

Symplectic Techniques for Semiclassical Completely Integrable Systems

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Abstract

This article is a survey of classical and quantum completely integrable systems from the viewpoint of local “phase space” analysis. It advocates the use of normal forms and shows how to get global information from glueing local pieces. Many crucial phenomena such as monodromy or eigenvalue concentration are shown to arise from the presence of non-degenerate critical points.

1 Foreword

This article is mainly an adaptation and a translation of the last chapters of my habilitation thesis defended in December 2003. Its aim is to describe, in a unified way, old and new results in the theory of classical and quantum (or rather semiclassical) completely integrable systems in the spirit of the famous Darboux-Carathéodory theorem. This theorem (which was essentially already known to Liouville) gives a *symplectic local normal form* for a classical completely integrable system near a regular point. From the viewpoint of modern geometers, it is very natural to build up the global theory from such local results. It was far less obvious to apply this idea to quantum systems, until the appearance of pseudodifferential operators, Fourier integral operators and microlocalisation techniques in the late 1960's. Nowadays, if one is reasonably familiar with both geometric and microlocal techniques, it seems evident that, as we did for classical systems, one should be able to discover the global theory of semiclassical integrable systems from local semiclassical analogues of the Darboux-Carathéodory theorem. This, in essence, is what this article is about. Although it is not my purpose to give the reader many details (especially about semiclassical theories), I still hope that the text manages to convey the right intuition. A more complete treatment will be published elsewhere [54].

2 What is a completely integrable system ?

While the notion of completely integrable systems is now perfectly standard, it is not obvious at all to define the *moduli space* of all integrable systems. More simply put: when shall we say that two completely integrable systems are equivalent ?

2.1 Classical mechanics

Let M be a C^∞ symplectic manifold of dimension $2n$. The algebra $C^\infty(M)$ of classical observables (or Hamiltonians) is equipped with the symplectic Poisson bracket $\{\cdot, \cdot\}$. Any function $H \in C^\infty(M)$ gives rise to a vector field \mathcal{X}_H (the Hamiltonian vector field) which, as a derivation, is just the Poisson bracket by H ; in other words the evolution of a function f under the flow of \mathcal{X}_H is given by the equation

$$\dot{f} = \{H, f\}.$$

An *integral* of the Hamiltonian H is a function which is invariant under the flow of \mathcal{X}_H ; this means a function f such that $\{H, f\} = 0$. The Hamiltonian H is called *completely integrable* if there exists $n - 1$ independent functions f_2, \dots, f_n which are integrals of H ($\{H, f_j\} = 0$) and moreover *pairwise commute*: $\{f_i, f_j\} = 0$. (This last condition is always a consequence of the former for “generic” H ; but since we are going to study particular models or *normal forms*, which are not generic, this condition is crucial.)

Actually, one sees from the definition that the function H does not play any distinguished role among the other functions f_2, \dots, f_n . Our point of view will always be to consider, as a whole, a *classical completely integrable system* to be the data of n functions f_1, \dots, f_n in involution: $\{f_i, f_j\} = 0$, which are independent in the sense that for almost every point $m \in M$, $df_1(m), \dots, df_n(m)$ are linearly independent. n is the largest number of such functions for which this is indeed possible.

We define the *momentum map* of the system to be the map $F = (f_1, \dots, f_n) : M \rightarrow \mathbb{R}^n$. In the language of Hamiltonian Lie group actions, this is indeed a momentum (or “moment”) map for a local action of \mathbb{R}^n on M .

It is tempting to say that two completely integrable systems are equivalent when their momentum maps are equivalent, in the sense that there exists a diffeomorphism g of \mathbb{R}^n such that $F_1 = g \circ F_2$. However, as we shall see later, this notion is too strong, especially due to the existence of flat functions in the C^∞ category.

The first natural attempt to weaken this equivalence is the following.

Definition 2.1 *Let U be an open subset of M . The **momentum algebra** $\mathfrak{f}|_U = \langle f_1, \dots, f_n \rangle$ is the linear span of the f_j 's, as an abelian subalgebra of $C^\infty(U)$. The **commutant** of \mathfrak{f} is the set of all $g \in C^\infty(U)$ Poisson-commuting with \mathfrak{f} . It is denoted by $\mathcal{C}_{\mathfrak{f}}(U)$.*

Notice that by Jacobi's identity, $\mathcal{C}_{\mathbf{f}}(U)$ is a Lie algebra under Poisson bracket.

Definition 2.2 Let $F = (f_1, \dots, f_n)$ and $G = (g_1, \dots, g_n)$ be two completely integrable systems with momentum algebras \mathbf{f} and \mathbf{g} . Then F and G are called **weakly equivalent** on an open subset U if

$$\mathcal{C}_{\mathbf{f}}(U) = \mathcal{C}_{\mathbf{g}}(U).$$

It is clear that the definition of weak equivalence does not depend on the particular choice of basis for \mathbf{f} and \mathbf{g} . Therefore, by a slight abuse of notation, the fact that the two systems in consideration are equivalent shall be denoted as

$$\mathbf{f} \sim_U \mathbf{g}.$$

The associations $U \rightarrow \mathbf{f}|_U$ or $U \rightarrow \mathcal{C}_{\mathbf{f}}(U)$ define typical presheaves over M . With this in mind, we shall often simply refer to \mathbf{f} or $\mathcal{C}_{\mathbf{f}}$ when the localisation on U is unimportant or clearly implicit. See also corollary 3.6.

The geometric interpretation of this definition is clear: The fibres of F define on M a *singular Lagrangian foliation*: when $c \in \mathbb{R}^n$ is a regular value of F (rank $dF(m) = n$ for all $m \in F^{-1}(c)$), then $F^{-1}(c)$ is a Lagrangian submanifold of M . The leaves of the singular Lagrangian foliation are all the connected components of the fibres of F . Then $\mathcal{C}_{\mathbf{f}}$ is just the algebra of smooth functions that are constant on the leaves. So $\mathbf{f} \sim \mathbf{g}$ says that the spaces of leaves of the corresponding singular foliations are the same for \mathbf{f} and \mathbf{g} , in a smooth way.

It turns out that the weak equivalence is not able, in some cases, to distinguish singularities of integrable systems; this is due to the fact that there is no requirement that the functions f_1, \dots, f_n always be a *reduced* set of equations for the foliation (in the algebraic geometry sense). For instance the functions x and x^3 on \mathbb{R}^2 give weakly equivalent systems, while as functions they obviously don't have the same singularity type.

Definition 2.3 Let $F = (f_1, \dots, f_n)$ and $G = (g_1, \dots, g_n)$ be two completely integrable systems with momentum algebras \mathbf{f} and \mathbf{g} . Let $m \in M$. Then F and G are called **strongly equivalent** at m if there exists a small neighbourhood of m on which

$$\mathcal{C}_{\mathbf{f}}(\mathbf{f} - \mathbf{f}(m)) = \mathcal{C}_{\mathbf{g}}(\mathbf{g} - \mathbf{g}(m)).$$

This will be denoted as $\mathbf{f} \stackrel{s}{\sim} \mathbf{g}$.

It is elementary (and probably standard in algebraic geometry) to see that two strongly equivalent systems have exactly the same singularity type. To do this, given a momentum algebra \mathbf{f} vanishing at m , we shall say that a basis (f_1, \dots, f_n) of \mathbf{f} is well ordered when there is a partition of $n = n_1 + \dots + n_d$, $n_k \geq 0$ such that for all $k \in [1..d]$, the vector space spanned by the f_j corresponding to the block n_k (which means $j \in [(n_1 + \dots + n_{k-1} + 1)..(n_1 + \dots + n_k)]$) consists of functions

vanishing exactly at order k at m (except for the zero function). The associated partition is unique. For instance if m is a regular point, every basis of \mathbf{f} is well ordered. If m is a singular point, then necessarily n_1 is the rank of $dF(m)$. Then one can show the following lemma:

Lemma 2.4 *If $\mathbf{f} \stackrel{s}{\sim} \mathbf{g}$ at a point m , then there exist well ordered basis of \mathbf{f} and \mathbf{g} associated to a common partition $n = n_1 + \dots + n_d$, and an $n \times n$ matrix N with coefficients in $\mathcal{C}_{\mathbf{f}} (= \mathcal{C}_{\mathbf{g}})$ such that $N(m)$ is block-diagonal (the i -eth block being of size n_i) and*

$$(g_1 - g_1(m), \dots, g_n - g_n(m)) = N \cdot (f_1 - f_1(m), \dots, f_n - f_n(m)).$$

2.2 Quantum mechanics

The quantum analysis will be performed using \hbar -pseudo-differential operators. This theory is now well established and very robust. It is simultaneously a quantum theory dealing with self-adjoint operators on a Hilbert space and a semiclassical theory, dealing with \hbar -deformations of classical Hamiltonians. The reader is invited to check the references [41], [18] or [9].

Let X be an n -dimensional smooth manifold equipped with a density $|dx|$. The Hilbert space is the corresponding $L^2(X, |dx|)$. \hbar -pseudodifferential operators on X act on $L^2(X, |dx|)$ and have symbols defined on the cotangent bundle $M = T^*X$. Locally in x , any such operator can be obtained from a *full symbol* $a(x, \xi; \hbar)$ by the Weyl quantisation formula

$$(Au)(x) = (\text{Op}_\hbar^w(a)u)(x) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} e^{i\hbar(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi; \hbar\right) u(y) |dy d\xi| \quad (1)$$

Such a full symbol is not invariant under coordinate changes, but if we restrict to volume preserving diffeomorphisms, and to symbols admitting an asymptotic expansion⁽¹⁾ of the form $\hbar^k a_0(x, \xi) + \hbar^{k+1} a_1(x, \xi) + O(\hbar^{k+2})$, then k is called the *order* of the operator, and the *principal symbol* a_0 and the *subprincipal symbol* a_1 are intrinsically defined as functions on T^*X .

The space of \hbar -pseudodifferential operators is a graded algebra. We have a symbolic calculus, which means that the product of operators of order zero is still a pseudodifferential operator of order zero whose principal symbol is the product of the original principal symbols. Moreover the commutation bracket of operators of order zero is a pseudodifferential operator of order 1 whose principal symbol is $\frac{1}{i}$ times the Poisson bracket of the original principal symbols.

By definition, a *semiclassical completely integrable system* is the data of n self-adjoint \hbar -pseudodifferential operators P_1, \dots, P_n of order zero which pairwise commute modulo $\mathcal{O}(\hbar^\infty)$: $[P_i, P_j] = \mathcal{O}(\hbar^\infty)$ and whose principal symbols p_j have almost

⁽¹⁾Such semiclassical symbols are usually called ‘classical’...

everywhere independent differentials. Then of course these symbols (p_1, \dots, p_n) define a classical completely integrable system on T^*X .

We are mainly interested in the microlocal behaviour of the P_j 's to construct *joint quasimodes*, ie. solutions u of the system of equations

$$P_j u = \mathcal{O}(\hbar^\infty) \quad \forall j = 1, \dots, n, \quad (2)$$

where u is a distribution on X depending on \hbar in a temperate way (which we don't explicit here). More generally we shall deal with *microlocal solutions* of this system. Roughly speaking u is a microlocal solution at a point $m \in T^*X$ if (2) holds when both sides are multiplied on the left by a pseudodifferential operator whose principal symbol does not vanish at m . The set of points in T^*X where a distribution u does not vanish microlocally is called the *microsupport* of u . A microlocal solution of (2) has therefore a microsupport in the level set $p^{-1}(0)$.

As quantum operators, the P_j 's have a spectrum, which we shall always assume to be discrete (this is the case for instance when the momentum map $p = (p_1, \dots, p_n)$ is proper). If they commute exactly: $[P_i, P_j] = 0$ then they also have a *joint spectrum*, which is the set of n -uples of eigenvalues $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ associated to a common eigenfunction. Then the construction of joint quasimodes goes a long way in describing this joint spectrum modulo $\mathcal{O}(\hbar^\infty)$.

In analogy with the classical case, for each semiclassical completely integrable system P_1, \dots, P_n and for any open subset $U \subset M = T^*X$, we define the semiclassical momentum algebra $\mathbf{P}|_U$ to be the linear span of the P_j 's, and the semiclassical commutant $\mathcal{C}_{\mathbf{P}}(U)$ to be the Lie algebra of all \hbar -pseudodifferential operators commuting with \mathbf{P} microlocally in U . As before, \mathbf{P} and $\mathcal{C}_{\mathbf{P}}$ are presheaves over M .

The weak and strong equivalences for semiclassical systems are defined as in the classical case. For instance, we will use the following:

Definition 2.5 *Two semiclassical completely integrable systems with momentum algebra \mathbf{P} and \mathbf{Q} are called **strongly equivalent** at a point $m \in M$ if, microlocally near m ,*

$$\mathcal{C}_{\mathbf{P}}(\mathbf{P} - \mathbf{p}(m)) = \mathcal{C}_{\mathbf{Q}}(\mathbf{Q} - \mathbf{q}(m)).$$

This will be denoted by $\mathbf{P} \stackrel{s}{\sim} \mathbf{Q}$.

The ‘‘quantised’’ version of lemma 2.4 then holds, where N becomes a matrix with pseudodifferential coefficients. Notice that if $\mathbf{P} \stackrel{s}{\sim} \mathbf{Q}$ at m then $\mathbf{P} - \mathbf{p}(m)$ and $\mathbf{Q} - \mathbf{q}(m)$ share the same microlocal joint quasimodes. In a sense, the set (presheaf) of all joint quasimodes for a semiclassical system is the quantum analogue of the classical Lagrangian foliation.

Given a semiclassical system \mathbf{P} , we have an underlying classical system given by the principal symbols. The *subprincipal symbols* r_j can then be seen as a small deformation of the induced Lagrangian foliation; more precisely, they define a family of 1-forms κ_c on the leaves $p^{-1}(c)$ by the formula

$$\kappa_c(\mathcal{X}_{p_j}) = -r_j, \quad (3)$$

It is easy to check (using for instance the Darboux-Carathéodory theorem below) that near any regular point of $p^{-1}(c)$, κ_c is a smooth closed 1-form. It is called the *subprincipal form* of the semiclassical system.

2.3 Canonical transformations

The main strength of pseudodifferential operators is the possibility of transforming them according to any local symplectomorphism. If χ is a local symplectomorphism of T^*X near m then there exist a *Fourier integral operator* U which is a bounded operator on $L^2(X)$ such that for any pseudodifferential operator P with principal symbol p , the operator $U^{-1}PU$ is a pseudodifferential operator whose principal symbol near m is $p \circ \chi^{-1}$ (Egorov theorem [41, 9]).

Constructing more global Fourier integral operators using a partition of unity is not difficult, provided the following obstruction vanishes: let α be the canonical Liouville 1-form of T^*X . Then $\alpha - \chi^*\alpha$ is closed. The obstruction is its cohomology class. In other words χ should be “exact” in the sense that it preserves integrals of α along closed loops.

If \mathbf{P} is a quantum completely integrable system and U a Fourier integral operator associated to a canonical transformation χ then $U^{-1}\mathbf{P}U$ is a quantum completely integrable system with momentum algebra $\mathbf{p} \circ \chi^{-1}$. Moreover the subprincipal form κ_c is modified only by the addition of an exact 1-form.

3 Local study

The local behaviour of a completely integrable system can be very rich and is far from being thoroughly understood in general. We review here the current state of the art for the C^∞ category and show how it applies to quantum systems.

3.1 Regular points

Let (f_1, \dots, f_n) be a classical completely integrable system on a $2n$ -symplectic manifold M , with momentum map F .

Definition 3.1 *A point $m \in M$ is called **regular** for F if $dF(m)$ has maximal rank (n). In other words $df_1 \wedge \dots \wedge df_n(m) \neq 0$.*

By the local submersion theorem, the fibres $F^{-1}(c)$ for c close to $F(m)$ are locally n -dimensional submanifolds near a regular point m . The local structure of regular points of completely integrable systems is very simple. It is actually entirely described by the following classical theorem:

Theorem 3.2 (Darboux-Carathéodory) *If m is regular, F is symplectically conjugate near m to the linear fibration (ξ_1, \dots, ξ_n) on the linear symplectic space \mathbb{R}^{2n} with coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ and symplectic form $\sum_i d\xi_i \wedge dx_i$.*

In other words, there exists functions ϕ_1, \dots, ϕ_n on M such that

$$(\phi_1, \dots, \phi_n, f_1, \dots, f_n)$$

is a system of canonical coordinates in a neighbourhood of m .

In principle the name of Liouville should be associated with this theorem, since well before Darboux and Carathéodory, Liouville gave a very nice and explicit formula for the functions ϕ_j . This result published in 1855 [34] explains the local integration of the flow of any completely integrable Hamiltonian (possibly depending on time) near a regular point of the foliation in terms of the famous Liouville 1-form $\sum_i \xi_i dx_i$. With this respect it implies the Darboux-Carathéodory theorem, even if Liouville's formulation is more complicated.

The Darboux-Carathéodory theorem has some simple but very important corollaries.

Lemma 3.3 *Locally near a regular point, the commutant $\mathcal{C}_{\mathbf{f}}$ is the set of functions of the form $\varphi(f_1, \dots, f_n)$, where $\varphi \in C^\infty(\mathbb{R}^n)$.*

Proof. Apply Darboux-Carathéodory. □

Proposition 3.4 *$\mathcal{C}_{\mathbf{f}}$ is a commutative Lie-Poisson algebra.*

Proof. As we saw already, it is a Lie algebra due to the Jacobi identity. It is a Poisson algebra due to the Leibniz identity. By the preceding lemma, it is commutative near regular points, and hence everywhere. □

As a consequence of this proposition, we have the following useful lemma:

Lemma 3.5 *If \mathbf{f} and \mathbf{g} are two momentum algebras, then*

$$\mathbf{f} \sim \mathbf{g} \iff \mathbf{f} \subset \mathcal{C}_{\mathbf{g}}.$$

Corollary 3.6 *If $\mathbf{f} \sim_U \mathbf{g}$ and $V \subset U$ then $\mathbf{f} \sim_V \mathbf{g}$.*

Finally, lemma 3.3 implies the following characterisation (which will not hold in the singular case)

Proposition 3.7 *If m is regular for both \mathbf{f} and \mathbf{g} then $\mathbf{f} \sim \mathbf{g}$ near m if and only if $(f_1, \dots, f_n) = \varphi(g_1, \dots, g_n)$, where φ is a local diffeomorphism of \mathbb{R}^n .*

In this case, the weak equivalence is identical to the strong one (recall lemma 2.4 above).

Semiclassics — The Darboux-Carathéodory theorem admits a semiclassical analogue which is again simple but powerful. The first proof in the framework of homogeneous pseudodifferential operators is due to Colin de Verdière [7] (although the case $n = 1$ was already treated by Duistermaat and Hörmander [21]).

Let P_1, \dots, P_n be a semiclassical completely integrable system on $M = T^*X$ and let $p = (p_1, \dots, p_n)$ be the classical momentum map consisting of the principal symbols. We also use $P = (P_1, \dots, P_n)$ for the quantum momentum map, or “quantum fibration”.

Theorem 3.8 *If m is regular then P is microlocally conjugate near m to the fibration $(\frac{\hbar}{i} \frac{\partial}{\partial x_1}, \dots, \frac{\hbar}{i} \frac{\partial}{\partial x_n})$ acting on \mathbb{R}^n .*

In other words, there exists a Fourier integral operator U defined near m and microlocally unitary such that $U^{-1}P_jU = \frac{\hbar}{i} \frac{\partial}{\partial x_j}$.

Proof. Consider the symplectomorphism given by Darboux-Carathéodory’s theorem and U_0 a Fourier integral operator quantising it near m . The result is thus obtained modulo pseudodifferential operators of order 1. To correct this error one conjugates again by a pseudodifferential operator of the form $\exp(iA)$, where A is a pseudodifferential operator. Since we are only after a microlocal result (and hence modulo $\mathcal{O}(\hbar^\infty)$), it is enough to show that there is a neighbourhood of m in which one can solve to any order in \hbar , which is a simple exercise. \square

Remark 3.9 It is not necessary for P to be self-adjoint. As long as the principal symbol is real valued, the theorem still holds, but the unitariness of U is lost. \triangle

Using this theorem one can check that all the corollaries of the classical Darboux-Carathéodory theorem we have mentioned above still hold in the semiclassical framework. It is even more important to see that one can now describe microlocal joint quasimodes near regular points.

Proposition 3.10 ([56]) *If m is a regular point, the space of microlocal solutions of the system*

$$P_j u = O(\hbar^\infty) \quad \text{near } m \quad \forall j = 1, \dots, n$$

is a \mathbb{C}_\hbar -module of rank 1, generated by $U^{-1}\mathbf{1}$, where U is a Fourier integral operator as in Theorem 3.8 and $\mathbf{1}$ is a wave function microlocally equal to 1 near $0 \in \mathbb{R}^{2n}$.

Here \mathbb{C}_\hbar is the natural ring that acts on microlocal solutions; it is the set of complex numbers depending in a temperate way on \hbar (see [56]).

3.2 Singular points

The singularity theory of integrable systems is certainly not completely understood, even for classical systems. I present here my personal perception of it.

The study of singularities of integrable systems is fundamental for various reasons. On the one hand, because of the way an integrable system is defined : n functions on a manifold, it is expected (apart from exceptional cases) that singularities will necessarily occur. On the other hand these functions define a dynamical system such that their singularities correspond to fixed points and relative equilibria of the system, which are of course one of the main characteristics of the dynamics. From a semiclassical viewpoint, we know furthermore that important wave functions such as eigenfunctions of the system have a microsupport which is invariant under the classical dynamics; therefore, in a sense that I shall not present here (one should talk about semiclassical measures), they concentrate near hyperbolic singularities (see for instance [11] and the work of Toth [48])). This concentration entails not only the growth in norm of eigenfunctions (see for instance [49]) but also a higher local density of eigenvalues (see figure 4 in section 4.2 below and the articles [12, 56, 15]).

The singularities of a Hamiltonian system can be approached either through the study of the flow of the vector fields — this is the “dynamical systems” viewpoint — or through the study of the Hamiltonian functions themselves — this is the “foliation” perspective. In the case of completely integrable systems, both aspects are equivalent because the vector fields of the n functions f_1, \dots, f_n form a basis of the tangent spaces of the leaves of the foliation $f_i = \text{const}_i$, at least for regular points. I shall always tend to be on the foliation side, which displays more clearly the geometry of the problem.

However, the foliations we are interested in are *singular*, and the notion of a singular foliation is already delicate. Generally speaking these foliations are of Stefan-Süßmann type [45] : the leaves are defined by an integrable distribution of vector fields. But they are more than that: they are Hamiltonian, and they are *almost regular* in the sense that the singular leaves cannot fill up a domain of positive measure. The precise way of dealing with these singular foliations is encoded in the way we define two equivalent foliations; for us, this will be the strong equivalence of definition 2.3.

Non-degenerate singularities — In singularity theory for differentiable functions, “generic” singularities are Morse singularities. In the theory of completely integrable systems there exists a very natural analogue of the notion of Morse singularities (or more generally of Morse-Bott singularities if one allows critical submanifolds). These so-called *non-degenerate* singularities are now well defined (and “exemplified”) in the literature, so I will only recall briefly the definition.

Let $F = (f_1, \dots, f_n)$ be a completely integrable system on M with momentum algebra \mathfrak{f} .

Definition 3.11 ([53]) *A fixed point $m \in M$ is called **non-degenerate** if the Hessians $d^2 f_j(m)$ span a Cartan subalgebra of the Lie algebra of quadratic forms on $T_m M$ (equipped with the linearised Poisson bracket).*

A friendlier characterisation is that a generic linear combination of the linearised vector fields at m (these are Hamiltonian matrices : in $\mathfrak{sp}(2n, \mathbb{R})$) should admit $2n$ distinct eigenvalues. The definition above applies to a fixed point. But more generally if $dF(m)$ has corank r one can assume that $df_1(m), \dots, df_{n-r}(m)$ are linearly independent; then we consider the restriction of f_{n-r+1}, \dots, f_n to the symplectic manifold Σ obtained by local symplectic reduction under the action of f_1, \dots, f_{n-r} . We shall say that m is non-degenerate (or *transversally non-degenerate*) whenever m is a non-degenerate fixed point for this restriction of the system to Σ .

The linear approximation or *linear model*⁽²⁾ of such a critical point is the system $\mathbf{f}_0 := (\xi_1, \dots, \xi_{n-r}, q_1, \dots, q_r)$ on $T^*\mathbb{R}^{n-r} \times T_m\Sigma$, where the q_j 's form a basis of the aforementioned Cartan subalgebra.

Theorem 3.12 (Eliasson's theorem [24, 23]) *Non-degenerate critical points are linearisable: there exists a local symplectomorphism χ in the neighbourhood of m such that*

$$\chi^*\mathbf{f} \stackrel{s}{\sim} \mathbf{f}_0.$$

In order to use this theorem one has to understand the linear classification of Cartan subalgebras of $\mathfrak{sp}(2n, \mathbb{R})$. This follows from the work of Williamson [64], which shows that any such Cartan subalgebra has a basis build with three type of blocks: two uni-dimensional ones (the elliptic block: $q = x^2 + \xi^2$ and the real hyperbolic one: $q = x\xi$) and a two-dimensional block called focus-focus or loxodromic or complex hyperbolic: $q_1 = x\xi + y\eta$, $q_2 = x\eta - y\xi$. Notice that over \mathbb{C} the classification is trivial since everything can be conjugate to the "hyperbolic" case $x\xi$. The analytic case of Eliasson's theorem was proved by Rübmann [43] for two degrees of freedom systems and by Vey [53] in any dimension. In the C^∞ category the *lemme de Morse isochore* of Colin de Verdière and Vey [14] implies Eliasson's result for one degree of freedom systems. Eliasson's proof of the general case was somewhat loose at a crucial step, but this has been recently completely clarified [37].

The strong equivalence relation for non-degenerate singularities is equal to the weak equivalence and is fairly well understood. In particular in case of real hyperbolic blocks it does not imply the functional equivalence (which would be " $\chi^*\mathbf{f} = \varphi \circ \mathbf{f}_0$ ") while this is indeed the case otherwise. For more details refer to [57].

Semiclassics — It is possible to prove a semiclassical version of Eliasson's theorem, which is crucial for further study of singular Bohr-Sommerfeld conditions as in [56] or for the estimate by Zelditch and Toth of norms of eigenfunctions [50]. The new semiclassical feature is the appearance of formal series in \hbar of microlocal invariants.

⁽²⁾The term "linear" refers to the linearisation of the vector fields \mathcal{X}_{f_i} at a fixed point; of course the functions f_j themselves do not become linear, but quadratic.

Theorem 3.13 ([57]) *If m is non-degenerate of corank r , there exists a Fourier integral operator U defined near m and microlocally unitary and there exists formal series $\alpha_j(\hbar) \in \mathbb{C}[[\hbar]]$, $j = 1, \dots, r$ such that*

$$U \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix} U^{-1} \underset{\mathcal{S}}{\sim} \begin{pmatrix} \hat{\xi}_1 \\ \vdots \\ \hat{\xi}_{n-r} \\ \hat{q}_1 - \hbar \alpha_1(\hbar) \\ \vdots \\ \hat{q}_r - \hbar \alpha_r(\hbar) \end{pmatrix}$$

acting on $\mathbb{R}^{n-r} \times \mathbb{R}^r$, microlocally near m .

In this statement we have used the hat for standard Weyl quantisation (for instance $\hat{\xi}_j = \frac{\hbar}{i} \frac{\partial}{\partial x_i}$ and $\widehat{x\xi} = \frac{\hbar}{i} (x \frac{\partial}{\partial x} + \frac{1}{2})$). The introduction of the series $\alpha_j(\hbar)$ is necessary to go from the weak equivalence to the strong one.

This theorem is well adapted to the microlocal resolution of the system $P_j u = \mathcal{O}(\hbar^\infty)$ since the latter is transformed into the system $(\hat{q}_j - \alpha_j)u = \mathcal{O}(\hbar^\infty)$, which can be solved explicitly. Since the model system is uncoupled, one just has to study separately each block, and one can show the following facts: for an elliptic block, the space of microlocal solutions (in the sense of proposition 3.10) has dimension 1; for a real hyperbolic block, it has dimension 2 [12]; for a focus-focus block, it has dimension 1 [56].

3.3 More degenerate singularities

For the moment very little is known concerning degenerate singularities. The most natural approach seems to be via algebraic geometry, as in [26, 27]. For one degree of freedom analytic systems, a more concrete method is presented in [10], which explicitly displays the relevant versal unfoldings. For a general linearisation result in the analytic category, see also [68]. I am not aware of similar results in the C^∞ category.

4 Semi-global study

If one aims at understanding the classical geometry of a completely integrable foliation or its microlocal analysis, the *semi-global* aspect is probably the most fundamental. The terminology semi-global refers to anything that deals with an invariant neighbourhood of a leaf of the foliation. This semi-global study is what allows for instance the construction of quasimodes associated to a Lagrangian submanifold. Sometimes semi-global merely reduces to local, when the leaf under consideration is a critical point with only elliptic blocks.

4.1 Regular fibres

The analysis of neighbourhoods of regular fibres, based on the Liouville-Arnold-Mineur theorem (also known as action-angle theorem) is now routine and fully illustrated in the literature, for classical aspects as well as for quantum ones. It is the foundation of the whole modern theory of completely integrable systems (in the spirit of Duistermaat's article [20]) but also of KAM-type perturbation theorems. The microlocal analysis of action-angle variables starts with the work of Colin de Verdière [8], followed in the \hbar semiclassical theory by Charbonnel [5], and more recently by myself and various articles by Zelditch, Toth, Popov, Sjöstrand and many others. The case of compact symplectic manifolds has recently started, using the theory of Toeplitz operators [6].

Let (f_1, \dots, f_n) be an integrable system on a symplectic manifold M . In the rest of this article we shall always assume the momentum map F to be *proper*: all fibres are compact. Let c be a regular value of F . If we restrict to an adequate invariant open set, we can always assume that the fibres of F are connected. Let $\Lambda_c := F^{-1}(c)$. Fibres being compact and parallelisable (by means of the vector fields \mathcal{X}_{f_i}), they are tori.

Theorem 4.1 (Liouville-Arnold-Mineur) *If Λ_c is regular, there exists a symplectomorphism χ from $T^*\mathbb{T}^n$ into M sending the zero section onto Λ_c such that*

$$\chi^*\mathbf{f} \sim \mathbf{f}_0,$$

where \mathbf{f}_0 is the linear system (ξ_1, \dots, ξ_n) on $T^*\mathbb{T}^n$.

Here and in what follows we identify $T^*\mathbb{T}^n$ with $\mathbb{T}^n \times \mathbb{R}^n$ (where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) equipped with coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ such that the canonical Liouville 1-form is $\sum_i \xi_i dx_i$. It is easy to see that the theorem actually implies $\chi^*\mathbf{f} = \varphi(\mathbf{f}_0)$ for φ a local diffeomorphism of \mathbb{R}^n ; this is usually the way the result is stated. It is important to remark that $d\varphi$ is an *invariant* of the system since it is determined by *periods* of periodic trajectories if the initial system. Regarded as functions on M the ξ_j 's are called *actions* of the system for one can find a primitive α of ω in a neighbourhood of Λ_c such that the ξ_j 's are integrals of α on a basis of cycles of Λ_c depending smoothly on c .

For semiclassical purposes, one needs an “exact” version of the Liouville-Arnold-Mineur theorem in the sense that, given a primitive α of ω , a symplectomorphism is called exact when it preserves the integrals of α along closed paths (action integrals). We get immediately:

Theorem 4.2 *If Λ_c is regular, there exists an exact symplectomorphism χ from $T^*\mathbb{T}^n$ into M sending the section $(\xi_1, \dots, \xi_n) = (a_1, \dots, a_n) = \text{const}$ onto Λ_c such that*

$$\chi^*\mathbf{f} \sim \mathbf{f}_0$$

if and only if $a_i = \int_{\gamma_i} \alpha$, where γ_i is the cycle on Λ_c corresponding via χ to the i -th canonical cycle on \mathbb{T}^n .

Semiclassics — Let (P_1, \dots, P_n) be a semiclassical completely integrable system whose principal symbols define a proper momentum map. We still denote by Λ_c the Lagrangian leaves. α is the canonical 1-form of $M = T^*X$ and as before we let $a = (a_1, \dots, a_n)$ be the action integrals along the basis of cycles of Λ_c defined by the chosen actions ξ_j .

Theorem 4.3 (Semiclassical action-angle) *If Λ_c is regular there exists formal series $\lambda_j(\hbar) \in \mathbb{C}[[\hbar]]$ and a Fourier integral operator U associated to an exact symplectomorphism from $T^*\mathbb{T}^n$ to M sending the “ $\xi = a$ ” section onto Λ_c such that, microlocally near the “ $\xi = a$ ” section, we have*

$$U(P_1, \dots, P_n)U^{-1} \stackrel{s}{\sim} (\hat{\xi}_1 - \hbar\lambda_1(\hbar), \dots, \hat{\xi}_n - \hbar\lambda_n(\hbar)),$$

acting on \mathbb{T}^n .

Explicitly this means

$$U(P_1 - p_1(m), \dots, P_n - p_n(m))U^{-1} = N \cdot (\hat{\xi}_1 - \tilde{\lambda}_1(\hbar), \dots, \hat{\xi}_n - \tilde{\lambda}_n(\hbar)),$$

where m is any point of Λ_c , N is an $n \times n$ microlocally invertible matrix of pseudodifferential operators and $\tilde{\lambda}_j(\hbar) = a_j + \hbar\lambda_j(\hbar)$. One can see that the action a_j can be considered as the first semiclassical invariant. The second term is given by integrals of the *subprincipal form* (see definition (3) page 5) and of the *Maslov cocycle* on the Lagrangian Λ_c (see also [56]).

The following statement is a direct consequence of the theorem.

Theorem 4.4 (Regular Bohr-Sommerfeld quasimodes [56]) *There is a non trivial microlocal solution of the system $P_j u = O(\hbar^\infty)$ (which is therefore microlocalised on Λ_0) if and only if $\tilde{\lambda}_j(\hbar) \in 2\pi\hbar\mathbb{Z}$. The solution is unique (in the sense of proposition 3.10).*

From this one can deduce the *regular Bohr-Sommerfeld conditions* in the following way. Suppose $\mathbf{P} = \mathbf{P}^E$ depends on a parameter $E \in \mathbb{R}^n$ in such a way that for any m near $\Lambda_0^E = (p^E)^{-1}(0)$, the principal symbols map $(E_1, \dots, E_n) \rightarrow (p_1^E, \dots, p_n^E)(m)$ is a local diffeomorphism.

Definition 4.5 *We call **microlocal joint spectrum** the set $\Sigma_\hbar(P_1^E, \dots, P_n^E)$ of all $E \in \mathbb{R}^n$ such that the system $P_j^E u_\hbar = O(\hbar^\infty)$, $j = 1, \dots, n$ admits a non trivial microlocal solution on the whole fibre Λ_0^E .*

The typical case is of course $P_j^E = P_j - E_j$. From our perspective it is often wiser to forget the linear dependence on E , which is not invariant under strong equivalence.

The regular Bohr-Sommerfeld conditions are obtained from theorem 4.4 if one remarks that it still holds “with parameters”. They can be stated as follows:

Theorem 4.6 ([56]) *The microlocal joint spectrum consists of solutions E of*

$$\tilde{\lambda}_j^E(\hbar) \in 2\pi\hbar\mathbb{Z},$$

where

$$\tilde{\lambda}_j^E(\hbar) = \int_{\gamma_i^E} \alpha + \hbar \int_{\gamma_i^E} \kappa^E + \hbar \frac{\mu(\gamma^E)\pi}{2} + O(\hbar^2). \quad (4)$$

Here κ^E is the subprincipal 1-form on Λ_0^E and μ the Maslov cocycle. $(\gamma_1^E, \dots, \gamma_n^E)$ is any basis of cycles on Λ_0^E .

4.2 Singular fibres

This section is devoted to the semi-global structure of fibres with non-degenerate singularities. I am not aware of any semi-global result for more degenerate singularities. The topological analysis of non-degenerate singular fibres was mainly initiated by Fomenko [25], and successfully expanded by a number of his students. See [4].

Elliptic case — Near an elliptic fixed point, the fibres are small tori and are entirely described by the local normal form, for classical systems as well as for semiclassical ones (the system is reduced to a set of uncoupled harmonic oscillators). Therefore I shall not talk about this type of singularity any further... even if strictly speaking the semi-global semiclassical study has not been fully carried out for *transversally elliptic* singularities. But no particular difficulties are expected in that case.

Focus-focus case — Eliasson's theorem gives the local structure of focus-focus singularities. Several people have noticed (in the years 1996-1997) that this was enough to determine the *monodromy* of the foliation around the singular fibre; I'll expand on this in section 5. Actually this local structure is a starting point for understanding much more: the semi-global classification of a singular fibre of focus-focus type. Unlike monodromy which is a topological invariant, already observed in torus fibrations without Hamiltonian structure, the semi-global classification involves purely symplectic invariants.

Let $F = (f_1, f_2)$ be a completely integrable system with two degrees of freedom on a 4-dimensional symplectic manifold M . Let m be a critical point of focus-focus type; we assume for simplicity that $F(m) = 0$, and that the (compact, connected) fibre Λ_0 does not contain other critical points. One can show that Λ_0 is a "pinched" torus (Lagrangian immersion of a sphere S^2 with a transversal double point), surrounded by regular fibres which are standard \mathbb{T}^2 tori. What are the semi-global invariants associated to this singular fibration ?

One of the major characteristics of focus-focus singularities is the existence of a *Hamiltonian action of S^1* that commutes with the flow of the system, in a

neighbourhood of Λ_0 . Indeed, let us start by applying Eliasson's theorem near m to reduce to a momentum map $F = (f_1, f_2)$ which is equal near m to the canonical focus-focus basis $(x\xi + y\eta, x\eta - y\xi)$. Then f_2 is the periodic Hamiltonian we are looking for; it can also be identified with an action integral associated to the vanishing cycle of the pinched torus (cf. fig. 1).

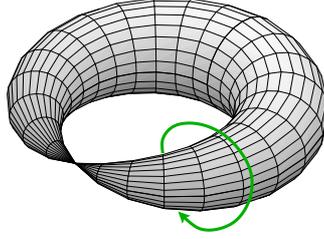


Figure 1: Vanishing cycle for the pinched torus

Let c be a regular value for F , close to 0. Given a point A on Λ_c , we define $\tau_1(c) > 0$ to be the time of first return for the \mathcal{X}_{f_1} -flow on the orbit of A under the \mathcal{X}_{f_2} -flow, and $\tau_2(c) \in \mathbb{R}/2\pi\mathbb{Z}$ to be the time it takes to return to A under the action of \mathcal{X}_{f_2} . Let us define

$$\sigma_1(c) = \tau_1(c) + \Re(\ln c) \quad \text{et} \quad \sigma_2(c) = \tau_2(c) - \Im(\ln c),$$

where we identified $c = c_1 + ic_2$. One shows that $\sigma := \sigma_1(c)dc_1 + \sigma_2(c)dc_2$ is a closed C^∞ 1-form in a neighbourhood of this origin. Let s be the primitive of σ vanishing at the origin. Let $[\mathbf{f}]$ be the foliation associated to the system in a neighbourhood of Λ_0 (ie. the equivalence class of \mathbf{f} modulo strong equivalence). Let $S([\mathbf{f}])$ be the Taylor expansion of s .

Theorem 4.7 ([61]) *$S([\mathbf{f}])$ completely characterises the singular foliation in a neighbourhood of Λ_0 , which means:*

- *$S([\mathbf{f}])$ is well-defined (it does not depend on the choice of Eliasson's local chart and is invariant under strong equivalence);*
- *If $[\mathbf{f}]$ and $[\mathbf{g}]$ are two singular foliations in a neighbourhood of focus-focus leaves, and satisfy $S([\mathbf{f}]) = S([\mathbf{g}])$, then there exists a semi-global symplectomorphism χ such that $\chi^*\mathbf{f} \stackrel{s}{\sim} \mathbf{g}$;*
- *If T is any formal series in $\mathbb{R}[[X, Y]]$ without constant term, then there exists a singular foliation of focus-focus type \mathbf{f} such that $T = S([\mathbf{f}])$.*

Remark 4.8 The fact that two focus-focus fibrations are always semi-globally topologically conjugate was already proved by Zung [66], who introduced various

topological notions of equivalence. In our language this means that the topological class is invariant by strong equivalence. The theorem shows that this is no longer the case for the symplectic class. \triangle

Remark 4.9 S can be interpreted as a *regularised* (or *desingularised*) *action*. Indeed if γ_c is the loop on Λ_c defined just as in the description of τ_j above, and if α is a semi-global primitive of the symplectic form ω , let $\mathcal{A}(c) = \int_{\gamma_c} \alpha$; then

$$S(c) = \mathcal{A}(c) - \mathcal{A}(0) + \Re(c \ln c - c). \quad \triangle$$

The semiclassical study of focus-focus fibres was carried out in the article [56]. *Singular Bohr-Sommerfeld conditions* are stated there which allow a complete description of the microlocal joint spectrum in a neighbourhood of the critical value of the momentum map. Unlike the case of standard action-angle variables, it is not possible to proceed merely by applying a semiclassical semi-global normal form; indeed, the classification theorem does not provide us with an explicit model. And all the examples that I know of, that can reasonably claim to be a “typical model of focus-focus singularity”, are not explicitly solvable. The strategy of that paper is to see the microlocal solutions as global sections of a sheaf which, because of the local normal forms, is a locally flat bundle; then such a global section exists if and only if the *holonomy* of the sheaf is trivial. This approach may of course be used in the regular case as well; the phase of this holonomy is then identified to the *semiclassical action integrals* of formula (4). In the singular case the adequate holonomy is a *regularisation* of the usual semiclassical action, in the sense of remark 4.9 above. As a matter of fact the first term of this holonomy is exactly the classical invariant described in theorem 4.7 above, which entails that the symplectic equivalence class of the foliation is a *spectral invariant* of the quantum system. See [56, 58] or more details.

Besides semiclassics, theorem 4.7 leads to a number of various applications. One can for instance exploit the fact that the set of symplectic equivalence classes of these foliations acquires a vector space structure. That is what Symington does in [47] to show that neighbourhoods of focus-focus fibres are always symplectomorphic (after forgetting the foliation, of course). For this one introduces functions S_0 and S_1 whose Taylor expansions give the invariants of the two foliations, and constructs a “path of foliations” by interpolating between S_0 and S_1 . Then a Moser type argument yields the result (since the symplectic forms are cohomologous).

The theorem is also very useful for doing almost explicit calculations in a neighbourhood of the fibre. For instance it is possible in this way to determine the validity of non-degeneracy conditions that appear in KAM type theorems⁽³⁾, for a perturbation of a completely integrable system with a focus-focus singularity (see also [65]).

Theorem 4.10 ([22]) *Let H be a completely integrable Hamiltonian with a loxo-*

⁽³⁾A nice discussion about these various conditions can be found in [42]

dromic singularity at the origin (ie. H admits a singular Lagrangian foliation of focus-focus type at the origin). Then in a neighbourhood of 0,

- Kolmogorov non-degeneracy condition if fulfilled on all tori close to the critical fibre;
- the “isoenergetic turning frequencies” condition is fulfilled except on a 1-parameter family of tori corresponding to a curve through the origin in the image of the momentum map which is transversal to the lines of constant energy H .

Hyperbolic case — Just as elliptic blocks, hyperbolic blocks have dimension 1 (normal form $q_i = x_i \xi_i$); however they turn out to be more complicated and display a richer structure, due to two main reasons. The first one is fundamental: the singular fibres are not localised near the singularity; instead they consist of stable and unstable manifolds that can connect several singular points. the second reason, more technical, is that the natural C^∞ structure of the space of leaves is more involved. For instance in the case of the figure “8” (fig. 2) the “topological” space of leaves is a “Y”; nevertheless from the real analytic viewpoint (when H is analytic) the space of leaves is just an interval (functions that commute with $x\xi$ are functions of $x\xi$). In the C^∞ category the space of leaves is still a “Y”, but whose hands have all derivatives equal at the branching point; C^∞ functions that commute with $x\xi$ are locally described by two smooth functions $f^+(x\xi)$ and $f^-(x\xi)$ (for instance associated to the half-spaces $\pm x > 0$) such that $f^+ - f^-$ is flat at the origin.

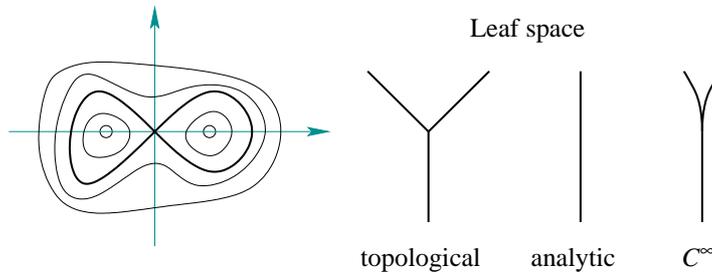


Figure 2: Leaf space of a real hyperbolic foliation

Concerning the semi-global aspect, the classification of hyperbolic foliations has been carried out only for 1 degree of freedom, in Toulet’s thesis [51, 19]. As a matter of fact, it is possible to give a proof of the statement provided in that note with similar methods to those that were used in [61]. In a slightly weaker context (orbital equivalence) Bolsinov [3] has studied the case of transversally hyperbolic singularities (codimension 1) in two degrees of freedom.

For one degree of freedom systems, the critical fibre is a *graph* whose vertices have degree 4 (if all singularities of the fibre are hyperbolic). The invariant is the

graph itself, properly *decorated*. The corresponding semiclassical analysis was treated by Colin de Verdière and Parisse [11, 12, 13]. The authors use the graph of Dufour-Molino-Toulet to read the correct quantisation conditions. Here again, a holonomy method is employed.

In the article [15] we have studied in details the case of two degrees of freedom with transversally hyperbolic singularities. The set of critical points in the critical fibre is a union of circles. In a neighbourhood of each circle, we reduce to a model situation which may have a $\mathbb{Z}/2\mathbb{Z}$ symmetry (this is the case for instance of the Birkhoff normal form in $1 : 2$ resonance, for a non vanishing energy; the critical fibre is displayed in figure 3). As for the unidimensional case, we construct a

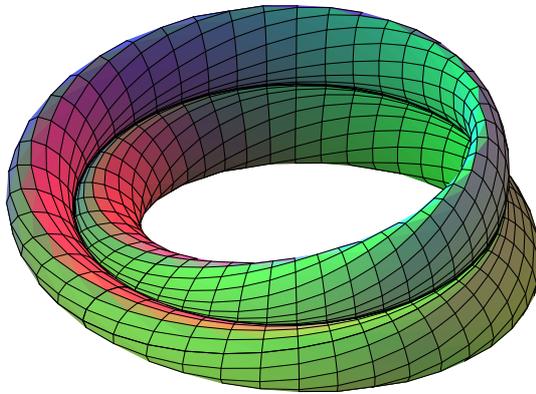


Figure 3: Hyperbolic critical fibre with $\mathbb{Z}/2\mathbb{Z}$ symmetry (case of the $1 : 2$ resonance)

graph whose homology serves to state the singular Bohr-Sommerfeld conditions. This graph is abstractly the *reduction* of the critical fibre by an S^1 action that we construct and which leave the foliation invariant. The “subtlety” is that critical circles with non-trivial $\mathbb{Z}/2\mathbb{Z}$ symmetry become vertices of *degree 2* (instead of degree 4). On the other hand the issue of the delicate C^∞ structure on the graph cannot be avoided either and leads to a somewhat involved proof of the validity of these Bohr-Sommerfeld conditions. (This difficulty was avoidable in dimension 1.)

In this way one obtains in a very precise way the *universal behaviour* of the microlocal joint spectrum near a transversally hyperbolic separatrix, which yields amongst others the calculation of the local density of eigenvalues (Weyl type formulas). As expected, the distance between two points in the joint spectrum is of order \hbar in the direction given by the periodic Hamiltonian, while it is of order $\hbar/|\ln \hbar|$ in the transversal direction (cf. fig. 4).

By looking at a picture similar to figure 4, Sadovskii et Zhilinskií had the idea that one could define (and calculate) a *fractional monodromy* [40], that reflects the homology with rational coefficients of the singular torus fibration. Using our

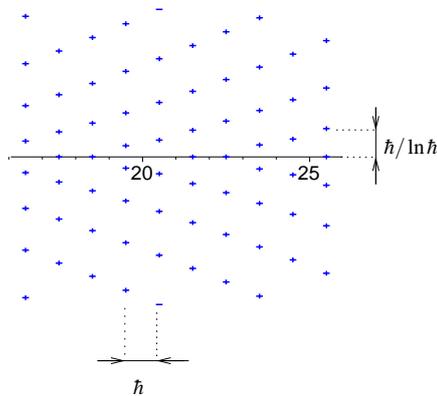


Figure 4: Part of the joint spectrum for a transversally hyperbolic singularity with $\mathbb{Z}/2\mathbb{Z}$ symmetry. Case of the 1 : 2 resonance. The horizontal axis carries the energy values and the vertical axis the values of the additional integral K .

work one should be able to write it in a rigorous way for a general transversally hyperbolic singularity and calculate it by means of the graph of the foliation. The “singular rational affine structure” of the basis should be closely compared to the joint spectrum as well.

Remaining cases — To complete the study of two degrees of freedom integrable systems one should still include the elliptic-hyperbolic case and the hyperbolic-hyperbolic case.

The case of a system with a critical point splitting into an elliptic block and a hyperbolic block is probably the more simple. It can be regarded as a limit case of a transversally hyperbolic case whose critical circles degenerate into a point, excluding the possibility of a $\mathbb{Z}/2\mathbb{Z}$ symmetry. The critical fibre is therefore a graph of degree 4 embedded in M .

Concerning semiclassics, one should obtain two quantisation conditions: one related to the cohomology of the graph; the other to the vanishing cycle.

The hyperbolic-hyperbolic case is certainly a very good open problem. It can be seen as a branching of four transversally hyperbolic singularities, possibly coupled. The various possible topologies are described in [32].

Higher degrees of freedom — In principle no essential difficulty should arise in higher degrees of freedom with non-degenerate singularities, since all the building blocks have at most two degrees of freedom. I still believe this should be very interesting to explore. On the classical side, the topological theory was written by Zung [66, 69], but it is not sufficient to turn it into semiclassics. On the other hand the generalisation of the normal form near transversally hyperbolic orbits (with

discrete symmetry) has been recently achieved in [38] and should be well suited for semiclassical purposes.

5 Global study

Up to now, we have considered various properties of integrable systems in small neighbourhoods of minimal invariants objects: orbits of points in M . How to get a global picture from them ? The adjective “global” covers several aspects, some qualitative and some quantitative.

For instance, the aim of Duistermaat was to globalise properties given by the Liouville-Arnold-Mineur theorem. He was thus interested in the fibration over the set of *regular points*, analysing the role of monodromy, Chern class, and cohomology class of the symplectic form.

On the other hand the most natural way of globalising is to look for a description of the symplectic manifold relying on formulas that provide a “localisation” of global objects on *singularities* of the system — just as Morse theory. Under the non-degeneracy hypothesis for critical points, Zung made a thorough study and displayed the importance of the *integral affine structure* on the base of the fibration [69].

This integral affine structure is another angle for dealing with the global problem. In the most simple case of *toric* completely integrable systems (those whose flow defines an effective action of \mathbb{T}^n) one can completely characterise the system consisting of the symplectic manifold M and the momentum map F by means of the image of F which, in the integral affine manifold \mathbb{R}^n , is a convex rational polytope (Delzant’s theorem [17]). It seems that a natural way of generalising toric systems is to allow only non-degenerate singularities of elliptic or focus-focus types. These systems are called *almost toric* (the terminology was probably introduced for the first time by Symington).

From the semiclassics viewpoint, the “globalisation” may refer to the semiclassical analogues of the above geometrical globalisations. It is also natural to consider the issue of passing from the microlocal to the “exact”: how to use microlocal constructions to obtain results concerning the “true” Schrödinger operator acting on the “true” Hilbert space $L^2(X)$? What is the relationship between the microlocal spectrum and the exact spectrum ?

5.1 The exact spectrum

On $M = T^*X$ we are given a quantum completely integrable system $P = (P_1, \dots, P_n)$.

Definition 5.1 *The joint spectrum of P is the set of all $(E_1, \dots, E_n) \in \mathbb{R}^n$ such that there exists a normalised element $\psi \in L^2(X)$ such that*

$$\forall i, \quad P_i \psi = E_i \psi.$$

The passage from microlocal to exact is based on two points. The first one is very general and the second one is more adapted to our vision of integrable systems.

The first point deals with the geometry and analysis at infinity: the part of the Lagrangian foliation under study should be well separated from further possible connected components of the fibration $p = (p_1, \dots, p_n)$. This shall be granted by the assumption of local properness of p : there exists a compact $K \subset \mathbb{R}^n$ such that $p^{-1}(K)$ is compact. If X is compact, or $X = \mathbb{R}^n$ and the p_j have a good behaviour at infinity (for example all their derivatives are bounded by a weight function in the sense of Hörmander [29]), one can show that in any compact K' whose interior is inside K , the joint spectrum is discrete (of finite multiplicity) [5].

The second point relies on the construction of microlocal quasimodes and their microlocal *multiplicity*, as in proposition 3.10. The microlocal uniqueness of solutions of the system not only shows that these quasimodes are good approximations of eigenfunctions but also demonstrates that they form a complete system, since they are microlocally orthogonal to each other [56]. This ensures that the microlocal spectrum is really a perturbation of order $\mathcal{O}(\hbar^\infty)$ of the exact spectrum, including multiplicities.

5.2 Regular fibrations: the case of monodromy

I recall here the definition of monodromy and its semiclassical consequences. I shall be very brief, referring for instance to [55, 56, 59] for more details.

The Liouville-Arnold-Mineur theorem defines actions variables in a neighbourhood of regular values of the momentum map F . Seen as local charts for the open set B_r of all regular values of F , they endow B_r with the structure of an integral affine manifold with structure group the affine group $\mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n$. By definition, the *affine monodromy* of the system is the holonomy of this affine structure. Another way of defining an integral affine structure is to specify a distribution of *lattices* of maximal rank in each tangent space. This lattice is the dual of the *period lattice*, which is the set of all (τ_1, \dots, τ_n) such that the Hamiltonian vector field $\tau_i \mathcal{X}_{f_i}$ is 1-periodic. The linear part of the affine monodromy, simply called *monodromy*, is also the holonomy of the flat bundle of homology groups of the fibres of the torus fibration over B_r .

Following an idea of Cushman and Duistermaat, I have shown how this monodromy can be read off from the microlocal *joint spectrum* of a corresponding quantum system. One just has to apply the regular Bohr-Sommerfeld conditions which locally describe the joint spectrum as a part of a lattice of type $\hbar\mathbb{Z}^n$, whose mesh size tends to 0. One can then define an asymptotic integral affine structure and show that for \hbar small enough it coincides with the classical affine structure on B_r [55].

5.3 Focus-focus and monodromy

A remarkable properties of focus-focus singularities for a two degrees of freedom system is that they imply the presence of a universal nontrivial monodromy for the regular fibration around the critical fibre. This results holds in general for topological torus fibrations with a generic isolated critical fibre [39, 35]; in the Hamiltonian case it was rediscovered by Nguyễn Tiên Zung [67], Matveev [36], and some others. The particular feature of the Hamiltonian situation is that the monodromy is oriented: while, in the topological case, the monodromy matrix is $\begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$, in the Hamiltonian case the sign is prescribed, and always positive.

Indeed if one chooses an orientation of \mathbb{R}^2 it induces through the momentum map F and the symplectic form a natural orientation on each Lagrangian torus, and hence on their homology [16]. Concerning the quantum case, Zhilinskií suggests that the sign should be interpreted as the fact that the “lattice” of eigenvalues around a focus-focus critical value has a *point defect* (in the sense that a number of points was removed, and not added). This assertion can be verified by the singular Bohr-Sommerfeld conditions which give the precise position of eigenvalues near the singularity [56].

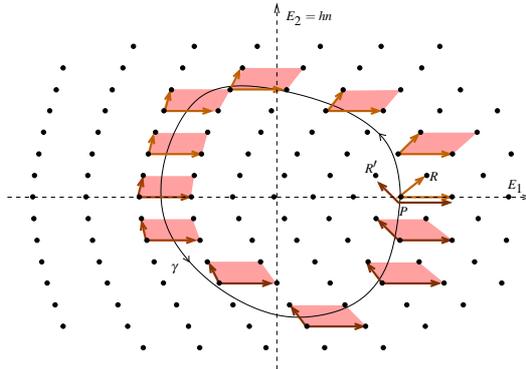


Figure 5: A joint spectrum with monodromy, and the calculation of the latter as an asymptotic affine holonomy

5.4 Toric systems and polyads

The simplest case of integrable systems whose global geometry is perfectly understood is the toric one. Let (f_1, \dots, f_n) be a completely integrable system whose momentum map F is proper.

Definition 5.2 *The system F is of toric type if there exists an effective Hamiltonian action of \mathbb{T}^n with momentum map Φ of the form $\Phi = \varphi \circ F$ where φ is a local diffeomorphism on the image of F .*

Recall that by the Atiyah-Guillemin-Sternberg theorem [1, 28] the fibres of a momentum map for a Hamiltonian torus action are connected and the image is a rational convex polytope; and by Delzant's theorem [17] this image fully determines the manifold M and the momentum map Φ (up to isomorphism) in the completely integrable case. Actually the connexity/convexity theorem is stated for compact M , but still holds if the momentum map is proper [31] (the "polytope" being not necessarily bounded). In this text we shall use abusively the terms polytopes and polygons for possibly non-compact polyhedral sets.

One can show that definition 5.2 entails that the fibres of F are connected and φ is a diffeomorphism from the image of F into the image of Φ . Hence the structure of the fibration is perfectly known. In particular the singularities are all of (transversally) elliptic type. Thus by glueing local descriptions one can easily obtain, in the semiclassical framework, a global description of the joint spectrum.

Theorem 5.3 ([62]) *Let $P = (P_1, \dots, P_n)$ be a quantum completely integrable system whose classical limit is of toric type, and let $\Sigma(P)$ be its joint spectrum. There exists a map φ_\hbar from \mathbb{R}^n to \mathbb{R}^n such that for any compact $K \subset \mathbb{R}^n$,*

- *the restriction of φ_\hbar to K is a classical symbol: $\varphi_\hbar = \varphi_0 + \hbar\varphi_1 + \dots$ whose principal term φ_0 is a local diffeomorphism;*
- *the components of $\varphi_0 \circ F$ are classical action variables;*
- *$\varphi_\hbar(K \cap \Sigma(P)) = \hbar\mathbb{Z}^n \cap \varphi_\hbar(K) + O(\hbar^\infty)$.*

The joint spectrum is thus transformed into a straight lattice associated to the moment polytope, possibly shifted from the latter because of the appearance of subprincipal terms.

Once a basis vector of the lattice in which sits the joint spectrum is chosen, one can define a grouping of eigenvalues by associating those that are on the same affine line directed by this vector. These "packets" of eigenvalues are called **polyads**, in reference to the case of the harmonic oscillator in the physico-chemistry literature. From the classical point of view, such a basis vector defines a sub-action of S^1 ; the corresponding polyads are then Weinstein's *clusters* [63]. Notice that in our completely integrable situation, Weinstein's assumption on the subprincipal symbols is no more necessary.

This theorem yields an amusing proof of the following corollary (which can be proved by more usual means): when $M = T^*X$ (which is obviously the case of the theorem, unless we state it in a Toeplitz context), there can be only one critical point of maximal corank. This indicates that in general the true interest of the theorem is actually not the global description of the spectrum, which would hold for a very limited set of examples, but instead the description, for a more general joint spectrum, of all parts (convex, polyhedral) which correspond to a sub-system of toric type.

5.5 Almost toric systems

A system of toric type is essentially a completely integrable system all of whose singularities are non-degenerate and of (transversally) elliptic type (although as it is, the assertion is not true: see proposition 5.5 below).

A mild way of generalising toric systems is to allow isolated singularities in the momentum image.

Definition 5.4 *A completely integrable system with proper momentum map is **almost toric** when its singularities are all non-degenerate, without real hyperbolic block.*

In other words an almost toric system admits only elliptic or focus-focus blocks. From now on we restrict to two degrees of freedom systems (symplectic 4-manifolds). The classification up to diffeomorphism of compact symplectic manifolds of dimension 4 admitting an almost toric system has just been carried out by Leung and Symington [46, 33].

Amongst elementary properties of almost toric systems, one can state:

- Proposition 5.5**
- *If F is a proper momentum map with non-degenerate singularities and the set of regular values of F is connected, then F is almost toric;*
 - *if F is almost toric then F is of toric type if and only if the set of regular values of F is connected and simply connected.*

Generalised polytopes — At the time of writing this article, no general result about semiclassics of almost toric systems is known. To start with, let us consider a simple sub-class of two degrees of freedom almost toric systems, namely those whose deficiency index is 1:

Definition 5.6 *An almost toric integrable system $F = (f_1, f_2)$ on a symplectic manifold of dimension 4 has **deficiency index** equal to 1 if there exists a local diffeomorphism $\varphi = (\varphi_1, \varphi_2)$ on the image of F such that $\varphi_1 \circ F$ is a proper momentum map for an effective Hamiltonian action of S^1 .*

Hamiltonian actions of S^1 on compact 4-manifolds have been classified, both topologically [2] and symplectically [30]. One can show that systems with deficiency index 1 are often compact and hence subject to this classification. However the symplectic manifold itself is not what really matters here, considering that it is often obtained by a “symplectic cutting” adapted to the part of the system under study. From the semiclassics viewpoint, the essential object is the image of the momentum map, endowed with its structure of integral affine manifold (with singularities).

Now let $F = (f_1, f_2)$ be an almost toric system with deficiency index 1; denote by $B \subset \mathbb{R}^2$ the image of F , B_r the set of regular values, and m_f the number of critical values c_1, \dots, c_{m_f} of focus-focus type. Let $\vec{\varepsilon} \in \{-1, +1\}^{m_f}$. Denote by ℓ_i the vertical half-line from c_i in the direction given by ε_i . Let k_i be the *monodromy index* of c_i (it was shown in [16] that in this situation the monodromy is *abelian* and can be identified to an integer valued index). Let $\mathbb{A}_{\mathbb{Z}}^2$ be the space \mathbb{R}^2 equipped with the standard integral affine structure.

Theorem 5.7 ([60]) *There exists a homeomorphism ψ from B to $\psi(B) \subset \mathbb{A}_{\mathbb{Z}}^2$ of the form $\psi(x, y) = (x, \psi^{(2)}(x, y))$ such that*

1. *in the complement of $\cup_i \ell_i$, ψ is an affine diffeomorphism (ie. the components of $\psi \circ F$ are local action variables)*
2. *ψ extends to a C^∞ multivalued map from B_r to $\mathbb{A}_{\mathbb{Z}}^2$ (with branching at all ℓ_i) and for all $i = 1, \dots, m_f$ and all $c \in \ell_i$,*

$$\lim_{\substack{(x,y) \rightarrow c \\ x < x_i}} d\psi(x,y) = \begin{pmatrix} 1 & 0 \\ \varepsilon_i k_i & 1 \end{pmatrix} \lim_{\substack{(x,y) \rightarrow c \\ x > x_i}} d\psi(x,y),$$

3. *The image of ψ is a rational convex polygon.*

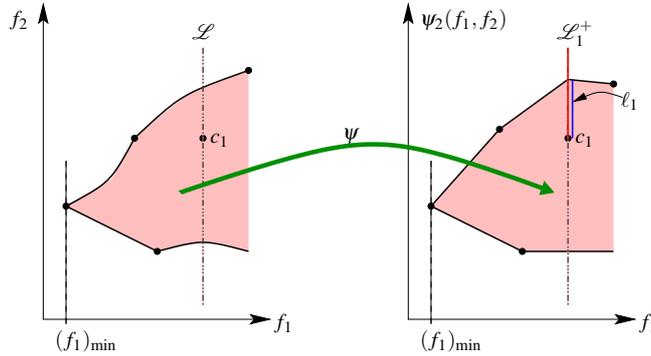


Figure 6: Construction of the polygon

Although the system F is not toric, the theorem still provides us with a way of associating to it a rational convex polygon, which turns out to be very useful to study the system. One can for instance write localisation formulas that express Duistermaat-Heckman measures associated with the action of the system in terms of the monodromy index [60].

The difference with the toric case, which of course is the main thrust for these systems, is that the “generalised” moment polygon is not unique; on the contrary, it is parameterised by a multi-sign $\vec{\varepsilon}$, which endows the class of possible polygons with an abelian group structure and expresses the non-uniqueness of action variables.

5.6 Bifurcations and redistribution of eigenvalues

Consider the following general question. Let an integrable quantum system $P(t)$ depend on a parameter $t \in [0, 1]$ such that $P(0)$ and $P(1)$ are of toric type. We know how to describe the joint spectrum of $P(0)$ and $P(1)$ thanks to theorem 5.3. What is the relation between both spectra ?

To be more precise let us assume that there is way to define a particular rational direction in the image of the joint principal symbol $p(t)$, independently of t (this is the case for instance if there exists a sub-action of S^1 independent of t). One can then define the corresponding *polyads* for $P(0)$ and $P(1)$. How do the eigenvalues rearrange from a set of polyad to another ? This is the so-called *redistribution problem*. According to theorem 5.3 it is enough to study the transformation of the moment polygon to obtain the asymptotic behaviour of the number of eigenvalues in each polyad.

By looking at the example of the coupling of two spins (it is a Hamiltonian system on $S^2 \times S^2$ which satisfies our hypothesis) Sadovskii and Zhilinskiĭ conjectured that this redistribution was related to the appearance of monodromy for certain intermediate values of t [44].

Now assume that the system $p(t)$ is almost toric with deficiency index 1, except for a finite number of t 's which we shall call bifurcation times. Under the assumption that the only bifurcations that the system undergoes are Hamiltonian Hopf bifurcations⁽⁴⁾ (which correspond to a transformation elliptic-elliptic \leftrightarrow focus-focus and are generic⁽⁵⁾[52]), the conjecture is confirmed in the following way: the polygons associated to $P(0)$ and $P(1)$ are generalised polygons for a common system and the $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_{m_f})$ corresponding to their difference is determined by the sequence of Hopf bifurcations. In other words one passes from a polygon to the other by a piecewise affine transformation characterised by the position of bifurcating critical values and their monodromy index.

This result was the main motivation for introducing these generalised polytopes, since they perfectly describe in a geometrical and combinatorial way the eigenvalue redistribution amongst polyads. It is natural however to imagine other applications of these polytopes. Combined with the semi-global classification of theorem 4.7 they may turn out to be the right tool for a global classification theorem à la Delzant.

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⁽⁴⁾All these conditions are satisfied for the example of two spins.

⁽⁵⁾They are generic for instance in the class of Hamiltonians that commute with a fixed S^1 action

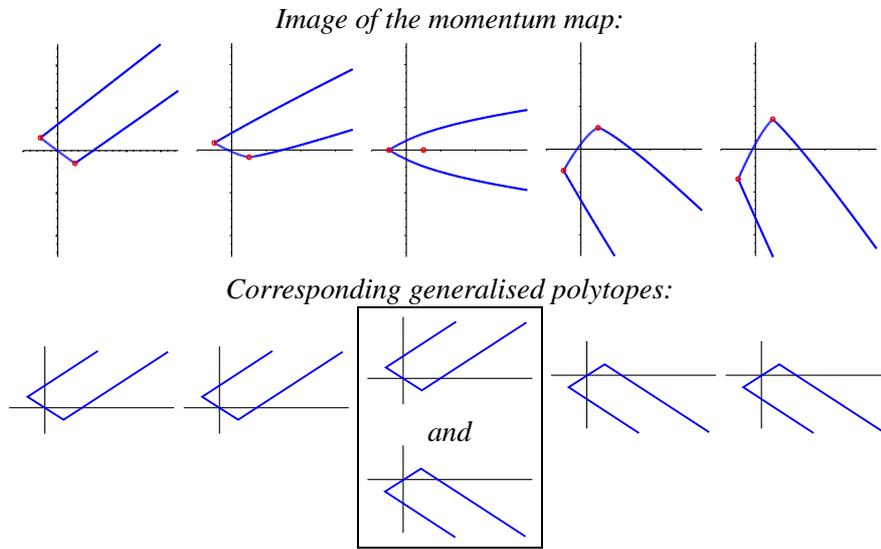


Figure 7: Bifurcation of the image of the momentum map for the coupling between a spin and a harmonic oscillator ($M = S^2 \times \mathbb{R}^2$)

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