boldsymbol{e}\}, \rightarrow)$. It is therefore natural to introduce *boundary elements* of $\mathbf{A}$ as infinite paths in the very same graph.

In order to uniformly treat elements and boundary elements of the monoid, we extend the notion of normal form as follows. If x is an element of $\mathbf{A}$, with height $k \geq 1$ and normal form $(x_1, \ldots, x_k)$, we put $x_j = \boldsymbol{e}$ for all integers $j > k$. The sequence $(x_j)_{j \geq 1}$ thus obtained is the *generalized normal form* of x.

We say that an infinite sequence $(x_k)_{k \geq 1}$ of elements of $\mathcal{S}$ is *normal* if $x_k \rightarrow x_{k+1}$ holds for all $k \geq 1$. Among infinite normal sequences, those hitting $\boldsymbol{e}$ at least once actually stay in $\boldsymbol{e}$ forever because of (2.4), and these sequences correspond bijectively to the usual elements of $\mathbf{A}$. And those normal sequences never hitting $\boldsymbol{e}$ correspond to the boundary elements of $\mathbf{A}$.

Definition 2.15. The *generalized elements* of an Artin–Tits monoid $\mathbf{A}$ are the infinite normal sequences of simple elements of $\mathbf{A}$. Their set is called the *completion of* $\mathbf{A}$, and is denoted by $\overline{\mathbf{A}}$. The *boundary elements of* $\mathbf{A}$ are the generalized elements that avoid the unit $\boldsymbol{e}$. Their set is called the *boundary at infinity of* $\mathbf{A}$, or shortly *boundary of* $\mathbf{A}$, and is denoted by $\partial\mathbf{A}$. Identifying elements of $\mathbf{A}$ with their generalized normal form, we have thus: $\overline{\mathbf{A}} = \mathbf{A} \cup \partial\mathbf{A}$.

Both sets $\partial\mathbf{A}$ and $\overline{\mathbf{A}}$ are endowed with their natural topologies, as subsets of the countable product $\mathcal{S} \times \mathcal{S} \times \cdots$, where $\mathcal{S}$ is the smallest Garside subset of $\mathbf{A}$, and with the associated Borelian σ-algebras.

For every element x of $\mathbf{A}$, of height k and with normal form $(x_1, \ldots, x_k)$, the *Garside cylinder* of base x is the open and closed subset of $\overline{\mathbf{A}}$ defined by:

$$\mathcal{C}_x = \{y = (y_j)_{j \geq 1} \in \overline{\mathbf{A}} \bigm| y_1 = x_1, \ldots, y_k = x_k\}.$$

Note that $\mathcal{C}_{\boldsymbol{e}} = \{\boldsymbol{e}\}$.

The partial ordering $\leq_{\mathrm{l}}$ is extended on $\overline{\mathbf{A}}$ by putting, for $x = (x_k)_{k \geq 1}$ and $y = (y_k)_{k \geq 1}$:

$$x \leq_{\mathrm{l}} y \iff (\forall k \geq 1 \quad \exists j \geq 1 \quad x_1 \cdot \ldots \cdot x_k \leq_{\mathrm{l}} y_1 \cdot \ldots \cdot y_j).$$

For every $x \in \mathbf{A}$, the *visual cylinder* $\uparrow x \subseteq \partial\mathbf{A}$ and the *full visual cylinder* $\Uparrow x \subseteq \overline{\mathbf{A}}$ are:

$$\uparrow x = \{y \in \partial\mathbf{A} \bigm| x \leq_{\mathrm{l}} y\}, \qquad \Uparrow x = \{y \in \overline{\mathbf{A}} \bigm| x \leq_{\mathrm{l}} y\}.$$

Both spaces $\overline{\mathbf{A}}$ and $\partial\mathbf{A}$ are metrisable and compact, and $\overline{\mathbf{A}}$ is the topological closure of $\mathbf{A}$. More generally, for any $x \in \mathbf{A}$, the full visual cylinder $\Uparrow x$ is the topological closure in $\overline{\mathbf{A}}$ of $\{y \in \mathbf{A} \bigm| x \leq_{\mathrm{l}} y\}$.

The collection $\{\mathcal{C}_x \mid x \in \mathbf{A}\} \cup \{\emptyset\}$ is closed under intersection and generates the σ-algebra on $\overline{\mathbf{A}}$. Therefore, any finite measure ν on $\overline{\mathbf{A}}$ is entirely determined by the family of values $\big(\nu(\mathcal{C}_x)\big)_{x\in\mathbf{A}}$.

The following lemma provides an alternative description of the relation $x \leq_l y$ for $x \in \mathbf{A}$ and $y \in \overline{\mathbf{A}}$.

Lemma 2.16. *Let x be an element of an Artin–Tits monoid $\mathbf{A}$, and let $k = \tau(x)$. Let $y = (y_j)_{j\geq 1}$ be a generalized element of $\mathbf{A}$. Then $x \leq_l y$ in $\overline{\mathbf{A}}$ if and only if $x \leq_l y_1 \cdot \ldots \cdot y_k$ in $\mathbf{A}$. In particular, if $y \in \mathbf{A}$, then the relation $x \leq_l y$ holds in $\mathbf{A}$ if and only if it holds in $\overline{\mathbf{A}}$.*

Proof. Let $x \in \mathbf{A}$, $k = \tau(x)$ and $y \in \overline{\mathbf{A}}$ with $y = (y_j)_{j\geq 1}$. Assume that $x \leq_l y$. Then there is an integer $j \geq k$ such that $x \leq_l y'$, with $y' = y_1 \cdot \ldots \cdot y_j$. Applying Lemma 2.9, point 1, we obtain that $x \leq_l y_1 \cdot \ldots \cdot y_k$, which is what we wanted to prove. The converse implication also follows from Lemma 2.9: if $x \leq_l y_1 \cdot \ldots \cdot y_k$ then $x_1 \cdot \ldots \cdot x_j \leq_l y_1 \cdot \ldots \cdot y_j$ for all $j \leq k$, and $x_1 \cdot \ldots \cdot x_j = x \leq_l y_1 \cdot \ldots \cdot y_k \leq_l y_1 \cdot \ldots \cdot y_j$ for all $j > k$. □

2.3.2. Relating Garside cylinders and visual cylinders

Visual cylinders are natural from the point of view of the algebraic structure of the monoid, whereas Garside cylinders have a more operational presentation as they rely on the normal form of elements. Since both points of view are interesting, it is important to relate the two kinds of cylinders to one another, which we do now.

Given any two Garside cylinders $\mathcal{C}_x$ and $\mathcal{C}_y$, either $\mathcal{C}_x \cap \mathcal{C}_y = \emptyset$ or $\mathcal{C}_x \subseteq \mathcal{C}_y$ or $\mathcal{C}_y \subseteq \mathcal{C}_x$. Furthermore, for all $x \in \mathbf{A}$, the Garside cylinder $\mathcal{C}_x$ is contained into the full visual cylinder $\Uparrow x$. Consequently, for every open set $\mathcal{A} \subseteq \overline{\mathbf{A}}$, *i.e.*, every union of sets of the form $\Uparrow x$ with $x \in \mathbf{A}$, there exists a unique set $\mathcal{B}$ such that $\mathcal{A}$ is the disjoint union of the Garside cylinders $\mathcal{C}_x$ for $x \in \mathcal{B}$. We say that $\mathcal{B}$ is a *Garside base* of $\mathcal{A}$.

A case of special interest is that of the set $\mathcal{A} = \Uparrow x$, where $x \in \mathbf{A}$.

Definition 2.17. Let $\mathbf{A}$ be an Artin–Tits monoid. For every $x \in \mathbf{A}$, we denote by $\mathbf{A}[x]$ the Garside base of the set $\Uparrow x$, *i.e.*, the unique set such that the full cylinder $\Uparrow x$ has the following decomposition as a disjoint union:

$$\Uparrow x = \bigcup_{y\in\mathbf{A}[x]} \mathcal{C}_y\,. \tag{2.6}$$

Note that, equivalently, we might define the set $\mathbf{A}[x]$ as

$$\mathbf{A}[x] = \big\{y \in \mathbf{A} \cap \Uparrow x \;\big|\; \forall z \in \Uparrow x \quad \mathcal{C}_y \subseteq \mathcal{C}_z \Rightarrow y = z\big\}. \tag{2.7}$$

It follows from Lemma 2.18 below that the set $\mathbf{A}[x]$ is finite, and therefore that the disjoint union of (2.6) is finite.

Lemma 2.18. *Let* $\mathbf{A}$ *be an Artin–Tits monoid. For every* $x \in \mathbf{A}$ *and* $y \in \mathbf{A}[x]$*, it holds:* $\tau(y) \le \tau(x)$*.*

Proof. From $x \le_1 y$ with $y = (y_j)_{j\ge1}$, we know from Lemma 2.16 that $x \le y_1 \cdot \ldots \cdot y_k$ with $k = \tau(x)$, whence the result. □

If $x \in \mathcal{S}$, then every $y \in \mathbf{A}[x]$ has height exactly 1, and thus $\mathbf{A}[x] = \{y \in \mathcal{S} \mid x \le_1 y\}$. However, if x is of height at least 2, elements $y \in \mathbf{A}[x]$ may satisfy the *strict* inequality $\tau(y) < \tau(x)$. This is illustrated in Section 2.3.4. This situation, valid for general Artin–Tits monoids, contrasts with the case of specific monoids such as heap monoids and braid monoids, or more generally monoids of type FC; this is investigated in Section 2.3.4.

In addition, the very definition of $\mathbf{A}[x]$ has immediate consequences when considering finite measures on $\overline{\mathbf{A}}$.

Proposition 2.19. *Let* ν *be a finite measure on the completion of an Artin–Tits monoid* $\mathbf{A}$*. Then the measures of full visual cylinders and of Garside cylinders are related by the following formulas, for* x *ranging over* $\mathbf{A}$*:*

$$\nu(\Uparrow x) = \sum_{y\in\mathbf{A}[x]} \nu(\mathcal{C}_y). \tag{2.8}$$

Assume given a finite measure ν for which the values $\nu(\Uparrow x)$ are known—this will hold indeed for a family of probability measures that we shall construct later in Sections 2.4 and 4. Let $f, h : \mathbf{A} \to \mathbb{R}$ be the functions defined by $f(x) = \nu(\Uparrow x)$ and $h(x) = \nu(\mathcal{C}_x)$. Then, in view of (2.8), expressing the quantities $h(x) = \nu(\mathcal{C}_x)$ by means of the values $f(x) = \nu(\Uparrow x)$ amounts to inverting the linear operator T^* defined by:

$$\mathrm{T}^* h(x) = \sum_{y\in\mathbf{A}[x]} h(y). \tag{2.9}$$

Giving an explicit expression for the inverse of T^* is the topic of the next section.

2.3.3. Graded Möbius transform

Measure-theoretic reasoning provides a hint for guessing the right transformation. Let ν be a finite measure defined on the boundary $\partial\mathbf{A}$ of an Artin–Tits monoid $\mathbf{A} = \mathbf{A}(\Sigma, \ell)$. For x an element of $\mathbf{A}$, we put:

$$\begin{aligned} \mathcal{E}(x) &= \bigl\{u \in \mathbf{A} \bigm| \tau(x\cdot u) \le \tau(x)\bigr\} \setminus \{\boldsymbol{e}\}, \\ \mathcal{D}(x) &= \{\le_1\text{-minimal elements of } \mathcal{E}(x)\}. \end{aligned} \tag{2.10}$$

The set $\mathcal{E}(x)$ may be empty, in which case $\mathcal{D}(x) = \emptyset$ as well.

We claim that, for every $x \in \mathbf{A}$, the following equality of sets holds:

$$\mathcal{C}_x = \Uparrow x \setminus \bigcup_{u \in \mathcal{D}(x)} \Uparrow (x \cdot u). \tag{2.11}$$

Proof of (2.11). Let $(x_1, \ldots, x_k)$ be the normal form of x, and let $y \in \mathcal{C}_x$. Let $(y_j)_{j \geq 1}$ be the extended normal form of y, with $y_j = x_j$ for all $j \in \{1, \ldots, k\}$. Hence $x \leq_1 (y_1 \cdot \ldots \cdot y_j)$ for all $j \geq k$, and thus $y \in \Uparrow x$ according to the definition of the partial ordering $\leq_1$ on $\overline{\mathbf{A}}$ given in Definition 2.15. Let $u \in \mathbf{A}$ be such that $\tau(x \cdot u) \leq \tau(x)$ and $y \in \Uparrow (x \cdot u)$. It follows from Lemma 2.16 that $x \cdot u \leq_1 y_1 \cdot \ldots \cdot y_k$, and since $y_1 \cdot \ldots \cdot y_k = x$ it implies that $u = \boldsymbol{e}$. Hence $u \notin \mathcal{D}(x)$ and the $\subseteq$ inclusion in (2.11) follows.

Conversely, let y be an element of the right-hand set of (2.11), and let $(y_j)_{j \geq 1}$ be the generalized normal form of y. Let us prove that $y_1 \cdot \ldots \cdot y_k = x$, which entails that $y \in \mathcal{C}_x$. For this, we use the characterization given in Lemma 2.9, point 1:

$$y_1 \cdot \ldots \cdot y_k = \bigvee\nolimits_1 U, \qquad \text{with } U = \{z \in S^k \bigm| z \leq_1 y\}.$$

We have $x \in U$ since $x \leq_1 y$ and $\tau(x) = k$ by assumption. Let $z = \bigvee_1 U$ and, seeking a contradiction, assume that $x \neq z$. Then $z = x \cdot u$ with $u \neq \boldsymbol{e}$ and $\tau(x \cdot u) \leq \tau(x)$, thus $u \in \mathcal{D}(x)$. But then $y \in \Uparrow (x \cdot u)$, contradicting the assumption on y. $\square$

We observe that for any elements $u, u' \in \mathcal{D}(x)$, one has that $(x \cdot u) \vee_1 (x \cdot u')$ exists in $\mathbf{A}$ if and only if $u \vee_1 u'$ exists in $\mathbf{A}$, if and only if $\Uparrow (x \cdot u) \cap \Uparrow (x \cdot u') \neq \emptyset$. If these conditions are fulfilled, we have:

$$\Uparrow (x \cdot u) \cap \Uparrow (x \cdot u') = \Uparrow \big(x \cdot (u \vee_1 u')\big),$$

and the latter generalizes to intersections of the form $\Uparrow (x \cdot u_1) \cap \cdots \cap \Uparrow (x \cdot u_k)$ for $u_1, \ldots, u_k \in \mathcal{D}(x)$. Taking the measure of both members in (2.11), and applying the inclusion-exclusion principle, which is basically the essence of Möbius inversion formulas, we obtain thus:

$$\nu(\mathcal{C}_x) = \nu(\Uparrow x) - \sum_{D \Subset \mathcal{D}(x),\ D \neq \emptyset} (-1)^{|D|+1} \nu\big(\Uparrow \big(x \cdot \bigvee\nolimits_1 D\big)\big),$$

where $D \Subset \mathcal{D}(x)$ means that D is a subset of $\mathcal{D}(x)$ *and* that $\bigvee_1 D$ exists in $\mathbf{A}$. Observing that $\emptyset \Subset \mathcal{D}(x)$ with $\bigvee_1 \emptyset = \boldsymbol{e}$, we get:

$$\nu(\mathcal{C}_x) = \sum_{D \Subset \mathcal{D}(x)} (-1)^{|D|} \nu\big(\Uparrow \big(x \cdot \bigvee\nolimits_1 D\big)\big). \tag{2.12}$$

We are thus brought to introduce the following definition.

Definition 2.20. Let $\mathbf{A}$ be an Artin–Tits monoid. The *graded Möbius transform* of a function $f : \mathbf{A} \to \mathbb{R}$ is the function $\mathrm{T}f : \mathbf{A} \to \mathbb{R}$, defined as follows for every $x \in \mathbf{A}$:

$$\mathrm{T}f(x) = \sum_{D \Subset \mathcal{D}(x)} (-1)^{|D|} f\bigl(x \cdot \bigvee\nolimits_{l} D\bigr),$$

where $\mathcal{D}(\cdot)$ has been defined in (2.10) and $D \Subset \mathcal{D}(x)$ means that D is a subset of $\mathcal{D}(x)$ such that $\bigvee_l D$ exists in $\mathbf{A}$.

The *inverse graded Möbius transform* of a function $h : \mathbf{A} \to \mathbb{R}$ is the function $\mathrm{T}^*h : \mathbf{A} \to \mathbb{R}$, defined as follows for every $x \in \mathbf{A}$:

$$\mathrm{T}^*h(x) = \sum_{y \in \mathbf{A}[x]} h(y),$$

where $\mathbf{A}[\cdot]$ has been defined in (2.7).

Section 2.3.4 below details the expression of the graded Möbius transform for some specific examples. Before that, we bring additional information on the range of summation in the definition of the graded Möbius transform.

Lemma 2.21. *In an Artin–Tits monoid $\mathbf{A}$, for any $x \in \mathbf{A}$, one has $\mathcal{D}(x) = \mathcal{D}(u)$, where u is the* last *simple element in the normal form of x.*

Proof. Let $(x_1, \ldots, x_k)$ be the normal form of x. We first show the following:

$$\forall y \in \mathcal{S} \quad \mathcal{D}(x) \cap \downarrow y = \emptyset \iff \mathcal{D}(x_k) \cap \downarrow y = \emptyset, \tag{2.13}$$

where $\downarrow y = \{z \in \mathbf{A} \mid z \leq_l y\}$. Indeed, for $y \in \mathcal{S}$, one has according to Lemma 2.9:

$$\begin{aligned} x_k \to y &\iff x = \bigvee\nolimits_l \{z \in \mathcal{S}^k \mid z \leq_l x \cdot y\} \\ &\iff \mathcal{E}(x) \cap \downarrow y = \emptyset \iff \mathcal{D}(x) \cap \downarrow y = \emptyset. \end{aligned}$$

Applying the above equivalence with $x = x_k$ on the one hand, and with x on the other hand, yields (2.13). From (2.13), and since $\mathcal{D}(x)$ and $\mathcal{D}(x_k)$ are two $\leq_l$-antichains, we deduce that $\mathcal{D}(x) = \mathcal{D}(x_k)$, as expected. □

With Definition 2.20, we reformulate (2.12) as follows.

Proposition 2.22. *Consider a finite measure ν on the completion of an Artin–Tits monoid $\mathbf{A}$. For every $x \in \mathbf{A}$, let $f(x) = \nu(\Uparrow x)$ and let $h(x) = \nu(\mathcal{C}_x)$. Then h is the graded Möbius transform of f.*

If ν and ν' are two finite measures such that $\nu(\Uparrow x) = \nu'(\Uparrow x)$ for all $x \in \mathbf{A}$, then $\nu = \nu'$.

Proof. The first part of the statement is a simple rephrasing of (2.12). For the second part, if $\nu(\Uparrow x) = \nu'(\Uparrow x)$ for all $x \in \mathbf{A}$, then $\nu(\mathcal{C}_x) = \nu'(\mathcal{C}_x)$ for all $x \in \mathbf{A}$. We have already observed that this implies $\nu = \nu'$. □

Propositions 2.17 and 2.22 show that the transformations T and T* are inverse of each other when operating on functions f and h of the form $f(x) = \nu(\Uparrow x)$ and $h(x) = \nu(\mathcal{C}_x)$. We actually have the following more general result, of a purely combinatorial nature.

Theorem 2.23. *Let* $\mathbf{A}$ *be an Artin–Tits monoid. Then the graded Möbius transform* T *and the inverse graded Möbius transform* T* *are two endomorphisms inverse of each other, defined on the space of functions* $\mathbf{A} \to \mathbb{R}$*.*

We first prove the following lemma.

Lemma 2.24. *Let* $\mathbf{A}$ *be an Artin–Tits monoid, let* $x, y \in \mathbf{A}$*, and let:*

$$\mathcal{B}(x,y) = \big\{D \Subset \mathcal{D}(x) \;\big|\; y \in \mathbf{A}\big[x \cdot \textstyle\bigvee_1 D\big]\big\}.$$

Then:

$$\sum_{D \in \mathcal{B}(x,y)} (-1)^{|D|} = \mathbf{1}(x = y). \tag{2.14}$$

Proof. For $x, y \in \mathbf{A}$, we put:

$$\begin{aligned} K(x,y) &= \big\{z \in \mathbf{A} \;\big|\; x \cdot z \leq_1 y \text{ and } \tau(x \cdot z) \leq \tau(x)\big\}, \\ \mathcal{K}(x,y) &= \big\{D \Subset \mathcal{D}(x) \;\big|\; \textstyle\bigvee_1 D \in K(x,y)\big\}. \end{aligned}$$

Then we claim:

$$\sum_{D \in \mathcal{K}(x,y)} (-1)^{|D|} = \begin{cases} 1, & \text{if } y \in \mathcal{C}_x\,, \\ 0, & \text{otherwise.} \end{cases} \tag{2.15}$$

Let us first prove (2.15). If $x \leq_1 y$ does not hold, then $K(x,y) = \mathcal{K}(x,y) = \emptyset$ and (2.15) is true. Hence we assume that $x \leq_1 y$. Let $(y_1, \ldots, y_k)$ be the normal form of y. We put $y' = y_1 \cdot \ldots \cdot y_j$, where $j = \min\{i \geq 1 \;\big|\; x \leq_1 y_1 \cdot \ldots \cdot y_i\}$, and thus $x \leq_1 y'$ holds, with equality if and only if $y \in \mathcal{C}_x$. We observe that $K(x,y)$ is a lattice: this follows from Lemma 2.9; and thus $\mathcal{K}(x,y)$ is a Boolean lattice.

We discuss two cases. If $x = y'$, then $K(x,y) = \{\boldsymbol{e}\}$ and $\mathcal{K}(x,y) = \{\emptyset\}$, and thus (2.15) is true. However, if $x \neq y'$, then Lemma 2.16 proves that $\tau(y') \leq \tau(x)$. Hence, the set $K(x,y')$ contains a minimal non-unit element z. This element satisfies $\{z\} \in \mathcal{K}(x,y)$, which is thus a non-trivial Boolean lattice. Hence (2.15) is true in all cases.

We now come to the proof of (2.14). The case where $x = y$ or $\neg(x \leq_1 y)$ holds is easily settled, hence we assume below that $x <_1 y$. In particular, observe that $y \neq \boldsymbol{e}$. Let $(y_1, \ldots, y_k)$ be the normal form of y, and let $\widetilde{y} = y_1 \cdot \ldots \cdot y_{k-1}$. Then we have $\mathcal{K}(x, \widetilde{y}) \subseteq \mathcal{K}(x, y)$ and $\mathcal{B}(x, y) = \mathcal{K}(x, y) \setminus \mathcal{K}(x, \widetilde{y})$. Therefore, using (2.15), we have:

$$\sum_{D \in \mathcal{B}(x,y)} (-1)^{|D|} = A(x, y) - A(x, \widetilde{y}), \tag{2.16}$$

$$\text{with } A(u, v) = \begin{cases} 1, & \text{if } v \in \mathcal{C}_u, \\ 0, & \text{otherwise.} \end{cases}$$

If $\widetilde{y}$ belongs to the Garside cylinder $\mathcal{C}_x$, so does y. Conversely, since $x \neq y$, if $y \in \mathcal{C}_x$ then the normal form of x is a strict prefix of the normal form of y, and therefore $\widetilde{y} \in \mathcal{C}_x$. Together with (2.16) this completes the proof of (2.14). $\square$

Proof of Theorem 2.23. Let $h : \mathbf{A} \to \mathbb{R}$ be given. We compute, exchanging the order of summation:

$$\mathrm{T}\big(\mathrm{T}^*(h)\big)(x) = \sum_{y \in \mathbf{A}} h(y) \Big(\sum_{D \in \mathcal{B}(x,y)} (-1)^{|D|} \Big) = h(x),$$

where we have put $\mathcal{B}(x, y) = \big\{ D \Subset \mathcal{D}(x) \bigm| y \in \mathbf{A}[x \cdot \bigvee D] \big\}$, and where the last equality follows from Lemma 2.24.

Let E denote the vector space of functions $\mathbf{A} \to \mathbb{R}$. Then T is an endomorphism of E, and it remains to prove that T is injective. Let f be a non-zero function $f \in E$, let k be the smallest integer such that f is non-zero on $\mathcal{S}^k$, and let x be a maximal element of $\mathcal{S}^k$ such that $f(x) \neq 0$. For all non-empty sets $D \Subset \mathcal{D}(x)$, we have $x <_1 x \cdot \bigvee_1 D$, and $x \cdot \bigvee_1 D \in \mathcal{S}^\ell$ for some $\ell \leq k$, hence $f(x \cdot \bigvee_1 D) = 0$. It follows that $\mathrm{T}(f)(x) = f(x) \neq 0$, which completes the proof. $\square$

2.3.4. Examples and particular cases

In this section, we first develop the expression of the graded Möbius transform for monoids of type FC, pointing out the simplification that arises in this case. By contrast, we introduce then the example of an Artin–Tits monoid which is not of type FC. Finally, we give the expression of the graded Möbius transform and its inverse evaluated on the simple elements of a general Artin–Tits monoid.

Let us first examine the case of Artin–Tits monoids of type FC.

Proposition 2.25. *Let $\mathbf{A}$ be an Artin–Tits monoid of type FC. Then, for every $x \in \mathbf{A}$, the sets $\mathbf{A}[x]$ and $\mathcal{D}(x)$ have the following expressions:*

$$\begin{aligned} \mathbf{A}[x] &= \{ y \in \mathbf{A} \mid x \leq_1 y \text{ and } \tau(y) \leq \tau(x) \} \\ &= \{ y \in \mathbf{A} \mid x \leq_1 y \text{ and } \tau(y) = \tau(x) \} \\ \mathcal{D}(x) &= \{ \sigma \in \Sigma \mid \tau(x \cdot \sigma) = \tau(x) \}. \end{aligned}$$

Proof. Let $x, y \in \mathbf{A}$ such that $x \leq_l y$ and $\tau(y) \leq \tau(x)$. Let $k = \tau(y)$, and let $(y_1, \ldots, y_k)$ be the normal form of y. The following statement is proved in [11, Prop. 2.95]:

(‡) *Let* $\mathbf{A}$ *be an Artin–Tits monoid of type FC. For all* $a, b \in \mathbf{A}$ *such that* $a \leq_l b$, *we have* $\tau(a) \leq \tau(b)$.

It follows from (‡) that $\tau(x) = \tau(y) = k$, *i.e.*, that $x \leq_l y_1 \cdot \ldots \cdot y_k$ and that $\neg(x \leq_l y_1 \cdot \ldots \cdot y_j)$ for all $j \leq k-1$. This proves the inclusion $\mathbf{A}[x] \supseteq \{y \in \mathbf{A} \mid x \leq_l y \text{ and } \tau(y) \leq \tau(x)\}$, whereas the converse inclusion follows from Lemma 2.18. Then, the claim (‡) also proves the inclusion $\{y \in \mathbf{A} \mid x \leq_l y \text{ and } \tau(y) \leq \tau(x)\} \subseteq \{y \in \mathbf{A} \mid x \leq_l y \text{ and } \tau(y) = \tau(x)\}$, and the converse inclusion is immediate.

Finally, assume that $\mathcal{D}(x)$ is non-empty, and let z be an element of $\mathcal{D}(x)$. Since $z \neq \boldsymbol{e}$, we can write $z = \sigma \cdot z'$ for some $\sigma \in \Sigma$ and $z' \in \mathbf{A}$. Since $x \leq_l x \cdot \sigma \leq_l x \cdot z$ and $k = \tau(x) = \tau(x \cdot z)$, it follows from (‡) that $k = \tau(x \cdot \sigma)$, *i.e.*, that $x \cdot \sigma \in \mathbf{A}[x]$. By minimality of z, we must have $z = \sigma$. It follows that $\mathcal{D}(x) \subseteq \{\sigma \in \Sigma \mid x \cdot \sigma \in \mathbf{A}[x]\}$, whereas the converse inclusion is immediate. Recalling that $\mathbf{A}[x] = \{y \in \mathbf{A} \mid x \leq_l y \text{ and } \tau(y) = \tau(x)\}$, the proof is complete. □

Recalling that $\mathcal{D}(x) = \mathcal{D}(u)$ according to Lemma 2.21, where u is the last element in the normal form of x, it follows that the graded Möbius transform, in a monoid of type FC, has the following form:

$$\mathrm{T}f(x) = \sum_{D \Subset \Sigma \,:\, u \cdot (\bigvee_l D) \in \mathcal{S}} (-1)^{|D|} f\big(x \cdot \bigvee\nolimits_l D\big).$$

Compared to monoids of type FC, general Artin–Tits monoids have the pathology that the smallest Garside subset $\mathcal{S}$ is not necessarily closed by left divisibility. It entails that the strict inequality $\tau(x \cdot \sigma) < \tau(x)$ may occur. This is illustrated in the following example, where we point out two consequences of this fact.

Let $\mathbf{A}$ be the Artin–Tits monoid defined below, known as the Artin–Tits groups of type affine $\tilde{A}_2$—see, for instance, [14]. It admits the set $\mathcal{S}$ described below as smallest Garside subset:

$$\mathbf{A} = \langle a, b, c \mid aba = bab,\ bcb = cbc,\ cac = aca \rangle$$
$$\mathcal{S} = \{\boldsymbol{e}, a, b, c, ab, ac, ba, bc, ca, cb, aba, bcb, cac, abcb, bcac, caba\}.$$

Hence, the normal forms of $x = abc$ and $y = abcb$ are, respectively, (ab, c) and $(abcb)$, which shows that $\tau(x \cdot b) = 1 < \tau(x) = 2$. The Hasse diagram of $(\mathcal{S}, \leq_l)$ is depicted on Fig. 1.

As a first consequence of $\mathcal{S}$ not being closed by left divisibility, the Garside cylinders $\mathcal{C}_y$ may not be disjoint, for $x \in \mathbf{A}$ fixed and for y ranging over $\{z \in \mathbf{A} \mid x \leq_l z \text{ and } \tau(z) \leq$

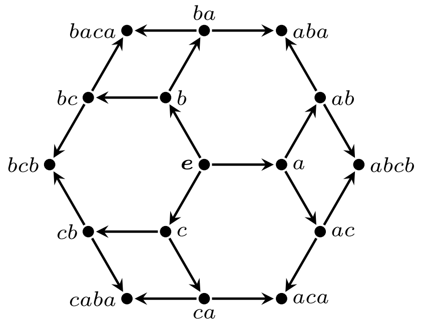

Fig. 1. Hasse diagram of $(\mathcal{S}, \leq_1)$ for the Artin–Tits monoid $\mathbf{A} = \langle a, b, c \mid aba = bab,\ bcb = cbc,\ cac = aca \rangle$. In this example, $\mathcal{S}$ is not closed by left divisibility since, for instance, $caba \in \mathcal{S}$ whereas $cab \notin \mathcal{S}$.

$\tau(x)\}$. Indeed, consider again $x = abc$, of height 2, and $y = abcb$ and $y' = abcba$, of height 1 and 2 respectively, the latter since the normal form of y' is $(abcb, a)$. Then both y and y' belong to the set described above; yet, $\mathcal{C}_y \subseteq \mathcal{C}_{y'}$ and thus $\mathcal{C}_y \cap \mathcal{C}_{y'} \neq \emptyset$. Whereas, combining Proposition 2.17 and Proposition 2.25, one sees that this situation cannot happen in monoids of type FC.

As a second consequence of $\mathcal{S}$ not being closed by left divisibility, the sets $\mathcal{D}(x)$ are not in general subsets of Σ, but of $\mathcal{S}$ itself. Indeed, one has for instance: $\mathcal{D}(ab) = \{a, cb\}$, contrary to the second statement of Proposition 2.25 for monoids of type FC.

We note however that $\mathcal{D}(\boldsymbol{e}) \subseteq \Sigma$ in any Artin–Tits monoid. Therefore, evaluated at $\boldsymbol{e}$, the graded Möbius transform has the following expression:

$$\mathrm{T}f(\boldsymbol{e}) = \sum_{D \Subset \Sigma} (-1)^{|D|} f\big(\textstyle\bigvee_1 D\big). \tag{2.17}$$

Consider in particular the case of a function f of the form $f(x) = p^{|x|}$, for some real number p. Then:

$$\mathrm{T}f(\boldsymbol{e}) = \sum_{D \Subset \Sigma} (-1)^{|D|} p^{|\bigvee_1 D|}. \tag{2.18}$$

This is a polynomial expression in p. We shall see in Section 4.1.2 that this polynomial corresponds to the Möbius polynomial of the monoid, which is a simplified version of the Möbius function in the sense of Rota [15] associated with the partial order.

Furthermore, if x is simple in a general Artin–Tits monoid, then $\mathbf{A}[x]$ has the following expression: $\mathbf{A}[x] = \{y \in \mathcal{S} \mid x \leq_1 y\}$. Therefore, restricted to $\mathcal{S}$, the inverse graded Möbius transform takes the following form:

$$\text{for } x \in \mathcal{S} \qquad \mathrm{T}^* h(x) = \sum_{y \in \mathcal{S}\,:\, x \leq_1 y} h(y). \tag{2.19}$$

2.4. Multiplicative probability measures on the boundary

In this section, we introduce a particular class of measures defined on the boundary of Artin–Tits monoids, the class of *multiplicative measures*. Measures defined on the boundary of an Artin–Tits monoid $\mathbf{A}$ can be seen as those measures ν on the completion $\overline{\mathbf{A}}$ with support in $\partial\mathbf{A}$, *i.e.*, such that $\nu(x) = 0$ for all $x \in \mathbf{A}$. In Section 2.4.2, we entirely characterize multiplicative measures through a finite family of probabilistic parameters. Each multiplicative measure induces a natural finite Markov chain. Finally, we investigate in Section 2.4.3 the particular case of the uniform measure.

2.4.1. Multiplicative measures and valuations

Multiplicative measures on the boundary of Artin–Tits monoids are probability measures that generalize the classical Bernoulli measures on infinite sequences of letters.

Definition 2.26. Let $\mathbf{A}$ be an Artin–Tits monoid. A finite measure ν on the boundary $\partial\mathbf{A}$ is *multiplicative* if it satisfies the two following properties:

$$\forall x \in \mathbf{A} \quad \nu(\uparrow x) > 0$$
$$\forall x \in \mathbf{A} \quad \forall y \in \mathbf{A} \quad \nu(\uparrow (x \cdot y)) = \nu(\uparrow x) \cdot \nu(\uparrow y).$$

A *valuation on* $\mathbf{A}$ is any function $f : \mathbf{A} \to (0, +\infty)$ satisfying $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in \mathbf{A}$.

If ν is a multiplicative measure on $\partial\mathbf{A}$, then the *valuation associated with* ν is the function $f : \mathbf{A} \to (0, +\infty)$ defined by $f(x) = \nu(\uparrow x)$ for all $x \in \mathbf{A}$.

Remark 2.27.

1. Assuming that multiplicative measures exist, which is not obvious to prove, any such measure satisfies $\nu(\partial\mathbf{A}) = \nu(\uparrow \boldsymbol{e}) = 1$, hence is a probability measure.
2. If $\mathbf{A}$ is a free monoid, multiplicative measures correspond to the usual Bernoulli measures characterizing *i.i.d.* sequences. If $\mathbf{A}$ is a heap monoid or a braid monoid, multiplicative measures have been introduced respectively in [16] and in [1].

Let us put aside a trivial multiplicative measure which is found in Artin–Tits monoids of spherical type.

Definition 2.28. Let $\mathbf{A}$ be an Artin–Tits monoid of spherical type. Let Δ_∞ denote the boundary element $\Delta_\infty = (\Delta, \Delta, \ldots)$, where Δ is the Garside element of $\mathbf{A}$. Then the constant valuation $f(x) = 1$ on $\mathbf{A}$ corresponds to the multiplicative measure $\nu = \delta_{\Delta_\infty}$, which we call *degenerate*. Any other multiplicative measure on the boundary of an Artin–Tits monoid, either of spherical type or not, is *non-degenerate*.

By Proposition 2.22, a multiplicative measure is entirely determined by its associated valuation. We note that valuations always exist: it suffices to consider for instance f to be constant on the set Σ of generators of $\mathbf{A}$, say equal to r. Then f extends uniquely to a valuation on $\mathbf{A}$, given by $f(x) = r^{|x|}$ for all $x \in \mathbf{A}$. Since we will give a special attention to this kind of valuation and associated multiplicative measures whenever they exist, we introduce the following definition.

Definition 2.29. A valuation $f : \mathbf{A} \to (0, +\infty)$ on an Artin–Tits monoid $\mathbf{A} = \mathbf{A}(\Sigma, \ell)$ is *uniform* if it is constant on Σ, or equivalently if it is of the form $f(x) = p^{|x|}$ for some $p \in (0, +\infty)$. A multiplicative measure is *uniform* if it is non-degenerate and if its associated valuation is uniform.

2.4.2. Characterization and realization of multiplicative measures

Our aim is to characterize the valuations that correspond to multiplicative measures. To this end, we introduce the following definition.

Definition 2.30. Let $f : \mathbf{A} \to (0, +\infty)$ be a valuation on an Artin–Tits monoid $\mathbf{A}$, and let $h = \mathrm{T}f$ be the graded Möbius transform of f. We say that f is a *Möbius valuation* whenever:

$$h(\boldsymbol{e}) = 0, \qquad \text{and} \qquad \forall x \in \mathcal{S} \setminus \{\boldsymbol{e}\} \quad h(x) > 0,$$

where $\mathcal{S}$ denotes as usual the smallest Garside subset of $\mathbf{A}$.

This definition is motivated by the following result, which shows in particular that being Möbius is a necessary and sufficient condition for a valuation to be associated with some non-degenerate multiplicative measure. It also details the nature of the probabilistic process associated with the boundary elements.

Theorem 2.31. *Let $\mathbf{A}$ be an irreducible Artin–Tits monoid, and let ν be a non-degenerate multiplicative measure on $\partial\mathbf{A}$. Then the valuation $f(\cdot) = \nu(\uparrow \cdot)$ is a Möbius valuation.*

Conversely, if f is a Möbius valuation on $\mathbf{A}$, then there exists a non-degenerate multiplicative measure on $\partial\mathbf{A}$, necessarily unique, say ν, such that $f(\cdot) = \nu(\uparrow \cdot)$. Hence, non-degenerate multiplicative measures on $\partial\mathbf{A}$ correspond bijectively to Möbius valuations on $\mathbf{A}$.

Furthermore, let (f, ν) be such a corresponding pair. Let h be the graded Möbius transform of f. For every integer $n \geq 1$, let $X_n : \partial\mathbf{A} \to \mathcal{S} \setminus \{\boldsymbol{e}\}$ denote the nth canonical projection of boundary elements, which maps a boundary element $\xi = (y_j)_{j\geq 1}$ to the simple element y_n. Then, under the probability measure ν, the sequence of random variables $(X_n)_{n\geq 1}$ is a homogeneous Markov chain with values in the finite set $\mathcal{S} \setminus \{\boldsymbol{e}\}$, with initial distribution and with transition matrix P given by:

$$\forall x \in \mathcal{S} \setminus \{\boldsymbol{e}\} \quad \nu(X_1 = x) = h(x),$$

$$\forall x, y \in \mathcal{S} \setminus \{\boldsymbol{e}\} \quad P_{x,y} = \mathbf{1}(x \to y) f(x) \frac{h(y)}{h(x)} .$$

This Markov chain is ergodic if $\mathbf{A}$ *is not of spherical type; and has* $\mathcal{S} \setminus \{\boldsymbol{e}, \Delta\}$ *as unique ergodic component if* $\mathbf{A}$ *is of spherical type.*

The following lemmas are central in the proof of Theorem 2.31, and also for subsequent results.

Lemma 2.32. *Let* $\mathbf{A}$ *be an Artin–Tits monoid, of smallest Garside subset* $\mathcal{S}$*. Let* $f : \mathbf{A} \to (0, +\infty)$ *be a valuation, and let* $h = \mathrm{T}f$ *be the graded Möbius transform of* f*. For an element* $x \in \mathbf{A}$*, let* $u \in \mathcal{S}$ *be the last element in the normal form of* x*, and let* $\widetilde{x} \in \mathbf{A}$ *be such that* $x = \widetilde{x} \cdot u$*. It holds that:*

$$h(x) = f(\widetilde{x}) h(u).$$

Proof. Lemma 2.21 proves that $\mathcal{D}(x) = \mathcal{D}(u)$, and since f is a valuation it follows that

$$\begin{aligned} h(x) &= \sum_{D \Subset \mathcal{D}(x)} (-1)^{|D|} f\bigl(x \cdot \textstyle\bigvee_1 D\bigr) \\ &= \sum_{D \Subset \mathcal{D}(u)} (-1)^{|D|} f(\widetilde{x}) f\bigl(u \cdot \textstyle\bigvee_1 D\bigr) = f(\widetilde{x}) h(u). \quad \square \end{aligned}$$

Lemma 2.33. *Let* $\mathbf{A}$ *be an Artin–Tits monoid, of smallest Garside subset* $\mathcal{S}$*. Let* $f : \mathbf{A} \to (0, +\infty)$ *be a valuation, and let* $h = \mathrm{T}f$ *be the graded Möbius transform of* f*. Let also* $g : \mathbf{A} \to \mathbb{R}$ *be the function defined by:*

$$g(x) = \sum_{y \in \mathcal{S} \,:\, u \to y} h(y),$$

where $u \in \mathcal{S}$ *is the last element in the normal form of* x*. Then* $h(x) = f(x) \cdot g(x)$ *holds for all* $x \in \mathbf{A}$*.*

Proof. Let $\mathcal{P}(\mathcal{S})$ denote the powerset of $\mathcal{S}$, and let $F, G : \mathcal{P}(\mathcal{S}) \to \mathbb{R}$ be the two functions defined by:

$$F(U) = \sum_{D \Subset U} (-1)^{|D|} f\bigl(\textstyle\bigvee_1 D\bigr), \qquad G(U) = \sum_{y \in \mathcal{S}} \mathbf{1}(U \cap \downarrow y = \emptyset) h(y),$$

where $\downarrow y = \{z \in \mathbf{A} \mid z \leq_1 y\}$.

We first prove the equality $F(U) = G(U)$ for all $U \in \mathcal{P}(\mathcal{S})$. For any $x \in \mathcal{S}$, we have $f(x) = \mathrm{T}^* h(x)$ according to Theorem 2.23. According to (2.19), this writes as follows:

$$f(x) = \sum_{y \in \mathcal{S} \,:\, x \leq_1 y} h(y).$$

This works in particular for $x = \bigvee_1 D$ for any $D \Subset \mathcal{S}$, since then $x \in \mathcal{S}$, yielding:

$$\begin{aligned} F(U) &= \sum_{D \Subset U} (-1)^{|D|} \Big(\sum_{y \in \mathcal{S}} \mathbf{1}\big(\bigvee\nolimits_1 D \leq_1 y\big) h(y) \Big) \\ &= \sum_{y \in \mathcal{S}} \Big(\sum_{D \Subset U} \mathbf{1}\big(\bigvee\nolimits_1 D \leq_1 y\big)(-1)^{|D|} \Big) \cdot h(y) \qquad (2.20) \\ &= \sum_{y \in \mathcal{S}} \Big(\sum_{D \subseteq (U \cap \downarrow y)} (-1)^{|D|} \Big) \cdot h(y) = G(U). \qquad (2.21) \end{aligned}$$

Next, we observe that for every $u, y \in \mathcal{S}$, one has $u \to y$ if and only if $\mathcal{D}(u) \cap \downarrow y = \emptyset$, which implies:

$$g(u) = G\big(\mathcal{D}(u)\big). \qquad (2.22)$$

And, using that f is a valuation, we have:

$$\begin{aligned} h(u) &= \sum_{D \Subset \mathcal{D}(u)} (-1)^{|D|} f\big(u \cdot \bigvee\nolimits_1 D\big) \\ &= f(u) \cdot \Big(\sum_{D \Subset \mathcal{D}(u)} (-1)^{|D|} f\big(\bigvee\nolimits_1 D\big) \Big) = f(u) \cdot F\big(\mathcal{D}(u)\big). \qquad (2.23) \end{aligned}$$

Putting together (2.23), (2.22) and (2.20) with $U = \mathcal{D}(u)$, we obtain $h(u) = f(u) \cdot g(u)$.

So far, we have proved the statement of the lemma for $x = u \in \mathcal{S}$. For an element $x \in \mathbf{A}$, let $u \in \mathcal{S}$ be the last element in the normal form of x, and let $\widetilde{x} \in \mathbf{A}$ be such that $x = \widetilde{x} \cdot u$. Lemma 2.32 states that $h(x) = f(\widetilde{x}) \cdot h(u)$. We also have $g(x) = g(u)$, and thus by the first part of the proof: $h(x) = f(\widetilde{x}) \cdot f(u) \cdot g(u) = f(x) \cdot g(x)$, which completes the proof. $\square$

Lemma 2.34. *Let $f : \mathbf{A} \to (0, +\infty)$ be a valuation defined on an irreducible Artin–Tits monoid $\mathbf{A}$ with at least two generators. Let $h = \mathrm{T}f$ be the graded Möbius transform of f, that we assume to satisfy $h(\boldsymbol{e}) = 0$.*

We consider the Charney graph $(\mathscr{C}, \to)$, and the non-negative square matrix $B = (B_{x,y})_{(x,y) \in \mathscr{C} \times \mathscr{C}}$ defined by $B_{x,y} = \mathbf{1}(x \to y) f(y)$. Let also $g : \mathbf{A} \to \mathbb{R}$ be the function defined as in Lemma 2.33. Then:

1. *The matrix B is primitive, and the column vector $\overline{g} = (g(x))_{x \in \mathscr{C}}$ satisfies $B \cdot \overline{g} = \overline{g}$.*
2. *Furthermore, if $f(\cdot) = \nu(\uparrow \cdot)$ for some multiplicative measure ν, then the assumption $h(\boldsymbol{e}) = 0$ is necessarily satisfied, and one and only one of the following two propositions is true:*
   (a) *h and g are identically zero on $\mathscr{C}$. In this case, $\mathbf{A}$ is necessarily of spherical type, and ν is the degenerate measure δ_{Δ_∞}.*

(b) *h and g are positive on $\mathscr{C}$, B has spectral radius* 1, *$\overline{g}$ is the Perron eigenvector of B and ν is non-degenerate.*

Proof. The graph $(\mathscr{C}, \to)$ is non-empty since $\mathbf{A}$ is assumed to have at least two generators. The function f is positive on $\mathscr{C}$ by definition of valuations; the graph $(\mathscr{C}, \to)$ is strongly connected according to Theorem 2.12, and has loops since every element $\sigma \in \Sigma$ belongs to $\mathscr{C}$ and satisfies $\sigma \to \sigma$. It follows that B is indeed primitive. Finally, to see that $\overline{g}$ is a fixed point of B, we compute:

$$(B \cdot \overline{g})_x = \sum_{y \in \mathscr{C} \,:\, x \to y} f(y)g(y) = \sum_{y \in \mathscr{C} \,:\, x \to y} h(y),$$

the latter equality by Lemma 2.33. Now the set $\mathscr{C}$ differs from $\mathcal{S}$ by at most two elements: either $\boldsymbol{e}$, if $\mathbf{A}$ is not of spherical type, or $\boldsymbol{e}$ and Δ if $\mathbf{A}$ is of spherical type. Since $h(\boldsymbol{e}) = 0$ by assumption, and since $x \to \Delta$ does not hold for any $x \in \mathscr{C}$, the above equality writes $(B \cdot \overline{g})_x = \sum_{y \in \mathcal{S} \,:\, x \to y} h(y) = g(x)$, which completes the proof of point 1.

For point 2, assume that $f(\cdot) = \nu(\uparrow \cdot)$ for some multiplicative measure ν on $\partial\mathbf{A}$. Write $\xi = (X_1, X_2, \dots)$ for a generic element $\xi \in \partial\mathbf{A}$. Then, by Proposition 2.22 applied to elements of $\mathcal{S}$, one has $h(x) = \nu(X_1 = x)$ for all $x \in \mathcal{S}$. It implies on the one hand that h is non-negative on $\mathcal{S}$. On the second hand, since X_1 takes its values in $\mathcal{S} \setminus \{\boldsymbol{e}\}$ only, the total probability law yields:

$$\sum_{y \in \mathcal{S} \setminus \{\boldsymbol{e}\}} h(y) = 1. \tag{2.24}$$

Applying formula (2.19) to $x = \boldsymbol{e}$ yields:

$$\sum_{y \in \mathcal{S}} h(y) = \nu(\uparrow \boldsymbol{e}) = \nu(\partial\mathbf{A}) = 1. \tag{2.25}$$

Comparing (2.24) and (2.25) yields $h(\boldsymbol{e}) = 0$.

Since h is non-negative, the vector $\overline{g}$ is also non-negative. Since B is primitive, it follows from the Perron–Frobenius Theorem for primitive matrices [17] that $\overline{g}$ is either identically zero or positive.

Assume that $\overline{g} = 0$. Then Lemma 2.33 implies that $h = 0$ on $\mathscr{C}$. It follows from (2.24) that $\mathscr{C} \setminus \{\boldsymbol{e}\} \neq \emptyset$, and thus that $\mathbf{A}$ is of spherical type and that $h(\Delta) = 1$, where Δ is the Garside element of $\mathbf{A}$. For any element $x \in \mathbf{A}$, if $u \in \mathcal{S}$ is the last element of the normal form of x, and if $\widetilde{x} \in \mathbf{A}$ is such that $x = \widetilde{x} \cdot u$, Lemma 2.32 states that $h(x) = f(\widetilde{x})h(u)$. Hence, for all normal sequences $(x_1, \dots, x_k)$, if $x_i \neq \Delta$ for some $i \leq k$, then by Proposition 2.22 we have:

$$\nu(X_1 = x_1, \dots, X_k = x_k) \leq \nu(X_1 = x_1, \dots, X_i = x_i) = h(x_1 \cdot \ldots \cdot x_i) = 0,$$

from which follows $\nu\big(\xi = (\Delta, \Delta, \dots)\big) = \nu\big(\partial\mathbf{A}\big) = 1$, or in other words, $\nu = \delta_{\Delta_\infty}$, as claimed in point 2a.

Assume now that $\overline{g} \neq 0$, and thus that $\overline{g}$ is positive. Since $\overline{g}$ is a fixed point of B, it follows from the Perron–Frobenius Theorem for primitive matrices that $\overline{g}$ is a Perron eigenvector of B, and thus B is of spectral radius 1. From $\overline{g} > 0$ and from Lemma 2.33, we deduce that $h > 0$ on $\mathscr{C}$, and this implies that ν is non-degenerate as claimed in point 2b. □

Proof of Theorem 2.31. We first prove that, if ν is a non-degenerate multiplicative measure on $\mathbf{A}$, then $f : \mathbf{A} \to (0, \infty)$ defined by $f(\cdot) = \nu(\uparrow \cdot)$ is Möbius. Let $h = \mathrm{T}f$. It follows from Lemma 2.34 that $h(\boldsymbol{e}) = 0$, and since ν is non-degenerate, point 2b shows that $h > 0$ on $\mathscr{C}$. To obtain that f is a Möbius valuation, it remains only to prove, in case where $\mathbf{A}$ is of spherical type, that $h(\Delta) > 0$ for Δ the Garside element of $\mathbf{A}$. But, since Δ satisfies $\Delta \to x$ for all $x \in \mathcal{S}$, one has $\mathcal{D}(\Delta) = \emptyset$ and thus $h(\Delta) = f(\Delta) > 0$. This proves the first statement of Theorem 2.31.

Conversely, let f be a Möbius valuation on $\mathbf{A}$. Define a non-negative matrix $Q = (Q_{x,y})_{(x,y) \in (\mathcal{S}\setminus\{\boldsymbol{e}\}) \times (\mathcal{S}\setminus\{\boldsymbol{e}\})}$ by

$$Q_{x,y} = \mathbf{1}(x \to y) f(x) \frac{h(y)}{h(x)}.$$

Since $h(\boldsymbol{e}) = 0$, Lemma 2.33 shows that Q is stochastic. Furthermore, the non-negative vector $(h(x))_{x \in \mathcal{S}\setminus\{\boldsymbol{e}\}}$ satisfies, using that $h(\boldsymbol{e}) = 0$ and Theorem 2.23:

$$\sum_{x \in \mathcal{S}\setminus\{\boldsymbol{e}\}} h(x) = \sum_{x \in \mathcal{S}} h(x) = \mathrm{T}^* h(\boldsymbol{e}) = f(\boldsymbol{e}) = 1.$$

It is thus a probability vector.

Consider the canonical probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ associated to the Markov chain $(X_n)_{n \geq 1}$ with values in $\mathcal{S}\setminus\{\boldsymbol{e}\}$, with initial distribution $(h(x))_{x \in \mathcal{S}\setminus\{\boldsymbol{e}\}}$ and with transition matrix Q. Let also $\pi : \Omega \to \partial\mathbf{A}$ be the canonical mapping, defined with $\mathbb{P}$-probability 1, and let $\nu = \pi_* \mathbb{P}$, the image probability measure on $\partial\mathbf{A}$. Then we claim that $\nu(\uparrow \cdot) = f(\cdot)$.

Indeed, by Theorem 2.23 and Proposition 2.22, it is enough to prove that, for every integer $n \geq 1$, the law of $(X_1, \dots, X_n)$ satisfies: $\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = h(x_1 \cdot \dots \cdot x_n)$ for every sequence $(x_1, \dots, x_n)$. If the sequence $(x_1, \dots, x_n)$ is not normal, then both members vanish. And if the sequence is normal, then the consecutive cancellations yield:

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = f(x_1 \cdot \dots \cdot x_{n-1}) \cdot h(x_n) = h(x_1 \cdot \dots \cdot x_n),$$

the latter equality using Lemma 2.32. This proves that $\nu(\uparrow \cdot) = f(\cdot)$, as expected.

This also proves that, for any non-degenerate multiplicative measure ν on $\partial\mathbf{A}$, the sequence $(X_n)_{n \geq 1}$ defined in the statement of Theorem 2.31 is indeed a Markov chain with the specified transition matrix and initial distribution.

If $\mathbf{A}$ is of spherical type, then $f(\Delta) = \nu(\uparrow \Delta) < 1$. Indeed, otherwise one would have $\nu(\uparrow \Delta^n) = \nu(\uparrow \Delta)^n = 1$, and therefore $\nu = \delta_{\Delta^\infty}$, contradicting its non-degeneracy. The ergodicity statements derive at once from the fact that the Charney graph is irreducible on the one hand, and that Δ is initial with $Q_{\Delta,\Delta} = f(\Delta) < 1$ if $\mathbf{A}$ is of spherical type, on the other hand. The proof is complete. $\square$

2.4.3. Uniqueness of the uniform measure for Artin–Tits monoids

Although we have not yet proved the existence of non-degenerate uniform measures on the boundary of Artin–Tits monoids—which is deferred to Section 4.2—, we are ready to prove the following uniqueness result.

Theorem 2.35. *Let* $\mathbf{A}$ *be an irreducible Artin–Tits monoid. Then there exists at most one uniform measure on* $\partial\mathbf{A}$.

Recall that, by Definition 2.29, uniform measures are non-degenerate.

Proof of Theorem 2.35. Let ν and ν' be two uniform measures, say associated to the two uniform valuations $f_1(x) = p_1^{|x|}$ and $f_2(x) = p_2^{|x|}$. Without loss of generality, we may assume that $p_1 \leq p_2$.

Consider the primitive matrices B_1 and B_2 constructed as in Lemma 2.34, associated to f_1 and to f_2 respectively. Then $B_1 \leq B_2$ since $p_1 \leq p_2$, and both matrices have spectral radius 1. It follows from the Perron–Frobenius Theorem [17] that $B_1 = B_2$, and thus $p_1 = p_2$ and $f_1 = f_2$. Then Proposition 2.22 implies that $\nu_1 = \nu_2$. $\square$

3. Conditioned weighted graphs

3.1. General framework

3.1.1. Non-negative matrices

Although we already appealed to the concept of non-negative matrix and to the Perron–Frobenius theory, we recall now some standard definitions from this theory, see for instance [17]. A real square matrix M is *non-negative*, denoted $M \geq 0$, if all its entries are non-negative, and *positive*, denoted $M > 0$, if all its entries are positive. The same definitions apply to vectors. If $M \geq 0$, it is *primitive* if $M^K > 0$ for some integer power $K > 0$, and then $M^k > 0$ for all $k \geq K$. The matrix M is *irreducible* if for every pair (i, j) of indices, there exists an integer $k > 0$ such that $M^k_{i,j} > 0$.

We interpret non-negative matrices as labeled oriented graphs, which we simply call *graphs* for brevity; vertices are represented by the indices of the matrix, and there is an edge from x to x', labeled by the entry $M_{x,x'}$, whenever $M_{x,x'} > 0$. A *path* in the graph is any non-empty sequence $(x_0, \dots, x_k)$ of indices such that $M_{x_i,x_{i+1}} > 0$ for all $i = 0, \dots, k-1$. The non-negative integer k is the *length* of the path. The path is a *circuit* if, in addition, $x_k = x_0$.

In this representation, irreducible matrices correspond to strongly connected graphs and primitive matrices correspond to strongly connected graphs such that circuits have 1 as greatest common divisor of their lengths.

The *spectral radius*, denoted $\rho(M)$, of a non-negative matrix M is defined as the largest modulus of its complex eigenvalues. In the seek of completeness, we establish the following elementary lemma, which is probably found elsewhere as a textbook exercise.

Lemma 3.1. *Let M be a non-negative square matrix of size $N > 1$. Assume that M has a unique eigenvalue of maximal modulus, which is simple. Let λ be this eigenvalue. Then:*

1. *λ is real and positive, and thus $\lambda = \rho(M)$.*
2. *There exists a pair (ℓ, r) of non-negative vectors, such that ℓ is a left (row) λ-eigenvector and r is a right (column) λ-eigenvector of M, and such that $\ell \cdot r = 1$. Any other such pair is of the form $(t\ell, t^{-1}r)$ for some $t > 0$.*
3. *With (ℓ, r) a pair of non-negative vectors as above, the matrix M has the following decomposition:*

$$M = \lambda\Pi + Q, \quad \textit{with } \Pi = r \cdot \ell,$$

and where Q is a matrix satisfying $\Pi \cdot Q = Q \cdot \Pi = 0$ and $\rho(Q) < \lambda$. It entails the following convergence:

$$\lim_{k\to\infty} \Big(\frac{1}{\lambda} M\Big)^k = \Pi. \tag{3.1}$$

Proof. Observe first that $\lambda \neq 0$, otherwise M could not have other eigenvalues than $\lambda = 0$, contradicting that λ is simple. Hence, considering $(1/|\lambda|)M$, which is still non-negative, instead of M, we assume without loss of generality that $|\lambda| = 1$. Secondly, the spectral decomposition of M entails that M writes as $M = \lambda\Pi + Q$, where Π is the matrix of a projector of rank 1, Q is a matrix with all eigenvalues less than 1 in modulus, and $\Pi \cdot Q = Q \cdot \Pi = 0$. Consequently, the powers M^k form a bounded sequence of matrices.

Now, let x be a right λ-eigenvector of M, and let $|x|$ denote the vector with $|x|_i = |x_i|$ for all $i \in \{1, \ldots, N\}$. Then $\sum_j M_{i,j} x_j = \lambda x_i$ yields $|x|_i \leq \sum_j M_{i,j} |x_j|$, or in other words: $M \cdot |x| \geq |x|$. Since M is non-negative, it follows that $M^{k+1} \cdot |x| \geq M^k \cdot |x|$, so that each coordinate of $M^k \cdot |x|$ is a non-decreasing sequence of reals. Since $(M^k)_{k>0}$ is bounded, it follows that $(M^k \cdot |x|)_{k>0}$ converges toward a non-negative vector r, satisfying $r \geq |x|$ and $M \cdot r = r$. Since $|x| \neq 0$, in particular $r \neq 0$ and thus 1 is an eigenvalue of M, which implies that $\lambda = 1$.

We have already found that r is a non-negative fixed point of M for its right action on vectors. The same reasoning applied to the transpose of M (or to the left action of M on vectors) yields the existence of a non-negative left fixed point, say ℓ, of M.

The decomposition seen at the beginning of the proof now writes as $M = \Pi + Q$ with $\rho(Q) < 1$ and $\Pi \cdot Q = Q \cdot \Pi = 0$. It implies at once $M^k = \Pi + Q^k$ for all integers $k > 0$ and thus $\lim_{k\to\infty} M^k = \Pi$.

To obtain the existence of the pair (ℓ, r) with the normalization condition $\ell \cdot r = 1$, it suffices to prove that $\ell \cdot r > 0$. For this, we observe first that r and ℓ are respectively right and left fixed point of Π, since Π is associated to the λ-characteristic subspace of M. Being a rank 1 projector with r as right fixed point, Π writes as $\Pi = r \cdot \ell'$ for some non-zero row vector ℓ'. Hence $\ell = \ell \cdot \Pi = (\ell \cdot r)\ell'$. This implies that $\ell \cdot r \neq 0$, which was to be proved.

Assuming now that the normalization condition $\ell \cdot r = 1$ holds, we obtain $\ell' = \ell$ and thus $\Pi = r \cdot \ell$, completing the proof. □

3.1.2. Conditioned weighted graphs

The central object of study of this section is the following.

Definition 3.2 *(*CWG*).* A *conditioned weighted graph (*CWG*)* is given by a triple $\mathcal{G} = (M, w^-, w^+)$, where M is a non-negative matrix of size $N \times N$ with $N > 1$, and

$$w^- : \{1, \ldots, N\} \to \mathbb{R}^+, \qquad w^+ : \{1, \ldots, N\} \to \mathbb{R}^+,$$

are two real-valued and non-negative functions, respectively called *initial* and *final*. We identify w^- with the corresponding row vector of size N, and we identify w^+ with the corresponding column vector of size N. Furthermore, we assume that the triple (M, w^-, w^+) satisfies the following conditions.

1. M has a unique eigenvalue of maximal modulus, which is simple. Let λ be this eigenvalue, which is real and positive according to Lemma 3.1.
2. Let (ℓ, r) be a pair of nonzero non-negative left and right λ-eigenvectors of M. We assume that $w^- \cdot r > 0$ and $\ell \cdot w^+ > 0$ both hold.

3.1.3. Two particular cases

For the study of Artin–Tits monoids, we shall be interested in triples (M, w^-, w^+) falling into one of the two following cases.

Case A. M is primitive, and the functions w^- and w^+ are non-identically zero.

Case B. For some integer $0 < K < N$ and for some non-negative matrices A, T and $\widetilde{M}$ of sizes $K \times K$, $K \times (N - K)$, and $(N - K) \times (N - K)$ respectively, M has the following form:

$$M = \begin{pmatrix} A & T \\ 0 & \widetilde{M} \end{pmatrix} \tag{3.2}$$

where $\widetilde{M}$ is primitive, and the spectral radii $\rho(A)$ and $\rho(\widetilde{M})$ satisfy $\rho(A) < \rho(\widetilde{M})$. The functions w^- and w^+ are assumed to be non-identically zero on the $N-K$ last indices.

Proposition 3.3. *In either case A or B described above, the triple* (M, w^-, w^+) *is a* CWG.

Proof. In case A, this is a direct application of the Perron–Frobenius Theorem for primitive matrices. In case B, let M be as in (3.2). Let λ be the Perron eigenvalue of $\widetilde{M}$, *i.e.*, according to the Perron–Frobenius Theorem for primitive matrices, the simple, unique eigenvalue of $\widetilde{M}$ of largest modulus. Then λ is the unique eigenvalue of M of maximal modulus since $\rho(A) < \rho(\widetilde{M})$, and it is simple as an eigenvalue of M.

It remains only to prove the existence of a pair (ℓ, r) of λ-eigenvectors of M such that $w^- \cdot r > 0$ and $\ell \cdot w^+ > 0$. For this, let $(\widetilde{\ell}, \widetilde{r})$ be a pair of positive left and right λ-eigenvectors of $\widetilde{M}$, and consider the vectors ℓ and r defined by:

$$\ell = \begin{pmatrix} 0 & \widetilde{\ell} \end{pmatrix} \qquad r = \begin{pmatrix} (\lambda I - A)^{-1} \cdot T \cdot \widetilde{r} \\ \widetilde{r} \end{pmatrix}$$

The hypothesis $\rho(A) < \rho(\widetilde{M})$ implies that $\lambda I - A$ is invertible, hence r is well defined, and ℓ and r are left and right λ-eigenvectors of M, which, due to Lemma 3.1, have non-negative entries. Since w^- and w^+ are assumed to be non-identically zero on their last $(N-K)$ coordinates, and since $\widetilde{\ell}$ and $\widetilde{r}$ are positive, they satisfy $w^- \cdot r > 0$ and $\ell \cdot w^+ > 0$. □

3.2. Weak convergence of weighted distributions

Let $\mathcal{G} = (M, w^-, w^+)$ be a CWG. Given a path $x = (x_0, \ldots, x_k)$ associated to M, we define its *weight* $w(x)$ as the following non-negative real:

$$w(x) = w^-(x_0) \cdot M_{x_0,x_1} \cdot \ldots \cdot M_{x_{k-1},x_k} \cdot w^+(x_k)\,.$$

From now on, our study of conditioned weight graphs focuses on paths and on probability distributions over sets of paths. As a first elementary result, we show that paths of positive weight and of length k exist for all k large enough.

Lemma 3.4. *Let* $\mathcal{G} = (M, w^-, w^+)$ *be a conditioned weighted graph. Then there exists an integer* K *such that, for each* $k \geq K$*, the set of paths of length* k *and with positive weight is non-empty.*

Proof. Let Z_k be the sum of the weights of all paths of length k. Then, identifying the functions w^- and w^+ with the corresponding row and column vectors, one has: $Z_k = w^- \cdot M^k \cdot w^+$. Let (ℓ, r) be a pair of left and right λ-eigenvectors of M and satisfying $\ell \cdot r = 1$, where λ is the eigenvalue of maximal modulus of M. Then, putting $\Pi = r \cdot \ell$, and according to Lemma 3.1:

$$\frac{1}{\lambda^k} Z_k = w^- \cdot \frac{1}{\lambda^k} M^k \cdot w^+ \xrightarrow[k\to\infty]{} w^- \cdot \Pi \cdot w^+ = (w^- \cdot r)(\ell \cdot w^+) > 0.$$

It follows that $Z_k > 0$ for all k large enough, and in particular the set of paths with positive weight is non-empty. $\square$

Definition 3.5. Let $\mathcal{G} = (M, w^-, w^+)$ be a conditioned weighted graph, and for each integer $k \geq 1$, let G_k denote the set of paths of length k in $\mathcal{G}$. The *weighted distribution* on G_k is the probability distribution μ_k on G_k defined by:

$$\forall z \in G_k \quad \mu_k(z) = \frac{w(z)}{Z_k}, \qquad \text{with } Z_k = \sum_{z \in G_k} w(z),$$

which is well defined at least for k large enough according to Lemma 3.4.

For each integer $j \geq 1$, and for k large enough, we denote by $\mu^-_{k,j}$ the joint law of the first $j+1$ elements $(x_0, \ldots, x_j)$ of a path $(x_0, \ldots, x_k)$ of length k distributed according to μ_k. We call $\mu^-_{k,j}$ the *left j-window distribution with respect to μ_k.*

Let $\mathcal{G} = (M, w^-, w^+)$ be a conditioned weighted graph, and for each integer $k \geq 0$, let Ω_k denote the set of paths of length *at most* k. Let also Ω be the set of finite or infinite paths, with its canonical topology (for which it is a compact space). The set of finite paths $\bigcup_k \Omega_k$ is dense in Ω.

In general, the collection $(\mu_k)_{k\geq 0}$ *is not a projective system of probability measures*, since the measure induced by μ_{k+1} on the set G_k of paths of length k does not coincide with μ_k. Hence, the projective limit of $(\mu_k)_{k\geq 0}$ is not defined in general.

Yet, weak limits of measures are an adequate tool to replace projective limits in this case. Indeed, each G_k is naturally embedded into Ω. Through this embedding, the distribution μ_k identifies with a discrete probability measure, still denoted by μ_k, on the space Ω equipped with its Borel σ-algebra.

Theorem 3.6. *Let $\mathcal{G} = (M, w^-, w^+)$ be a conditioned weighted graph, and consider as in Definition 3.2 the eigenvalue λ of maximal modulus together with the pair (ℓ, r) of associated eigenvectors.*

The sequence $(\mu_k)_{k\geq 0}$ of weighted distributions converges weakly toward a probability measure μ on Ω, which is concentrated on the set $\Xi \subseteq \Omega$ of infinite paths.

For each integer $k \geq 0$, let $X_k : \Xi \to \{1, \ldots, N\}$ denote the kth natural projection. Then, under μ, $(X_k)_{k\geq 0}$ is a Markov chain. Its initial distribution, denoted h, and its transition matrix, denoted $P = (P_{i,j})$ where $P_{i,j}$ is the probability to jump from state i to state j, are given by:

$$h(i) = \frac{w^-(i)r(i)}{w^- \cdot r}, \qquad P_{i,j} = \lambda^{-1} M_{i,j} \frac{r(j)}{r(i)}, \quad \textit{if } r(i) \neq 0,$$

independently of the choice of r. If $r(i) = 0$, the line $P_{i,\bullet}$ is defined as an arbitrary probability vector. The chain can only reach the set of states i such that $r(i) \neq 0$.

Restricted to the set of reachable states, the chain has a unique stationary measure, say π, given by:

$$\forall i \in \{1, \dots, N\} \quad \pi(i) = \ell(i)r(i).$$

In Case A introduced in Section 3.1.3, the chain $(X_k)_{k\geq 1}$ is ergodic. In Case B, the chain has a unique ergodic component, namely the $N - K$ last indices $\{K+1, \dots, N\}$, and hence the states in $\{1, \dots, K\}$ are all transient.

In view of the above result, we introduce the following definition.

Definition 3.7. The probability measure μ on the space of infinite sequences characterized in Theorem 3.6 is called the *limit weighted measure* of the conditioned weighted graph $\mathcal{G}$.

Proof of Theorem 3.6. For x a finite path of length j, denote by $\mathcal{C}_x$ the elementary cylinder of base x, *i.e.*, the set of finite or infinite words that start with x. For all integers k such that $k \geq j$ and such that μ_k is well defined, one has:

$$\mu_k(\mathcal{C}_x) = \frac{1}{Z_k} \sum_{z \in G_k \,:\, \theta_j(z) = x} w(z), \tag{3.3}$$

where θ_j is the truncation map that only keeps the first j steps of a path.

Let $\widetilde{w}$ be the real-valued function defined on finite paths by:

$$\widetilde{w}(x_0, \dots, x_j) = w^-(x_0) M_{x_0,x_1} \cdots M_{x_{j-1},x_j}.$$

Then both terms of the quotient in (3.3) can be written through powers of the matrix M:

$$\mu_k(\mathcal{C}_x) = \frac{1}{w^- \cdot M^k \cdot w^+} \widetilde{w}(x) \mathbf{1}_{x_j} \cdot M^{k-j} \cdot w^+,$$

where $\mathbf{1}_{x_j}$ denotes the row vector filled with 0s, except for the entry x_j where it has a 1.

According to Lemma 3.1, the following asymptotics holds for the powers of M:

$$M^k = \lambda^k (r \cdot \ell)\big(1 + o(1)\big), \quad k \to \infty.$$

Therefore $(\mu_k(\mathcal{C}_x))_{k\geq 0}$ is convergent, with limit given by:

$$\lim_{k\to\infty} \mu_k(\mathcal{C}_x) = \lambda^{-j} \widetilde{w}(x) \frac{(\mathbf{1}_{x_j} \cdot r)(\ell \cdot w^+)}{(w^- \cdot r)(\ell \cdot w^+)} = \lambda^{-j} \widetilde{w}(x) \frac{r(x_j)}{w^- \cdot r}. \tag{3.4}$$

Elementary cylinders, together with the empty set, are stable under finite intersections and generate the Borel σ-algebra on Ω. Elementary cylinders are clopen sets and thus of empty topological boundary. And finally, Ω is a compact metric space. According to [18, Th. 25.8], this is enough to deduce the weak convergence of $(\mu_k)_{k\geq 0}$ toward a probability measure μ on Ω such that $\mu(\mathcal{C}_x)$ coincides with the value of the limit in (3.4).

It is clear that the support of μ only contains infinite paths since, for every finite path $z = (x_0, \ldots, x_j)$, one has $\mu_k(\{z\}) = 0$ for k large enough. It follows that $\mu\big(\{z\}\big) = 0$, and thus $\mu(\Xi) = 1$ since finite paths are countably many.

The vector h and the matrix P defined in the statement are indeed respectively a probability vector and a stochastic matrix on $\{1, \ldots, N\}$. The Markov chain with initial law h and transition matrix P gives to the cylinder $\mathcal{C}_x$ the following probability:

$$\begin{aligned}
&h(x_0)P_{x_0,x_1}P_{x_1,x_2}\cdots P_{x_{j-1},x_j}\\
&\quad= \frac{w^-(x_0)r(x_0)}{w^-\cdot r}\cdot\lambda^{-1}M_{x_0,x_1}\frac{r(x_1)}{r(x_0)}\cdot\lambda^{-1}M_{x_1,x_2}\frac{r(x_2)}{r(x_1)}\cdots\lambda^{-1}M_{x_{j-1},x_j}\frac{r(x_j)}{r(x_{j-1})}\\
&\quad= \frac{w^-(x_0)r(x_j)}{w^-\cdot r}\lambda^{-j}M_{x_0,x_1}\cdots M_{x_{j-1},x_j} = \mu(\mathcal{C}_x),
\end{aligned}$$

by (3.4). This shows that this Markov chain has the same joint marginals as $(X_k)_{k\geq 0}$ under μ, or equivalently, that $(X_k)_{k\geq 0}$ under μ is the Markov chain with initial law h and transition matrix P.

By the normalization condition $\ell\cdot r = 1$, the vector π is indeed a probability distribution, which is readily seen to be left invariant for P. Furthermore, left invariant vectors θ for P and left λ-eigenvectors θ' for M, with support within the set of reachable states, correspond to each other by $\theta'(i) = \theta(i)/r(i)$. Since M has a unique left λ-eigenvector ℓ, the unique ergodic component of the chain corresponds to the support of ℓ. In case A, $\ell > 0$ hence the chain is ergodic. In Case B, the unique ergodic component corresponds to the last $N-K$ states. □

Corollary 3.8. *We keep the same notations as in Theorem 3.6. Let $j \geq 1$ be an integer. Then, with respect to μ_k, as $k\to\infty$, the left j-window distributions $\mu^-_{k,j}$ converge toward the joint law of $(X_0, \ldots, X_j)$ under the uniform distribution at infinity μ.*

Proof. This is a rephrasing of the weak convergence stated in Theorem 3.6. □

3.3. Related notions found in the literature

The notion of conditioned weighted graph is often found in the literature under disguised forms. For instance, the transition matrix P of the Markov chain introduced in Theorem 3.6 corresponds to the transformation of an incidence matrix first introduced by Parry [19–21] in its construction of a stationary Markov chain reaching the maximum entropy.

This matrix P also has the very same form as the transition matrix of the *survival process* of a discrete time, finite states absorbing Markov chain [22,23]. Actually, discrete time, finite states absorbing Markov chains can be interpreted as a particular case of conditioned weighted graph, as we briefly explain now.

A finite absorbing Markov chain $(Y_i)_{i\geq 0}$ is a Markov chain on $\{0,\ldots,N\}$ such that $P_{0,0}=1$, and such that 0 can be reached in a finite number of states from any other state. Usually, it is assumed that all states in $\{1,\ldots,N\}$ are strongly connected, which we assume too. For each $x\in\{1,\ldots,N\}$, we consider the conditioned weighted graph $\mathcal{G}_x=(M,w_x,w^+)$ defined as follows: M is the restriction of P to the entries in $\{1,\ldots,N\}$; w^+ is the constant vector with entries 1; and w_x is the indicator function of x.

Assume that $(Y_i)_{i\geq 0}$ starts from 1. Let T be the first hitting time of 0 of the chain $(Y_i)_{i\geq 0}$. Then it is clear that the marginal law of $(Y_0,\ldots,Y_j)$ conditioned on $\{T>k\}$ corresponds to our left j-window distribution for the conditioned weighted graph $\mathcal{G}_1$. The survival process, if it exists, is a process $(X_i)_{i\geq 0}$ such that:

$$\mathbb{P}(X_0=x_0,\ldots,X_j=x_j)=\lim_{k\to\infty}\mathbb{P}(Y_0=x_0,\ldots,Y_j=x_j|T>k)\,. \tag{3.5}$$

We recover thus the existence of the survival process, and its form as a Markov chain, through Theorem 3.6 (or Corollary 3.8); this is established for instance in [23, Sections 3.1 and 3.2] for continuous-time Markov chains.

4. Application to Artin–Tits monoids

We apply the notion of Conditioned Weighted Graphs (CWG) introduced in Section 3 to the counting of elements of Artin–Tits monoids, maybe with a multiplicative positive weight. The limit of the associated weighted measures is found to be concentrated on the boundary of the monoid and to be multiplicative. This yields another representation of multiplicative measures, introduced in Section 2, as weak limits of finite probability distributions, and provides a proof of existence for multiplicative measures. It also yields a parametrization of multiplicative measures.

4.1. CWG *associated to an irreducible Artin–Tits monoid*

4.1.1. Uniform case

Definition 4.1. Let $\mathbf{A}$ be an irreducible Artin–Tits monoid, and let $\mathcal{S}$ be the smallest Garside subset of $\mathbf{A}$. Let $J=\{(x,i)\bigm| x\in\mathcal{S}\setminus\{\boldsymbol{e}\}\text{ and }1\leq i\leq|x|\}$ and $N=\#J$. The CWG associated with $\mathbf{A}$ is the triple (M,w^-,w^+), where M is the non-negative square matrix of size $N\times N$ indexed by J, and w^- and w^+ are defined by:

$$M_{(x,i),(y,j)} = \begin{cases} 1, & \text{if } x = y \text{ and } j = i+1 \\ 1, & \text{if } x \to y \text{ and } i = |x| \text{ and } j = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$w^-(x,i) = \mathbf{1}(i=1), w^+(x,i) = \mathbf{1}(i=|x|).$$

The motivation behind this definition is that elements of $\mathbf{A}$ correspond bijectively to paths in the graph associated with the matrix M, or more precisely with the triple (M, w^-, w^+). Indeed, let x be an element of $\mathbf{A}$, with $x \neq \boldsymbol{e}$, and let $(x_1, \ldots, x_j)$ be the normal form of x. We associate to x the sequence $\widetilde{x}$ defined by:

$$\widetilde{x} = \big((x_1,1),\ldots,(x_1,|x_1|),(x_2,1),\ldots,(x_2,|x_2|),\ldots,(x_j,1),\ldots,(x_j,|x_j|)\big). \tag{4.1}$$

Then $\widetilde{x}$ is indeed a path in the graph corresponding to the triple (M, w^-, w^+), of length $|x_1| + \cdots + |x_j| - 1 = |x| - 1$.

This correspondence is a bijection between elements of $\mathbf{A}$ of length $k > 0$ and paths in (M, w^-, w^+) of length $k-1$, whence:

$$\#\{x \in \mathbf{A} \mid |x| = k\} = w^- \cdot M^{k-1} \cdot w^+. \tag{4.2}$$

Proposition 4.2. *Let $\mathbf{A}$ be an irreducible Artin–Tits monoid with at least two generators. Then the triple (M, w^-, w^+) introduced in Definition 4.1 is indeed a* CWG*, corresponding either to Case A of Section 3.1.3 if $\mathbf{A}$ is not of spherical type, or to case B if $\mathbf{A}$ is of spherical type.*

Proof. Assume that $\mathbf{A}$ is not of spherical type. Then the graph $(\mathcal{S} \setminus \{\boldsymbol{e}\}, \to)$ is strongly connected according to Theorem 2.12, which yields that the graph associated with M is strongly connected. Since $x \to x$ holds for every $x \in \Sigma$, the diagonal element $M\big((x,1),(x,1)\big)$ is 1, hence M is primitive. The conditions on w^- and w^+ are trivially satisfied, and thus (M, w^-, w^+) is of type A.

If $\mathbf{A}$ is of spherical type, let $J_\Delta = \{(\Delta, i) \mid 1 \leq i \leq |\Delta|\}$. Then the matrix M has the following form:

$$M = \begin{pmatrix} A & T \\ 0 & \widetilde{M} \end{pmatrix} \qquad \text{with} \quad A = \begin{pmatrix} 0 & 1 & 0 & & \cdots & 0 \\ & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ & & & \cdots & 0 & 1 \\ 1 & 0 & & & \cdots & 0 \end{pmatrix}$$

where A has size $\#J_\Delta \times \#J_\Delta$, and $\widetilde{M}$ is irreducible according to Theorem 2.12 and by the same reasoning as above. The matrix T is filled with 0s, except for its last line where it has a 1 at each column indexed by $(x,1)$ for $x \in \mathcal{S} \setminus \{\boldsymbol{e}, \Delta\}$.

Since $A^{\#J_\Delta} = I$, all the eigenvalues of A have modulus 1 and thus the spectral radius of A is 1. On the other hand, since $\mathbf{A}$ is assumed to have at least two generators and to be irreducible, there exist elements $x, y, z \in \mathcal{S}\setminus\{\boldsymbol{e}, \Delta\}$ such that $y \neq z$, $x \to y$ and $x \to z$. Hence, $\widetilde{M}$ is greater than a permutation matrix; being primitive, $\widetilde{M}$ has a spectral radius greater than 1 by the Perron–Frobenius Theorem. Finally, the conditions on w^- and w^+ are trivially satisfied, hence (M, w^-, w^+) falls into case B. □

Proposition 4.3. *Let* $\mathbf{A}$ *be an irreducible Artin–Tits monoid with at least two generators. For each integer* $k \geq 0$*, let* $\lambda_k = \#\{x \in \mathbf{A} \mid |x| = k\}$*. Then there exist two real constants* $C > 0$ *and* $p_0 \in (0,1)$*, that depend on* $\mathbf{A}$*, such that:*

$$\lambda_k \sim_{k\to\infty} C p_0^{-k}. \tag{4.3}$$

The real p_0 *is the inverse of the Perron eigenvalue of the* CWG *associated to* $\mathbf{A}$*.*

Proof. By (4.2), we have $\lambda_k = w^- M^{k-1} w^+$. Hence, putting $p_0 = \lambda^{-1}$, where λ is the Perron eigenvalue of M, we obtain the expected form according to Lemma 3.1, point 3. □

4.1.2. Möbius polynomial

There is a nice combinatorial interpretation of the real p_0 introduced in Proposition 4.3. It is much similar to the case of other monoids such as braid monoids or trace monoids; see [1] or [16] for more details.

The *Möbius polynomial* of an Artin–Tits monoid $\mathbf{A} = \mathbf{A}(\Sigma, \ell)$ is the polynomial $\mu_{\mathbf{A}} \in \mathbb{Z}[T]$ defined by:

$$\mu_{\mathbf{A}} = \sum_{D \Subset \Sigma} (-1)^{|D|} T^{|\bigvee_1 D|},$$

where the notation $D \Subset \Sigma$ has been introduced in Definition 2.20. Note that this is nothing but the polynomial expression found for $h(\boldsymbol{e})$ in Section 2.3.4, where $h(\cdot)$ was the graded Möbius transform of the uniform valuation $f(x) = p^{|x|}$ on $\mathbf{A}$.

Let also the *growth series* $G \in \mathbb{Z}[[T]]$ be the formal series defined by:

$$G = \sum_{x \in \mathbf{A}} T^{|x|} = \sum_{k \geq 0} \lambda_k T^k. \tag{4.4}$$

Then G is a rational series, inverse of the Möbius polynomial: $G(T) = 1/\mu_{\mathbf{A}}(T)$; see a proof for a slightly more general result below in Section 4.1.3. Since G has non-negative terms, its radius of convergence is one of its singularities by Pringsheim's Theorem [24]. Since G is rational with coefficients of the form (4.3), provided that $\mathbf{A}$ is irreducible with at least two generators, this singularity is necessarily of order 1, and there is no other singularity of G with the same modulus.

These facts reformulate as follows: *If* $\mathbf{A}$ *is an irreducible Artin–Tits monoid with at least two generators, the Möbius polynomial of* $\mathbf{A}$ *has a unique root of smallest modu-*

lus. This root is simple, real, lies in $(0,1)$*, and coincides with the real* p_0 *introduced in Proposition 4.3.*

4.1.3. Multiplicative case

More generally, assume given a multiplicative and positive weight on the elements of an Artin–Tits monoid $\mathbf{A}$, hence what we called a valuation $\omega : \mathbf{A} \to (0, +\infty)$. We associate to the pair $(\mathbf{A}, \omega)$ the following square matrix M with the same indices as in the uniform case (Definition 4.1), and the initial and final vectors w^- and w^+ given by:

$$M_{(x,i),(y,j)} = \begin{cases} 1, & \text{if } x = y \text{ and } j = i+1 \\ \omega(y), & \text{if } x \to y \text{ and } i = |x| \text{ and } j = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$w^-(x,i) = \mathbf{1}(i=1)\omega(x) \qquad w^+(x,i) = \mathbf{1}(i=|x|)$$

Then we claim that (M, w^-, w^+) is a CWG.

Indeed, if $\mathbf{A}$ is not of spherical type, then the same arguments used in the proof of Proposition 4.2 show that (M, w^-, w^+) is a CWG since M is primitive. Assume now that $\mathbf{A}$ is of spherical type. If ω is constant on Σ, so that $\omega(x) = p^{|x|}$ for all $x \in \mathbf{A}$ and for some positive real number p, then the same arguments used in the proof of Proposition 4.2 show that (M, w^-, w^+) is a CWG. If, however, ω is non-constant on Σ, introduce again $J_\Delta = \{(\Delta, i) \bigm| i \leq |\Delta|\}$, A the restriction of M to $J_\Delta \times J_\Delta$ and $\widetilde{M}$ the restriction of M to $(J \setminus J_\Delta) \times (J \setminus J_\Delta)$. Let $x \in \Sigma$ be such that $\omega(x) = \max \omega(\Sigma)$. Since ω is a valuation, and since Δ is divisible by some element $y \in \Sigma$ such that $\omega(y) < \omega(x)$, it comes at once that $\omega(\Delta) < \omega(x)^{|\Delta|}$. Due to the loop $x \to x$, it follows that $\rho(\widetilde{M}) \geq \omega(x) > \omega(\Delta)^{1/|\Delta|} = \rho(A)$, which proves that (M, w^-, w^+) is a CWG in this case too.

Let G_ω be the generating series:

$$G_\omega = \sum_{x \in \mathbf{A}} \omega(x) T^{|x|} = \sum_{k \geq 0} Z_\omega(k) T^k, \qquad \text{with } Z_\omega(k) = \sum_{x \in \mathbf{A}\,:\,|x|=k} \omega(x).$$

Then the coefficients $Z_\omega(k)$ have the following expression, for all $k > 0$:

$$Z_\omega(k) = w^- \cdot M^{k-1} \cdot w^+. \tag{4.5}$$

The uniform case seen in Section 4.1.1 corresponds to the constant valuation $\omega(x) = 1$, in which case G_ω is the growth series (4.4) of the monoid. We note that $G_\omega = 1/\mu_\omega$, hence is rational, where $\mu_\omega \in \mathbb{Z}[T]$ is the following polynomial:

$$\mu_\omega = \sum_{D \Subset \mathcal{S}} (-1)^{|D|} \omega\Big(\bigvee\nolimits_{\mathbf{l}} D\Big) T^{|\bigvee_{\mathbf{l}} D|}.$$

To prove the equality $G_\omega = 1/\mu_\omega$, one has $\mu_\omega G_\omega = \sum_{x \in \mathbf{A}} a_x T^{|x|}$, where a_x is computed by:

$$
\begin{aligned}
a_x &= \sum_{\substack{D\Subset\mathcal{S},\ y\in\mathbf{A}\ :\\ (\bigvee_1 D)\cdot y=x}} (-1)^{|D|}\omega\big(\textstyle\bigvee_1 D\big)\omega(y)\\
&= \omega(x)\Big(\sum_{D\Subset\mathcal{S}\ :\ \bigvee_1 D\leq_1 x}(-1)^{|D|}\Big) = \mathbf{1}(x=\boldsymbol{e}).
\end{aligned}
$$

4.2. Parametrization of multiplicative measures

Assume given a valuation $\omega : \mathbf{A} \to (0, +\infty)$ defined on an irreducible Artin–Tits monoid with at least two generators. For each integer $k \geq 1$, let $\mathbf{A}_k = \{x \in \mathbf{A} \mid |x| = k\}$ and let $m_{\omega,k}$ be the probability distribution on $\mathbf{A}_k$ proportional to ω:

$$
m_{\omega,k}(x) = \frac{\omega(x)}{Z_\omega(k)} \qquad \text{for } x \in \mathbf{A}_k\,.
$$

Then the finite probability space $(\mathbf{A}_k, m_{\omega,k})$ is isomorphic to the finite probability space of all paths of length $k-1$ in the CWG (M, w^-, w^+) equipped with the associated probability distribution from Definition 3.5 (Section 3.2). Furthermore, the space of infinite paths in the CWG (M, w^-, w^+) is homeomorphic to the boundary $\partial\mathbf{A}$. By Theorem 3.6, we deduce that the sequence $(m_{\omega,k})_{k\geq 0}$ converges weakly toward a probability measure $m_{\omega,\infty}$ on $\partial\mathbf{A}$. The following result gives the form of this limit measure.

Theorem 4.4. *Let $\omega : \mathbf{A} \to (0, +\infty)$ be a valuation defined on an irreducible Artin–Tits monoid with at least two generators. Then the weak limit $m_{\omega,\infty}$ of the sequence of finite probability distributions $(m_{\omega,k})_{k\geq 0}$ is a multiplicative measure on $\partial\mathbf{A}$. Its associated valuation is given as follows, for any $x \in \mathbf{A}$:*

$$
m_{\omega,\infty}(\uparrow x) = \lambda^{-|x|}\omega(x),
$$

where λ is the Perron eigenvalue of the CWG associated to ω.

Proof. Recall that $\overline{\mathbf{A}} = \mathbf{A} \cup \partial\mathbf{A}$ denotes the completion of $\mathbf{A}$, and $\Uparrow x$ denotes the full visual cylinder with base x (see Definition 2.15). Since the support of $m_{\omega,\infty}$ is a subset of $\partial\mathbf{A}$, one has $m_{\omega,\infty}(\uparrow x) = m_{\omega,\infty}(\Uparrow x)$. Since $\Uparrow x$ is both open and closed in $\overline{\mathbf{A}}$, its topological boundary is empty, and therefore by [18, Th. 25.8]:

$$
m_{\omega,\infty}(\Uparrow x) = \lim_{k\to\infty} m_{\omega,k}(\Uparrow x).
$$

Next, using that $\mathbf{A}$ is left cancellative, we compute for $k \geq |x|$:

$$
m_{\omega,k}(\Uparrow x) = \frac{1}{Z_\omega(k)}\Big(\sum_{y\in\mathbf{A}_k\ :\ x\leq_1 y}\omega(y)\Big) = \omega(x)\frac{Z_\omega(k-|x|)}{Z_\omega(k)}\,.
$$

Given the expression (4.5) for $Z_\omega(\cdot)$ on the one hand, and the asymptotics from Lemma 3.1 for the powers of M on the other hand, we deduce $m_{\omega,\infty}(\uparrow x) = \lambda^{-|x|}\omega(x)$. This is indeed a valuation, hence $m_{\omega,\infty}$ is a multiplicative measure. □

Corollary 4.5. *Let* $\mathbf{A}$ *be an irreducible Artin–Tits monoid with at least two generators. Then there exists a unique non-degenerate uniform measure* ν *on* $\partial\mathbf{A}$. *It is characterized by* $\nu(\uparrow x) = p_0^{|x|}$ *for all* $x \in \mathbf{A}$, *where* p_0 *is the unique root of smallest modulus of the Möbius polynomial of* $\mathbf{A}$.

Proof. The uniqueness of the non-degenerate uniform measure has already been proved in Theorem 2.35. For the existence, let $\omega(x) = 1$ be the constant uniform valuation on $\mathbf{A}$, and let $\nu = m_{\omega,\infty}$. Then, according to Theorem 4.4, we have $\nu(\uparrow x) = \lambda^{-|x|}$ for all $x \in \mathbf{A}$ and for λ the Perron eigenvalue of the CWG associated with ω. We have seen in Section 4.1.2 that $\lambda = p_0^{-1}$, whence the result. □

As illustrated by the above corollary, Theorem 4.4 provides a mean for proving the existence of multiplicative measures. We shall see that all multiplicative measures can be obtained as weak limits of such finite 'multiplicative distributions'. This yields in Theorem 4.7 below a parametrization of all multiplicative measures on the boundary.

From the operational point of view however, expressing a measure on the boundary $\partial\mathbf{A}$ as a weak limit of finite distributions on $\mathbf{A}$ does not provide a realization result similar to Theorem 2.31. It is therefore not much useful for simulation purposes for instance. Nevertheless, it yields a way to obtain asymptotic information on these finite 'multiplicative distributions', which are of interest *per se*. The latter aspect will be developed in Section 5.2.

Let us first investigate the structure of valuations on an Artin–Tits monoid: this task does not present any difficulty, and therefore the proof of the following result is omitted.

Proposition 4.6. *Let* $\mathbf{A} = \mathbf{A}(\Sigma, \ell)$ *be an Artin–Tits monoid. Let* $\overline{R}$ *be the reflexive and transitive closure of the symmetric relation* $R \subseteq \Sigma \times \Sigma$ *defined by:*

$$R = \big\{(x, y) \in \Sigma \times \Sigma \;\big|\; x \neq y \text{ and } \ell(x, y) < \infty \text{ and } \ell(x, y) = 1 \mod 2\big\},$$

and let $\mathcal{R}$ *be the set of equivalence classes of* $\overline{R}$.

Then, for any valuation $f : \mathbf{A} \to (0, +\infty)$, *and for any equivalence class* $r \in \mathcal{R}$, *the value* $f(a)$ *is constant for* a *ranging over* r. *Conversely, if* $x_r \in (0, +\infty)$ *is arbitrarily fixed for every* $r \in \mathcal{R}$, *then there exists a unique valuation* $f : \mathbf{A} \to (0, +\infty)$ *such that* $f(a) = x_{r(a)}$ *for every* $a \in \Sigma$, *where* $r(a)$ *is the equivalence class of* a.

Valuations on $\mathbf{A}$ are thus in bijection with the product set $\mathcal{F} = (0, +\infty)^K$, where K is the number of equivalence classes of $\overline{R}$. Let $\mathcal{M}$ denote the subset of $\mathcal{F}$ corresponding to parameters of Möbius valuations, hence those associated with a non-degenerate multiplicative measure. Theorem 4.7 below shows that $\mathcal{M}$ has a familiar topological structure.

We first introduce the following notation. If $\omega : \mathbf{A} \to (0, \infty)$ is a valuation, and if κ is a positive real number, then $\kappa\omega$ denotes the valuation on $\mathbf{A}$ defined by $(\kappa\omega)(x) = \kappa^{|x|}\omega(x)$ for all $x \in \mathbf{A}$. The half-line $(0, \infty)\omega$ is the set of valuations of the form $\kappa\omega$ for κ ranging over $(0, \infty)$.

Theorem 4.7. *Let $\mathbf{A}$ be an irreducible Artin–Tits monoid with at least two generators. Let K be the number of equivalence classes of the relation $\overline{R}$ from Proposition 4.6. Then the set $\mathcal{M} \subseteq (0, +\infty)^K$ of parameters of non-degenerate multiplicative measures defined on $\partial\mathbf{A}$ intersects any half-line $(0, +\infty)\omega$ in exactly one point. This gives a homeomorphism between $\mathcal{M}$ and the open simplex $\mathbb{P}\big((0, +\infty)^K\big)$ of dimension $K - 1$ (if $K = 1$, the open simplex of dimension 0 reduces to a singleton).*

We first need the following lemma, which generalizes the uniqueness result of uniform measures proved in Theorem 2.35 with the same technique of proof.

Lemma 4.8. *Let $\mathbf{A}$ be an irreducible Artin–Tits monoid with at least two generators. Let ν and ν' be two multiplicative non-degenerate measures on $\partial\mathbf{A}$, with associated valuations f and f'. Assume that there exists a constant $\kappa > 0$ such that $f' = \kappa f$. Then $f = f'$.*

Proof. Without loss of generality, we assume that $\kappa \le 1$. Let $\mathcal{S}$ be the smallest Garside subset of $\mathbf{A}$. Let B and B' be the square non-negative matrices, indexed by $(\mathcal{S} \setminus \{\boldsymbol{e}, \Delta\}) \times (\mathcal{S} \setminus \{\boldsymbol{e}, \Delta\})$, with Δ to be ignored if $\mathbf{A}$ is not of spherical type, and defined by:

$$B_{x,x'} = \mathbf{1}(x \to x')f(x'), \qquad B'_{x,x'} = \mathbf{1}(x \to x')f'(x').$$

Then $B' \le B$. According to Lemma 2.34, point 2b, both matrices are primitive of the same spectral radius 1. According to Perron–Frobenius Theorem [17], it implies $B = B'$ and thus $f = f'$. $\square$

Proof of Theorem 4.7. Let K be defined as in the statement, and let $\mathcal{F} = (0, +\infty)^K$. We identify the set of valuations on $\mathbf{A}$ with the product set $\mathcal{F}$, which is justified by Proposition 4.6. Let ω be a valuation on $\mathbf{A}$, and let (M, w^-, w^+) be the CWG associated to ω. Let also λ be the Perron eigenvalue of M, and let $f = \lambda^{-1}\omega$. The valuation f corresponds to a non-degenerate multiplicative measure according to Theorem 4.4. This association defines thus a mapping $\Phi : \mathcal{F} \to \mathcal{M}$.

We first prove that $\Phi(r) = r$ for every $r \in \mathcal{M}$. If $r \in \mathcal{M}$, then both r and $\Phi(r)$ correspond to non-degenerate multiplicative measures, and they are related for some constant $\kappa > 0$ by $\Phi(r) = \kappa r$. According to Lemma 4.8, it implies that $\Phi(r) = r$, as claimed. In particular, we deduce that Φ is onto. By the same lemma, we also observe that Φ is constant on all half-lines $(0, \infty)\omega$, for any $\omega \in \mathcal{F}$.

Let $\mathcal{R}$ be the set of equivalence classes of $\overline{R}$, and let S be the open simplex of dimension $K-1$ defined by:

$$S = \big\{ r = (r_a)_{a\in\mathcal{R}} \in \mathcal{F} \bigm| \sum_{a\in\mathcal{R}} r_a = 1 \big\},$$

and let $\Psi = \Phi\big|_S$. Since Φ is onto, and constant on the half-lines of $\mathcal{F}$, it follows at once that Ψ is onto. If $\Psi(f) = \Psi(f')$ for f and f' in S, then f and f' are related by a relation of the form $f' = \kappa f$ for some $\kappa > 0$, and thus obviously $f = f'$. Hence Ψ is one-to-one and thus bijective.

The mapping which associates to a family $r \in S$ the corresponding matrix M is obviously continuous, as well as the one mapping a primitive matrix to its Perron eigenvalue (see [25, Fact 10 of Section 9.2]). Hence Ψ is continuous. And Ψ^{-1} is continuous since, for $f \in \mathcal{M}$, the unique $\omega \in S$ such that $\Psi(\omega) = f$ is given by:

$$\forall a \in \mathcal{R} \quad \omega(a) = \frac{f(a)}{\sum_{b\in\mathcal{R}} f(b)}.$$

Hence Ψ is a homeomorphism. $\square$

5. Asymptotics: concentration of ergodic means and Central Limit Theorem

5.1. Case of conditioned weighted graphs

Consider a conditioned weighted graph $\mathcal{G} = (M, w^-, w^+)$, with M of order N, and a cost function $f : \{1,\ldots,N\} \to \mathbb{R}$. The *ergodic sums* $S_k f(z)$ and *ergodic means* $R_k f(z)$ of f along a path $z = (x_0,\ldots,x_k)$ of length k are defined by:

$$S_k f(z) = \sum_{i=0}^{k} f(x_i)\,, \qquad R_k f(z) = \frac{1}{k+1}\sum_{i=0}^{k} f(x_i)\,. \tag{5.1}$$

For each integer $k \geq 0$, let G_k denote the set of paths in $\mathcal{G}$ of length k, equipped with the associated weighted distribution as in Definition 3.5. The function $R_k f : G_k \to \mathbb{R}$ can be seen as a random variable on the discrete probability space (G_k, μ_k). Since this collection of random variables are not defined on the same probability spaces, the only way we have to compare them is to consider their laws and their convergence in law.

A weak variant of the Law of large numbers, adapted to the framework of conditioned weighted graphs, is the following concentration result—to be refined in a Central Limit Theorem next.

Theorem 5.1. *Let $\mathcal{G} = (M, w^-, w^+)$ be a conditioned weighted graph, with M of order N. Let π be the stationary measure on $\{1,\ldots,N\}$ associated with the limit weighted measure of $\mathcal{G}$.*

Then, for every function $f : \{1, \ldots, N\} \to \mathbb{R}$, the sequence of ergodic means $(R_k f)_{k\geq 0}$, with $R_k f$ defined on (G_k, μ_k) by (5.1), converges in distribution toward the Dirac measure δ_{γ_f}, where γ_f is the constant defined by:

$$\gamma_f = \sum_{j=1}^{N} \pi(j) f(j) \,. \tag{5.2}$$

Proof. Using the characteristic functions, it is enough to show the following convergence, for every real t:

$$\lim_{k\to\infty} \mathbb{E}_{\mu_k}(e^{\mathbf{i}tR_k f}) = e^{\mathbf{i}t\gamma_f} \,, \tag{5.3}$$

where $\mathbb{E}_{\mu_k}(\cdot)$ denotes the expectation with respect to μ_k, and γ_f is the constant defined in (5.2).

We express the above expectation in matrix terms, as follows. For each complex number u, let $M_f(u)$ be the complex-valued matrix defined by:

$$M_f(u) = D_f(u)M \,, \qquad D_f(u) = \mathrm{Diag}(e^{uf}) \,,$$

where the last matrix is the diagonal matrix with entry $e^{uf(i)}$ at position (i, i).

Let also $v_f(u)$ be the column vector defined by $v_f(u) = D_f(u)w^+$. Then:

$$\mathbb{E}_{\mu_k}(e^{\mathbf{i}tR_k f}) = \frac{1}{Z_k} w^- \big(M_f(u)\big)^k v_f(u) \,, \qquad \text{with } u = \frac{\mathbf{i}t}{k+1} \,,$$

where $Z_k = w^- \cdot M^k \cdot w^+$ is the normalization factor.

For small values of u, $D_f(u)$ is an analytic perturbation of the identity. By assumption, M has a unique eigenvalue of maximal modulus, say λ, and it is simple; the same spectral picture persists for $M_f(u)$ for small values of u. Hence, according to [26, Theorem III.8], denoting by $\lambda(u)$ the eigenvalue of highest modulus of $M_f(u)$, by $\ell(u)$ and by $r(u)$ the unique left and right associated eigenvectors normalized by the conditions $\ell \cdot r(u) = 1$ and $\ell(u) \cdot r(u) = 1$, all these quantities are analytic in u around zero, and for u small enough:

$$M_f(u) = \lambda(u)\Pi(u) + Q(u) \,, \qquad \text{with} \quad \Pi(u) = r(u) \cdot \ell(u),$$

where the spectrum of $Q(u)$ is included in a fixed disk of radius $< \lambda$, and $\Pi(u) \cdot Q(u) = Q(u) \cdot \Pi(u) = 0$. Raising to the power k yields:

$$\big(M_f(u)\big)^k = \lambda(u)^k \Pi(u) + Q(u)^k \,, \quad \|Q(u)^k\| = O\big((\lambda\varepsilon)^k\big)$$

for any spectral norm $\|\cdot\|$ and for some $0 < \varepsilon < 1$.

Put $\ell = \ell(0)$ and $r = r(0)$, consistently with our previous notation for the pair (ℓ, r). For t fixed and for k large enough, $u = \frac{t}{k+1}$ eventually reaches the region of validity of the above estimate. Hence, recalling that $Z_k = w^- \cdot M^k \cdot w^+ = w^- \cdot \big(M_f(0)\big)^k \cdot w^+$,

$$\mathbb{E}_{\mu_k}(e^{\mathbf{i}tR_kf}) =_{k\to\infty} \left(\frac{\lambda(u)}{\lambda}\right)^k \frac{w^- \cdot \Pi(u) \cdot v_f(u)}{w^- \cdot \Pi(0) \cdot v_f(0)} + o(1)\,.$$

As the last fraction on the right tends to 1 when $u \to 0$, this gives

$$\mathbb{E}_{\mu_k}(e^{\mathbf{i}tR_kf}) =_{k\to\infty} \left(\frac{\lambda(u)}{\lambda}\right)^k (1 + o(1)) + o(1)\,.$$

Passing to the limit as $k \to \infty$ above yields, using the development of $\lambda(\cdot)$ around zero at order 1:

$$\lim_{k\to\infty} \mathbb{E}_{\mu_k}(e^{\mathbf{i}tR_kf}) = \lim_{k\to\infty} \left(\frac{\lambda\big(\frac{\mathbf{i}t}{k+1}\big)}{\lambda}\right)^k = e^{\mathbf{i}\frac{\lambda'(0)}{\lambda}t}\,. \tag{5.4}$$

It remains to evaluate $\lambda'(0)$. For this, we first differentiate the equality $\ell \cdot r(u) = 1$ at 0 and obtain $\ell \cdot r'(0) = 0$. We also differentiate the equality $M_f(u) \cdot r(u) = \lambda(u)r(u)$ at 0 and obtain:

$$M_f'(0) \cdot r + M \cdot r'(0) = \lambda'(0)r + \lambda r'(0)\,.$$

We multiply both members of the above equality by ℓ on the left to derive:

$$\ell \cdot M_f'(0) \cdot r = \lambda'(0)\,, \qquad \text{since } \ell \cdot r'(0) = 0,\ \ell \cdot M = \lambda\ell \text{ and } \ell \cdot r = 1.$$

By the definition $M_f(u) = D_f(u) \cdot M$, we have $M_f'(0) \cdot r = \operatorname{Diag}(f) \cdot M \cdot r = \lambda \operatorname{Diag}(f) \cdot r$, and thus:

$$\lambda'(0) = \lambda\ell \cdot \operatorname{Diag}\big(f\big) \cdot r = \lambda \sum_{j=1}^{N} f(j)\ell(j)r(j) = \lambda\gamma_f\,.$$

Returning to (5.4), we deduce the validity of (5.3), which completes the proof. □

Extending the analysis to the next order yields a Central Limit Theorem.

Theorem 5.2. *Let $\mathcal{G} = (M, w^-, w^+)$ be a conditioned weighted graph, with M of order N. Let π be the stationary measure on $\{1, \ldots, N\}$ associated with the uniform measure at infinity of $\mathcal{G}$. Let $f : \{1, \ldots, N\} \to \mathbb{R}$ be a function.*

Then there exists a non-negative σ^2 such that the following convergence in law with respect to μ_k toward a Normal law holds:

$$\frac{1}{\sqrt{k+1}}\big(S_k f - (k+1)\gamma_f\big) \xrightarrow[k\to\infty]{\mathcal{L}} \mathcal{N}(0,\sigma^2)\,.$$

Moreover, $\sigma^2 = 0$ if and only if there exists a function $g : \{1,\dots,N\} \to \mathbb{R}$ such that $f(i) = \gamma_f + g(i) - g(j)$ whenever $M_{i,j} > 0$ and $i, j \in \operatorname{supp}\pi$.

Proof. We keep the same notations introduced in the proof of Theorem 5.1. Replacing f by $f - \gamma_f$ if necessary, we assume without loss of generality that $\gamma_f = 0$. We evaluate the characteristic function of $(S_k f)/\sqrt{k+1}$:

$$\mathbb{E}_{\mu_k}\big(e^{\mathbf{i}\frac{t}{\sqrt{k+1}}S_k f}\big) =_{k\to\infty} \left(\frac{\lambda\big(\frac{\mathbf{i}t}{\sqrt{k+1}}\big)}{\lambda}\right)^k (1+o(1)) + o(1)\,. \tag{5.5}$$

Since $\gamma_f = 0$, it follows from the computations performed previously that $\lambda'(0) = 0$, and thus, for $\kappa = \lambda''(0)/(2\lambda)$, around zero:

$$\lambda(u) = \lambda(1+\kappa u^2) + o(u^2)\,.$$

Passing to the limit in (5.5) yields:

$$\lim_{k\to\infty} \mathbb{E}_{\mu_k}\big(e^{\mathbf{i}\frac{t}{\sqrt{k+1}}S_k f}\big) = e^{-\kappa t^2}\,.$$

Since the left member in the above equation is uniformly bounded in t and in k, it entails that $\kappa = \sigma^2 \geq 0$, and this proves the convergence in law of $(S_k f)/\sqrt{k+1}$ toward $\mathcal{N}(0,\sigma^2)$.

To prove the non-degeneracy criterion, we need to compute the second derivative of λ. First, we claim that we can assume that $r(0)(i) > 0$ for all i. Indeed, in the general case, we can partition $\{1,\dots,N\}$ into the set where $r(0) > 0$ and the set where $r(0) = 0$. This gives rise to a block-triangular decomposition of M, with diagonal blocks A (on $r(0) > 0$) and B (on $r(0) = 0$). The spectrum of M is the union of the spectra of A and B, hence A has λ as a unique eigenvalue of maximal modulus, while B has strictly smaller spectral radius. When one perturbs M into $M(u)$, the block-triangular decomposition survives. By continuity of the spectrum, the dominating eigenvalue $\lambda(u)$ of $M(u)$ is also the dominating eigenvalue of $A(u)$, which is of the same type except that $r(0)$ is everywhere positive for A.

From now on, we assume that $r(0)(i) > 0$ for all i. To compute the second derivative of λ, it is more convenient to reduce by conjugation to a situation where $r'(0) = 0$, as follows. The vector $\widetilde{r}(u) = \operatorname{Diag}(e^{-ur'(0)/r(0)}) \cdot r(u)$ is equal to $r + O(u^2)$, and it is an eigenvector of the matrix

$$\widetilde{M_f}(u) = \operatorname{Diag}\big(e^{-ur'(0)/r(0)}\big) \cdot M_f(u) \cdot \operatorname{Diag}\big(e^{ur'(0)/r(0)}\big)\,,$$

for the eigenvalue $\lambda(u)$. Differentiating twice the equality $\lambda(u)\widetilde{r}(u) = \widetilde{M_f}(u) \cdot \widetilde{r}(u)$ yields

$$\lambda''(0)r + 2\lambda'(0)\widetilde{r}'(0) + \lambda\widetilde{r}''(0) = \widetilde{M}''_f(0) \cdot r + 2\widetilde{M}'_f(0) \cdot \widetilde{r}'(0) + \widetilde{M}_f(0) \cdot \widetilde{r}''(0)\,.$$

By construction, $\widetilde{r}'(0) = 0$. Multiplying this equation on the left by ℓ, the terms $\lambda\widetilde{r}''(0)$ and $\widetilde{M}_f(0) \cdot \widetilde{r}''(0)$ cancel each other as $\ell \cdot \widetilde{M}_f(0) = \ell \cdot M = \lambda\ell$. Thus,

$$\lambda''(0) = \ell \cdot \widetilde{M}''_f(0) \cdot r.$$

Moreover:

$$\big(\widetilde{M}_f(u)\big)_{i,j} = M_{i,j} \cdot \exp u\Big(f(i) - \frac{r'(0)(i)}{r(0)(i)} + \frac{r'(0)(j)}{r(0)(j)}\Big)\,.$$

Finally, writing $g(i) = r'(0)(i)/r(0)(i)$, we get the formula

$$\lambda''(0) = \sum_{i,j} \ell(i)\big(f(i) - g(i) + g(j)\big)^2 M_{i,j} r(j).$$

Note that g is real-valued, as the matrix $D_f(u)$ for small real u is real-valued and has a dominating real-valued eigenvector.

If follows from this expression that, if the variance vanishes, then $f(i) = g(i) - g(j)$ whenever $M_{i,j} > 0$ and $\ell(i) > 0$ and $r(j) > 0$ (these last two conditions are satisfied if i and j belong to the support of π).

Conversely, assume that there exists g such that $f(i) = g(i) - g(j)$ whenever $i, j \in \operatorname{supp}\pi$ and $M_{i,j} > 0$. Then, along any path $x = (x_0, \ldots, x_k)$ in the graph with nonzero weight and in the support of π, the quantity $S_k f = g(x_0) - g(x_k) + f(x_k)$ is uniformly bounded. For the limit weighted measure μ, almost every path enters the support of π by ergodicity. Hence, $S_k f$ remains bounded along almost every path. In particular, for any $\epsilon > 0$, there is a clopen set K_ϵ of μ-measure $> 1 - \epsilon$ on which $S_k f$ is bounded for all k by a constant $C(\epsilon)$. As μ_k converges weakly to μ by Theorem 3.6, it follows that $\mu_k(K_\epsilon) > 1 - \epsilon$ for large enough k. Hence, $S_k f$ is also bounded by $C(\epsilon)$ with μ_k-probability $1 - \epsilon$. This shows that $S_k f/\sqrt{k+1}$ converges in distribution with respect to μ_k towards the Dirac mass at 0. □

5.2. Asymptotics for Artin–Tits monoids

Let an Artin–Tits monoid $\mathbf{A}$, that we assume to be irreducible and with at least two generators, be equipped with a Möbius valuation ω. Let (M, w^-, w^+) be the associated CWG. We have already observed that, for every integer $k \geq 0$, the finite probability distribution $m_{\omega,k}$ on $\mathbf{A}_k = \{x \in \mathbf{A} \mid |x| = k\}$ which is proportional to ω corresponds to the weighted distribution on the set of paths of length $k - 1$ in the CWG. Let $m_{\omega,\infty}$ be the weak limit on $\partial\mathbf{A}$ of $(m_{\omega,k})_{k\geq 0}$.

Let Ξ denote the space of infinite paths in the CWG, equipped with the limit weighted measure $\widetilde{m}$. Then the natural correspondence between $\partial\mathbf{A}$ and Ξ makes the two probability measures $m_{\omega,\infty}$ and $\widetilde{m}$ image of each other.

By Theorems 2.31 and 3.6, both measures correspond to finite homogeneous Markov chains, but on two different finite sets of states: the set $\mathcal{S}\setminus\{\boldsymbol{e}\}$ for $m_{\omega,\infty}$ and the set $J=\big\{(x,i)\in(\mathcal{S}\setminus\{\boldsymbol{e}\})\times\mathbb{N}\;\big|\;1\leq i\leq|x|\big\}$ for $\widetilde{m}$. We wish to relate the stationary measures of these chains, that is to say, the finite probability distributions which are left invariant with respect to the transition matrices.

Lemma 5.3. *Let $f:\mathbf{A}\to(0,+\infty)$ be the valuation corresponding to a non-degenerate multiplicative measure m on the boundary at infinity of an Artin–Tits monoid $\mathbf{A}$. Let P be the transition matrix of the Markov chain on $\mathcal{S}\setminus\{\boldsymbol{e}\}$ associated to m, and let θ be the unique probability vector on $\mathcal{S}\setminus\{\boldsymbol{e}\}$ left invariant for P.*

Let (M,w^-,w^+) be the CWG *associated with f as described in Section 4.1.3, and let $\widetilde{P}$ be the transition matrix on $J=\big\{(x,i)\in(\mathcal{S}\setminus\{\boldsymbol{e}\})\times\mathbb{N}\;\big|\;1\leq i\leq|x|\big\}$ of the Markov chain corresponding to the limit weighted measure of the* CWG *(see Definition 3.7).*

Then the probability vector $\widetilde{\theta}$ on J defined by:

$$\widetilde{\theta}(x,i)=\frac{1}{\kappa}\theta(x),\qquad \text{with}\quad \kappa=\sum_{x\in\mathcal{S}\setminus\{\boldsymbol{e}\}}|x|\theta(x),$$

is left invariant for $\widetilde{P}$.

Proof. We put $\mathcal{S}'=\mathcal{S}\setminus\{\boldsymbol{e}\}$ to shorten the notations. Let h be the Möbius transform of f, and let $g:\mathcal{S}'\to\mathbb{R}$ be the normalization vector defined on $\mathcal{S}'$ by:

$$\forall y\in\mathcal{S}'\qquad g(x)=\sum_{y\in\mathcal{S}'\;:\;x\to y}h(y).$$

It follows from Theorem 2.31 that f is a Möbius valuation, hence $h(\boldsymbol{e})=0$. Therefore g coincides on $\mathcal{S}'$ with the function g defined in Lemma 2.33, and thus $h(x)=f(x)g(x)$ holds in particular for all $x\in\mathcal{S}'$. Furthermore, since m is non-degenerate, $g>0$ according to Lemma 2.34.

Writing down the equation $\theta P=\theta$ yields, for every $x\in\mathcal{S}'$, and using the expression for P given by Theorem 2.31:

$$\theta(x)=\sum_{y\in\mathcal{S}'}\theta(y)P_{y,x}=\sum_{y\in\mathcal{S}'\;:\;y\to x}f(y)\frac{\theta(y)}{h(y)}h(x)\,.$$

Using the identities $h(x)=f(x)g(x)$ and $f(y)/h(y)=1/g(y)$, we obtain:

$$f(x)\cdot\left(\sum_{y\in\mathcal{S}'\;:\;y\to x}\frac{\theta(y)}{g(y)}\right)=\frac{\theta(x)}{g(x)}\,. \tag{5.6}$$

Now we claim:

(†) *The vectors* u *and* v *defined, for* $(x,i) \in J$, *by* $u_{(x,i)} = \theta(x)/g(x)$ *and* $v_{(x,i)} = g(x)$, *are respectively left and right invariant for* M, *hence satisfy* $u \cdot M = u$ *and* $M \cdot v = v$.

The claim follows from the following computations, referring to the definition for the matrix M given in Section 4.1.3:

$$(u \cdot M)_{(x,1)} = \sum_{y \in \mathcal{S}' \,:\, y \to x} \frac{\theta(y)}{g(y)} f(x) = u_{(x,1)} \,,$$

where the last equality derives from (5.6). And for $i > 1$:

$$(u \cdot M)_{(x,i)} = u_{(x,i-1)} = u_{(x,i)} \,.$$

This proves that u is left invariant for M. To prove the right invariance of v, we compute as follows:

$$\text{for } i = |x|: \qquad (M \cdot v)_{(x,|x|)} = \sum_{y \in \mathcal{S}' \,:\, x \to y} f(y)g(y) = \sum_{y \in \mathcal{S}' \,:\, x \to y} h(y) = g(x) = v_{(x,|x|)}$$

$$\text{for } i < |x|: \qquad (M \cdot v)_{(x,i)} = g(x) = v_{(x,i)}$$

This proves the claim (†).

Thus, the stationary distribution of $\widetilde{P}$ is proportional to the vector $u(x,i)v(x,i) = \theta(x)$ and is a probability vector; whence the result. $\square$

Definition 5.4. Let $\mathbf{A}$ be an irreducible Artin–Tits monoid with at least two generators, equipped with a valuation $\omega : \mathbf{A} \to (0, +\infty)$. The *speedup* of ω is the quantity:

$$\kappa = \sum_{x \in \mathcal{S} \setminus \{\boldsymbol{e}\}} |x| \theta(x),$$

where $\mathcal{S}$ is the smallest Garside subset of $\mathbf{A}$, and θ is the stationary distribution of the Markov chain on $\mathcal{S} \setminus \{\boldsymbol{e}\}$ associated with the multiplicative measure $m_{\omega,\infty}$ on $\partial\mathbf{A}$.

We note that $\theta(\Delta) = 0$ if $\mathbf{A}$ is of spherical type. We now come to the study of asymptotics for combinatorial statistics defined on Artin–Tits monoids.

Definition 5.5. Let $\mathbf{A}$ be an Artin–Tits monoid. A function $F : \mathbf{A} \to \mathbb{R}$ is said to be:

1. *Additive* if $F(x \cdot y) = F(x) + F(y)$ for all $x, y \in \mathbf{A}$.

2. *Normal-additive* if $F(x_1 \cdot \ldots \cdot x_n) = F(x_1) + \ldots + F(x_n)$ whenever $(x_1, \ldots, x_n)$ is a normal sequence of $\mathbf{A}$.

Additive functions are normal additive, but the converse needs not to be true. For instance, the height function is normal additive without being additive in general.

Theorem 5.6. *Let $\mathbf{A}$ be an irreducible Artin–Tits monoid with at least two generators. Let $\omega : \mathbf{A} \to (0, +\infty)$ be a valuation, and let $F : \mathbf{A} \to \mathbb{R}$ be a normal-additive function.*

If $k > 0$ is an integer, we let x denote a random element in $\mathbf{A}_k$ distributed according to the finite distribution $m_{\omega,k}$. Let θ be the stationary distribution on $\mathcal{S} \setminus \{\boldsymbol{e}\}$ of the Markov chain associated with the weak limit $m_{\omega,\infty} = \lim_{k\to\infty} m_{\omega,k}$.

Then the following convergence in distribution holds:

$$\frac{F(x)}{|x|} \xrightarrow[k\to\infty]{\mathcal{L}} \delta_\gamma\,, \qquad \text{with} \quad \gamma = \frac{1}{\kappa} \sum_{x\in\mathcal{S}\setminus\{\boldsymbol{e},\Delta\}} \theta(x)F(x),$$

where κ is the speedup of ω, and where Δ is to be ignored if $\mathbf{A}$ is not of spherical type.

Assume furthermore that: (1) $\mathbf{A}$ has at least three generators in case that it is of spherical type, and (2) F is not proportional on $\mathcal{S} \setminus \{\boldsymbol{e}, \Delta\}$ to the length function (with Δ to be ignored if $\mathbf{A}$ is not of spherical type). Then there exists a constant $s^2 > 0$ such that the following convergence in distribution toward a Normal law holds:

$$\frac{1}{\sqrt{k}}\big(F(x) - k\gamma\big) \xrightarrow[k\to\infty]{\mathcal{L}} \mathcal{N}(0, s^2).$$

Proof. We assume without loss of generality that ω is a Möbius valuation. For otherwise, we normalize it by putting $\omega' = \kappa\omega$, for the unique positive real κ such that, according to Theorem 4.7, the resulting valuation ω' is a Möbius valuation. Then the weighted distributions associated with ω' are the same as the weighted distributions associated with ω, and the convergences that we shall establish for ω' correspond to the convergences for ω.

Let (M, w^-, w^+) be the CWG associated to ω. We express $F(x)$ for $x \in \mathbf{A}_k$ as an ergodic sum in order to apply Theorem 5.1. For this, let $\widetilde{F} : J \to \mathbb{R}$, with $J = \{(x, i) \in (\mathcal{S} \setminus \{\boldsymbol{e}\}) \times \mathbb{N} \;\big|\; 1 \le i \le |x|\}$, be defined by:

$$\widetilde{F}(x, i) = \begin{cases} 0, & \text{if } i < |x| \\ F(x) & \text{if } i = |x| \end{cases}$$

Then, for $x \in \mathbf{A}_k$, if $\widetilde{x}$ is the corresponding path in the CWG as in (4.1), one has $F(x) = S_{k-1}\widetilde{F}(\widetilde{x})$, where $S_{k-1}\widetilde{F}$ denotes the ergodic sums associated to $\widetilde{F}$, and thus $F(x)/|x| = R_{k-1}\widetilde{F}(\widetilde{x})$ where $R_{k-1}\widetilde{F}$ denotes the ergodic means associated to $\widetilde{F}$. Let $\widetilde{\theta}$ be the stationary distribution given by Theorem 3.6 applied to the CWG (M, w^-, w^+). Then Theorem 5.1 entails the convergence in distribution:

$$\frac{F(x)}{|x|} \xrightarrow[k\to\infty]{\mathcal{L}} \delta_\gamma\,, \qquad \text{with} \quad \gamma = \sum_{(x,i)\in J} \widetilde{\theta}(x,i)\widetilde{F}(x,i) = \frac{1}{\kappa} \sum_{x\in\mathcal{S}\setminus\{\boldsymbol{e}\}} \theta(x)F(x),$$

where the last equality comes from Lemma 5.3. If $\mathbf{A}$ is of spherical type, then $\theta(\Delta) = 0$ hence the sum ranges over $\mathcal{S} \setminus \{\boldsymbol{e}, \Delta\}$.

We now aim at applying the Central Limit Theorem 5.2. For this, let $H(x) = F(x) - \gamma|x|$ and let $\widetilde{H}$ be defined with respect to H in the same way than $\widetilde{F}$ was defined with respect to F. We prove that $\widetilde{H}$ satisfies the non-degeneration criterion stated in Theorem 5.2. For this, since $\gamma_H = 0$, assume that there exists a function $g(\cdot,\cdot)$ on J such that $\widetilde{H}(x,i) = g(x,i) - g(y,j)$ whenever $M_{(x,i),(y,j)} > 0$ and $(x,i),(y,j)$ belong to the support of the invariant measure of the Markov chain, *i.e.*, $x, y \in \mathcal{S} \setminus \{\boldsymbol{e}, \Delta\}$ (with Δ appearing only if $\mathbf{A}$ is of spherical type).

For any $x \in \mathcal{S} \setminus \{\boldsymbol{e}, \Delta\}$, and for $i < |x|$, one has $\widetilde{H}(x,i) = 0 = g(x,i) - g(x,i+1)$, hence the value of $g(x,i)$ is independent of $i \in \{1,\ldots,|x|\}$. And for all y such that $x \to y$ holds, $\widetilde{H}(x,|x|) = g(x,|x|) - g(y,1)$. Therefore the value of $g(y,1)$ is independent of y, provided that $x \to y$ holds.

To prove that g is globally constant, it is thus enough to show that the transitive closure on $\mathcal{S} \setminus \{\boldsymbol{e}, \Delta\}$ of the relation relating x and y if there exists z such that $z \to x$ and $z \to y$, is $\mathcal{S} \setminus \{\boldsymbol{e}, \Delta\}$. In turn, this derives from the following claim:

(†) *For every $x \in \mathcal{S} \setminus \{\boldsymbol{e}, \Delta\}$ (with Δ appearing only if $\mathbf{A}$ is of spherical type), let $D(x) = \big\{y \in \mathcal{S} \setminus \{\boldsymbol{e}, \Delta\} \bigm| x \to y\big\}$. Then there exists a subset I of $\mathcal{S} \setminus \{\boldsymbol{e}, \Delta\}$ such that:*

$$\forall x, y \in I \quad x \neq y \implies D(x) \cap D(y) \neq \emptyset, \tag{5.7}$$

$$\forall y \in \mathcal{S} \setminus \{\boldsymbol{e}, \Delta\} \quad \exists x \in I \quad y \in D(x). \tag{5.8}$$

We prove (†). If $\mathbf{A}$ is not of spherical type, any singleton $I = \{x\}$ is suitable if x is maximal in the finite poset $(\mathcal{S}, \leq_l)$. Indeed, (5.7) is trivially true, and for (5.8), by maximality of x, any $y \in \mathcal{S} \setminus \{\boldsymbol{e}\}$ satisfies $x = \bigvee_l \{\zeta \in \mathcal{S} \bigm| \zeta \leq_l x \cdot y\}$ and thus $y \in D(x)$ by Lemma 2.8.

For $\mathbf{A}$ of spherical type, we put $I = \big\{\Delta_A \bigm| \exists a \in \Sigma \quad A = \Sigma \setminus \{a\}\big\}$, where $\Delta_A = \bigvee_l A$. Let $A = \Sigma \setminus \{a\}$ and $B = \Sigma \setminus \{b\}$ with $a \neq b$. Then there exists $c \notin \{a, b\}$ since $|\Sigma| > 2$, and then $L(c) = \{c\} \subseteq R(\Delta_A), R(\Delta_B)$, hence $c \in D(\Delta_A) \cap D(\Delta_B)$ using Corollary 2.11. This proves (5.7). To prove (5.8), consider any $x \in \mathcal{S} \setminus \{\boldsymbol{e}, \Delta\}$. Then $L(x) \neq \Sigma$ hence $L(x) \subseteq R(\Delta_A)$ for any $A = \Sigma \setminus \{a\}$ such that $a \notin L(x)$, and thus $x \in D(\Delta_A)$.

Hence we have proved the claim (†), and thus that g is globally constant. It entails that $\widetilde{H}$ is itself constant, equal to zero, and that F is proportional on $\mathcal{S} \setminus \{\boldsymbol{e}, \Delta\}$ to the length function, contradicting our assumption. Consequently, Theorem 5.2 applies and we derive the stated convergence. $\square$

Remark 5.7. Let $\mathcal{N}(0,0)$ denote the Dirac measure on $\mathbb{R}$ concentrated on 0. We remark that, in the cases excluded in the last paragraph of Theorem 5.6, the same convergence holds with $s^2 = 0$, hence toward $\mathcal{N}(0,0)$.

For functions F proportional to the length function, the quantity $F(x) - k\gamma$ identically vanishes, hence the convergence is trivial. The statement also excludes the case of an Artin–Tits monoid $\mathbf{A}$ of spherical type and with two generators only. Since $\mathbf{A}$ is assumed to be irreducible, it is the case of $\mathbf{A} = \mathbf{A}\big(\{a,b\},\ell\big)$ with $3 \leq \ell(a,b) < \infty$. Such a monoid is easily investigated. Putting $n = \ell(a,b)$, one has $\mathcal{S} = \{\boldsymbol{e}, x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}, \Delta\}$ with $x_i = abab\cdots$, $y_i = baba\cdots$ and $|x_i| = |y_i| = i$, and $\Delta = x_n = y_n$. Furthermore, putting $X = \{x_j \bigm| 1 \leq j < n\}$ and $Y = \{y_j \bigm| 1 \leq j < n\}$, one has:

$$\begin{aligned} &\text{for } 2i+1 < n: \qquad D(x_{2i+1}) = X, \qquad D(y_{2i+1}) = Y,\\ &\text{for } 2i < n: \qquad D(x_{2i}) = Y, \qquad D(y_{2i}) = X. \end{aligned}$$

Consider a Möbius valuation ω, and let G be the normal-additive function defined by $G(x_{2i}) = 1$, $G(y_{2i}) = -1$, $G(x_{2i+1}) = G(y_{2i+1}) = 0$. Observe that $\omega(x_{2i}) = \omega(y_{2i})$, hence $\gamma = 0$ for symmetry reasons. For any $x \in \mathbf{A}$, with normal form $x = \Delta^{i_x} \cdot s_1 \cdot \ldots \cdot s_m$, one has $G(x) = i_x G(\Delta) + \varepsilon_m$ with $\varepsilon_m \in \{-1, 0, 1\}$, where i_x is the number of Δs in the normal form of x. Hence, for any $a > 0$:

$$\omega_k\Big(\Big|\frac{G(x)}{\sqrt{k}}\Big| \geq a\Big) \leq \omega_k\Big(i_x \geq \frac{a\sqrt{k} - 1}{1 + |G(\Delta)|}\Big).$$

Since i_x converges in law toward a geometric law, it follows that the right-hand member above, and thus the left-hand member, converges toward 0, and so $G(x)/\sqrt{k}$ converges in law toward $\mathcal{N}(0,0)$, as expected.

An inspection of the proof of Theorem 5.6 shows that the functions for which the non-degeneracy criterion for the convergence applies, are exactly those in the two-dimensional vector space generated by the length function and G. Hence, for normal-additive functions outside this vector space, the convergence stated in Theorem 5.6 applies with $s^2 > 0$.

Let us apply Theorem 5.6 to obtain information on the following statistics: the ratio height over length of large elements in an irreducible Artin–Tits monoid.

Corollary 5.8. *Let $\mathbf{A}$ be an Artin–Tits monoid. We assume that $\mathbf{A}$ is irreducible, has at least two generators and is not a free monoid. Let κ be the speedup associated to the constant valuation $\omega(x) = 1$ on $\mathbf{A}$.*

Then, for all reals $a < b$ distinct from κ^{-1}, one has:

$$\lim_{k\to\infty} \frac{\#\{x \in \mathbf{A}_k \bigm| a < \tau(x)/k < b\}}{\#\mathbf{A}_k} = \mathbf{1}(a < \kappa^{-1} < b),$$

and for a and b distinct from κ:

$$\lim_{k\to\infty} \frac{\#\{x \in \mathbf{A}_k \mid a < k/\tau(x) < b\}}{\#\mathbf{A}_k} = \mathbf{1}(a < \kappa < b).$$

It entails the convergence of the following expectations, where $\mathbb{E}_k$ denotes the expectation with respect to the uniform distribution on $\mathbf{A}_k$:

$$\lim_{k\to\infty} \frac{1}{k}\mathbb{E}_k\tau(\cdot) = \kappa^{-1}, \qquad \lim_{k\to\infty} k\mathbb{E}_k\Big(\frac{1}{\tau(\cdot)}\Big) = \kappa.$$

Furthermore, there is a positive constant s^2 such that, for any two reals $a < b$, one has:

$$\lim_{k\to\infty} \frac{\#\big\{x \in \mathbf{A}_k \mid a\sqrt{k} < \tau(x) - k\kappa^{-1} < b\sqrt{k}\big\}}{\#\mathbf{A}_k} = \frac{1}{\sqrt{2\pi s^2}} \int_a^b \exp\Big(-\frac{t^2}{2s^2}\Big)\, dt, \tag{5.9}$$

$$\lim_{k\to\infty} \frac{\#\big\{x \in \mathbf{A}_k \mid a < \sqrt{k}\big(k/\tau(x) - \kappa\big) < b\big\}}{\#\mathbf{A}_k} = \frac{1}{\sqrt{2\pi s^2\kappa^{-4}}} \int_a^b \exp\Big(-\frac{t^2}{2s^2\kappa^{-4}}\Big)\, dt. \tag{5.10}$$

Proof. We apply Theorem 5.6 with the height function $F(x) = \tau(x)$, which is normal-additive. It is not proportional on $\mathcal{S} \setminus \{\boldsymbol{e}, \Delta\}$ to the length function since $\mathbf{A}$ is not a free monoid. If $\mathbf{A}$ is of spherical type with only two generators, the function F does not belong to the vector space generated by the length function and the function G described in the above Remark. Hence, Theorem 5.6 applies.

The ratios $\tau(x)/|x|$ converge in distribution toward δ_γ with:

$$\gamma = \frac{1}{\kappa} \sum_{x\in\mathcal{S}\setminus\{\boldsymbol{e}\}} \theta(x) = \frac{1}{\kappa}\,.$$

Since it is a constant, it entails the convergence in distribution of the inverse ratios $|x|/\tau(x)$ toward the constant $\gamma^{-1} = \kappa$. The two first points derive at once, as well as the convergence of the expectations.

The convergence (5.9) is the reformulation of the convergence in distribution $\sqrt{k}\big(\tau(x)/|x| - \kappa^{-1}\big) \xrightarrow{\mathcal{L}} \mathcal{N}(0, s^2)$. It is then well know how to derive the following convergence:

$$\sqrt{k}\Big(\frac{|x|}{\tau(x)} - \kappa\Big) \xrightarrow[k\to\infty]{\mathcal{L}} \mathcal{N}(0, s^2\gamma^4),$$

based on the Delta method, see [18, p. 359]. The convergence (5.10) follows. $\square$

6. Similar results for other monoids

So far, our results only focused on Artin–Tits monoids and their Garside normal forms, with the help of conditioned weighted graphs. However, the arguments developed above may apply to other monoids as well, provided that their combinatorial structure is similar to that of Artin–Tits monoids.

In Section 6.1, we list the different properties on which our arguments rely, and then we state the corresponding results. In Section 6.2, we underline that the main objects that we construct—the boundary at infinity and the class of multiplicative probability measures on the boundary—are intrinsically attached to the monoid, although the construction uses non-intrinsic objects such as a particular Garside subset. Finally, we identify in Section 6.3 several examples of monoids fitting into this more general framework, and not falling into the class of Artin–Tits monoids.

6.1. A more general working framework

The properties that must be satisfied by the monoid $\mathbf{A}$ to follow the sequence of arguments that we developed are the following:

(P1) There exists a *length* function on the monoid $\mathbf{A}$, *i.e.*, a function $|\cdot| : \mathbf{A} \to \mathbb{Z}_{\geq 0}$ such that:

$$\forall x, y \in \mathbf{A} \quad |x \cdot y| = |x| + |y| \qquad \text{and} \qquad \forall x \in \mathbf{A} \quad x = \boldsymbol{e} \iff |x| = 0.$$

(P2) The monoid $\mathbf{A}$ is both left and right cancellative, meaning:

$$\forall x, y, z \in \mathbf{A} \quad (z \cdot x = z \cdot y \implies x = y) \text{ and } (x \cdot z = y \cdot z \implies x = y).$$

(P3) The ordered set $(\mathbf{A}, \leq_{\mathrm{l}})$ is a lower semi-lattice, *i.e.*, any non-empty set has a greatest lower bound in $\mathbf{A}$.

(P4) There exists a finite Garside subset $\mathcal{S}$, *i.e.*, a finite subset of $\mathbf{A}$ which generates $\mathbf{A}$ and is closed under existing $\bigvee_{\mathrm{l}}$ and downward closed under $\leq_{\mathrm{r}}$ (the condition that $\mathcal{S}$ contains Σ is dropped: it does not make sense in our context as we have not singled out a generating set Σ in our assumptions).

(P5) The Charney graph $(\mathscr{C}, \to)$ is strongly connected, where $\mathscr{C}$ is the subset of $\mathcal{S}$ and $\to$ is the relation on $\mathcal{S} \times \mathcal{S}$ defined by:

$$\mathscr{C} = \begin{cases} \mathcal{S} \setminus \{\Delta, \boldsymbol{e}\}, & \text{if } (\mathcal{S}, \leq_{\mathrm{l}}) \text{ has a maximum } \Delta \\ \mathcal{S} \setminus \{\boldsymbol{e}\}, & \text{otherwise} \end{cases}$$

$$x \to y \iff x = \bigvee\nolimits_{\mathrm{l}} \big\{ z \in \mathcal{S} \bigm| z \leq_{\mathrm{l}} x \cdot y \big\}.$$

(P6) The integers in the set:

$$\bigl\{|z_1| + \ldots + |z_k| \bigm| k \geq 1, \quad z_1, \ldots, z_k \in \mathcal{S}, \quad z_1 \to z_2 \to \ldots \to z_k \to z_1\bigr\},$$

are setwise coprime, *i.e.*, have 1 as greatest common divisor.

(P7) If $(\mathcal{S}, \leq_1)$ has a maximum, then the Charney graph $(\mathscr{C}, \to)$ has at least one vertex with out-degree two or more.

We observe that all irreducible Artin–Tits monoids with at least two generators satisfy the above axioms.

Property (P1) states the conditions that must be met by the notion of length of an element, and then Properties (P2) to (P4) lead to the notion of Garside normal form and its variants, such as the generalized Garside normal form.

Property (P5) is then equivalent with Theorem 2.12. Property (P6) is used to obtain aperiodic, and therefore primitive matrices. Property (P7) ensures that the Δ component, if it exists, will have a small spectral radius. In the case of irreducible Artin–Tits monoids, (P7) holds for monoids with at least two generators, and only for them.

The results obtained in the previous sections for irreducible Artin–Tits monoids with at least two generators, and regarding the construction of multiplicative probability measures at infinity, generalize to all monoids satisfying Properties (P1) to (P7).

Theorem 6.1. *Let* $\mathbf{A}$ *be a monoid satisfying Properties* (P1)–(P7)*. Then Theorem* 2.35*, Proposition* 4.3*, Theorem* 4.4*, Corollary* 4.5*, Theorem* 4.7 *and Theorem* 5.6 *hold for the monoid* $\mathbf{A}$*.*

In view of Theorem 6.1, two questions naturally arise:

- Assume given a monoid $\mathbf{A}$ satisfying (P1)–(P7). The central objects that we consider are the boundary at infinity of $\mathbf{A}$, the multiplicative measures and the uniform measure on the boundary. How much of these objects are intrinsic to $\mathbf{A}$? And on the contrary, how much depend on the specific length function and on the Garside subset that were chosen?
- What are typical examples of monoids satisfying (P1)–(P7) outside the family of Artin–Tits monoids?

We answer the first question below in Section 6.2, showing that most of our objects of interest are intrinsic to the monoid. The answer to the second question is the topic of Section 6.3.

6.2. The boundary at infinity and the multiplicative measures are intrinsic

We have constructed in Section 2.3.1 a compactification $\overline{\mathbf{A}}$ of an Artin–Tits monoid $\mathbf{A}$ as the set of infinite paths in the graph $(\mathcal{S}, \to)$, where $\mathcal{S}$ is the smallest finite Garside subset of $\mathbf{A}$. Then $\mathbf{A}$ identifies with the set of paths that hit $\boldsymbol{e}$, whereas the *boundary* is the set of paths that never hit $\boldsymbol{e}$.

The same construction is carried over for an arbitrary monoid $\mathbf{A}$ equipped with a finite Garside subset $\mathcal{S}$, and this was implicitly understood in the statement of Theorem 6.1. We have thus an operational description of the compactification and of the boundary at infinity relative to the Garside subset $\mathcal{S}$, say $\overline{\mathbf{A}}_{\mathcal{S}}$ and $\partial\mathbf{A}_{\mathcal{S}}$. It is not obvious however to see that the two compact spaces thus obtained are essentially independent of $\mathcal{S}$. We sum up an alternative construction of the boundary at infinity, already used in [16], and the fact that it is equivalent to the previous construction in the following result, the proof of which we omit.

Proposition 6.2. *Let $\mathbf{A}$ be a monoid, and define the preorder $\leq_1$ on $\mathbf{A}$ by putting $x \leq_1 y \iff (\exists z \in \mathbf{A}\ y = x \cdot z)$. Let $\mathcal{H} = \bigl\{(x_i)_{i\geq 0} \bigm| \forall i \geq 0\ x_i \in \mathbf{A}\ x_i \leq_1 x_{i+1}\bigr\}$. Equip $\mathcal{H}$ with the preordering relation defined, for $x = (x_i)_{i\geq 0}$ and $y = (y_i)_{i\geq 0}$, by:*

$$x \sqsubseteq y \iff \bigl(\forall i \geq 0 \quad \exists j \geq 0 \quad x_i \leq_1 y_j\bigr).$$

Let finally $(\overline{\mathbf{A}}, \leq_1)$ be the collapse *of $(\mathcal{H}, \sqsubseteq)$. That is to say, $\overline{\mathbf{A}}$ is the partially ordered set obtained as the quotient of $\mathcal{H}$ by the equivalence relation $\sqsubseteq \cap (\sqsubseteq)^{-1}$.*

Assume that $\mathbf{A}$ satisfies Properties (P1), (P2) *and* (P4).

1. *Then $(\mathbf{A}, \leq_1)$ is a partial order that identifies with its image in $\overline{\mathbf{A}}$ through the composed mapping $\mathbf{A} \to \mathcal{H} \to \overline{\mathbf{A}}$, where the mapping $\mathbf{A} \to \mathcal{H}$ sends an element x to the constant sequence $(x, x, \ldots)$.*

One equips $\overline{\mathbf{A}}$ with the smallest topology containing as open sets all sets of the form $\uparrow x$ with $x \in \mathbf{A}$ and all sets of the form $\overline{\mathbf{A}} \setminus \uparrow x$ with $x \in \overline{\mathbf{A}}$ (in Domain theory, this corresponds to the Lawson topology [27, p. 211]*). Finally, the* boundary at infinity *of $\mathbf{A}$ is the topological space defined by $\partial\mathbf{A} = \overline{\mathbf{A}} \setminus \mathbf{A}$.*

2. *There is a canonical mapping between $\overline{\mathbf{A}}_{\mathcal{S}}$ and $\overline{\mathbf{A}}$. This mapping makes $(\overline{\mathbf{A}}_{\mathcal{S}}, \leq_1)$ and $(\overline{\mathbf{A}}, \leq_1)$ isomorphic as partial orders, and the topological spaces homeomorphic. Its restriction to $\partial\mathbf{A}_{\mathcal{S}}$ induces a homeomorphism from $\partial\mathbf{A}_{\mathcal{S}}$ to $\partial\mathbf{A}$.*

Multiplicative measures on the boundary are those measures m on $\partial\mathbf{A}$ satisfying $m\bigl(\uparrow (x \cdot y)\bigr) = m(\uparrow x) \cdot m(\uparrow y)$. We deduce the following corollary.

Corollary 6.3. *Let $\mathbf{A}$ be a monoid satisfying Properties* (P1) *to* (P7)*. Then the notions of boundary at infinity and of multiplicative measure are intrinsic to $\mathbf{A}$, and do not depend either on the specific length function nor on the specific finite Garside subset considered.*

By contrast, the notion of uniform measure does depend on the specific length function one considers. Nevertheless, the uniform measure associated to any length function belongs to the class of multiplicative measures—and this class is intrinsic to **A** according to Corollary 6.3.

6.3. Examples outside the family of Artin–Tits monoids

We now mention two families of monoids matching the general working framework introduced in Section 6.1, and yet outside the family of Artin–Tits monoids.

6.3.1. Dual braid monoids

Braid monoids, which are among the foremost important Artin–Tits monoids, can be seen as sub-monoids of *braid groups*. The braid group with n strands, for $n \geq 2$, is the group B_n defined by the following presentation:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1,\ \sigma_i\sigma_j\sigma_i = \sigma_i\sigma_j\sigma_i \text{ if } i = j \pm 1 \right\rangle .$$

The associated braid monoid is just the sub-monoid positively generated by the family $\{\sigma_i \bigm| 1 \leq i \leq n\}$.

The *dual braid monoid* is another sub-monoid of the braid group, strictly greater than the braid monoid when $n \geq 3$. This monoid was introduced in [28]. It is the sub-monoid of B_n generated by the family $\{\sigma_{i,j} \bigm| 1 \leq i < j \leq n\}$, where $\sigma_{i,j}$ is defined by:

$$\begin{aligned}
&\sigma_{i,j} = \sigma_i\,, && \text{for } 1 \leq i < n \text{ and } j = i+1,\\
&\sigma_{i,j} = \sigma_i\sigma_{i+1}\ldots\sigma_{j-1}\sigma_{j-2}^{-1}\sigma_{j-3}^{-1}\ldots\sigma_i^{-1}\,, && \text{for } 1 \leq i < n-1 \text{ and } i+2 \leq j \leq n\,.
\end{aligned}$$

Alternatively, the dual braid monoid is the monoid generated by the elements $\sigma_{i,j}$ for $1 \leq i < j \leq n$, and subject to the relations:

$$\begin{cases}
\sigma_{i,j} \cdot \sigma_{j,k} = \sigma_{j,k} \cdot \sigma_{i,k} = \sigma_{i,k} \cdot \sigma_{i,j} & \text{for } 1 \leq i < j < k \leq n;\\
\sigma_{i,j} \cdot \sigma_{k,\ell} = \sigma_{k,\ell} \cdot \sigma_{i,j} & \text{for } 1 \leq i < j < k < \ell \leq n;\\
\sigma_{i,j} \cdot \sigma_{k,\ell} = \sigma_{k,\ell} \cdot \sigma_{i,j} & \text{for } 1 \leq i < k < \ell < j \leq n.
\end{cases}$$

It is proved in [1] that dual braid monoids satisfy Properties (P1) to (P7), where the length of an element x of the monoid is the number of generators $\sigma_{i,j}$ that appear in any word representing x, and where $\mathcal{S}$ is the smallest Garside subset of the monoid.

In fact, this construction can be generalized to all irreducible Artin–Tits monoids of spherical type. Indeed, for any such monoid **A**, Bessis also developed in [29] a notion of dual monoid, which subsumes the notion of dual braid monoid in case **A** is the braid monoid B_n^+. In particular, Bessis proved that these dual monoids satisfy Properties (P1) to (P4), (P6) and (P7), with the same definitions of element length and of Garside set $\mathcal{S}$.

However, Property (P5) was not investigated. Hence, we prove here the following result.

Proposition 6.4. *Let* **A** *be an irreducible Artin–Tits monoid of spherical type, with at least two generators. The dual monoid associated with* **A** *satisfies Properties* (P1) *to* (P7)*.*

Proof. Based on the above discussion, it only remains to prove that the dual monoid **B** satisfies Property (P5). It follows from the classification of finite Coxeter groups [9] that **A** must fall into one of the following finitely many families. Three families consist of so-called monoids of type A, B and D, one is the family of dihedral monoids $I_2(m)$ (with $m \geqslant 3$), and other families are finite (these are families of *exceptional type*).

Monoids of type A are braid monoids, which are already treated in [1]. Furthermore, using computer algebra (for instance the package CHEVIE of GAP [30]), it is easy to prove (P5) in the case of families of exceptional type. Hence, we focus on the three infinite families of monoids.

If **A** is of type $I_2(m)$, the monoid **B** has the following presentation (this presentation is the one given by [31], with reversed product order):

$$\mathbf{B} = \left\langle \sigma_1, \ldots, \sigma_m \mid \sigma_i \sigma_j = \sigma_m \sigma_1 \text{ if } j = i + 1 \right\rangle^+ .$$

Then, its Garside set is $\mathcal{S} = \{\boldsymbol{e}, \sigma_1, \ldots, \sigma_m, \Delta\}$, where $\Delta = \sigma_m \sigma_1$ is the only element of $\mathcal{S}$ with length 2. In particular, we have $\mathscr{C} = \{\sigma_1, \ldots, \sigma_m\}$, and $\sigma_i \to \sigma_j$ for all $j \neq i+1$, which proves (P5) in this case.

In case **A** is of type B or D, combinatorial descriptions of **B** are provided in [32,33,31, 29,34,35]. They include descriptions of the generating family Σ, of the Garside set $\mathcal{S}$, and of the multiplication relations in $\mathcal{S}$. These descriptions lead, in all dual braid monoids, to the fact that $(\mathcal{S}, \leq_\mathrm{l})$ is a lattice, and to the following property:

(†) *For all* $\sigma, \tau \in \Sigma$ *such that* $\sigma \cdot \tau \in \mathcal{S}$, *it holds that* $\tau \leq_\mathrm{l} \sigma \cdot \tau$ *and* $\sigma \leq_\mathrm{r} \sigma \cdot \tau$.

Property (†) leads to Lemma 6.5 and Corollary 6.6. From there, and by following a proof that is very similar to the one in [1] for monoids of type A, we show Lemmas 6.8 and 6.9, thereby demonstrating Proposition 6.4. □

Lemma 6.5. *The left and right divisibility relations coincide on* **B***, i.e.:*

$$\forall x, y \in \mathbf{B} \quad x \leq_\mathrm{l} y \iff x \leq_\mathrm{r} y. \tag{6.1}$$

Proof. We proceed by induction on $|y|$. The statement is immediate if $|x| = 0$ or $|x| = |y|$, hence we assume that $0 < |x| < |y|$.

If $x \leq_\mathrm{l} y$, let us factor x and y as products of the form $x = x_1 \cdot x_2$ and $y = x \cdot y_3 \cdot y_4$, where x_2 and y_4 belong to Σ. Using the induction hypothesis and Property (†), there exist elements y_3', y_4' and y_4'' of **B** such that

$$y = x_1 \cdot x_2 \cdot y_3 \cdot y_4 = y_3' \cdot x_1 \cdot x_2 \cdot y_4 = y_3' \cdot x_1 \cdot y_4' \cdot x_2 = y_3' \cdot y_4'' \cdot x_1 \cdot x_2,$$

which proves that $x \leq_\mathrm{r} y$. We prove similarly that, if $x \leq_\mathrm{r} y$, then $x \leq_\mathrm{l} y$. □

Corollary 6.6. *For all $x \in \mathcal{S}$, let $L(x) = \{\sigma \in \Sigma \mid \sigma \leq_l x\}$. Then, for all x and y in $\mathcal{S}$:*

$$\big(\forall \tau \in L(y) \quad \exists \sigma \in L(x) \quad \sigma \to \tau\big) \implies (x \to y).$$

Proof. Assume that the relation $x \to y$ does not hold. This means that we can factor y as a product $y = \tau \cdot y_2 \cdot y_3$, with $\tau \in \Sigma$, such that the element $z = x \cdot \tau \cdot y_2$ belongs to $\mathcal{S}$. Then, let $\sigma \in L(x)$ be such that $\sigma \to \tau$.

By Lemma 6.5, we can also factor x as a product $x = x_1 \cdot \sigma$. Hence, the element $\sigma \cdot \tau$ is a factor of z, and therefore $\sigma \cdot \tau \leq_r z$. Since $\mathcal{S}$ is downward closed under $\leq_r$, this means that $\sigma \cdot \tau \in \mathcal{S}$, contradicting the fact that $\sigma \to \tau$. □

Remark 6.7. The above corollary indicates how to infer a relation $x \to y$ for $x, y \in \mathcal{S}$ based on the knowledge of the restriction $\to \big|_{\Sigma \times \Sigma}$. In turn, the explicit presentations given in [31] for dual monoids of type B and D have the following property: for any two generators $\sigma, \tau \in \Sigma$, the relation $\sigma \to \tau$ holds if and only if the product $\sigma \cdot \tau$ does not appear as a member of the presentation rules of the dual monoid. In particular, $\sigma \to \sigma$ holds for all $\sigma \in \Sigma$.

Lemma 6.8. *Let $\mathbf{A}$ be an irreducible Artin–Tits monoid of type B, with two generators or more. The dual monoid associated with $\mathbf{A}$ satisfies Property* (P5).

Proof. Thanks to the combinatorial descriptions mentioned above, we identify elements of $\mathcal{S}$ with *type B non-crossing partitions* of size n, where n is the number of generators of $\mathbf{A}$. These are the partitions $\mathbf{T} = \{T^1, \ldots, T^m\}$ of $\mathbb{Z}/(2n)\mathbb{Z}$ such that, for every set T^i, the set $n + T^i$ is also in $\mathbf{T}$, and the sets $\{\exp(\mathbf{i}k\pi/n) \mid k \in T^i\}$ have pairwise disjoint convex hulls in the complex plane. Note that $\mathbf{T}$ is thus globally invariant by central symmetry. Both relations $\leq_l$ and $\leq_r$ then coincide with the partition refinement relation.

Thus, we respectively identify the extremal elements $\boldsymbol{e}$ and Δ of $\mathcal{S}$ with the partitions $\big\{\{1\}, \{2\}, \ldots, \{2n\}\big\}$ and $\big\{\{1, 2, \ldots, 2n\}\big\}$, and Σ consists of those partitions

$$\sigma_{i,j} = \big\{\{i, j\}, \{i+n, j+n\}\big\} \cup \big\{\{k\} \ : \ k \neq i, j, i+n, j+n\big\},$$

with $1 \leq i \leq n$ and $i < j \leq n+i$. They are pictured in Fig. 2 for $n = 3$. The left divisors of a partition $\mathbf{T}$, *i.e.*, the elements of $L(\mathbf{T})$, are then those partitions $\sigma_{i,j}$ that refine $\mathbf{T}$.

Finally, keep in mind the following observation. Let v be a maximal proper factor of Δ, *i.e.*, a type B non-crossing partition of which Δ is the immediate successor in the partition refinement order. After application of a bijection of the form $i \in \mathbb{Z}/(2n)\mathbb{Z} \mapsto i + j$, for some integer $j \in \{0, 2n-1\}$, v is of the form

$$v_i = \big\{\{1, \ldots, i\}, \{n+1, \ldots, n+i\}, \{i+1, \ldots, n, n+i+1, \ldots, 2n\}\big\}, \qquad (6.2)$$

for some integer $i \in \{1, \ldots, n\}$.

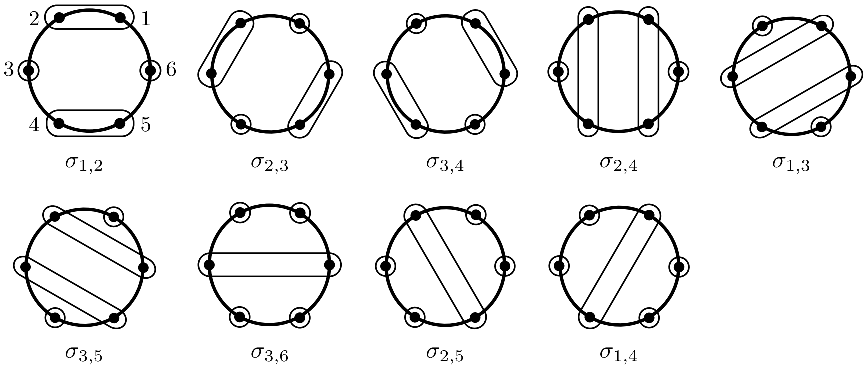

Fig. 2. Generators of the dual monoid of type B for $n = 3$.

Now, consider two elements y, z of $\mathcal{S} \setminus \{\boldsymbol{e}, \Delta\}$, and let us prove that $y \to^* z$. Since $z <_1 \Delta$, and since the map $\sigma_{i,j} \mapsto \sigma_{i+1,j+1}$ induces an automorphism of the monoid $\mathbf{B}$, corresponding to the rotation of the complex plane of angle π/n, we assume without loss of generality that z is a refinement of some partition v_i from (6.2), for an integer $i \in \{1, \ldots, n\}$. Corollary 6.6 implies in particular that $v_i \to z$ holds.

Based on Remark 6.7, one sees that:

$$\sigma_{i,j} \to \sigma_{u,v} \iff \big((u < i \leq v < j) \text{ or } (i \leq u < j \leq v) \text{ or } (i \leq u + n < j \leq v + n)\big).$$

Using Corollary 6.6, it implies: $v_j \to v_{j+1}$ when $1 \leq j \leq n-1$, and $v_n \to w \to v_1$ with $w = \big\{\{1, n+2, \ldots, 2n\}, \{2, \ldots, n+1\}\big\}$. It follows that holds: $v_n \to^* v_i \to z$.

Finally, let $\sigma_{a,b}$ be some generator in $L(y)$. It comes that $y \to \sigma_{1,a} \to \sigma_{1,n+1}$ if $1 < a$, or $y \to \sigma_{1,n+1}$ if $a = 1$. We then observe that $\sigma_{1,n+1} \to x \to v_n$, where $x = \big\{\{j, n+1-j\} \ : \ 1 \leq j \leq n\big\}$, and therefore that $y \to^* z$. $\square$

Lemma 6.9. *Let* $\mathbf{A}$ *be an irreducible Artin–Tits monoid of type* D*, with two generators or more. The dual monoid associated with* $\mathbf{A}$ *satisfies Property* (P5)*.*

Proof. Let $n+1$ be the number of generators of $\mathbf{A}$. If $n \leq 2$, $\mathbf{A}$ is also a braid monoid. Hence, we assume that $n \geq 3$.

We identify elements of $\mathcal{S}$ with *type* D *non-crossing partitions* of size $n+1$. Here, we associate every element k of $\mathbb{Z}/(2n)\mathbb{Z}$ with the complex point $p_k = \exp(\mathbf{i}k\pi/n)$. We also consider a two-element set $\{\bullet, \bullet + n\}$, with the convention that $\bullet + 2n = \bullet$, and associate both $\bullet$ and $\bullet + n$ with the complex point $p_\bullet = p_{\bullet+n} = 0$.

Then, type D non-crossing partitions of size n are the partitions $\mathbf{T} = \{T^1, \ldots, T^m\}$ of $\mathbb{Z}/(2n)\mathbb{Z} \cup \{\bullet, \bullet + n\}$ such that (i) $\mathbf{T}$ does *not* contain the set $\{\bullet, \bullet + n\}$, (ii) for every set T^i, the set $n + T^i$ is also in $\mathbf{T}$, and (iii) the sets $\{p_k \mid k \in T^i\}$ have pairwise disjoint convex hulls in the complex plane, with the exception that they may share the point $p_\bullet = p_{\bullet+n} = 0$.

Again, both relations $\leq_{\mathrm{l}}$ and $\leq_{\mathrm{r}}$ coincide with the partition refinement relation, extremal elements $\boldsymbol{e}$ and Δ of $\mathcal{S}$ are identified with the respective partitions

$$\big\{\{1\},\{2\},\ldots,\{2n\},\{\bullet\},\{\bullet+n\}\big\} \quad \text{and} \quad \big\{\{1,2,\ldots,2n,\bullet,\bullet+n\}\big\},$$

and Σ consists of those partitions

$$\begin{aligned}\sigma_{i,j} &= \big\{\{k\} \ : \ k \neq i,j,i+n,j+n\big\} \cup \big\{\{\bullet\},\{\bullet+n\},\{i,j\},\{i+n,j+n\}\big\}\\ \tau_\ell &= \big\{\{k\} \ : \ k \neq \ell,\ell+n\big\} \cup \big\{\{\ell,\bullet\},\{\ell+n,\bullet+n\}\big\},\end{aligned}$$

with $1 \leq i \leq n$, $i < j < n+i$, and $1 \leq \ell \leq 2n$.

Moreover, proceeding as in Remark 6.7, one finds that:

$$\begin{cases} \sigma_{i,j} \to \tau_w & \text{if } w \in \{i,\ldots,j-1\} \text{ or } w+n \in \{i,\ldots,j-1\};\\ \sigma_{i,j} \to \sigma_{u,v} & \text{if } u < i \leq v < j,\ i \leq u < j \leq v \text{ or } i \leq u+n < j \leq v+n;\\ \tau_\ell \to \tau_w & \text{if } w \in \{\ell,\ldots,\ell+n-1\};\\ \tau_\ell \to \sigma_{u,v} & \text{if } \{\ell,\ell+n\} \cap \{u+1,\ldots,v\} \neq \emptyset. \end{cases}$$

Now, consider two elements y,z of $\mathcal{S} \setminus \{\boldsymbol{e},\Delta\}$, and let us prove that $y \to^* z$. Since $z <_{\mathrm{l}} \Delta$, and since the map $\sigma_{i,j} \mapsto \sigma_{i+1,j+1}, \quad \tau_\ell \mapsto \tau_{\ell+1}$ induces an automorphism of the monoid $\mathbf{B}$, we assume without loss of generality that z is refinement of a partition v that is either equal to $\tau_{2n}^{-1}\Delta = \{\{1,\ldots,n,\bullet\},\{n+1,\ldots,2n,\bullet+n\}\}$ or to

$$\begin{aligned}\sigma_{n,n+i}^{-1}\Delta = \{\{1,\ldots,i\},&\{n+1,\ldots,n+i\},\\ &\{i+1,\ldots,n,n+i+1,\ldots,2n,\bullet,\bullet+n\}\}\end{aligned}$$

for some integer $i \in \{1,\ldots,n-1\}$.

Using Corollary 6.6, one checks that $\sigma_{n,n+j}^{-1}\Delta \to \sigma_{n,n+j+1}^{-1}\Delta$ when $1 \leq j \leq n-1$, and that $\sigma_{n,2n}^{-1}\Delta \to \tau_{2n}^{-1}\Delta \to \tau_1^{-1}\Delta \to \sigma_{n,n+1}^{-1}\Delta$, where

$$\tau_1^{-1}\Delta = \big\{\{2,\ldots,n+1,\bullet\},\{n+2,\ldots,1,\bullet+n\}\big\}.$$

It follows that $\tau_{2n}^{-1}\Delta \to^* v \to z$. Finally, let λ be some generator in $L(y)$. It comes that

$$\begin{cases} y \to \sigma_{1,a} \to \sigma_{1,n} & \text{if } \lambda = \sigma_{a,b} \text{ with } 2 \leq a;\\ y \to \sigma_{1,n} & \text{if } \lambda = \sigma_{a,b} \text{ with } a = 1;\\ y \to \sigma_{1,\ell} \to \sigma_{1,n} & \text{if } \lambda = \tau_\ell \text{ with } \ell \leq n;\\ y \to \sigma_{1,\ell+n} \to \sigma_{1,n} & \text{if } \lambda = \tau_\ell \text{ with } n+1 \leq \ell. \end{cases}$$

We then observe that $\sigma_{1,n+1} \to w \to \tau_{2n}^{-1}\Delta$, where

$$w = \begin{cases} \big\{\{j, 2n+1-j\} \ : \ 1 \le j \le n \text{ and } j \ne m\big\} \cup \big\{\{m, m+n, \bullet, \bullet + n\}\big\} \\ \quad \text{if } n \text{ is odd and } m = (n+1)/2; \\ \big\{\{j, 2n+1-j\} \ : \ 1 \le j \le n\big\} \cup \big\{\{\bullet\}, \{\bullet + n\}\big\} \quad \text{if } n \text{ is even.} \end{cases}$$

It follows that $y \to^* z$. □

6.3.2. Free products of Garside monoids

Garside monoids are extensively studied structures [36–38,8]. Their definition is as follows [37]. First recall that an *atom* of a monoid $\mathcal{M}$ is an element $x \in \mathcal{M}$, different from the unit element $\boldsymbol{e}$, and such that $x = yz \implies (y = \boldsymbol{e} \vee z = \boldsymbol{e})$. A monoid is *atomic* if it is generated by its atoms. *A Garside monoid is an atomic monoid $\mathcal{M}$, left and right cancellative, such that $(\mathcal{M}, \le_{\mathrm{l}})$ and $(\mathcal{M}, \le_{\mathrm{r}})$ are two lattices, and such that $\mathcal{M}$ contains a Garside element.* A *Garside element* is an element $\Delta \in \mathcal{M}$ such that $\{x \in \mathcal{M} \bigm| x \le_{\mathrm{l}} \Delta\} = \{x \in \mathcal{M} \bigm| x \le_{\mathrm{r}} \Delta\}$, and such that this set is finite and generates $\mathcal{M}$. By [37, Prop. 1.12], this set is then necessarily a Garside subset of $\mathcal{M}$.

Garside monoids do not necessarily have a length function as in (P1). Hence, in the following result, we have to assume its existence as an additional assumption in order to fit with our previous setting.

Proposition 6.10. *Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be non-trivial Garside monoids satisfying Property* (P1)*. Then the monoid $\mathbf{A}_0$ defined as the free product $\mathbf{A}_0 = \mathbf{A}_1 * \mathbf{A}_2$ satisfies Properties* (P1) *to* (P7)*.*

Proof. This is a consequence of the properties of Garside monoids recalled above, together with the following elementary lemma. □

Lemma 6.11. *Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two non-trivial monoids satisfying Properties* (P1)*–*(P4)*. Then the free product monoid $\mathbf{A}_0 = \mathbf{A}_1 * \mathbf{A}_2$ satisfies Properties* (P1) *to* (P7)*.*

Proof. Let $u = x_1 y_1 \ldots x_p y_p$, with $x_1, \ldots, x_p \in \mathbf{A}_1$ and $y_1, \ldots, y_p \in \mathbf{A}_2$, denote a generic element of $\mathbf{A}_0$. The length of u is defined by $|u|_0 = |x_1|_1 + \cdots + |x_p|_1 + |y_1|_2 + \cdots + |y_p|_2$, and the length function thus defined on $\mathbf{A}_0$ satisfies (P1). The free product of cancellative monoids is itself cancellative, hence $\mathbf{A}_0$ satisfies (P2). Let $u' = x'_1 y'_1 \ldots x'_q y'_q \in \mathbf{A}_0$. Put $k = \max\{j \le p, q \bigm| x_1 y_1 \ldots x_j y_j = x'_1 y'_1 \ldots x'_j y'_j\}$. If $k = p$ then $u \wedge u' = u$ and if $k = q$ then $u \wedge u' = u'$. Otherwise, it is readily seen that:

$$u \wedge u' = \begin{cases} x_1 y_1 \ldots x_k y_k (x_{k+1} \wedge x'_{k+1}), & \text{if } x_{k+1} \ne x'_{k+1}\,, \\ x_1 y_1 \ldots x_k y_k x_{k+1} (y_{k+1} \wedge y'_{k+1}), & \text{if } x_{k+1} = x'_{k+1}\,. \end{cases}$$

Hence $u \wedge u'$ exists in all cases, and $\mathbf{A}_0$ satisfies (P3).

Identify the two monoids $\mathbf{A}_1$ and $\mathbf{A}_2$ with their images in $\mathbf{A}_0$ through the canonical injections $\mathbf{A}_1 \to \mathbf{A}_0$ and $\mathbf{A}_2 \to \mathbf{A}_0$. Then the (disjoint) union $\mathcal{S}_1 \cup \mathcal{S}_2$ is a finite Garside subset of $\mathbf{A}_0$, which satisfies thus (P4).

Consider the *augmented* Charney graph $(\mathscr{C}_1', \to)$ of $\mathbf{A}_1$, where $\mathscr{C}_1' = \mathcal{S}_1 \setminus \{\boldsymbol{e}_{\mathbf{A}_1}\}$, *i.e.*, $\mathscr{C}_1' = \mathscr{C}_1 \cup \{\Delta_1\}$ if $\mathcal{S}_1$ has a maximum Δ_1, and $\mathscr{C}_1' = \mathscr{C}_1$ otherwise. We define the augmented Charney graph $(\mathscr{C}_2', \to)$ of $\mathbf{A}_2$ similarly. Then, the Charney graph $(\mathscr{C}_0, \to)$ of $\mathbf{A}_0$ is the bipartite complete graph with parts $(\mathscr{C}_1', \to)$ and $(\mathscr{C}_2', \to)$. Therefore $\mathbf{A}_0$ satisfies necessarily (P5). And since the two monoids $\mathbf{A}_1$ and $\mathbf{A}_2$ are assumed to be non-trivial, both parts contain edges. Hence $\mathbf{A}_0$ satisfies (P7).

Up to rescaling the length function $|\cdot|_0$ by a multiplicative factor, we assume that the integers in the set $\{|x|_0 \bigm| x \in \mathbf{A}_0\}$ are setwise coprime. We show below that $\mathbf{A}_0$ satisfies (P6) by proving that, for all prime numbers p, there exists a cycle in $(\mathscr{C}_0, \to)$ whose total length is not divisible by p.

We pick an element $a \in \mathcal{S}_1 \setminus \{\boldsymbol{e}_1\}$ such that p does not divide $|a|_1$. Such an element exists; otherwise, since $\mathcal{S}_1$ generates $\mathbf{A}_1$, it would contradict our assumption that the integers in $\{|x|_0 \bigm| x \in \mathbf{A}_0\}$ are setwise coprime. We also pick a cycle $x_1 \to x_2 \to \ldots \to x_k \to x_1$ in $(\mathscr{C}_0, \to)$, such that x_1 and x_k belong to $\mathcal{S}_2$. Then, both $x_1 \to x_2 \to \ldots \to x_k \to x_1$ and $x_1 \to x_2 \to \ldots \to x_k \to a \to x_1$ are cycles in the Charney graph $(\mathscr{C}_0, \to)$. Their total lengths differ by $|a|_1$, hence p does not divide both of them. This completes the proof. $\square$

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