Uniform spectral asymptotics for semiclassical wells on phase space loops

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Abstract

We consider semiclassical self-adjoint operators whose symbol, defined on a two-dimensional symplectic manifold, reaches a non-degenerate minimum b_0 on a closed curve. We derive a classical and quantum normal form which gives uniform eigenvalue asymptotics in a window $(-\infty, b_0 + \epsilon)$ for $\epsilon > 0$ independent on the semiclassical parameter. These asymptotics are obtained in two complementary settings: either an approximate invariance of the system under translation along the curve, which produces oscillating eigenvalues, or a Morse hypothesis reminiscent of Helffer-Sjöstrand's "miniwell" situation.

This paper is dedicated to the memory of Hans Duistermaat.

1 Introduction

The general framework of this article is the study of the discrete spectrum of semiclassical operators $(P_\hbar)_{\hbar>0}$ acting on the Hilbert space of a particle with one degree of freedom. Typical examples include electro-magnetic Schrödinger operators:

$$P_\hbar = \left(\frac{\hbar}{i}\frac{\mathrm{d}}{\mathrm{d}x} - \alpha(x)\right)^2 + V(x), \tag{1}$$

acting on $L^2(X)$ where X is a one-dimensional manifold, $X = \mathbb{R}$ or $X = S^1$, and the semiclassical parameter $\hbar > 0$ is very small. Here, the magnetic potential α and the electric potential V are smooth functions on X, and may

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be allowed to depend on $\hbar$. However, we don't want to restrict the discussion to operators of the form (1), and will more generally consider $(P_\hbar)_\hbar$ to be *pseudo-differential operators* of the form $P_\hbar = \mathrm{Op}_\hbar(p_\hbar)$ of the form (4) below, whose *symbol* $p_\hbar : T^*X \to \mathbb{R}$ may depend on $\hbar$. Here, the cotangent space T^*X is simply $X \times \mathbb{R}$ endowed with the symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}\xi$.

The relevance of general 1D pseudo-differential operators stems from the fact that they are often *effective* operators coming from higher dimensional settings. For instance, in the regime of strong magnetic fields, the spectrum of the purely magnetic Schrödinger operator in 2 or 3 dimensions can be reduced to the spectral study of a one degree of freedom pseudo-differential operator $P_\hbar = \mathrm{Op}_\hbar(p)$ (see for instance [40, 25, 38]), which is very rarely a differential operator, let alone a Schrödinger operator: in the simplest 2D case treated in [40] it shown that the *principal symbol* $p_0 = \lim_{\hbar\to 0} p_\hbar$ of the effective 1D operator is the magnetic field itself, and there is no reason why it should be polynomial in the second variable, which is a necessary and sufficient condition for $P_\hbar$ to be a differential operator.

It is natural to investigate cases where the exterior forces acting on the particle are able to *confine* it in a bounded region, leading to discrete spectrum for $P_\hbar$. This happens if p_0 has a global minimum on a compact set. Several interesting regimes have been abundantly studied; a key feature is that the various possible topologies of a level set $\{p_0 = \lambda\}$ give rise to very different eigenvalue asymptotics for $P_\hbar$ near λ.

Two cases are particularly well understood in 1D. The first one corresponds to a global non-degenerate minimum for p_0, reached at a single point in phase space. The second is a regular compact level set of p_0. In both cases, one can obtain [26, 43, 11, 42] "uniform asymptotics": one does not only have complete asymptotics for a finite number of eigenvalues (in a window of size $\hbar$ around the minimal value or the regular level of p_0, respectively), but for all eigenvalues in a window of $\hbar$-independent size, an expansion of the form

$$\lambda_j(\hbar) = f_0(\hbar j) + \hbar f_1(\hbar j) + \hbar^2 f_2(\hbar j) + \cdots \tag{2}$$

where j belongs to an interval of $\mathbb{Z}$ of size $\hbar^{-1}$. In the case of a single minimal point for p_0, this allows in particular to obtain asymptotics for the low-energy spectrum. These results notably use a quantum version of the "action-angle" coordinates [17]. They were recently extended to Berezin-Toeplitz 1D operators [9, 34].

In this article, we are interested in the case where the minimum of the principal symbol is reached on a connected compact submanifold γ (that is, a smooth topological circle) of the phase space, see hypothesis (3) below.

Under this hypothesis, we show how to obtain uniform asymptotics near the minimum of p_0. Operators with such a feature have been studied in the framework of hypoellipticity (see the seminal articles [30, 7]) but also spectrally [27]. To our knowledge however, the precise information (both geometric and analytic) gained from the complete integrability of the 1D situation was not investigated before.

Generically, due to the presence of a *subprincipal* term in our pseudo-differential operator, we expect a second-order quantum confinement *within* the minimal manifold, leading to a situation similar to (2), but on a smaller scale. This is the so-called "mini-well" phenomenon described for Schrödinger operators in [27], and recently extended to Berezin-Toeplitz operators in [14]. One of our main results is to obtain a precise and uniform description of this case, see Propositions 6.8 and 6.10.

The degenerate case (where the subprincipal term vanishes) turns out to be interesting as well, especially given the relationship with the strong magnetic field situation described above. Indeed, if a charged particle tends to be confined on a closed loop (for instance, the boundary of a 2D domain), the absence of subprincipal symbol will lead to an *oscillatory behaviour* of the low-lying eigenvalues, thus very different from what (2) describes; this is related to the "lack of strong diamagnetism" and the Little-Parks effect, see [24, 32]. Another goal of the present article is to present a general framework explaining this behaviour and the link with eigenvalue crossings (or pseudo-crossings: the gap between the first and the second eigenvalue periodically collapses at dominant order), see Theorem 2.2 and Figure 1.

Our results, both in the generic and degenerate cases, are consequences of a new "quantum folded action-angle theorem" (Theorem 2.1), and its classical version (Proposition 3.8).

The article is organised as follows: Section 2 introduces the notation, related to the geometric setting and its quantization, necessary to state the microlocal folded action-angle Theorem 2.1, whose proof is delayed in Sections 3, 4, and 5. Section 3 contains a classical normal form for functions admitting a non-degenerate well on a closed loop and a reminder on the "Bohr-Sommerfeld invariant" I_0. In preparation for the quantum normal form, Section 4 contains a treatment of formal perturbations of the normal form above. Then, in Section 5 we derive a corresponding quantum normal form, microlocally near the non-degenerate well. In Section 6 we apply this quantum normal form to obtain asymptotics of the low-lying eigenvalues. In an Appendix, we recall a few topological notions that we use in Section 3.

2 Wells on closed loops

2.1 The classical problem

Let (M, ω) be a symplectic surface without boundary, which will be our classical phase space. When introducing quantization, we will assume for simplicity that $M = T^*\mathbb{R} \approx \mathbb{R} \times \mathbb{R}$ or $M = T^*S^1 \approx S^1 \times \mathbb{R}$. Let $\gamma \subset M$ be a smooth embedded closed loop. We say that a smooth function $p \in C^\infty(M)$ — which will later be the principal symbol of a quantum operator — admits a non-degenerate well on the loop γ if there exists a neighbourhood Ω of γ in M such that

Assumption 1.

1. *$p_{\upharpoonright\Omega}$ is minimal on γ:*

$$p^{-1}(b_0) \cap \Omega = \gamma, \qquad \text{where } \inf_{\Omega} p = \min_{\Omega} p = b_0; \tag{3}$$

2. *and this minimum is Morse-Bott non-degenerate: at each point $m \in \gamma$, the restriction of the Hessian $p''(m)$ to a transversal direction to γ does not vanish.*

In particular, by the Morse-Bott lemma (see for instance [3, Theorem 2]), there exists a neighbourhood $\tilde{\Omega} \subset \Omega$ of γ, and coordinates $(z, t) : \tilde{\Omega} \to \gamma \times [-\delta, \delta]$ such that $\gamma = \{t = 0\}$ and $p = b_0 + t^2$.

An example of an operator satisfying Assumption 1 is an electro-magnetic Schrödinger of the form (1) with $X = S^1$ and $V = 0$. In fact, for operators of the form (1), since the principal symbol $p = (\xi - \alpha_0(x))^2 + V_0$ is convex in ξ, the hypothesis (3) imposes $X = S^1, V_0 = 0$. In this case, there is a well-known simplification of the problem: after the shear $(x, \xi) \mapsto (x, \xi - \alpha_0(x) + \overline{\alpha}_0)$, where $\overline{\alpha}_0 = \int_{S^1} \alpha_0$ is the magnetic flux, the function p depends only on ξ.

Our first result (Proposition 3.8) generalises the previous fact, by finding a symplectic change of coordinates near γ such that p depends only on one variable, locally near γ. The reduction to one variable will turn out to be important for having a manageable quantum normal form.

2.2 The quantum problem

Let $P = (P_\hbar)_{\hbar>0}$ be a semiclassical pseudo-differential operator on $X = \mathbb{R}$ or $X = S^1$, with a symbol $p_\hbar \in S^0(T^*X)$, where S^0 denotes any class of symbols for which Egorov theorem holds (see for instance [15, 45]). An electro-magnetic Schrödinger operator (1) is a good candidate as soon as the electro-magnetic fields α and V are smooth functions on X (with at most polynomial growth at infinity, together with their derivatives, in the case $X = \mathbb{R}$). We shall always assume that $p_\hbar$ is *classical*, in the sense that it admits an asymptotic expansion in integral powers of $\hbar$, in the topology of S^0. Without loss of generality, we may assume that P has order zero:

$$p_\hbar \sim p_0 + \hbar p_1 + \hbar^2 p_2 + \dots$$

It will be convenient to use Weyl quantization, which is valid for both $X = \mathbb{R}$ and $X = S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and which will be denoted by $P_\hbar = \mathrm{Op}_\hbar^W(p_\hbar)$:

$$P_\hbar u(x) = \frac{1}{2\pi\hbar}\int_{X\times\mathbb{R}} e^{\frac{i}{\hbar}(x-y)\xi} p_\hbar(\tfrac{x+y}{2}, \xi) u(y) \mathrm{d}y \mathrm{d}\xi. \tag{4}$$

Assume now that the principal symbol $p = p_0$ satisfies Assumption 1. Our next goal is to find a quantum equivalent for the simple description of the classical problem found in Proposition 3.8. This is far from being automatic: as in [12], the geometric hypothesis (3) is not stable under perturbations, so that the normal form of Proposition 3.8 cannot itself be stable by perturbation. Nevertheless, we are able to separate position and momentum variables in the quantum problem. Another subtlety is of topological nature: an invariant I_0, not present in Proposition 3.8, appears in its quantum equivalent and lies behind the oscillatory effects observed in [24, 32].

Theorem 2.1 (Quantum Folded Action-Angle Theorem)**.** *Let $P = (P_\hbar)_{\hbar>0}$ be a semiclassical pseudo-differential operator, as above, with principal symbol $p = p_0$ admitting a non-degenerate well on a loop γ — see Assumption 1. Let $\alpha = \xi \mathrm{d}x$ be the Liouville 1-form on T^*X, and let $I_0 = \int_\gamma \alpha$. There exists $\epsilon > 0$, a neighbourhood Ω of γ, and a Fourier integral operator $U_\hbar : L^2(X) \to L^2(S^1)$, uniformly bounded in operator norm, such that*

1. *$U_\hbar$ is microlocally unitary from Ω to $\{(\theta, I) \in T^*S^1, |I - I_0| < \epsilon\}$.*

2. *$U_\hbar$ microlocally conjugates $P_\hbar$ to a pseudo-differential operator $Q_\hbar$ of the form*

$$Q_\hbar := b_0 + \Big(g_\hbar(\frac{\hbar}{i}\frac{\partial}{\partial\theta})\Big)^2 + \hbar V_\hbar(\theta) + R_\hbar,$$

in the sense that

$$Q_\hbar = U_\hbar P_\hbar U_\hbar^* + R_\hbar,$$

where, $R_\hbar$ is such that, for every $u_\hbar \in L^2(S^1)$ with $WF_\hbar(u_\hbar) \subset \{(\theta, I) \in T^*S^1, |I - I_0| < \epsilon\}$, one has $\|R_\hbar u_\hbar\| = \mathcal{O}(\hbar^\infty)\|u_\hbar\|$.

In the expression of $Q_\hbar$, $V_\hbar$ is an $\hbar$-dependent potential on S^1 with an asymptotic expansion

$$V_\hbar(\theta) = V_0(\theta) + \hbar V_1(\theta) + \cdots ,$$

$g_\hbar \in C_0^\infty(\mathbb{R})$ is supported on an $\hbar$-independent set, with

$$g_\hbar(I) = g_0(I) + \hbar g_1(I) + \cdots ,$$

and g_0 is a local diffeomorphism from a neighbourhood of $I = I_0$ to a neighbourhood of $0 \in \mathbb{R}$.

As usual in semiclassical analysis, the asymptotic expansions for $V_\hbar$ and $g_\hbar$ hold in the C^∞ topology. The number I_0 is sometimes called the (first) Bohr-Sommerfeld invariant of γ (see Subsection 3.1). The notation $WF_\hbar(u_\hbar)$ stands for the semiclassical wavefront set of $u_\hbar$ (initially called the Frequency Set in [23], see also [35, Definition 2.9.1]).

Fourier Integral Operators were first introduced in a microlocal (homogeneous) context [29, 18] and a semiclassical theory of Canonical Operators was developed independently [36]. Duistermaat was the first to build the bridge between both theories [16], paving the way to modern semiclassical analysis. The construction of such quantum maps $U_\hbar$ in presence of non-trivial topology was discussed already at the time when Fourier Integral Operators were invented, see [44]. In the semiclassical setting, related constructions appear for instance in [42, 41].

In particular, Theorem 2.1 can be used to study the low-energy spectrum of 2D magnetic Laplacians, in the case where the magnetic field is positive everywhere and reaches a non-degenerate minimum along a curve, by Theorem 1.6 in [40].

An interesting consequence of Theorem 2.1 is that, if the subprincipal contribution V_0 is Morse, one can formulate Bohr-Sommerfeld conditions (in a folded covering) for the eigenvalues in a macroscopic window

$$[\min \operatorname{Spec}(P_\hbar), \min \operatorname{Spec}(P_\hbar) + c]$$

for c small (see Propositions 6.8 and 6.10), since we in fact reduced the problem to the case where p_0 is Morse. This leads to uniform asymptotics of the form (2), but with an expansion in integer powers of $\hbar^{\frac{1}{2}}$, which is known to be the effective semiclassical parameter in the study of operators whose symbol reach a Morse-Bott minimum on an isotropic manifold [27, 14]. We do not explicitly perform this analysis here: after Propositions 6.8 and 6.10, it simply remains to apply the results of [16, 11, 13].

Theorem 2.1 is even more useful in the opposite case where V_0 (and, possibly, higher-order terms in V) are constant. In this case, the asymptotics of low-lying eigenvalues for $P_\hbar$ acquire a particularly nice oscillating form, in which the invariant I_0 turns out to play a prominent role, as stated in the following theorem.

Theorem 2.2. *Let $k \geq 0$. Suppose that, in Theorem 2.1, the $k+1$ first terms $V_0, V_1, \ldots, V_k$ of the potential do not depend on θ. Suppose also that $P_\hbar - b_0$ is elliptic at infinity. Then the following is true.*

1. *There exists a smooth, non constant function $f : S^1 \to \mathbb{R}$ such that the first eigenvalue $e_0^\hbar$ of $P_\hbar$ satisfies:*

$$e_0^\hbar = b_0 + \hbar V_0(0) + \hbar^2 f\left(\frac{I_0}{\hbar} \bmod \mathbb{Z}\right) + \mathcal{O}(\hbar^{\max(k+2,3)}).$$

2. *Let $e_1^\hbar$ similarly denote the second eigenvalue of $P_\hbar$ (with multiplicity). There exists a sequence $(\hbar_j)_{j\in\mathbb{N}} \to 0$ such that*

$$e_1^{\hbar_j} - e_0^{\hbar_j} = \mathcal{O}(\hbar_j^{k+2}).$$

This oscillatory behaviour of the first eigenvalue was remarked in recent work on the magnetic Laplacian [24], but to our knowledge our result on the spectral gap is entirely new, even in the particular case of magnetic Laplacians. These phenomena are related to the topological nature of the problem and sometimes coined under the term "Aharonov-Bohm effect": low-energy eigenfunctions are microsupported on a non-contractible set (here, γ).

Generally, one cannot say anything about the actual gap between $e_1^{\hbar_j}$ and $e_0^{\hbar_j}$. Figure 1 shows the low-energy spectrum of the solvable, rotation-invariant example

$$H_{\text{sym}} = \mathrm{Op}_\hbar^W((x^2 + \xi^2 - 1)^2 - \hbar^2) : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \tag{5}$$

with actual eigenvalue crossings; these crossings are not stable under perturbations and a generic $\mathcal{O}(\hbar^{k+2})$ perturbation of p opens the gap by $\hbar^{k+2}$. This is the case in Figure 2, where we illustrate the case $k = 0$ with a numerical computation of the first two eigenvalues for the Hamiltonian

$$\frac{1}{2}\left[((x-2)^2 + 1)H_{\text{sym}} + H_{\text{sym}}((x-2)^2 + 1)\right].$$

In Theorem 2.2, if $k \geq 1$ then f is explicit and, piece-wise, a polynomial of degree 2 (it coincides, up to rescaling, with Figure 1). If $k = 0$ however, f

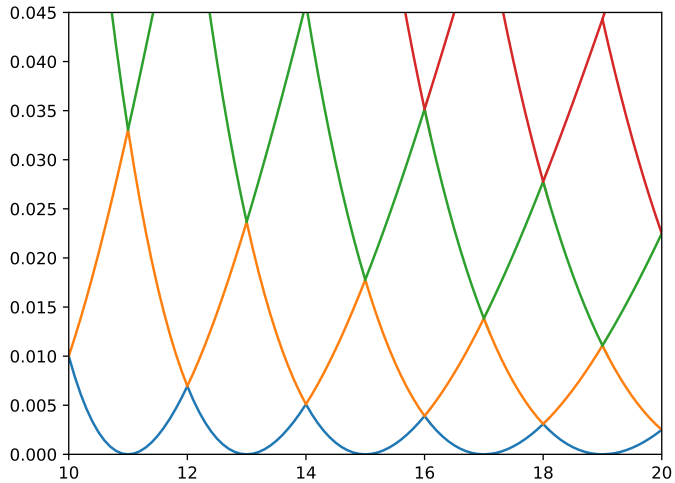

Figure 1: Small eigenvalues for the operator H_{sym} in (5), as a function of $1/\hbar$. With the notations of Theorem 2.1, one has $V_\hbar = 0$. The first eigenvalue jumps branches when $1/\hbar$ is a multiple of $\frac{1}{I_0} = 2$. The operator H_{sym} is a function of the harmonic oscillator, and its eigenvalues are explicitly given by $\{(\hbar(2k+1)-1)^2, k \in \mathbb{N}\}$.

is implicit and the fact that it is not constant is given by the end of the proof of Proposition 6.5.

How likely is it that the first few terms $V_0, V_1, \ldots$ are independent of θ? The dominant term $V_0(\theta)$ coincides with the Melin value: it is the dominant term, of order $\hbar$, of the minimal possible energy for a quantum state localised at the point of γ corresponding to θ [14]. In particular, if $P_\hbar = \mathrm{Op}_\hbar^W(p)$ where p satisfies Assumption 1 and is independent of $\hbar$, then $V_0 = 0$ everywhere.

In the analogy with strong constant magnetic fields on smooth 2D domains, V_0 corresponds to the curvature at boundary points in the domain. If V_0 is constant, then the domain is a Euclidian disk, and the model operator is perfectly invariant by rotation, not just at order $\hbar$.

In the strong magnetic field regime, oscillations (and crossings) of the first few eigenvalues also happen at a much finer scale on very symmetric domains. For instance, in the case of a strong constant magnetic field on a 2D domain, recent results on smooth domains [6, 31], and numerical simulations for the square [5], indicate that the first few eigenvalues oscillate at a scale

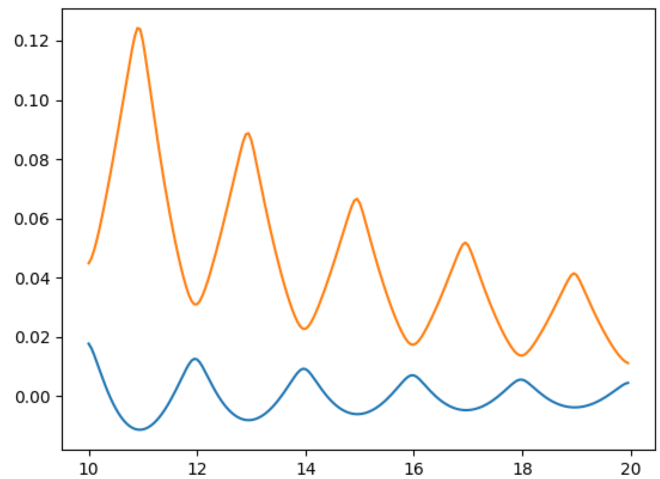

Figure 2: First two eigenvalues for the perturbed operator $\frac{1}{2}[((x-2)^2+1)H_{\mathrm{sym}}+H_{\mathrm{sym}}((x-2)^2+1)]$, as a function of $1/\hbar$. With the notations of Theorem 2.1, one has $V_0=0$ and V_1 not constant. The gaps are lifted by the symmetry breaking, at the same order as the amplitude of oscillations.

$e^{-S/\hbar^\alpha}$ for some $\alpha>0$. Usual tools in the analysis of pseudo-differential operators are limited to $\mathcal{O}(\hbar^\infty)$; the study of this phenomenon might require the use of analytic microlocal methods, which allow one to reach $\mathcal{O}(e^{-c/\hbar})$ precision. In the specific case of Schrödinger operators, to the explicit shear $(x,\xi)\mapsto(x,\xi-\alpha(x)+\overline{\alpha})$ corresponds an explicit quantum map, and one can hope to treat the tunnelling effect above without analytic microlocal tools.

In the generic case where V_0 is a Morse function (*i.e.* its critical points are non-degenerate), this oscillatory behaviour does not appear at the bottom of the spectrum: since V_0 varies along the circle, eigenfunctions with energies smaller than $b_0+\hbar\max(V_0)$ will microlocalise on a contractible set, and one can, in principle, build a quantum normal form independent on I_0.

Schrödinger operators of the form (1) may either belong to the scope of Propositions 6.8 and 6.10, or of Theorem 2.2, depending on the way V (the one in (1)) depends on $\hbar$. Recalling that $V=\mathcal{O}(\hbar)$, the term in V of order $\hbar$ corresponds to V_0 in Theorem 2.2. Magnetic Schrödinger operators in 2D have a low-energy spectrum given by a 1D pseudodifferential operator whose principal symbol is the magnetic field ([40], Theorem 1.6), times a supplementary factor $\hbar$; to decide whether we fall in the scope of Propositions 6.8 and 6.10, or of Theorem 2.2, one must study how the Fourier Integral Operator in [40], Theorem 1.6, acts at order $\hbar^2$.

The technical hypothesis that $P_\hbar - b_0$ be elliptic at infinity is simply here to ensure discrete spectrum in a neighbourhood of the ground state. Theorem 2.1 did not require this because that was a purely microlocal result. It would be interesting to apply it in the absence of discrete spectrum, for instance to the description of quantum resonances.

Another perspective is the quantum study of other (higher-dimensional) Hamiltonian invariants. The Bohr-Sommerfeld invariant generalises into an invariant of compact Lagrangean submanifolds. Can you hear this invariant by oscillations of the ground state of a quantum system? Does the quantum system need to be completely integrable in order to hear it?

3 Reduction of Morse-Bott functions

In this section we discuss the *classical* problem: given the Hamiltonian p on the symplectic manifold $M = T^*X$, we reduce the equations of motion given by p. We first review the *first Bohr-Sommerfeld invariant*, a real number associated with curves on M, invariant under Hamiltonian dynamics (but not necessarily under general symplectic changes of variables). Then, we create, in a neighbourhood $\widetilde{\Omega}$ of $\gamma = \{p = b_0\}$, action-angle coordinates, that is, a local change of coordinates simplifying p. Indeed, we construct (Proposition 3.8) a symplectic change of variables $(\theta, I) : \widetilde{\Omega} \to S^1_\theta \times (-\epsilon, \epsilon)_I$ from a neighbourhood $\widetilde{\Omega}$ of γ to a neighbourhood of the zero section in T^*S^1, such that

$$p = b_0 + g(I), \tag{6}$$

for some $g : \mathbb{R} \to \mathbb{R}^+$.

To conclude this section, we show (Propositions 3.10 and 3.12) that this symplectic change of variables (θ, I) can be extended from $\widetilde{\Omega}$ to the whole of T^*X after shifting I by I_0; the identity (6) will only hold in the vicinity of $\{I = I_0\}$, but the fact that this change of variables is global will allow us to associate with it a well-behaved quantum transformation.

Remark 3.1. Recall that the Morse-Bott Lemma mentioned in Section 2 gives $p = b_0 + f^2$, for some smooth function f defined in a neighbourhood of γ. However, in general it is *not* possible to find *symplectic* coordinates (θ, I) such that $f = I$; on the contrary, the function g in (6) is a symplectic invariant of the Hamiltonian p, and will be crucial in obtaining the behaviour of eigenvalues when we turn to the quantum problem, see Section 5.

3.1 The first Bohr-Sommerfeld invariant

Definition 3.2. Suppose $M = T^*X$ where either $X = \mathbb{R}$ or $X = S^1$. Let α be the standard Liouville 1-form on M, so that the canonical symplectic form is $\omega = \mathrm{d}\alpha$. Let γ be a simple curve in M. The first Bohr-Sommerfeld invariant is the action integral

$$I_0(\gamma) = \frac{1}{2\pi}\int_\gamma \alpha.$$

Remark 3.3. Using Stokes' theorem, one can define $I_0(\gamma)$ without referring to the Liouville 1-form, as follows.

1. If γ is contractible, it is the boundary of a close, compact surface $\Sigma \subset M$. Then $I_0 = \frac{1}{2\pi}\int_\Sigma \omega$.
2. If γ is not contractible, then $M = T^*S^1$ (with coordinates $(\theta,\xi) \in S^1 \times \mathbb{R}$) and γ is a curve with winding number 1 with respect to θ. For $K \in \mathbb{N}$ large enough, $\gamma \cup \{\xi = -K\}$ is the boundary of a close, compact surface $\Sigma \subset M$. Then $I_0 = \frac{1}{2\pi}(-K + \int_\Sigma \omega)$.

Indeed, Stokes' theorem implies that all definitions agree up to a constant term, and we can check that all definitions give $I_0 = 0$ on the zero section $\xi = 0$.

The following proposition is well known.

Proposition 3.4 ([19])**.** *$I_0(\gamma)$ is a Hamiltonian invariant of γ.*

Proof. If Y is a Hamiltonian vector field, then by Cartan's formula, $\mathcal{L}_Y\alpha$ is an exact 1-form and hence acts on the cohomology class of α restricted to γ (known as the Liouville class of γ). Therefore, a Hamiltonian flow preserves the Liouville class. Since $\gamma \simeq S^1$ this means that it preserves the integral $\int_\gamma \alpha$. □

Remark 3.5. In case 1 of Remark 3.3, I_0 is clearly a symplectic invariant, and the proposition above is obvious. However, in case 2 above, I_0 is *not* a *symplectic* invariant of γ; indeed any curve of the type $\{\xi = C\}$, for $C \in \mathbb{R}$ can be sent to $\{\xi = 0\}$ by the symplectic change of variables $(\theta, I) \mapsto (\theta, I - C)$. However, for this curve, $I_0 = C$.

Remark 3.6. The Liouville class I_0 is called the *first* Bohr-Sommerfeld invariant, because it is the principal term in the Bohr-Sommerfeld cocycle defined in [41] (the subprincipal terms involve Maslov indices and the 1-form induced by the subprincipal symbol of $P_\hbar$). In the case of Berezin-Toeplitz quantization, I_0 can be defined using parallel transport along γ on the prequantum bundle [10]. In this case, I_0 is defined up to a sign and modulo $\mathbb{Z}$, but the choice does not impact the oscillations in Theorem 2.2 since, for Toeplitz quantization, $\hbar^{-1}$ takes integer values.

Remark 3.7. The Liouville class can be defined on Lagrangian tori in higher dimensional completely integrable systems, giving rise to a vector of Bohr-Sommerfeld invariants. These invariants are important in the study of the spectrum of Laplace-Beltrami operators in the integrable of KAM regime, see [39], or for the joint spectrum of commuting operators, see for instance [1, 41].

3.2 Local symplectic normal form

Let us use Definition 3.2 to find symplectic coordinates simplifying p near the simple curve γ: after this change of variables, p depends only on one "action" variable.

Proposition 3.8. *If a smooth function p admits a non-degenerate well along a closed curve γ (see Assumption 1), then there exists smooth "folded action-angle" coordinates (θ, I) near γ that are adapted to p, in the sense that $\gamma = \{I = 0\}$ and*

$$p = b_0 + (g(I))^2,$$

for some smooth function $g : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ with non-vanishing derivative. The neighbourhood of γ where this holds can be chosen saturated with respect to the level sets of p, i.e. *of the form $\{(\theta, I); |I| < \epsilon\}$ for some $\epsilon > 0$.*

Proof. Recall that by the Morse-Bott Lemma there exists, on a neighbourhood Ω of γ, a change of variables $(z, f) : \Omega \to S^1 \times \mathbb{R}$ such that

$$p = b_0 + f^2.$$

Without loss of generality, Ω is an open sublevel set of p (that is, the image of f is an interval $[-f_0, f_0]$).

In particular, $\mathrm{d}f$ is everywhere non-zero in Ω; hence we can now view $f : \Omega \to \mathbb{R}$ as a non-critical Hamiltonian, and apply the action-angle theorem (see [17] and [28], Appendix A2): on a small enough sublevel set $\hat{\Omega}$ of p, there exists a smooth *symplectic* change of variables $(\theta, I) : \hat{\Omega} \to S^1 \times \mathbb{R}$ and a smooth diffeomorphism $g : \mathbb{R} \to \mathbb{R}$ such that $f = g(I)$. ☐

Remark 3.9. It follows that the set of leaves defined by p, *i.e.* the space of connected components of levels sets of p, is a smooth one-dimensional manifold $\mathcal{C}$ (parameterised by I or $\tilde{I} := g(I)$) , and the induced map $\bar{p} - b_0 : \mathcal{C} \to \mathbb{R}$ is a simple fold: $\tilde{I} \mapsto \tilde{I}^2$.

In the rest of this section, we use Proposition 3.8 to build normal forms on the whole phase space.

Proposition 3.10. *Let $p : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Suppose that p admits a non-degenerate well a curve γ, see Assumption 3. Let $I_0 = I_0(\gamma)$ be the first Bohr-Sommerfeld invariant, see Definition 3.2.*

There exists $\epsilon > 0$ and a smooth Hamiltonian change of variables $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$, equal to the identity outside of a compact set, such that, for all $(x, \xi) \in \mathbb{R}^2$,

$$|x|^2 + |\xi|^2 \in (2I_0 - \epsilon, 2I_0 + \epsilon) \Rightarrow p \circ \sigma(x, \xi) = b_0 + (g(x^2 + \xi^2 - 2I_0))^2.$$

Proof. Since here $M = \mathbb{R}^2$, we know that $2\pi I_0$ is the area inside the loop γ, and hence $I_0 > 0$. Let $r_0 = \sqrt{2I_0}$. We apply Proposition 3.8, and compose with symplectic polar coordinates $(\theta, \tilde{I}) \mapsto (x = \sqrt{2\tilde{I}} \cos\theta, \xi = \sqrt{2\tilde{I}} \sin\theta)$, where $\tilde{I} := I + I_0$ varies in a neighbourhood of I_0; this gives a symplectic change of variables σ_0 from a neighbourhood Ω_0 of γ to a neighbourhood of $\{x^2 + \xi^2 = r_0^2\}$, and a local diffeomorphism g of $(\mathbb{R}, 0)$ such that

$$p \circ \sigma_0(x, \xi) = b_0 + (g(x^2 + \xi^2 - r_0^2))^2.$$

In particular, σ_0 maps level sets of p to circles with centre 0.

By the Jordan curve theorem, $\mathbb{R}^2 \setminus \gamma$ consists in two connected components: a bounded "interior" component Ω_i and an unbounded "exterior" component Ω_e. Let $\gamma_i \subset \Omega_i$ be a connected component of a level set of p, close to $\partial\Omega_0$. Let D_i be the closure of the interior component of $\mathbb{R}^2 \setminus \gamma_i$; this is a closed topological disk with smooth boundary. We let $r_i > 0$ be such that

$$\pi r_i^2 = vol(D_i).$$

Note, in particular, that for all $(x, \xi) \in \gamma_i$, one has $\|\sigma_0(x)\|^2 = r_i^2$.

By Proposition A.5, there exists an orientation-preserving smooth diffeomorphism ϕ_i from D_i to the closed ball $\overline{B}_{\mathbb{R}^2}(0, r_i)$. In particular, by Proposition A.7, we can deform ϕ_i into another orientation-preserving smooth diffeomorphism $\widetilde{\phi_i}$ which coincides with σ_0 near the boundary.

We can play the same game on Ω_e with an additional condition of compact support, using Proposition A.6: this produces an orientation-preserving diffeomorphism $\widetilde{\phi_e}$ on the complement of an open topological ball in $\mathbb{R}^2$, equal to the identity outside a larger ball, and which coincides with σ_0 near the boundary.

Gluing $\sigma_0, \widetilde{\phi_i}$ and $\widetilde{\phi_e}$, we obtain a diffeomorphism $\phi : M \to M$ satisfying the following assumptions:

- There exists a neighbourhood Ω_1 of γ on which ϕ is a symplectomorphism and
$$p = [(x, \xi) \mapsto (b_0 + g(x^2 + \xi^2 - 2I_0))^2] \circ \phi.$$
- The domain bounded by γ is sent by ϕ to $B(0, r_0)$.
- ϕ is identity outside a large ball $B(0, R)$.

It only remains to modify ϕ into a volume-preserving transformation. To this end, we will apply the Moser-Weinstein argument (see for instance [8, Theorem 7.3]). On $\mathbb{R}^2$, the canonical symplectic form $\omega = \mathrm{d}\alpha$ is exact; moreover, there is a canonical choice of symplectic potential $\alpha = \xi \mathrm{d}x$ (the Liouville 1-form).

Consider the difference $\alpha - \phi^*\alpha$. It is a 1-form supported in $B(0, R)$, which is closed inside Ω_1. Since Ω_1 retracts to a circle, $\alpha - \phi^*\alpha$ is exact if and only if its integral along such a circle vanishes. But by assumption, $\int_\gamma \alpha = 2\pi I_0 = \pi r_0^2$. On the other hand, by construction $\int_\gamma \phi^*\alpha = \int_{\phi(\gamma)} \alpha = \int_{\Lambda_{r_0}} \alpha = \pi r_0^2$, where we used Stokes' theorem for the last equality. Hence there exists a smooth function $f : \Omega_1 \to \mathbb{R}$ such that $\alpha - \phi^*\alpha = \mathrm{d}f$ in Ω_1. Using a cut-off function, let $\tilde{f}$ be equal to f near γ and to zero outside of Ω_1. We now use the Moser-Weinstein argument with 1-form $\alpha - \mathrm{d}\tilde{f}$, which vanishes near γ and outside of $B(0, R)$. Since the support of α is compact, we may integrate along the Moser path and obtain a diffeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, which is the identity near γ and outside of $B(0, R)$ — because there we integrate the zero vector-field—, such that $\varphi^*(\phi^*\omega) = \omega$. Thus, the symplectomorphism $\phi \circ \varphi$ answers the question. To conclude, every symplectomorphism of $\mathbb{R}^2$ with compact support is Hamiltonian.

□

Remark 3.11. In Proposition 3.10, if a ball $B(0, c)$ lies inside the compact component of $\mathbb{R}^2 \setminus \gamma$, one can impose that σ is equal to identity on $B(0, c - \epsilon)$. Indeed, in this case, one can prescribe that ϕ_i is the identity on $B(0, c - \epsilon/2)$ using Proposition A.6 rather than Proposition A.5, and the corrections in the rest of the proof preserve the fact that ϕ_i is the identity on $B(0, c - \epsilon)$.

Proposition 3.12. *Let $p : T^*S^1 \to \mathbb{R}$ be a smooth function admitting a non-degenerate well along a curve γ. Suppose that γ is non-contractible.*

Then there exists $\epsilon > 0$ and a smooth Hamiltonian diffeomorphism $\sigma : T^*S^1 \to T^*S^1$, equal to the identity outside of a compact set, such that, for all $(x, \xi) \in T^*S^1$,

$$\xi \in (I_0 - \epsilon, I_0 + \epsilon) \Rightarrow p \circ \sigma(x, \xi) = b_0 + (g(\xi - I_0))^2.$$

Proof. Let $R > 0$; consider the following symplectomorphism from $S^1 \times [-2R, 2R]$ to $\{(x, \xi) \in \mathbb{R}^2, R \leq x^2 + \xi^2 \leq 9R\}$:

$$(\theta, I) \mapsto \{(\sqrt{2(I + 5R/2)} \cos(\theta), \sqrt{2(I + 5R/2)} \sin(\theta))\}.$$

Through this symplectomorphism, we are reduced to Proposition 3.10: because of the volume considerations, one can extend the symplectic normal form given by Proposition 3.8 to a Hamiltonian change of variables, equal to identity outside of $\{(x, \xi) \in \mathbb{R}^2, R \leq x^2 + \xi^2 \leq 9R\}$. $\square$

The symplectic change of variables at the beginning of the last proof can be quantised; this will allow us in Section to quantise the normal form 3.8 into a unitary operator, up to $\mathcal{O}(\hbar)$ error, but where I is replaced with $I - I_0$. Improving this $\mathcal{O}(\hbar)$ error is the topic of the next section.

4 Formal perturbations

Before giving a quantum equivalent to Proposition 3.8, we now spend some time on the symplectic reduction of small perturbations of a Hamiltonian p with a non-degenerate well along a curve. The Morse-Bott condition is not stable by perturbations: generic smooth perturbations of p have a single, non-degenerate, minimal point. In particular, the action-angle coordinates of Proposition 3.8 are not stable under perturbations. In this section, we study a perturbation of the action-angle coordinates which simplifies as much as possible a perturbation of p while staying close to the original ones. To our knowledge, this procedure was never performed for p satisfying Assumption 1; following the spirit of Poincaré-Birkhoff normal forms, we will introduce the decomposition of $C^\infty(\widetilde{\Omega}, \mathbb{R})$ into the kernel and image of the map $a \mapsto \{a, p\}$.

Suppose that p admits a non-degenerate well along γ, with $p(\gamma) = b_0$, and let

$$p_\epsilon := p + \epsilon p_1,$$

where p_1 is smooth. We consider infinitesimal Hamiltonian deformations of p, *i.e.* functions of the form $\exp(\epsilon \mathrm{ad}_a)p = p_\epsilon + \epsilon\{a, p_\epsilon\} + \mathcal{O}(\epsilon^2)$, where the generator of the deformation is the smooth function a and $\mathrm{ad}_a(h) := \{a, h\} = -\mathrm{ad}_p(a)$ (see [2], Appendix 2A, for details on the adjoint representation). We have

$$\exp(\epsilon \mathrm{ad}_a)p_\epsilon = p + \epsilon(p_1 + \{a, p\}) + \mathcal{O}(\epsilon^2).$$

This leads to the study of the cohomological equation $\{a, p\} = r$ where r is given and a is unknown. As in the previous section, we let f be a smooth branch of $\sqrt{p - b_0}$.

We use the notation $\mathcal{C}$ from Remark 3.9; all quantities that are invariant by the Hamiltonian flow of p can be viewed as functions on $\mathcal{C}$. In particular, for any $\delta \in \mathcal{C}$, and $h \in C^\infty(\hat{\Omega})$, we define the average

$$\langle h \rangle_\delta := \frac{1}{2\pi} \int_0^{2\pi} h(\theta, I(\delta)) \mathrm{d}\theta.$$

Given a Hamiltonian H, let us denote by φ_H^t the Hamiltonian flow of H at time t. We notice that, since the flow of the Hamiltonian $f = g(I)$ introduced in the proof of Proposition 3.8 is a time-reparametrisation of the flow of I, we get, for all $m \in \delta$,

$$\langle h \rangle_\delta = \frac{1}{2\pi} \int_0^{2\pi} (\varphi_I^t)^* h(m) \mathrm{d}t = \frac{1}{T_\delta} \int_0^{T_\delta} (\varphi_f^t)^* h(m) \mathrm{d}t,$$

where $T_\delta = \frac{2\pi}{g'(I(\delta))}$ is the period of the Hamiltonian flow of f on δ.

The following Lemma is standard for regular Hamiltonians; but we need here a version for our singular situation.

Lemma 4.1. *There exists a neighbourhood $\hat{\Omega}$ of γ on which, for any $h \in C^\infty(\hat{\Omega})$, the following holds.*

1. *$h \in \ker \mathrm{ad}_p$ if and only if $h = q \circ f$ for some smooth function q.*
2. *$h \in \mathrm{ad}_p(C^\infty(\hat{\Omega}))$ if and only if*
   (a) *for all $\delta \in \mathcal{C}$, $\langle h \rangle_\delta = 0$ and*
   (b) *$h_{\restriction \gamma} = 0$.*

Proof. We will work in the coordinates (θ, I) introduced in Proposition 3.8 and proceed by Fourier decomposition on θ. The fact that p does not depend on θ in these coordinates greatly simplifies the discussion because it simplifies the expression of ad_p.

1. Recall
$$p : (\theta, I) \mapsto b_0 + (g(I))^2,$$
where $g : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is a smooth diffeomorphism.

   On Ω_2, one has
$$\{p, h\} = 2g'(I)g(I)\partial_\theta h(\theta, I).$$
In particular, $\{p, h\} = 0$ if and only if h depends only on I, that is, $h = q \circ f$ for some $f \in C^\infty(\mathbb{R}, \mathbb{R})$.

2. Let us decompose $h \in C^\infty(\Omega_2, \mathbb{R})$ in Fourier series in θ:
$$h : (\theta, I) \mapsto \sum_{k \in \mathbb{Z}} h_k(I) e^{ik\theta}.$$
We search for $a \in C^\infty(\Omega_2, \mathbb{R})$, of the form
$$a : (\theta, I) \mapsto \sum_{k \in \mathbb{Z}} a_k(I) e^{ik\theta}.$$
such that
$$\{a, p\} = h.$$
One can compute
$$\{a_k(I) e^{ik\theta}, p\} = ikg'(I)g(I)a_k(I)e^{ik\theta}.$$
The action of ad_p is diagonal with respect to the Fourier series decomposition; h belongs to its image if and only if $h_0 = 0$ and for every $k \neq 0$, h_k belongs to the ideal generated by g, that is, $h_k(0) = 0$. This concludes the proof.

□

Let $\pi_\theta : \hat{\Omega} \to \gamma$ be given by $(\theta, I) \mapsto \theta$. The space of functions that depend only on θ is then denoted $\pi_\theta^* C^\infty(\gamma)$.

A corollary of Lemma 4.1 is that the decomposition
$$C^\infty(\hat{\Omega}) = \ker \mathrm{ad}_p \oplus \mathrm{ad}_p(C^\infty(\hat{\Omega}))$$
is explicit. Inside $\ker \mathrm{ad}_p$, let $(\ker \mathrm{ad}_p)_0$ denote the subspace of functions vanishing on γ. Let us make the decomposition above more precise.

Proposition 4.2. *Let $p : M \to \mathbb{R}$ be a Hamiltonian with a non-degenerate well along a curve γ. There exists a neighbourhood $\hat{\Omega}$ of γ on which the following direct sum decomposition holds:*
$$C^\infty(\hat{\Omega}) = (\ker \mathrm{ad}_p)_0 \oplus \mathrm{ad}_p(C^\infty(\hat{\Omega})) \oplus \pi_\theta^* C^\infty(\gamma).$$

Proof. Let us write again h as a Fourier series in θ:

$$h : (\theta, I) \mapsto \sum_{k\in\mathbb{Z}} h_k(I)e^{ik\theta}.$$

We decompose $h = h_1 + h_2 + h_3$, where

$$\begin{aligned}
(\ker \mathrm{ad}_p)_0 \ni h_1 &: (\theta, I) \mapsto h_0(I) - h_0(0) \\
\mathrm{ad}_p(C^\infty(\Omega_2)) \ni h_2 &: (\theta, I) \mapsto \sum_{k\in\mathbb{Z}^*} (h_k(I) - h_k(0))e^{ik\theta} \\
\pi_\theta^* C^\infty(\gamma) \ni h_3 &: (\theta, I) \mapsto \sum_{k\in\mathbb{Z}} h_k(0)e^{ik\theta}.
\end{aligned}$$

This concludes the proof. □

In particular, we obtain the following:

Proposition 4.3. *Let $p : M \to \mathbb{R}$ be a Hamiltonian with a non-degenerate well along a curve γ. There exists a neighbourhood $\hat{\Omega}$ of γ on which, given any $r \in C^\infty(\hat{\Omega})$, there exists $a \in C^\infty(\hat{\Omega})$, $q \in C^\infty(\mathbb{R}, b_0)$ with $q(0) = 0$, and $V \in \pi_\theta^* C^\infty(\gamma)$, such that*

$$\{p, a\} = r - q \circ f - V.$$

By induction, this leads to the following Birkhoff normal form.

Theorem 4.4. *Let $p : M \to \mathbb{R}$ be a Hamiltonian with a non-degenerate well along a curve γ (see Assumption 1). Let p_ϵ be a formal perturbation of p; that is, p_ϵ is the jet at order ∞ of a smooth family of smooth perturbations; we write*

$$p_\epsilon = p + \sum_{j=1}^{\infty} \epsilon^j p_j + \mathcal{O}(\epsilon^\infty),$$

where the p_j's are smooth functions.

There exists a symplectic diffeomorphism φ_ϵ in a neighbourhood of γ, depending smoothly on ϵ, such that

$$\varphi_\epsilon^* p_\epsilon = b_0 + (g_\epsilon \circ f)^2 + \epsilon V_\epsilon + \mathcal{O}(\epsilon^\infty),$$

where $g_\epsilon \in C^\infty(\mathbb{R}, 0)$, $V_\epsilon = \pi_\theta^* \widetilde{V}_\epsilon$ for some $\widetilde{V}_\epsilon \in C^\infty(\gamma)$; moreover both g_ϵ and $\widetilde{V}_\epsilon$ (and hence V_ϵ) admit an asymptotic expansion in integer powers of ϵ (for the C^∞ topology), and moreover $g_\epsilon = g + \mathcal{O}(\epsilon)$ and $g_\epsilon(0) = g(0)$.

In other words, there exists canonical coordinates $(\theta, I) \in T^*S^1$ in which

$$p_\epsilon(\theta, I) = b_0 + (g_\epsilon(I))^2 + \epsilon V_\epsilon(\theta) + \mathcal{O}(\epsilon^\infty).$$

Proof. By Proposition 3.8, there holds $p = b_0 + (g \circ I)^2$ where $I : \hat{\Omega} \to \mathbb{R}$ is smooth with $\mathrm{d}I = 0$ everywhere, and $g : \mathbb{R} \to \mathbb{R}$ with $g' \neq 0$ everywhere. Suppose by induction that

$$\varphi_\epsilon^* p_\epsilon = b_0 + (g_\epsilon \circ I)^2 + \epsilon V_\epsilon + \epsilon^N r,$$

for some $N \geq 1$ (if $N = 1$ we choose $g_\epsilon = g$ and $V_\epsilon = 0$).

Let (a, q, V) be as in Proposition 4.3. We have

$$\exp(\epsilon^N \mathrm{ad}_a)\varphi_\epsilon^* p_\epsilon = \varphi_\epsilon^* p_\epsilon + \epsilon^N \{a, \varphi_\epsilon^* p_\epsilon\} + \mathcal{O}(\epsilon^{2N}).$$

Hence

$$\exp(\epsilon^N \mathrm{ad}_a)\varphi_\epsilon^* p_\epsilon = b_0 + (g_\epsilon \circ I)^2 + \epsilon V_\epsilon + \epsilon^N (r + \{a, p\}) + \mathcal{O}(\epsilon^{N+1}),$$

with

$$r + \{a, p\} = q \circ I + V$$

where $q(0) = 0$.

Hence

$$\begin{aligned}\exp(\epsilon^N \mathrm{ad}_a)\varphi_\epsilon^* p_\epsilon &= b_0 + (g_\epsilon \circ I)^2 + \epsilon^N q \circ I + \epsilon(V_\epsilon + \epsilon^{N-1} V) + \mathcal{O}(\epsilon^{N+1}) \\ &= b_0 + \left[\left(g_\epsilon + \epsilon^N \frac{1}{2} q\right) \circ I\right]^2 + \epsilon(V_\epsilon + \epsilon^{N-1} V) + \mathcal{O}(\epsilon^{N+1}).\end{aligned} \tag{7}$$

Finally, since we assumed that φ_ϵ was the time-one flow of a Hamiltonian a_ϵ, we see that the left-hand side of (7) is the flow of the Hamiltonian $a_\epsilon + \epsilon^N a$ modulo $\mathcal{O}(\epsilon^{N+1})$. This proves the induction step. □

5 Semiclassical normal form

In this section, we use the discussion of Section 4 to give a quantum equivalent to Proposition 3.8. We rely on the properties of Weyl quantization, although those methods can be adapted to other contexts. Recall that Weyl quantization, defined by (4), associates with a function $p : M \to \mathbb{R}$ a *pseudo-differential operator*, which is a family of linear operators depending on a parameter $\hbar$; we refer to [45] for an introduction to pseudo-differential operators.

5.1 Quantum maps

In order to quantise the results of Section 3, we need a proper notion of quantum map corresponding to a symplectic change of variables.

In the whole of this section, to simplify notation, we will use the subscript $\hbar$ to denote that an object depends on a parameter $\hbar$ belonging to a punctured neighbourhood of zero within a closed subset of $\mathbb{R}^+$.

Definition 5.1. Let $(M^1, \sigma^1, H^1_\hbar, \mathrm{Op}^1_\hbar)$ and $(M^2, \sigma^2, H^2_\hbar, \mathrm{Op}^2_\hbar)$ be two quantization procedures: for $i = 1, 2$, (M^i, σ^i) are symplectic manifolds, $H^i_\hbar$ are ($\hbar$-dependent) Hilbert spaces and $\mathrm{Op}^i_\hbar : C^\infty_c(M^i, \mathbb{C}) \to B(H^i_\hbar)$ realise formal deformations of the Poisson algebras $C^\infty_c(M^i, \mathbb{C})$. The functors $\mathrm{Op}^i_\hbar$ yield natural notions of $\hbar$-wave front set for families of elements of $H^i_\hbar$.

A **quantum map** consists of the data $(U_\hbar, \Omega_1, \Omega_2, \sigma)$, where Ω_1, Ω_2 are respectively open subsets of M_1 and M_2, $\sigma : \Omega_1 \to \Omega_2$ is a smooth and proper symplectomorphism, and $U_\hbar : H^1_\hbar \to H^2_\hbar$ is uniformly bounded in operator norm and satisfies the following properties:

1. For every $K \subset\subset \Omega_1$, for every $u_\hbar \in H^1$ with $\|u_\hbar\|_{H^1} = 1$ such that
$$WF_\hbar(u_\hbar) \subset K,$$
one has
$$\|U_\hbar u_\hbar\|_{H^2} = 1 + \mathcal{O}(\hbar^\infty).$$
2. For every $K \subset\subset \Omega_2$, for every $v_\hbar \in H^2$ with $\|v_\hbar\|_{H^2} = 1$ such that
$$WF_\hbar(v_\hbar) \subset K,$$
one has
$$\|U^*_\hbar v_\hbar\|_{H^1} = 1 + \mathcal{O}(\hbar^\infty).$$
3. For every $a \in C^\infty_c(\Omega_2, \mathbb{R})$, there exists a sequence $(b_k)_{k\geq 0}$ of elements of $C^\infty_c(\Omega_1, \mathbb{R})$, such that $b_0 = a \circ \sigma$, $\mathrm{supp}(b_k) \subset \sigma^{-1}(\mathrm{supp}(a))$ for every k, and
$$U^*_\hbar \mathrm{Op}^2_\hbar(a) U_\hbar = \sum_{k=0}^{\infty} \hbar^{-k} \mathrm{Op}^1_\hbar(b_k) + \mathcal{O}(\hbar^\infty).$$

A linear operator $U_\hbar$ satisfying conditions 1 and 2 above will be called a microlocal unitary transform.

Note that condition 3 of the definition implies the symmetric property where the roles of 1 and 2 are flipped: one can reconstruct $a = \sum \hbar^{-k} a_k$ from b by induction on k.

A broad class of examples of quantum maps is given by the Egorov Theorem (see [45], Theorem 11.1). Indeed, if $(M^1,\omega^1)=(M^2,\omega^2)=T^*X$ where X is a smooth, compact manifold, if $\mathrm{Op}_\hbar^i$ is the Weyl quantization, and if σ is a global Hamiltonian transformation (corresponding to a time-dependent Hamiltonian $H(t)$ for $t\in[0,1]$), then one can construct $U_\hbar$ as follows: for $u_0\in L^2(X)$, $U_\hbar u_0$ is the solution at time $t=1$ of the differential equation $i\hbar\partial_t u(t)=\mathrm{Op}_\hbar^W(H(t))u(t)$ with initial value $u(0)=u_0$. This procedure also works in more general quantization contexts.

In this section, we will use two particular quantum maps from T^*S^1 to $\mathbb{R}^2$, which we define now.

Definition 5.2. Let $\Omega_1=S^1\times\mathbb{R}_*^+$ and $\Omega_2=\mathbb{R}^2\setminus\{0\}$, which are open sets of T^*S^1 and $\mathbb{R}^2$, respectively. Let $\sigma:\Omega_1\to\Omega_2$ be defined as

$$(\theta,I)\mapsto(\sqrt{2I}\cos(\theta),\sqrt{2I}\sin(\theta)).$$

For $\hbar>0$ and $k\in\mathbb{N}_0$, let $\phi_{k,\hbar}\in L^2(\mathbb{R})$ denote the k-th Hermite eigenfunction of the $\hbar$-harmonic oscillator, defined by the following induction relation:

$$\begin{aligned}\phi_{0,\hbar}&:x\mapsto\frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{x^2}{2\hbar}}\\ \phi_{k+1,\hbar}&=\frac{1}{\hbar\sqrt{2(k+1)}}(-\hbar\partial+x)\phi_{k,\hbar}\qquad\text{for } k\geq 0.\end{aligned}$$

The toric quantum map $(\mathcal{T}_\hbar,\Omega_1,\Omega_2,\sigma)$ is defined by its action on the Fourier basis as

$$\mathcal{T}_\hbar(\theta\mapsto e^{ik\theta})=\begin{cases}\phi_{k,\hbar} & \text{if } k\geq 0\\ 0 & \text{if } k<0.\end{cases}$$

Proposition 5.3. *The toric quantum map is indeed a quantum map.*

Proof. Points 1 and 2 of the definition are almost automatic: $\mathcal{T}_\hbar$ sends a Hilbert basis of $L^2(\mathbb{R})$ to a subset of a Hilbert basis $L^2(S^1)$, to which corresponds a projector $\Pi_\hbar$. Then, by definition of $WF_\hbar$, for all compact $K\subset S^1\times(0,+\infty)$, one has, uniformly for all sequences $(u_\hbar)_{\hbar>0}$ with wave front set in K,

$$\|(\Pi_\hbar-1)u_\hbar\|_{L^2}=\mathcal{O}(\hbar^\infty).$$

One can check from the definition of $\mathcal{T}_\hbar$ that, for all $0<I_1\leq I_2$,

$$WF(u_\hbar)\subset\{(\theta,I)\in T^*S^1,I\in[I_1,I_2]\}$$

is equivalent to

$$WF(\mathcal{T}_\hbar u_\hbar) \subset \{(x,\xi) \in T^*\mathbb{R}, x^2+\xi^2 \in [\sqrt{2I_1}, \sqrt{2I_2}]\}.$$

Let us use this property to check point 3. By definition, one has, for $k \geq 0$,

$$\mathcal{T}_\hbar^*(-\hbar\partial + x)\mathcal{T}_\hbar(\theta \mapsto e^{ik\theta}) = (\theta \mapsto \sqrt{2}\hbar\sqrt{k+1}e^{i(k+1)\theta}).$$

In other terms, if $\mathrm{Op}_\hbar^1$ denotes left quantization [45], one has the exact correspondence

$$\mathcal{T}_\hbar^*(-\hbar\partial + x)\mathcal{T}_\hbar = \mathrm{Op}_\hbar^1(\sqrt{2I}\mathbb{1}_{I\geq 0}e^{i\theta}).$$

Even though $(I,\theta) \mapsto \sqrt{2I}\mathbb{1}_{I\geq 0}e^{i\theta}$ is not smooth, it is the sum of a compactly supported L^1 function and an element of $S^{\frac{1}{2}}$, so that the associated pseudo-differential operator is well-defined.

Let now $K \subset S^1 \times (0,+\infty)$ be a compact set and let us study the action of $\mathcal{T}_\hbar^*(-\hbar\partial + x)\mathcal{T}_\hbar$ on states with wave front set in K. Let $\chi : T^*S^1 \to \mathbb{R}$ be a smooth cut-off, equal to 1 on K and with compact support in $S^1 \times \mathbb{R}_+^*$. Suppose that χ is invariant by rotation. Then, uniformly on families $(u_\hbar)_{\hbar>0}$ of normalised elements of $L^2(S^1)$ with wave front set in K one has

$$\mathrm{Op}_\hbar^1(\sqrt{2I}\mathbb{1}_{I\geq 0}e^{i\theta})u_\hbar = \mathrm{Op}_\hbar^1(\chi(I)\sqrt{2I}e^{i\theta})u_\hbar + \mathcal{O}(\hbar^\infty).$$

Weyl quantization and left quantization are equivalent for smooth symbols: given a classical symbol a, there exists a classical symbol b such that $\mathrm{Op}_\hbar^1(a) = \mathrm{Op}_\hbar^W(b) + \mathcal{O}(\hbar^\infty)$. In particular,for all $K \subset\subset S^1 \times \mathbb{R}_+^*$, for all $\chi \in C_c^\infty(S^1 \times \mathbb{R}_+^*)$ equal to 1 near K, there exists a sequence $(b_k)_{k\in\mathbb{N}_{>0}}$ of elements of $C_c^\infty(S^1 \times \mathbb{R}_+^*, \mathbb{R})$ such that, for all $u_\hbar \in L^2(S^1)$ normalised with $WF_\hbar(u_\hbar) \subset K$, one has

$$\mathcal{T}_\hbar^*(-\hbar\partial + x)\mathcal{T}_\hbar u_\hbar = \mathrm{Op}_\hbar^W\left(\chi(I)\sqrt{2I}e^{i\theta} + \sum_{k=1}^{+\infty}\hbar^{-k}b_k(\theta,I)\right)u_\hbar + \mathcal{O}(\hbar^\infty).$$

All the sequences (b_k) constructed in this fashion are unique near K.

Taking the symmetric and antisymmetric part yields, with the same hypotheses,

$$\begin{aligned}\mathcal{T}_\hbar^*\mathrm{Op}_\hbar^W(x)\mathcal{T}_\hbar u_\hbar = \mathrm{Op}_\hbar^W\left(\chi(I)\sqrt{2I}\cos(\theta) + \sum_{k=1}^{+\infty}\hbar^{-k}\mathrm{Re}(b_k)(\theta,I)\right)&u_\hbar \\ &+ \mathcal{O}(\hbar^\infty) \\ \mathcal{T}_\hbar^*\mathrm{Op}_\hbar^W(\xi)\mathcal{T}_\hbar u_\hbar = \mathrm{Op}_\hbar^W\left(\chi(I)\sqrt{2I}\sin(\theta) + \sum_{k=1}^{+\infty}\hbar^{-k}\mathrm{Im}(b_k)(\theta,I)\right)&u_\hbar \\ &+ \mathcal{O}(\hbar^\infty).\end{aligned}$$

Then, using the explicit composition rules of Weyl quantization, one can determine $\mathcal{T}_\hbar^* \mathrm{Op}_\hbar^W(Q(x,\xi))\mathcal{T}_\hbar$ for any polynomial Q. The equivalence takes the following form: there exists a sequence of differential operators (D_k), such that $D_0 = id$ and D_k has degree $2k$, such that, for every polynomial Q, for any compact set K, for any χ and $(u_\hbar)$ as above,

$$\mathcal{T}_\hbar^* \mathrm{Op}_\hbar^W(Q(x,\xi))\mathcal{T}_\hbar u_\hbar = \mathrm{Op}_\hbar^W\left(\chi(I)\sum_{k=0}^{+\infty}\hbar^k[(D_kQ)\circ\sigma](\theta,I)\right)u_\hbar + \mathcal{O}(\hbar^\infty).$$

Since, in the equation above, the wave front set of $\mathcal{T}_\hbar u_\hbar$ belongs to a compact set bounded away from zero, one can add a smooth cut-off χ_1 with compact support in the equation above:

$$\mathcal{T}_\hbar^* \mathrm{Op}_\hbar^W(\chi_1(x,\xi)Q(x,\xi))\mathcal{T}_\hbar u_\hbar = \mathrm{Op}_\hbar^W\left(\chi(I)\sum_{k=0}^{+\infty}\hbar^k[(D_kQ)\circ\sigma](\theta,I)\right)u_\hbar \\ + \mathcal{O}(\hbar^\infty).$$

Let now $K_1 \subset\subset \mathbb{R}^2 \setminus \{0,0\}$ and $0 < r < R$ be such that $K_1 \subset \{x^2+\xi^2 \in [r^2,R^2]\}$. Let us choose $K \subset\subset S^1 \times \mathbb{R}_+^*$ containing an open neighbourhood of $S^1 \times [r^2/2, R^2/2]$, then χ and χ_1 as previously. In particular, χ_1 on K_1.

Let $a \in C^\infty(\mathbb{R}^2,\mathbb{R})$ be supported on K_1. Let $(Q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials such that $(Q_n)_{n\in\mathbb{N}}$ converges towards a in the C^∞ topology, on a neighbourhood of the support of χ_1. Then, in particular, $Q_n\chi_1$ converges towards a in the topology of $\mathcal{S}$ so that, by the Calderon-Vaillancourt theorem ([45], Theorem 4.23), in operator norm,

$$\mathrm{Op}_\hbar^W(\chi_1(x,\xi)Q_n(x,\xi)) \to \mathrm{Op}_\hbar^W(a).$$

On the right-hand side, one has similarly, for every k in $\mathbb{N}$,

$$\mathrm{Op}_\hbar^W(\chi(I)[(D_kQ_n)\circ\sigma](\theta,I)) \to \mathrm{Op}_\hbar^W(\chi(I)[(D_ka)\circ\sigma](\theta,I)).$$

Thus, for any sequence $(u_\hbar)$ with wave front set in K, one has, by diagonal extraction of the Q_n's,

$$\mathcal{T}_\hbar^* \mathrm{Op}_\hbar^W(a)\mathcal{T}_\hbar u_\hbar = \mathrm{Op}_\hbar^W\left(\chi(I)\sum_{k=0}^{+\infty}\hbar^k[(D_ka)\circ\sigma](\theta,I)\right)u_\hbar + \mathcal{O}(\hbar^\infty).$$

On the other hand, if the wave front set of $(u_\hbar)$ does not intersect K, then both terms in the equation above are $\mathcal{O}(\hbar^\infty)$. We conclude that

$$\mathcal{T}_\hbar^* \mathrm{Op}_\hbar^W(a)\mathcal{T}_\hbar = \mathrm{Op}_\hbar^W\left(\chi(I)\sum_{k=0}^{+\infty}\hbar^k[(D_ka)\circ\sigma](\theta,I)\right) + \mathcal{O}(\hbar^\infty).$$

□

Remark 5.4. The operator $\mathcal{T}_\hbar$ acquires a somewhat closed expression through the Bargmann transform: given the power series

$$H : y \mapsto \sum_{k\geq 0} \frac{y^k}{\sqrt{k!}},$$

which converges on the whole complex plane, $\mathcal{T}_\hbar$ has the following integral kernel:

$$(x,\theta) \mapsto C\hbar^{-2} \int_{\mathbb{C}} \exp\left[-\frac{1}{\hbar}\left(|z|^2 + |x|^2 - 2\sqrt{2}z\cdot x\right)\right] H\left(\frac{ze^{-i\theta}}{\sqrt{\hbar}}\right) \mathrm{d}z,$$

where C is a universal constant. One can check that this is a Fourier Integral Operator with complex phase; however, this explicit form is not easily tractable because the function H is transcendental [21].

An alternative representation of $\mathcal{T}_\hbar$ uses the generative functions approach of Hörmander: with

$$G : (x,\theta) \mapsto \frac{1}{2}x^2 \tan(\theta),$$

then the symplectic polar change of coordinates σ can be written

$$\sigma : (\theta, \partial_\theta G(x,\theta)) \mapsto (x, \partial_x G(x,\theta))$$

so that $\mathcal{T}_\hbar$ is a Fourier Integral operator of the form

$$(x,\theta) \mapsto \hbar^{-1} e^{\frac{i}{\hbar}G(x,\theta)} a_\hbar(x,\theta),$$

where $a_\hbar$ is a classical symbol. However, the function G is singular at $\theta = \frac{\pi}{2}$, and one should cut off this integral in phase space in x and add another contribution from the vicinity of $\theta = \frac{\pi}{2}$.

Definition 5.5. Let $(x_0, \xi_0) \in \mathbb{R}^2$ and let $r < \pi$. Let

$$\Omega_1 = \{(\theta, I) \in S^1 \times \mathbb{R}, \operatorname{dist}(\theta + 2\pi\mathbb{Z}, x_0)^2 + (I - \xi_0)^2 < r\}$$
$$\Omega_2 = \{(x,\xi) \in \mathbb{R}^2, (x-x_0)^2 + (\xi - \xi_0)^2 < r\}.$$

Let $\sigma_{x_0,\xi_0,r} : \Omega_1 \to \Omega_2$ be defined by $(\theta, I) \mapsto (x_\theta, I)$ where $x_\theta \in \theta + 2\pi\mathbb{Z}$ and $dist(x_\theta, x_0) = dist(\theta + 2\pi\mathbb{Z}, x_0)$. Let $\chi : \mathbb{R} \mapsto [0,1]$ be a smooth function equal to 1 on a neighbourhood of $[-r, r]$ and to 0 on a neighbourhood of $\mathbb{R} \setminus [-\pi, \pi]$.

We then define $\mathcal{W}_{x_0,\xi_0,r} : L^2(S^1) \to L^2(\mathbb{R})$ as follows: for $u \in L^2(S^1)$,

$$\mathcal{W}_{x_0,\xi_0,r} u : x \mapsto \chi(x - x_0)\mathrm{Op}_\hbar^W(\mathbb{1}_{(\theta,I)\in\Omega_1})u(x \bmod 2\pi\mathbb{Z}),$$

and we define the developing quantum map as $(\mathcal{W}_{x_0,\xi_0,r}, \Omega_1, \Omega_2, \sigma)$.

The developing quantum map is a quantum map by definition of $\mathrm{Op}_\hbar^W$ on T^*S^1.

5.2 Quantization of the normal form

From now on, $M = T^*X$, with $X = \mathbb{R}$ or $X = S^1$; our semiclassical analysis will be concerned with Weyl quantization. The results can be transported to other geometrical settings (manifolds with asymptotically conic or hyperbolic ends, Berezin-Toeplitz quantization of compact manifolds, ...) as long as one has a good notion of ellipticity at infinity and a microlocal equivalence with Weyl quantization, and provided that one can make sense of the invariant I_0 above. One should note, however, that even the main term V_0 in Theorem 2.1, and in particular the Morse condition of Section 6.3 or the conditions in Theorem 2.2, are not invariant under a change of quantization.

Let $(P_\hbar)_{\hbar>0}$ be a semiclassical pseudo-differential operator on X with a classical symbol in a standard class: $P_\hbar = \mathrm{Op}_\hbar^W(p_\hbar)$, with

$$p_\hbar(x, \xi) = p_0(x, \xi) + \hbar p_1(x, \xi) + \cdots$$

See [45] We assume that the principal symbol p_0 admits a non-degenerate well on a loop γ.

We are now ready to prove Theorem 2.1.

Proof. One proceeds as in Theorem 4.4. The starting point is a quantization $(U_{0,\hbar})_{\hbar>0}$ of the symplectic normal form given by Proposition 3.8.

In our setting, there are three possible topological situations for γ, and we give the three corresponding constructions of U_0.

1. If $M = \mathbb{R}^2$, then γ is contractible and one can apply Proposition 3.10. Let H be a (time-dependent) Hamiltonian satisfying the conditions of Proposition 3.10 (in particular, H is constant near infinity, so it belongs to the symbol class S_0). We let $\exp(-i\hbar^{-1}\hat{H})$ be the corresponding quantum evolution. We now let, for all $\hbar > 0$,

$$U_{0,\hbar} = \mathcal{T}_\hbar^* \exp(i\hbar^{-1}\hat{H}).$$

2. If $M = T^*S^1$ and γ is contractible, we let Σ be the compact connected component of $M \setminus \gamma$, and we let $(B((\theta_i, \xi_i), r_i))_{i \in \mathcal{I}}$ be a finite covering of a contractible neighbourhood of Σ by disks of radius $< \pi$, and $(\chi_i)_{i \in \mathcal{I}}$ be an associated partition of unity. We then let $(x_i)_{i \in \mathcal{I}}$ be a family of real numbers such that $[x_i] = \theta_i$ and $(B((x_i, \xi_i), r_i))_{i \in I}$ is a covering of a connected preimage $\hat{\Sigma}$ of Σ by the developing map. Then, we define
$$\mathcal{V} = \sum_{i \in \mathcal{I}} \mathcal{W}_{x_i, \theta_i, r_i} \mathrm{Op}_\hbar^W(\chi_i).$$
Near $\hat{\Sigma}$, one can apply Proposition 3.10 as in the previous case, and we let
$$U_0 = \mathcal{T}_\hbar^* \exp(-i\hbar^{-1}\hat{H})\mathcal{V}.$$

3. If $M = T^*S^1$ and γ is not contractible, then we apply Proposition 3.12; if H is a (time-dependent) Hamiltonian satisfying Proposition 3.12, then we let
$$U_0 = \exp(-i\hbar^{-1}\hat{H}).$$

In all cases, by the Egorov theorem, there exists a classical symbol $q_\hbar = \sum_{k=0}^{+\infty} \hbar^{-k} q_k + \mathcal{O}(\hbar^\infty)$ such that, for all u microlocalised in a neighbourhood Ω of $\{\xi = I_0\}$, one has
$$Q_0 u := U_{0,\hbar} P_\hbar U_{0,\hbar}^* u = b_0 u + \left(g_0 \left(\frac{\hbar}{i} \frac{\partial}{\partial \theta} \right) \right)^2 u + \hbar \mathrm{Op}_\hbar^W(q) u + \mathcal{O}(\hbar^\infty).$$

It remains to correct U_0 by induction, in order to get an $\mathcal{O}(\hbar^\infty)$ remainder. To this end, we proceed by induction, exactly as in Theorem 4.4. Let $N \in \mathbb{N}$; suppose by induction that there exists a quantum map $(U_{N,\hbar})$ such that
$$U_{N,\hbar} P_\hbar U_{N,\hbar}^* = b_0 u + \mathrm{Op}_\hbar^W(g_{(N),\hbar}^2(I)) + \hbar \mathrm{Op}_\hbar^W(V_{(N),\hbar}(\theta)) + \hbar^N \mathrm{Op}_\hbar^W(r_\hbar) + \mathcal{O}(\hbar^\infty),$$
where $g_{(N),\hbar}$ and $V_{(N),\hbar}$ are, respectively, degree $N-1$ and $N-2$ polynomials in $\hbar$, and $r_\hbar$ is a classical symbol on Ω. (We start with $N = 1$, and by convention a polynomial with degree -1 is the zero function.) In particular, $r_\hbar = r_0 + \mathcal{O}(\hbar)$.

We now let a_N, q_N, V_{N-1} be as in Proposition 3.3 (replacing p with p_0 and r with r_0). By the Egorov theorem,
$$U_{N,\hbar} \exp(i\hbar^{N-1} \mathrm{Op}_\hbar^W(a)) P_\hbar \exp(-i\hbar^{N-1} \mathrm{Op}_\hbar^W(a)) U_{N,\hbar}^*$$

is, up to $\mathcal{O}(\hbar^\infty)$, a pseudo-differential operator with classical symbol. Moreover, this symbol is equal to

$$(I + \hbar^N ad_a)(b_0 + (g_{(N),\hbar}(I))^2 + \hbar V_{(N),\hbar}(\theta) + \hbar^N r_0) + \mathcal{O}(\hbar^{N+1}),$$

which, by the construction above, is equal to

$$b_0 + (g_{(N),\hbar}(I) + \hbar^N q_N(I))^2 + \hbar(V_{(N),\hbar} + \hbar^{N-1} V_{N-1})(\theta) + \mathcal{O}(\hbar^{N+1}).$$

Letting

$$\begin{aligned} g_{(N+1),\hbar} &= g_{(N),\hbar} + \hbar^N q_N \\ V_{(N+1),\hbar} &= V_{(N),\hbar} + \hbar^{N-1} V_N, \end{aligned}$$

we can conclude the induction. □

6 Low-energy spectrum under global ellipticity

Let $P_\hbar$ be a pseudo-differential operator whose principal symbol admit a non-degenerate well on a loop γ. If γ is a *global* minimum for p, then one can hope to describe the spectrum of $P_\hbar$ at low energies by a microlocal analysis in a neighbourhood of γ, which should allow us to use the normal form $Q_\hbar$ of Theorem 2.1. This section is devoted first to the proof that the spectrum of $P_\hbar$ can be very well approximated by the spectrum of $Q_\hbar$, and then to the spectral study of $Q_\hbar$ under two different assumptions.

1. Case where V_0 (in Theorem 2.1) is constant. When $\hbar$ varies, the eigenvalues are located on smooth branches (parabolas) and the smallest eigenvalue regularly "jumps" from one branch to the other (See Figure 1). In the case of Schrödinger operators with a strong magnetic field, this oscillatory effect is known as "Little-Parks", see Figure 1 in [32] and [20].

2. Generic subprincipal symbol. Then we can reduce to a Schrödinger-like operator with Morse potential V, but after a $\sqrt{\hbar}$ zoom in the variable I. We consider the following two interesting cases.

   (a) local minima of the potential: we get "mini-wells";

   (b) local maxima: we can describe the concentration on hyperbolic trajectories.

6.1 Microlocal confinement

From now on, in addition to Assumption 1, we make the following hypothesis:

Assumption 2. *The curve γ is a global minimum for p, with $p = b_0$ on γ. Moreover, there exist $m_1 \geq 0, m_2 \geq 0$ such that p satisfies the following conditions:*

$$\forall j, k, \ell \in \mathbb{N}^2, \exists C > 0, \forall (x, \xi) \in T^*X,$$
$$|\partial_x^j \partial_\xi^k p_\ell(x, \xi)| \leq C(1 + |x|)^{m_1}(1 + |\xi|)^{m_2 - k}$$

$$\exists K \subset\subset T^*X, \exists c > 0, \forall (x, \xi) \in T^*X \setminus K,$$
$$p_0(x, \xi) - b_0 \geq c(1 + |x|)^{m_1}(1 + |\xi|)^{m_2}.$$

Under the assumption above we say that $P_\hbar$ is elliptic. Our first result is that, under this assumption, the low-energy spectrum of $P_\hbar$ is given by the low-energy spectrum of a modification of its normal form $Q_\hbar$ (from Theorem 2.1), and reciprocally.

Proposition 6.1. *Suppose Assumption 2 holds. With the notations of Theorem 2.1, let $\widetilde{g_0} \in C^\infty(\mathbb{R}, \mathbb{R})$ be equal to g_0 near I_0 and to 1 near infinity. In particular, if we replace g_0 with $\widetilde{g_0}$ in the expression of $Q_\hbar$, we obtain an operator $\widetilde{Q_\hbar}$ which is elliptic, in the same sense as $P_\hbar$.*

By standard elliptic estimates, every sequence of eigenfunctions of $P_\hbar$ or $\widetilde{Q}_\hbar$ with low enough energy has a wave front set near γ or $\{I = I_0\}$, respectively.

In particular, there exists $E_0 > b_0$ such that, for any family of eigenpairs $(u_\hbar, E_\hbar)$ of $P_\hbar$ with $E_\hbar < E_0$ and $\|u_\hbar\|_{L^2(X)} = 1$, one has $\|U_\hbar u_\hbar\|_{L^2(S^1)} = 1 + \mathcal{O}(\hbar^\infty)$ and

$$\|\widetilde{Q_\hbar} U_\hbar u_\hbar - U_\hbar u_\hbar\|_{L^2(S^1)} = \mathcal{O}(\hbar^\infty).$$

Moreover, for any family of eigenpairs $(v_\hbar, E_\hbar)$ of $\widetilde{Q_\hbar}$ with $E_\hbar < E_0$ and $\|v_\hbar\|_{L^2(S^1)} = 1$, one has $\|U_\hbar^* v_\hbar\|_{L^2(X)} = 1 + \mathcal{O}(\hbar^\infty)$ and

$$\|P_\hbar U_\hbar^* v_\hbar - U_\hbar^* v_\hbar\|_{L^2(X)} = \mathcal{O}(\hbar^\infty).$$

Proof. Without loss of generality, $b_0 > 0$. Let $E_1 > E_0 > p(\gamma)$ be such that

$$\{p \leq E_1\} \subset\subset \Omega \qquad \phi_0^{-1}(\{p \leq E_1\}) \subset\subset \{|I - I_0| \leq \eta\}.$$

We let $\chi : \mathbb{R} \to [0, 1]$ be any function equal to 1 on $(-\infty, E_0]$ and to 0 on $[E_1, +\infty)$.

Any sequence of normalised elements in the range of $\chi(P_\hbar)$ has its wave front set on $\{p \leq E_1\}$ and a similar property holds for $\chi(\widetilde{Q_\hbar})$. The estimate

$$U_\hbar P_\hbar U_\hbar^* u_\hbar = \widetilde{Q_\hbar} u_\hbar + \mathcal{O}(\hbar^\infty)$$

holds uniformly on normalised sequences in the range of $\chi(\widetilde{Q_\hbar})$; indeed, assuming the converse was true, one could build by a diagonal extraction a counter-example to Theorem 2.1. In particular, one has

$$U_\hbar P_\hbar U_\hbar^* \mathbb{1}(\widetilde{Q_\hbar} \leq E_0) = \widetilde{Q_\hbar} \mathbb{1}(\widetilde{Q_\hbar} \leq E_0) + \mathcal{O}_{L^2 \to L^2}(\hbar^\infty),$$

and similarly

$$P_\hbar \mathbb{1}(P_\hbar \leq E_0) = U_\hbar^* \widetilde{Q_\hbar} U_\hbar \mathbb{1}(P_\hbar \leq E_0) + \mathcal{O}_{L^2 \to L^2}(\hbar^\infty).$$

In particular, eigenfunctions of $P_\hbar$ with energy less than E_0 give $\mathcal{O}(\hbar^\infty)$ quasimodes for $\widetilde{Q_\hbar}$, and reciprocally. □

Remark 6.2. The last couple of identities in the proof of Proposition 6.1 also yield Weyl laws for the low-energy spectrum of $P_\hbar$. Indeed, they imply, for every $c \leq E_0$, for every $k \in \mathbb{N}$,

$$\mathbb{1}(P_\hbar \leq c) = U_\hbar^* \mathbb{1}(\widetilde{Q_\hbar} \leq c + \hbar^k) U_\hbar \mathbb{1}(P_\hbar \leq c) + \mathcal{O}(\hbar^\infty),$$

so that

$$\operatorname{rank}(\mathbb{1}(P_\hbar \leq c)) \leq \operatorname{rank}(\mathbb{1}(\widetilde{Q_\hbar} \leq c + \hbar^k)),$$

and a symmetric inequality.

The $S_{\rho,\delta}$-calculus for $\rho + \delta < 1$ (See Chapter 3 in [22]) then leads to the following, more precise frequency localisation estimates.

Proposition 6.3. *Suppose Assumption 2 holds. Let $\delta > 0$ and $\delta' > 0$. For every $\hbar^{1-\delta} \leq E_\hbar \leq E_0$, where E_0 is as in Proposition 6.3, for every unit eigenfunction $v_\hbar$ of $Q_\hbar$ with eigenvalue $E_\hbar$, $\hat{v_\hbar}$ is $\mathcal{O}_{\delta,\delta'}(\hbar^\infty)$ on $\{|I - I_0| \geq \hbar^{\frac{1-\delta-\delta'}{2}}\}$.*

Here, for $v_\hbar \in L^2(S^1)$, $\hat{v}$ is the semiclassical discrete Fourier transform of $v_\hbar$, which we view as an element of $\ell^2(\hbar\mathbb{Z})$.

6.2 Case with a symmetry

In this section we prove Theorem 2.2, where in particular V_0 is assumed to be constant. We first give a proof in the simpler case when V_1 is constant as well. The following Proposition is valid for $k \geq 0$, and it allows us to complete the proof if $k \geq 1$.

Proposition 6.4. *Suppose that Assumption 2 holds and let $V_\hbar$ be as in Theorem 2.1. Let $k \geq 0$, and suppose that $V_0, \ldots, V_k$ do not depend on θ. Let E_0 be as in Proposition 6.1. The eigenvalues of $P_\hbar$ in the window $(-\infty, b_0 + E_0)$ are given up to a uniform $\mathcal{O}(\hbar^{k+2})$ error by*

$$\{b_0 + \hbar V_\hbar(0) + g_\hbar(\hbar j)^2 \cap [0, E_0),\ j \in \mathbb{Z}\}.$$

Proof. From Proposition 6.1, the eigenvalues of $P_\hbar$ in the window above are exactly given by eigenvalues of $Q_\hbar$ in the same window, up to an $\mathcal{O}(\hbar^\infty)$ error. Reciprocally, since low-energy eigenfunctions of $Q_\hbar$ are themselves microlocalised in $\{|\xi - I_0| < \epsilon\}$, small eigenvalues of $Q_\hbar$ are $\mathcal{O}(\hbar^\infty)$-close to the spectrum of $P_\hbar$.

Since $V_\hbar$ does not depend on θ up to $\mathcal{O}(\hbar^{k+1})$, $Q_\hbar$ is a Fourier multiplier up to $\mathcal{O}(\hbar^{k+2})$, and we can conclude. □

This concludes the proof of Theorem 2.2 if $k \geq 1$: the smallest eigenvalue is given by minimising $g_0(\hbar j)^2$, where g_0 has only one non-degenerate zero at I_0. For $k = 0$, this is not enough, since it only describes the spectrum modulo $\mathcal{O}(\hbar^2)$.

Proposition 6.5. *Suppose that Assumption 2 holds and that V_0 does not depend on θ. Then the first eigenvalue of $P_\hbar$ is given, up to $\mathcal{O}(\hbar^3)$, by $b_0 + \hbar(g_1(I_0) + V_0) + \hbar^2 f(I_0\hbar^{-1})$, where f is a non-constant, 1-periodic function.*

Proof. For all $k \in \mathbb{Z}$, let

$$\lambda_k = (k - I_0\hbar^{-1})g_1'(I_0) + (k - I_0\hbar^{-1})^2 g_0'(I_0).$$

Let us also write a Fourier decomposition of V_1 as

$$V_1 : \theta \mapsto \sum_{l \in \mathbb{Z}} v_l e^{il\theta}.$$

Then, by the ellipticity assumption, the first eigenvalue of $P_\hbar$ coincides, modulo $\mathcal{O}(\hbar^3)$, with the first eigenvalue of

$$b_0 + \hbar(V_0 + g_1(I_0)) + \hbar^2 A$$

where A is the following operator on $\ell^2(\mathbb{Z})$:

$$\forall (k,l) \in \mathbb{Z}^2, A_{k,l} = \begin{cases} \lambda_k + v_0 & \text{if } k = l \\ v_{l-k} & \text{if } k \neq l. \end{cases}$$

The spectrum of the operator A, as a set, is 1-periodic as a function of $\sigma = I_0\hbar^{-1}$. Indeed,

$$\lambda_k(\sigma) = \lambda_{k+1}(\sigma + 1).$$

In particular, the first eigenvalue of $P_\hbar$ has the requested form, but it remains to prove that f is not constant.

To this end, observe that A has compact resolvent and analytic dependence on σ, so that if its first eigenvalue is constant, the corresponding eigenspace E_0 is also constant.

However, we observe that $\partial_\sigma^2 A = g_0'(I_0)^2 \mathrm{Id}$, with $g_0'(I_0) \neq 0$. In particular, since E_0 does not depend on σ, $\partial_\sigma^2 A|_{E_0} = g_0'(I_0)^2 \mathrm{Id}$, so that the first eigenvalue cannot be constant. This concludes the proof. □

Remark 6.6. Since g_0^2 reaches a non-degenerate minimum at I_0, the first eigenvalue of $P_\hbar$ is, in this case,

$$b_0 + \hbar g_1(I_0) + \hbar(\hbar k_\hbar - I_0) g_1'(I_0) + (\hbar k_\hbar - I_0)^2 g_0'(I_0)^2 + \mathcal{O}(\hbar^3),$$

where

$$k_\hbar = \left\lfloor \frac{I_0}{\hbar} - \frac{1}{2} g_1'(I_0) - \frac{1}{2} \right\rfloor,$$

for typical values of $\hbar$ (unless $\frac{I_0}{\hbar} - \frac{1}{2} g_1'(I_0) - \frac{1}{2}$ is $\hbar$-close to an integer, in which case it might be $k_\hbar + 1$ or $k_\hbar - 1$). In particular, this proves Theorem 2.2.

The function V_0 is the pseudo-differential equivalent of the "Melin value" μ introduced and studied in [14]. In particular, if the subprincipal symbol p_1 of the original operator is identically zero, then so is V_0. However, the term V_1 is, in general, non-zero.

Example 6.7. Let $S \in \frac{1}{2}\mathbb{N}_{>0}$. Consider the normalised spin operator

$$S_z^2 = \frac{1}{4(S+1)^2} \begin{pmatrix} (-S)^2 & & & & \\ & (-S+1)^2 & & & \\ & & \ddots & & \\ & & & (S-1)^2 & \\ & & & & S^2 \end{pmatrix}.$$

This operator is the Berezin-Toeplitz quantization of the symbol $(x, y, z) \mapsto z^2 - \hbar$ on S^2, where the semiclassical parameter is $\hbar = \frac{1}{2S}$. This symbol vanishes on the equator in a Morse-Bott way; here $I_0 = \frac{1}{2}$. In this rotational invariant case, one has $V = 0$.

Even though $\hbar$ is a discrete parameter, the oscillation phenomenon of Figure 1 is also found here: for integer values of S, the lowest eigenvalue of S_z^2 is 0; whereas for half-integer values of S it is $\frac{1}{8(S+1)^2}$.

Spin operators are models for magnetism in solids. In some contexts, the behaviour of a spin system is expected to strongly depend on whether the spin is integer or half-integer (Haldane conjecture). These effects may be related to the model case above. Strictly speaking, the results of this article do not apply to Berezin-Toeplitz quantization, but it would be interesting to cover this case as well, using the construction of I_0 in [10].

6.3 Morse case

In this section we make the assumption of a generic subprincipal symbol. We give Bohr-Sommerfeld quantization rules in two overlapping regimes: the first one consists of energies smaller than $b_0 + C\hbar$ for any fixed $C > 0$. The second consists of energies in the window $[b_0 + C\hbar, b_0 + c]$ for $C > 0$ large enough and $c > 0$ small enough. Propositions 6.8 and 6.10 yield together the spectrum of $P_\hbar$ up to energies $b_0 + c$.

6.3.1 Small energies

Proposition 6.8. *Let the following unbounded operators act on $L^2(S^1)$:*

$$H_0 = g_0'(I_0)^2 \left(\frac{\sqrt{\hbar}}{i}\frac{\partial}{\partial\theta}\right)^2 + V_0(\theta)$$

$$H_1 = 2g_0'(I_0)\left[g_1(I_0) + g_0'(I_0)\left(\frac{I_0}{\hbar} - \left\lfloor \frac{I_0}{\hbar}\right\rfloor\right)\right]\frac{\sqrt{\hbar}}{i}\frac{\partial}{\partial\theta}.$$

Their respective domains are the Sobolev spaces $W^{2,2}(S^1)$ and $W^{1,2}(S^1)$. For every $\hbar > 0$, the operator $H_0 + \sqrt{\hbar}H_1$ is bounded from below and has compact resolvent.

Let $C > 0$ and $\epsilon > 0$. Then there exists $C_1 > 0$ such that the spectrum of $P_\hbar$, in the interval $[b_0, b_0 + C\hbar]$, is the spectrum of $H_0 + \sqrt{\hbar}H_1$ in the interval $[0, 2C]$, composed by the affine function $\lambda \mapsto b_0 + \hbar\lambda$, and up to an error uniformly bounded by $C_1\hbar^{2-\epsilon}$.

Remark 6.9. The operator $H_0 + \sqrt{\hbar}H_1$ is the quantization of a symbol on $L^2(S^1)$, with semiclassical parameter $\sqrt{\hbar}$; H_0 corresponds to the principal part and H_1 to the subprincipal part. The spectrum of this operator, on fixed intervals, can be described by Bohr-Sommerfeld rules if V is Morse: we refer to [16] for the regular case, [11] for the elliptic case, and [13] for the hyperbolic case.

In particular, away from the critical values of V_0, for instance on $[\max V_0 + c, C]$, the principal symbol of H_0 is regular and consists of two connected components. On each of these components, the Bohr-Sommerfeld rule yield $\mathcal{O}(\hbar)$-quasimodes for $H_0 + \sqrt{\hbar}H_1$, whose associated eigenvalues are separated by $\epsilon\sqrt{\hbar}$ for ϵ small enough depending on c. Eigenmodes corresponding to different components are microlocalised on disjoint regions of phase space (respectively $\{\xi > c\}$ and $\{\xi < -c\}$ so that they do not interact up to $\mathcal{O}(\hbar^\infty)$. In conclusion, for $\hbar$ small enough, by a perturbative argument, one can construct $\mathcal{O}(\hbar^\infty)$-quasimodes for Q in this spectral region, yielding $\mathcal{O}(\hbar^\infty)$-quasimodes for $P_\hbar$ in the region $[b_0 + \hbar(\max V_0 + c), b_0 + \hbar C]$.

Proof. First, by Proposition 6.1 we are reduced to the study of the spectrum $Q_\hbar$ in the same interval $[b_0, b_0 + C\hbar]$.

By Proposition 6.3, any eigenfunction v of $Q_\hbar$ in this interval is localised in frequency in $\{|\xi - I_0| \leq C\hbar^{\frac{1}{2}-\epsilon}\}$ for all $\epsilon > 0$. In particular, if the Taylor expansion of g_0 and g_1 around I_0 are

$$\begin{aligned}
g_0(I) &= g_0'(I_0)(I - I_0) + \frac{g_0''(I_0)}{2}(I - I_0)^2 + \mathcal{O}((I - I_0)^3) \\
g_1(I) &= g_1(I_0) + \mathcal{O}(I - I_0),
\end{aligned}$$

then

$$\begin{aligned}
&\left[g_0\left(\frac{\hbar}{i}\frac{\partial}{\partial\theta}\right) + \hbar g_1\left(\frac{\hbar}{i}\frac{\partial}{\partial\theta}\right)\right]^2 v \\
&= \left[g_0'(I_0)\left(\frac{\hbar}{i}\frac{\partial}{\partial\theta} - I_0\right) + \frac{g_0''(I_0)}{2}\left(\frac{\hbar}{i}\frac{\partial}{\partial\theta} - I_0\right)^2 + \hbar g_1(I_0) + \mathcal{O}(\hbar^{\frac{3}{2}-3\epsilon})\right]^2 v \\
&\qquad = \hbar\left[g_0'(I_0)^2 D_\hbar^2 + \sqrt{\hbar}g_0'(I_0)\left(2g_1(I_0) + g_0''(I_0)D_\hbar^2\right)D_\hbar + \mathcal{O}(\hbar^{1-3\epsilon})\right] v
\end{aligned}$$

where we introduce

$$D_\hbar = \frac{\sqrt{\hbar}}{i}\frac{\partial}{\partial\theta} - \frac{I_0}{\sqrt{\hbar}}.$$

Notice that, the unitary conjugation on $L^2(S^1)$ given by multiplication by

$$x \mapsto \exp\left(i\left\lfloor \frac{I_0}{\hbar}\right\rfloor\right)$$

amounts to replacing $D_\hbar$ with

$$\widetilde{D_\hbar} = \frac{\sqrt{\hbar}}{i}\frac{\partial}{\partial\theta} - \sqrt{h}\{I_0\}_\hbar$$

where

$$\{I_0\}_\hbar = \frac{I_0}{\hbar} - \left\lfloor \frac{I_0}{\hbar}\right\rfloor = O_{\hbar\to 0}(1).$$

In conclusion, the eigenvalues of $Q_\hbar$ in the interval $[b_0, b_0 + C\hbar]$ are given, up to $\mathcal{O}(\hbar^{2-3\epsilon})$, by the eigenvalues of

$$\left[g_0'(I_0)^2\widetilde{D_\hbar}^2 + V_0(\theta)\right] + \hbar^{\frac{1}{2}}g_0'(I_0)\left[2g_1(I_0) + g_0''(I_0)\widetilde{D}_\hbar^2\right]\widetilde{D}_\hbar$$

in the window $[0, C]$, pushed by the map $\lambda \mapsto b_0 + \hbar\lambda$. This concludes the proof. □

6.3.2 Large energies

It remains to study the spectrum of $Q_\hbar$ in the window $[b_0 + C\hbar, b_0 + c_1]$ for C large enough. For any $c_2 > 0$, in the window $[b_0 + c_2, b_0 + c_1]$, the principal symbol p_0 of $P_\hbar$ has no degenerate point and one can apply the usual Bohr-Sommerfeld rules. We prove here that, in fact, this approach works as long as the level sets of $p_0 + \hbar p_1$ are two topological circles, one on each side of γ, that is, for energies above $b_0 + C\hbar$.

To this end, let $E \in [2C\hbar, c_1]$; we will determine the eigenvalues of $Q_\hbar$ in the window $[b_0 + \frac{E}{2}, b_0 + 2E]$ up to an error $\mathcal{O}(\hbar^2)$ uniform in E. Since $g_0(I_0) = 0$ and $g_0 \in C^\infty([I_0 - c, I_0 + c], \mathbb{R})$, there exists $\widetilde{g_0} \in C^\infty([-c, c], \mathbb{R})$ such that

$$g_0(I) = (I - I_0)\widetilde{g_0}(I).$$

In particular, the following function belongs to $C^\infty([-c, c] \times [-c, c], \mathbb{R})$:

$$f : (x, y) \mapsto \frac{1}{x}g_0(xy + I_0) = y\widetilde{g_0}(xy + I_0).$$

In particular, $f(0, y) = (g_0'(I_0)y)$.

The function

$$h_0^{E,t} : (\theta, \eta) \mapsto f^2(\sqrt{E}, \eta) + tV_0(\theta),$$

is then a continuous deformation of $h_0^{0,0} = f^2(0,\eta)$, whose Hamiltonian trajectories are circles.

We also let

$$h_1^E : (\theta,\eta) \mapsto 2f(\sqrt{E},\eta)g_1(\eta\sqrt{E}+I_0).$$

We let $c_1 > 0, c_2 > 0$ be such that, for $0 \le E \le c_1$ and $0 \le t \le c_2$, the hamiltonian trajectories of $h_0^{E,t}$ of energies in the window $\left[\frac{1}{3},3\right]$ are nondegenerate circles.

Now

$$\frac{1}{E}(Q_\hbar - b_0) = \frac{1}{E}g_0\left(\frac{\hbar}{i}\frac{\partial}{\partial\theta}\right)^2 + 2\frac{\hbar}{E}g_0\left(\frac{\hbar}{i}\frac{\partial}{\partial\theta}\right)g_1\left(\frac{\hbar}{i}\frac{\partial}{\partial\theta}\right) + \frac{\hbar}{E}V_0(\theta) + O\left(\frac{\hbar^2}{E}\right)$$

where

$$\frac{1}{E}g_0\left(\frac{\hbar}{i}\frac{\partial}{\partial\theta}\right)^2 + \frac{h}{E}V_0(\theta) = \mathrm{Op}^W_{\frac{\hbar}{\sqrt{E}}}\left(h_0^{E,\frac{\hbar}{E}}\left(\theta,\eta-\frac{I_0}{\sqrt{E}}\right)\right)$$

and

$$2\frac{\hbar}{E}g_0\left(\frac{\hbar}{i}\frac{\partial}{\partial\theta}\right)g_1\left(\frac{\hbar}{i}\frac{\partial}{\partial\theta}\right) = \frac{\hbar}{\sqrt{E}}\mathrm{Op}^W_{\frac{\hbar}{\sqrt{E}}}\left(h_1^E\left(\theta,\eta-\frac{I_0}{\sqrt{E}}\right)\right).$$

As previously, after unitary conjugation with $x \mapsto \exp\left(-ix\left\lfloor\frac{I_0}{\hbar}\right\rfloor\right)$, one can replace $\frac{I_0}{\sqrt{E}}$ with $\frac{\hbar}{\sqrt{E}}\{I_0\}_\hbar$.

Proposition 6.10. *Let $E \in \left[\frac{1}{c_2}\hbar, c_1\right]$. The eigenvalues of $P_\hbar$ in the window $\left[b_0+\frac{E}{2}, b_0+2E\right]$ are given by the eigenvalues of*

$$\mathrm{Op}^W_{\frac{\hbar}{\sqrt{E}}}\left(h_0^{E,\frac{\hbar}{E}}\right) + \frac{\hbar}{\sqrt{E}}\mathrm{Op}^W_{\frac{\hbar}{\sqrt{E}}}\left(h_1^E\right)$$

in the window $\left[\frac{1}{2},2\right]$, by the transformation

$$\lambda \mapsto b_0 + \frac{\lambda}{E},$$

up to an error $\mathcal{O}(\hbar^2)$, uniform in E.

By definition of c_2, the Hamiltonian trajectories of $h_0^{E,\frac{\hbar}{E}}$ are non-degenerate circles, so that the eigenvalues and eigenfunctions of the model operator are given by the Bohr-Sommerfeld rules.

Again, the error $\mathcal{O}(\hbar^2)$ is very small compared to the spectral gap of the model operator in each branch, which is $\hbar\sqrt{E}$, as long as $\hbar$ is small enough. Hence, in practical cases one can determine $\mathcal{O}(\hbar^\infty)$-quasimodes for P by perturbation theory. The expansion is rather technical: we perturb (via a power series in $\hbar$) an operator with semiclassical parameter $\frac{\hbar}{\sqrt{E}}$ whose symbol depends smoothly on the parameter E: a complete expansion for the eigenvalues and the quasimodes involves positive powers of $\hbar$, $\frac{\hbar}{\sqrt{E}}$ and E.

A Appendix

In this Appendix we recall a few "classical" results in the topological study of smooth curves on surfaces, and we provide either a direct proof or an explicit citation.

Definition A.1. Let us identify an oriented circle with $\{z \in \mathbb{C}, |z| = 1\}$ with counterclockwise orientation.

The *winding number* of a smooth map ρ between oriented circles is defined as

$$\omega(\rho) = \frac{1}{2i\pi} \int_{\theta=0}^{2\pi} \frac{\rho'(e^{i\theta})}{e^{i\theta}} \mathrm{d}\theta.$$

Thus, the winding number of the identity map is 1.

Proposition A.2 (See [33], Section 4.4.4, and the examples in Chapter 5 of [37])**.** *The winding number of a smooth map between oriented circles is an integer. If this map is a diffeomorphism, then the winding number is* ± 1.

The winding number of a smooth diffeomorphism of $\{z \in \mathbb{C}, |z| = 1\}$ is equal to $+1$ if this diffeomorphism preserves the orientation and -1 if it flips the orientation. In particular, by the chain rule, the winding number of a map between oriented topological circles is independent of the way we identify them with $\{z \in \mathbb{C}, |z| = 1\}$.

Recall that the orientation of a manifold with boundary induces an orientation of its boundary.

Proposition A.3. *Let M and N be closed oriented topological disks. An orientation-preserving smooth diffeomorphism from M to N induces a diffeomorphism from ∂M to ∂N with winding number* 1.

Proof. This is a direct consequence of the previous remark; indeed the restriction to the boundary of an orientation-preserving smooth diffeomorphism is an orientation-preserving smooth diffeomorphism. □

Proposition A.4. *Let M, N be oriented circles. The set of smooth diffeomorphisms from M to N with winding number 1 is connected by smooth paths.*

Proof. Let us identify M and N with $\{z \in \mathbb{C}, |z| = 1\}$. To an orientation-preserving diffeomorphism of the unit circle, we can associate a smooth, 2π-periodic map $f : \mathbb{R} \to (0, +\infty)$ such that $\rho'(e^{i\theta}) = f(\theta)ie^{i\theta}$ and $\int_0^{2\pi} f = 2\pi$.

Reciprocally, to each such map f one can clearly associate an orientation-preserving diffeomorphism of $\{z \in \mathbb{C}, |z| = 1\}$.

The association $\rho \leftrightarrow f$ is a C^∞-diffeomorphism between Fréchet spaces, and the target space is a convex subset of $C^\infty(\mathbb{R}, \mathbb{R})$, hence the claim. □

Proposition A.5. *Let $D \in \mathbb{R}^2$ be a closed topological disk with smooth boundary. There exists a smooth, orientation preserving diffeomorphism between D and $\{z \in \mathbb{C}, |z| \leq 1\}$.*

Proof. One example of such a map is given by the famous Riemann mapping theorem (identifying $\mathbb{R}^2$ with $\mathbb{C}$). For a proof that, in the case above, the Riemann mapping and its reciprocal can be smoothly extended to the boundary, see Theorem 8.2 in [4]. □

Proposition A.6. *Let $A \in \mathbb{R}^2$ be a closed topological annulus. There exists a smooth, orientation preserving diffeomorphism between A and $\{z \in \mathbb{C}, 1 \leq |z| \leq 2\}$.*

Proof. This is a variant of the Riemann mapping theorem; see pp. 83 and following in [4]. □

Proposition A.7. *Let $D = \{z \in \mathbb{C}, |z| \leq 1\}$. Let $\phi : D \to D$ be a smooth, orientation-preserving diffeomorphism. For all $r_1 < 1$, there exists $r_2 \in (1, r_1)$ and $\widetilde{\phi} : D \to D$ a smooth, orientation-preserving diffeomorphism such that*

$$|z| \leq r_1 \Rightarrow \widetilde{\phi}(z) = \phi(z) \qquad\qquad |z| \in [r_2, 1] \Rightarrow \widetilde{\phi}(z) = z.$$

Proof. Let W_1 and W_2 be two closed neighbourhoods of ∂D such that $\phi(W_1) = W_2$ and such that $0 \notin W_1 \cup W_2$.

Without loss of generality, r_1 is such that $\{|z| \in [r_1, 1]\} \subset W_1 \cap W_2$. We use polar coordinates on W_1 and W_2 to write ϕ as

$$\phi : (r_1, \theta_1) \mapsto (r_2(r_1, \theta_1), \theta_2(r_1, \theta_1)).$$

On the boundary $\{r_1 = 1\}$, one has $\frac{\partial r_2}{\partial r_1} > 0$ and $\frac{\partial r_2}{\partial \theta_1} = 0$. Since the map is orientation-preserving, the Jacobian determinant is positive, so that $\frac{\partial \theta_2}{\partial \theta_1} > 0$ at the boundary. By continuity, the inequalities

$$\frac{\partial r_2}{\partial r_1} > 0 \qquad\qquad \frac{\partial \theta_2}{\partial \theta_1} > 0$$

hold in a neighbourhood of the boundary. Let W_3 be a closed neighbourhood of the boundary and $c > 0$ be such that $\frac{\partial r_2}{\partial r_1} \geq c$ on all of W_3.

Let now $\epsilon > 0$ and $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function, supported on $[1-\epsilon, 1+\epsilon]$, equal to 1 on $[1-\epsilon/3, 1+\epsilon/3]$, and such that $\sup |\chi'| \leq 2\epsilon^{-1}$. We also impose that χ is non-decreasing on $[0,1]$.

We now define the following map from W_1 to D:

$$\phi_1 : (r_1, \theta_1) \mapsto (\chi(r_1)(1 + \frac{3c}{4}(z-1)) + (1-\chi(r_1))r_2(r_1,\theta_1), \theta_2(r_1,\theta_1)).$$

This smooth maps coincides with ϕ on $\{|z| \leq 1-\epsilon\}$, so that we can glue it with ϕ outside of W_1.

Let us prove that ϕ_1 is a diffeomorphism. The derivative of the second component with respect to θ_1 is positive for ϵ small enough. The derivative of the first component with respect to r_1 yields

$$\frac{3c}{4}\chi(r_1) + (1-\chi(r_1))\partial_{r_1} r_2(r_1,\theta_1) + \chi'(r_1)(1 + \frac{3c}{4}(r_1 - 1) - r_2(r_1,\theta_1)).$$

We claim that this quantity is positive for every $(r_1, \theta_1) \in \Omega_3$.

Indeed, by definition of c, on Ω_3 one has

$$\frac{3c}{4}\chi(r_1) + (1-\chi(r_1))\partial_{r_1} r_2(r_1,\theta_1) \geq \frac{3c}{4}.$$

Moreover $\chi'(r_1) \in [0, \frac{2}{\epsilon}]$ is supported on $[1-\epsilon, 1+\epsilon]$ and $1 - r_2(r_1,\theta_1) \geq c(1-r_1)$, so that

$$\chi'(r_1)(1 + \frac{3c}{4}(r_1-1) - r_2(r_1,\theta_1)) \geq -\frac{2}{\epsilon}\frac{c}{4}(1-r_1) \geq -\frac{c}{2};$$

in particular, the sum is larger than $\frac{c}{4}$.

The diffeomorphism ϕ_1 is not equal to the identity, but it maps the circle $\{|z| = r\}$ to the circle $\{|z| = 1 + \frac{3c}{4}(r-1)\}$ for all r close to 1. One can easily modify ϕ_1 near the boundary into ϕ_2 such that the circle $\{|z| = r\}$ is mapped to the circle $\{|z| = r\}$ for all $r \in [r_0, 1]$.

For all such r, the restriction of ϕ_2 to the disk $\{|z| = r\}$ is the restriction to the boundary of an orientation-preserving diffeomorphism of this disk. By

Proposition A.3 it has winding number 1, so that, by Proposition A.4, it is smoothly isotopic to the identity. Let $(\rho_r)_{r\in[r_0,1]}$ be a smooth family of smooth diffeomorphisms of the circle, such that the $\rho_r = \phi$ for r close to r_0 and $\rho_r = I$ for r close to 1. Then, using ρ_r, we can modify ϕ_2 into $\widetilde{\phi}$ satisfying the conditions in the claim. □

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References

[1] C. Anné and A.-M. Charbonnel. Bohr-Sommerfeld conditions for several commuting Hamiltonians. *Cubo*, 6(2):15–34, 2004.

[2] V. Arnold. *Mathematical Methods of Classical Mechanics Graduate Texts in Mathematics, 60.* Springer-Verlag, New York, 1989.

[3] A. Banyaga and D. E. Hurtubise. A proof of the Morse-Bott lemma. *Expo. Math.*, 22(4):365–373, 2004.

[4] S. Bell. *The Cauchy Transform, Potential Theory and Conformal Mapping.* CRC press, 2015.

[5] V. Bonnaillie-Noël, M. Dauge, D. Martin, and G. Vial. Computations of the first eigenpairs for the Schrödinger operator with magnetic field. *Computer methods in applied mechanics and engineering*, 196(37-40):3841–3858, 2007.

[6] V. Bonnaillie-Noël, F. Hérau, and N. Raymond. Purely magnetic tunneling effect in two dimensions. *arXiv preprint arXiv:1912.04035*, 2019.

[7] L. Boutet de Monvel. Hypoelliptic operators with double characteristics and related pseudo-differential operators. *Communications on Pure and Applied Mathematics*, 27(5):585–639, 1974.

[8] A. Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[9] L. Charles. Quasimodes and Bohr-Sommerfeld conditions for the Toeplitz operators. *Comm. Partial Differential Equations*, 28(9-10), 2003.

[10] L. Charles. Symbolic calculus for Toeplitz operators with half-form. *Journal of Symplectic Geometry*, 4(2):171–198, 2006.

[11] Y. Colin de Verdière. Spectre conjoint d'opérateurs pseudo-différentiels qui commutent II. Le cas intégrable. *Mathematische Zeitschrift*, 171(1):51–73, 1980.

[12] Y. Colin de Verdière. Singular lagrangian manifolds and semiclassical analysis. *Duke Math. J.*, 116(2):263–298, 2003.

[13] Y. Colin de Verdière and B. Parisse. Singular Bohr-Sommerfeld rules. *Commun. Math. Phys.*, 205:459–500, 1999.

[14] A. Deleporte. Low-energy spectrum of Toeplitz operators with a miniwell. *Communications in Mathematical Physics*, (to appear), 2019.

[15] M. Dimassi and J. Sjöstrand. *Spectral asymptotics in the semi-classical limit*, volume 268 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1999.

[16] J. J. Duistermaat. Oscillatory integrals, Lagrange immersions and unfolding of singularities. *Communications on Pure and Applied Mathematics*, 27(2):207–281, 1974.

[17] J. J. Duistermaat. On global action-angle coordinates. *Comm. Pure Appl. Math.*, 33(6):687–706, 1980.

[18] J. J. Duistermaat and L. Hörmander. Fourier integral operators II. *Acta Math.*, 128:183–269, 1972.

[19] A. Einstein. Zum Quantensatz von Sommerfeld und Epstein. *Verhandlungen der deutsche physikalische Gesellschaft*, 19:82–92, 1917.

[20] S. Fournais and M. P. Sundqvist. Lack of diamagnetism and the Little–Parks effect. *Communications in Mathematical Physics*, 337(1):191–224, 2015.

[21] S. Gerhold. On some non-holonomic sequences. *the electronic journal of combinatorics*, pages R87–R87, 2004.

[22] A. Grigis and J. Sjöstrand. *Microlocal Analysis for Differential Operators: An Introduction*, volume 196. Cambridge University Press, 1994.

[23] V. Guillemin and S. Sternberg. *Geometric asymptotics*. American Mathematical Society, Providence, R.I., 1977. Mathematical Surveys, No. 14.

[24] B. Helffer and A. Kachmar. Thin domain limit and counterexamples to strong diamagnetism. *arXiv:1905.06152*, May 2019.

[25] B. Helffer, Y. Kordyukov, N. Raymond, and S. Vũ Ngọc. Magnetic Wells in Dimension Three. *Analysis & PDE*, 9(7):1575–1608, Nov. 2016.

[26] B. Helffer and D. Robert. Comportement semi-classique du spectre des hamiltoniens quantiques elliptiques. *Ann. Inst. Fourier (Grenoble)*, 31(3):xi, 169–223, 1981.

[27] B. Helffer and J. Sjöstrand. Puits multiples en limite semi-classique V: Étude des minipuits. In *Current Topics in Partial Differential Equations*, pages 133–186. Kinokuniya Company Ltd., Tokyo, ohya, y., kasahara, k., and shimakura, n. edition, 1986.

[28] H. Hofer and E. Zehnder. *Symplectic Invariants and Hamiltonian Dynamics*. Birkhäuser, 2012.

[29] L. Hörmander. Fourier integral operators I. *Acta Math.*, 127:79–183, 1971.

[30] L. Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.

[31] A. Kachmar and N. Raymond. Tunnel effect in a shrinking shell enlacing a magnetic field. *Rev. Mat. Iberoam.*, 35(7):2053–2070, 2019.

[32] A. Kachmar and M. P. Sundqvist. Counterexample to strong diamagnetism for the magnetic Robin Laplacian. *arXiv:1910.12499*, Nov. 2019.

[33] S. G. Krantz. *Handbook of Complex Analysis*. Birkhäuser, 1999.

[34] Y. Le Floch. Singular Bohr-Sommerfeld conditions for 1D Toeplitz operators: elliptic case. *Comm. Partial Differential Equations*, 39(2):213–243, 2014.

[35] A. Martinez. *An introduction to semiclassical and microlocal analysis.* Universitext. Springer-Verlag, New York, 2002.

[36] V. P. Maslov. *Théorie des perturbations et méthodes asymptotiques.* Dunod, Paris, 1972.

[37] J. Milnor and D. W. Weaver. *Topology from the Differentiable Viewpoint.* Princeton university press, 1997.

[38] L. Morin. A semiclassical Birkhoff normal form for symplectic magnetic wells. preprint `hal-02173445`, July 2019.

[39] G. S. Popov. Length spectrum invariants of Riemannian manifolds. *Math. Z.*, 213(2):311–351, 1993.

[40] N. Raymond and S. Vũ Ngọc. Geometry and spectrum in 2D magnetic wells. *Annales de l'Institut Fourier*, 65:137–169, 2015.

[41] S. Vũ Ngọc. Bohr-Sommerfeld conditions for integrable systems with critical manifolds of focus-focus type. *Comm. Pure Appl. Math.*, 53(2):143–217, 2000.

[42] S. Vũ Ngọc. Formes normales semi-classiques des systèmes complètement intégrables au voisinage d'un point critique de l'application moment. *Asymptotic Analysis*, 24(3,4):319–342, 2000.

[43] S. Vũ Ngọc. Systèmes intégrales semi-classiques: Du local au global. *Panoramas et synthèses-Société mathématique de France*, 22:1–151, 2006.

[44] A. Weinstein. Fourier integral operators, quantization, and the spectra of Riemannian manifolds. In *Géométrie symplectique et physique mathématique (Colloq. Internat. CNRS, No. 237, Aix-en-Provence, 1974)*, pages 289–298. Éditions C.N.R.S., Paris, 1975.

[45] M. Zworski. *Semiclassical Analysis*, volume 138. American Mathematical Soc., 2012.