

Moment polytopes for symplectic manifolds with monodromy

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Prépublication de l'Institut Fourier n^o 670 (2005)

<http://www-fourier.ujf-grenoble.fr/prepublications.html>

Abstract

A natural way of generalising Hamiltonian toric manifolds is to permit the presence of generic isolated singularities for the moment map. For a class of such “almost-toric 4-manifolds” which admits a Hamiltonian S^1 -action we show that one can associate a group of convex polygons that generalise the celebrated moment polytopes of Atiyah, Guillemin-Sternberg. As an application, we derive a Duistermaat-Heckman formula demonstrating a strong effect of the possible monodromy of the underlying integrable system.

Keywords : moment polytope, circle action, semi-toric, Duistermaat-Heckman, monodromy, symplectic geometry, Lagrangian fibration, completely integrable systems.

Math. Class. : 53D05, 53D20, 37J15, 37J35, 57R45

1 Introduction

Let M be a compact connected symplectic manifold, equipped with an effective Hamiltonian action of a torus \mathbb{T}^k . A moment map for this action is a map $\Phi : M \rightarrow \mathbb{R}^k$ (where \mathbb{R}^k is viewed as the dual of the Lie algebra of \mathbb{T}^k) whose components generate commuting Hamiltonian flows which are independent almost everywhere and thus define the given effective \mathbb{T}^k action. In 1982, Atiyah [1] and Guillemin-Sternberg [9] discovered independently that the image of Φ is very special: it is a convex polytope. This polytope encodes many pieces of information about (M, Φ) ; if the action is completely integrable in the sense that $2k$ is the dimension of M then Delzant [4] actually proved that the moment polytope completely determines (M, Φ) , thereby showing that M is in fact a toric variety.

The theory of Hamiltonian actions on symplectic manifolds has more recently been extended to include non-compact manifolds, provided the momentum map is proper. Then all the results essentially persist.

From the point of view of classical mechanics and applications to quantum mechanics, one is generally more interested in the particular Hamiltonian function under study than in the underlying manifold. Toric manifolds are perfectly good phase spaces for many relevant examples, but the class of toric Hamiltonians or toric momentum maps is by far too narrow.

Mechanical systems usually will show up more complicated singularities than those allowed by toric momentum maps. A much more flexible notion to use instead of completely integrable toric actions is completely integrable systems, which means that one is given a “momentum map” $\Phi = (f_1, \dots, f_n)$ with the only requirement that $\{f_i, f_j\} = 0$ for all i, j and df_1, \dots, df_n are independent almost everywhere. In other words Φ is a momentum map for a local Hamiltonian action of \mathbb{R}^n , which is locally free almost everywhere. In this generality, the image of the momentum map (sometimes called the bifurcation diagram) is still of great interest but has a much more complicated structure. Even with the requirement that all singularities be non-degenerate à la Morse-Bott, the global picture is much richer than a convex polytope (see for instance [7] for 2 degrees of freedom). Nevertheless, under the assumption that the momentum map is proper (and a submersion almost everywhere), the Liouville-Arnold-Mineur theorem (or action-angle theorem) still says that each regular orbit of Φ is an n -torus in a neighbourhood of which the action is toric. Hence the main question is how to globalise this Liouville-Arnold-Mineur theorem and has two related facets. First is the study of the topological invariants of the restriction of the momentum map to regular points: this was explained in Duistermaat’s paper [5]. Secondly one has to study the singularities of Φ and how they show up in topological or symplectic invariants. A global picture for this was developed by Nguyễn Tiên Zung [26].

In our paper we bring both theories (toric actions and integrable systems) together in the sense that we construct moment polytopes with some of the usual properties (rationality, convexity) for momentum maps that are *not* toric. Our initial motivation was that these polytopes happen to be excellent tools for the semi-

classical study of the eigenvalues of quantised Hamiltonians [19].

We deal here with symplectic 4-manifolds endowed with a completely integrable system $\Phi = (J, H)$, $\{J, H\} = 0$, such that J alone is a proper momentum map for an S^1 -action on M . Such a Φ will be called *semi-toric*. Then we will assume that all singularities are non-degenerate (in the sense of Eliasson) *without hyperbolic component*. In other words we allow – in addition to tori – singular fibres of focus-focus type, which are pinched tori. A torus pinched once is an immersion of a sphere with one double point and is considered as the “simplest” singular fibre for a 2-torus fibration (see [15]). We prove the following result

Theorem 1 (proposition 2.9 and theorem 3.4)

The image of Φ is simply connected, Φ has connected fibres, and the critical values of Φ are exactly the points in the boundary of the image, plus a finite number of isolated points corresponding to the focus-focus fibres.

Then we show that, in spite of the fact that focus-focus fibres imply non-trivial monodromy and hence the impossibility of constructing global action variables and a \mathbb{T}^2 -action, one can naturally transform the image of Φ into a rational convex polygon which is almost everywhere the image of a (local) momentum map for a 2-torus action with the same foliation by tori as Φ . This is the content of theorem 3.8. Such generalised “moment polytopes” are not unique; on the contrary the set of all possible polytopes for a given system has a natural structure of an abelian group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{m_f}$, where m_f is the number of focus-focus fibres (proposition 4.1).

This construction will be used to give a simple formula for the Duistermaat-Heckman function associated to the S^1 action generated by J , which shows clearly the role played by the possible *monodromy* of the integrable system.

Theorem 2 (theorem 5.3) *If $\alpha^+(x)$ (resp. $\alpha^-(x)$) denotes the slope of the top (resp. bottom) boundary of a generalised moment polytope for Φ , then the derivative of the Duistermaat-Heckman function is*

$$\rho'_J(x) = \alpha^+(x) - \alpha^-(x)$$

and is piecewise constant on $J(M)$. Discontinuities appear at the abscissæ x of critical values of Φ of maximal corank and are given by the jump formula :

$$\rho'_J(x+0) - \rho'_J(x-0) = -k(x) - e^+(x) - e^-(x), \quad (1)$$

where $k(x) \in \mathbb{N}^$ is an associated monodromy index, and $e^\pm(x)$ are non-negative contribution of corners of the polytope, of the form*

$$e^\pm = -\frac{1}{a^\pm b^\pm} \geq 0,$$

where a^\pm, b^\pm are the isotropy weights for the S^1 action at the corresponding vertices.

Since quantities in the right-hand side of (1) are negative, we see that singularities — and especially those inducing monodromy — have a strong effect on the geometry of the polygon. In particular this yields, as a corollary, the striking result:

Theorem 3 (corollary 5.8) *If M admits a semi-toric momentum map $\Phi = (J, H)$ with at least two focus-focus critical fibres and such that J has a unique minimum (or maximum) then M is compact.*

2 Almost toric momentum maps

Before studying semi-toric momentum map we shall need some general results about a wider class of integrable systems which are not far from defining a torus action on M , in a suitable sense. The main result of this section which will be crucial for our purposes is the description of the image of the moment map for such “almost-toric” systems, when the fibres are connected (proposition 2.9).

Although we are interested here in two degrees of freedom, the results can probably be extended (*mutatis mutandis*) to an arbitrary dimension.

Let M be a connected symplectic 4-manifold, and (J, H) a completely integrable system on M : $\{J, H\} = 0$, such that $\Phi := (J, H) : M \rightarrow \mathbb{R}^2$ is a proper map.

Definition 2.1 *A proper Φ will be said of **toric type** if there exists an effective, completely integrable Hamiltonian \mathbb{T}^2 -action on M whose momentum map is of the form $F = f \circ \Phi$, where f is a local diffeomorphism on the image of Φ .*

The topology and the symplectic geometry of Hamiltonian \mathbb{T}^2 -actions are a classical subject, described by what we call the *convexity theorem* by Atiyah [1], Guillemin-Sternberg [9], the *connectedness theorem*, which is generally tied to the former [1], and the *uniqueness theorem* by Delzant [4]. Note that these results have been generalised for non-compact manifolds in case of proper momentum maps by Lerman & al [14]). We will use in this work the following statements:

Theorem 2.2 ([14]) *If F is a proper momentum map for a Hamiltonian \mathbb{T}^k -action on a symplectic manifold M then*

- *the fibres of F are connected;*
- *the image of F is a rational convex polyhedron*

A rational convex polyhedron is by definition a set which can be obtained near each point by a finite intersection of closed half-spaces whose boundary hyperplanes admit normal vectors with integer coefficients.

Proposition 2.3 *In the definition above, f is a diffeomorphism from the image of Φ into the image of F . Therefore the fibres of Φ are connected.*

Proof. f is surjective by definition. Let us show that it is injective. Let c in the image of F . Since $F^{-1}(c)$ is connected and $F^{-1}(c) = \Phi^{-1}(f^{-1}(c))$, $f^{-1}(c)$ must be connected. Since f is a local diffeomorphism, $f^{-1}(c)$ is just a point; hence f is injective. \square

Remark 2.4 A weaker definition would be that there exists an effective, completely integrable \mathbb{T}^2 -action on M which leaves Φ invariant. This is indeed strictly weaker since this would allow $\Phi = g \circ F$ where g is any local diffeomorphism (=immersion), but not necessarily a global one (for instance g can send a square to an annulus). See also proposition 2.12 below. \triangle

We shall be interested here in momentum maps that sometimes fail to be of toric type.

Definition 2.5 A proper Φ is called *almost-toric* if all the singularities are non-degenerate in the sense of Eliasson without hyperbolic blocks.

Note that Symington [18] has independently introduced the same definition, and discussed many of its consequences of topological nature. For a discussion and references on the notion of Eliasson's non-degeneracy condition, see for instance [21]. At a critical point of rank zero ($d\Phi(m) = 0$) this means that a generic linear combination of the Hessians $J''(m)$ and $H''(m)$ defines a Hamiltonian matrix (via multiplication by the linearised symplectic form) that has pairwise distinct eigenvalues. Then Eliasson's theorem says that the Lagrangian foliation near such a critical point can be linearised in the C^∞ category.

Proposition 2.6 ([4]) *If Φ is of toric type then Φ is almost-toric (with only elliptic singularities).*

Proof. This is a standard argument. Let F be a momentum map for the \mathbb{T}^2 -action. By definition the singularities of Φ are the same as those of F . Now the result follows from the fact that a torus action is linearisable near a fixed point. Details can be found for instance in [4] \square

Proposition 2.7 *If all the singularities of Φ are non-degenerate and the set of regular values of Φ is connected then Φ is almost-toric.*

Proof. If a singular point of Φ has a hyperbolic block, then because of the normal form for non-degenerate singularities, there is an embedded line segment of critical values in the interior of the image of Φ . We conclude by the following lemma. \square

Lemma 2.8 *Assume all the singularities of Φ are non-degenerate. If there is an embedded line segment of critical values in the interior of the image of Φ , then the set B_r of regular values of Φ is not connected.*

Proof. Let γ be this segment. Choose an orientation in \mathbb{R}^2 and along γ : since the set B_r of regular values is open and dense in $\Phi(M)$, there exists small disjoint open balls on each side of γ . Because all singularities are non-degenerate, γ can be extended (in both directions) until it reaches a singular value of rank zero: elliptic-elliptic, hyperbolic-elliptic or hyperbolic-hyperbolic. In all cases γ is connected to one or several other branches of critical values. Choose one arbitrarily, and continue forever (in both directions). Since Φ is proper the set of critical values is compact in any compact of \mathbb{R}^2 , therefore only two things can happen: either γ intersects itself, or γ goes to infinity (goes out of any compact) in both directions. In both cases γ disconnects B_r . \square

In general fibres of almost-toric momentum maps need not be connected. For instance if F is a toric momentum map and f is a non-injective immersion of the image of F into \mathbb{R}^2 , then $f \circ F$ is almost-toric with non-connected fibres. However we have the important proposition below:

Proposition 2.9 *Assume Φ is almost-toric. Consider the following statements:*

1. *The fibres of Φ are connected;*
2. *the set B_r or regular values of Φ is connected;*
3. *B_r is “locally connected”: for any value c of Φ , for any sufficiently small ball D centred at c , $B_r \cap D$ is connected;*
4. *$B_r = \mathring{B} \setminus \{c_1, \dots, c_{m_f}\}$, where $B = \Phi(M)$, $m_f \leq \infty$ and c_j 's are the (isolated) values by Φ of the focus-focus singularities.*

Then we have $1 \Rightarrow 2$, and 2, 3, 4 are equivalent.

Proof. Recall that if c is a critical value of Φ , we call $\Phi^{-1}(c)$ a critical fibre. Sometimes we say also a singular fibre.

$1 \Rightarrow 2$: Since Φ is almost-toric, the singular fibres are either points (elliptic-elliptic), circles (codimension 1 elliptic) or pinched tori (focus-focus). They do not include regular tori since the fibres are assumed to be connected. Only codimension 1 elliptic critical values can appear in 1-dimensional families, and elliptic-elliptic critical values appear at the end of these families. Focus-focus pinched tori are isolated. Therefore the union of all critical fibres is a locally finite union of points, cylinders and pinched tori, and therefore of codimension 2. Hence the complementary set is connected, and therefore its image by Φ also.

$2 \Rightarrow 3$: Because of the normal forms of the singularities, the only way to disconnect a small disc $D \subset \Phi(M)$ is by an embedded segment of critical values. But then B_r would not be connected by Lemma 2.8.

3 \Rightarrow 4 : If there is a critical value c in the interior of B , then it is either isolated (then it must be the image of a focus-focus point) or inside an embedded line segment of critical values (which would come from codimension 1 elliptic singularities). But the latter case is obviously in contradiction with the hypothesis of local connectedness.

4 \Rightarrow 2 : B is pathwise connected since M is a connected manifold. Suppose c and c' are in \mathring{B} . They can be connected by a path in B . If this path meets the boundary ∂B (recall that B is closed since F is proper), it can be pushed inside \mathring{B} using the normal form of elliptic singularities. Hence \mathring{B} is connected, and the result follows. \square

Remark 2.10 In the proposition above, $2 \Rightarrow 1$ is not true. One can imagine a torus bundle over an annulus, where the fibre consists of two 2-dimensional tori which swap when going round the annulus. Note however that $2 \Rightarrow 1$ is true in case B_r is simply connected, as shown in Proposition 2.12 below. One might also conjecture that it is true also when B is simply connected. \triangle

Remark 2.11 The points c_i are called *nodes* in the terminology introduced by Symington [18]. \triangle

In the next section, moment polyhedrons will be defined for some almost-toric actions. This would happen obviously if the action were actually toric:

Proposition 2.12 *If Φ is almost-toric then Φ is of toric type if and only if the set of regular values of Φ is connected and simply connected.*

Proof. Assume B_r is connected and simply connected. Using the connectedness we know from point 4 of proposition 2.7 that $B_r = \mathring{B} \setminus \{c_1, \dots, c_{m_f}\}$. By the simple connectedness we must have $m_f = 0$. The fibres corresponding to the values in the boundary ∂B can only contain elliptic-elliptic fixed points and codimension 1 elliptic circles (otherwise Φ would take values in a small ball centred at our point in the boundary...). Therefore the union of all these fibres is of codimension 2 so $\Phi^{-1}(B_r) = \Phi^{-1}(\mathring{B})$ is connected.

Now, since $\pi_0(B_r) = 1$ and $\pi_1(B_r) = 1$, the homotopy sequence of the fibration $\Phi|_{\Phi^{-1}(B_r)}$, implies that $\pi_0(\Phi^{-1}(B_r)) \simeq \pi_0(\mathcal{F})$, where \mathcal{F} is the generic fibre of Φ . Hence $\pi_0(\mathcal{F}) = 1$: the fibres are connected.

Now let $B \subset \mathbb{R}^2$ be the image of Φ . For each $c \in B$ we define the \mathbb{Z} -module of germs of basic action variables at c , ie germs of functions f such that $f \circ \Phi$ has a 2π -periodic flow near $\Phi^{-1}(c)$ (the primitive period may be any $2\pi/k$, where $k \in \mathbb{N}^*$). This defines a sheaf over B . By Liouville-Arnold-Mineur, and since the fibre $\Phi^{-1}(c)$ is connected, the stalk over a regular value is isomorphic to \mathbb{Z}^2 . By Eliasson's normal form, this also holds near an elliptic critical value. Since no other type of critical point occur, our sheaf is just a flat bundle over B , and since B is simply connected, there is no obstruction to the existence of a global section of the associated frame bundle, which is by definition a smooth map $g : B \rightarrow \mathbb{R}^2$ which

is a local diffeomorphism and such that $g \circ \Phi$ defines an effective Hamiltonian \mathbb{T}^2 -action on M .

Conversely, if Φ is of toric type, we know from proposition 2.3 that the fibres are connected and the image $B = \Phi(M)$ (and even \hat{B}) is connected and simply connected. Now proposition 2.6 tells us that no focus-focus singularities are present. Hence by proposition 2.9 we have $B_r = \hat{B}$ and hence is connected and simply connected. \square

3 Moment polygons for semi-toric momentum maps

In this section we come to our main point, defining moment polyhedrons (here, polygons) for a particular class of almost-toric momentum maps, roughly speaking those for which an S^1 action persists.

To be precise, what we shall call a polygon is a closed subset of \mathbb{R}^2 whose boundary is a continuous, piecewise linear curve with a finite number of vertices in any compact. A convex polygon is equivalently the convex hull of isolated points in \mathbb{R}^2 . A polygon is rational if the difference of the slopes of consecutive edges is always rational.

We assume throughout that Φ is almost-toric (which, we recall, requires Φ proper). By Liouville-Arnold-Mineur, The image of Φ is naturally endowed with an integral affine structure with boundary (which means that the boundary is a piecewise linear curve, where linear means geodesic with respect to the affine structure): by definition, affine charts are action variables, *ie.* maps $f : U \rightarrow \mathbb{R}^2$, where U is a small open subset of the image of Φ and $f \circ \Phi$ generates a Hamiltonian \mathbb{T}^2 -action (more precisely, each component of $f \circ \Phi$ need have a 2π -periodic Hamiltonian flow.) This affine structure is integral because any two such charts differ by the action of the integral affine group $\text{GA}(n, \mathbb{Z}) := \mathbb{R}^2 \rtimes \text{GL}(n, \mathbb{Z})$. Many more details can be found in [5, 26, 22] or even [12].

Moreover by Eliasson's normal form at elliptic-elliptic singularities the corners are convex and rational in any affine chart (we will show below that the fibres are connected, hence by proposition 2.9 the boundary is exactly the set of elliptic critical values). This is just due to the fact that a germ of convex sector near its summit is sent by a local diffeomorphism to a germ of convex sector. So in this sense the image of Φ is always a kind of rational convex polygon (with focus-focus critical values inside). To have a *true* polygon in \mathbb{R}^2 we need to find a natural projection of the universal cover of B_r onto \mathbb{R}^2 , respecting the affine structure.

Definition 3.1 *We say that an almost-toric Φ is **semi-toric** if there is a local diffeomorphism $f = (f^{(1)}, f^{(2)})$ on the image of Φ such that $f^{(1)} \circ \Phi$ is a proper momentum map for an effective action of S^1 .*

The terminology *semi-toric* may be confusing, with the risk of being mistaken for *almost-toric*. A more precise phrase would be “almost-toric with deficiency index

one” or “almost-toric with complexity one” [11]. We shall keep semi-toric for its shortness.

Proposition 3.2 *In the definition above, f is a diffeomorphism from the image of Φ into the image of $f \circ \Phi$.*

Proof. The proof is the same as that of proposition 2.3, provided we show that the fibres of $f \circ \Phi$ are connected. But this is shown by theorem 3.4 below. \square

Remark 3.3 The condition that the momentum map for the S^1 action is proper is very strong. In our situation this implies in many cases that M is compact (this is due to the presence of focus-focus singularities — see corollaries 5.6 and 5.8), and compact symplectic 4-manifolds with such an action are classified by [2] and [10]. On the other hand many situations in classical mechanics when focus-focus singularities appear do have a global S^1 action but with non-proper momentum map : a famous example is the spherical pendulum. Our results are still relevant to these cases when one can perform a preliminary reduction, or symplectic cutting [13], or more generally some integrable surgery [26], which isolates the interesting part of the manifold, making the induced S^1 momentum map proper. Since we are more interested in the momentum map rather than in the symplectic manifold itself, this is a quite harmless operation. \triangle

Assume now that Φ is semi-toric. We switch to the new momentum map $f \circ \Phi$, which we call Φ again, and denote by (J, H) its components. We define J_{\min} (resp. J_{\max}) to be the (possibly infinite) minimum (resp. maximum) of J on M .

Theorem 3.4 *1. The functions $H^+(x) := \max_{J^{-1}(x)} H$ and $H^-(x) := \min_{J^{-1}(x)} H$ are continuous;*

2. The image $B = \Phi(M)$ is the domain defined by

$$B = \{(x, y) \in \mathbb{R}^2, \quad J_{\min} \leq x \leq J_{\max} \text{ and } H^-(x) \leq y \leq H^+(x)\}. \quad (2)$$

(Therefore B is simply connected.)

3. The fibres $\Phi^{-1}(c)$ are connected (and therefore the critical values of Φ are described as in proposition 2.9).

Proof. *1.* By standard Morse theory, a discontinuity of $H^{+, -}$ could only appear at a critical value of J . But since B is closed this means that ∂B would have a vertical segment in it. By hypothesis a segment of critical values can only correspond to a family of codimension 1 elliptic singularities of Φ . Hence the preimage of this segment by Φ would be a locally maximal or minimal manifold for J , which is impossible (except for $x = J_{\min}$ or $x = J_{\max}$): by Morse-Bott theory (see [1]) J has a unique locally maximal (resp. minimal) manifold.

2. Since J is a proper momentum map for a Hamiltonian S^1 action on M , the fibre $J^{-1}(x)$ is compact and connected. Hence $H(J^{-1}(x))$ is compact and connected. Since by definition

$$B = \bigsqcup_{x \in [J_{\min}, J_{\max}]} \{x\} \times H(J^{-1}(x)),$$

we have the description (2). In particular B is contractible to a line segment and hence is simply connected.

Finally, to prove the connectedness statement 3, we still proceed similarly to [1] (even if we are not in the toric case). For a regular value x of J , $Z := J^{-1}(x)$ is a smooth compact connected manifold. By the non-degeneracy hypothesis, $H|_Z$ is a Morse-Bott function with index 0 or 2. Hence the fibres of $H|_Z$ are connected. By continuity all fibres of Φ are connected. \square

Notice that it is quite remarkable that semi-toric implies connectedness of the fibres, as in the standard toric theorem where *both* Hamiltonians H and J needed to be periodic.

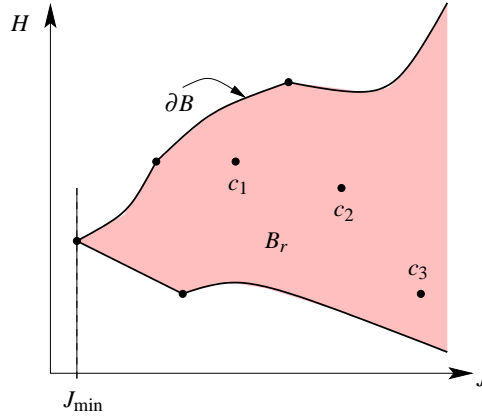


Figure 1: Image of Φ

Corollary 3.5 *If Φ is semi-toric then Φ is of toric type if and only if it has no focus-focus singularity.*

Proof. Combine the theorem with propositions 2.9 and 2.12. \square

Remark 3.6 Some of the proofs above could be made even more natural by considering the symplectic reduction by J , and also the so-called symplectic cutting. If x is a regular value of J , then we restrict Φ to the symplectic submanifold equal to $J^{-1}([x - \varepsilon, x + \varepsilon])$ with its boundary collapsed by the S^1 action. This manifold would be toric by proposition 2.12, and everything would follow from the standard toric theory. This would just require to state everything in the orbifold setting

(which is probably not a big trouble in principle), since the action defined by J is not necessarily free. \triangle

Monodromy of focus-focus points. — By the theorem, the local topological structure of the fibration by Φ can be read off from the image B (together with the focus-focus critical values). It is true for the local symplectic structure as well if one takes into account the integral affine structure of B . As we said before, the affine structure on set B_r of regular values of Φ comes from standard action variables; it is extended on the boundary ∂B using elliptic normal forms. And its behaviour at focus-focus singularities is well understood. In particular one can compute the holonomy of this affine structure around a focus-focus critical value c_i . Recall from (for instance) [20] that this holonomy (usually called the *affine monodromy*) μ_A of B_r is defined from a developing map as follows. On the universal cover $\tilde{B}_r \xrightarrow{\pi} B_r$, one can define a global set of action variables (ie. a global affine chart, or a developing map) $\tilde{f}: \tilde{B}_r \rightarrow \mathbb{R}^2$. Let $\gamma: [0, 1] \rightarrow B_r$ be a loop starting at a point c and $\tilde{\gamma}$ a lift to \tilde{B}_r . Then $\mu_A(\gamma, c)$ is defined to be the element in $\text{Aff}(2, \mathbb{Z})$ such that

$$\tilde{f}(\tilde{\gamma}(0)) = \mu_A(\gamma, c)\tilde{f}(\tilde{\gamma}(1)) \quad (3)$$

We know that the affine monodromy around a focus-focus critical value c_i has a unique line \mathcal{L} of fixed points in B (by line we mean a geodesic of the affine structure). But given any distribution of affine directions in B we can associate a 1-dimensional vector space of locally Hamiltonian vector fields on M : if β is a closed 1-form on B whose kernel gives the affine directions, then we choose the symplectic dual of $\Phi^*\beta$. In our case the smallest integral vector field \mathcal{X}_1 corresponding to the direction of \mathcal{L} is the unique (up to sign) invariant Hamiltonian vector field generating an S^1 action in a neighbourhood of the critical fibre. More precisely, suppose c is a point close to c_i and use such an $\mathcal{X}_1 = \mathcal{X}_1(c)$ to construct a basis $(\mathcal{X}_1(c), \mathcal{X}_2(c))$ corresponding to an integral affine basis \mathcal{B} of $T_c B_r$; next endow \mathbb{R}^2 with the affine structure characterised by the origin $f(c_1)$ and the basis $df(c) \cdot \mathcal{B}$. Then the affine monodromy of an oriented loop γ starting at c and winding once around c_i has no translation component and its linear part is equal to the matrix

$$T^k := \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \quad (4)$$

for some $k \in \mathbb{N}$, which is the number of focus-focus critical points in the critical fibre [16, 24, 3]. The fact that there is no translation component follows from the existence of a symplectic potential in a neighbourhood of the critical fibre $\Phi^{-1}(c_i)$ — which in turn is due to the Lagrangian nature of the critical fibre.

Remark 3.7 While the holonomy of the affine structure thus determines the topology of the critical fibre, the semi-global symplectic classification of the fibration near $\Phi^{-1}(c_i)$ is a much harder issue. In this case it is non-trivial and given in the article [23]. \triangle

Structure of the image of Φ . — A developing map \tilde{f} on \tilde{B}_r can be uniquely extended to the boundary $\pi^{-1}(\partial B)$. Using the definition of the affine structure on the boundary (given by the normal form of elliptic singularities) we see that the boundary $\pi^{-1}(\partial B)$ is sent by \tilde{f} to a piecewise linear curve in \mathbb{R}^2 . But the image of \tilde{f} is in general too non-injective to be of interest. In our case, instead of going to the universal cover, it is easier to make B_r simply connected by suitable *cuts*, and the image obtained thereby becomes simple to interpret.

Let $\{c_i = (x_i, y_i), i = 1, \dots, m_f\} \in \mathbb{R}^2$ the set of focus-focus critical values, ordered in such a way that $x_1 \leq x_2 \leq \dots \leq x_{m_f}$. For simplicity we have assumed that $m_f < \infty$, otherwise just label $\{c_i\}$ with $i \in \mathbb{Z}, \mathbb{Z}^+$ or \mathbb{Z}^- and the rest would go through. But we prove in corollary 5.10 below that m_f is actually always finite...

For each i and for some $\varepsilon \in \{-1, +1\}$ we define $\mathcal{L}_i^\varepsilon$ to be the vertical half line starting at c_i and going to $\varepsilon\infty$: $\mathcal{L}_i^\varepsilon = \{(x_i, y), \varepsilon y \geq \varepsilon y_i\}$. Given $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_{m_f}) \in \{-1, +1\}^{m_f}$, we define the line segment $\ell_i := B \cap \mathcal{L}_i^{\varepsilon_i}$, and

$$\ell^{\vec{\varepsilon}} = \bigcup_i \ell_i, \quad (5)$$

where in addition we decorate each ℓ_i with the multiplicity $\varepsilon_i k_i$, where k_i is the number of critical points in the fibre $\Phi^{-1}(c_i)$. More precisely, if several c_i 's have the same x_i -coordinate, ℓ_i is the union of all corresponding segments and we decide that each point c in the union (5) acquires the sum of the multiplicities involved, which we denote by $k(c)$. A point with multiplicity zero is omitted.

Let $\mathbb{A}_{\mathbb{Z}}^2$ be the plane \mathbb{R}^2 equipped with its standard integral affine structure. The group of automorphisms of $\mathbb{A}_{\mathbb{Z}}^2$ is the integral affine group $\text{Aff}(2, \mathbb{Z}) = \text{GL}(2, \mathbb{Z}) \ltimes \mathbb{R}^2$. We denote by \mathcal{T} the subgroup of $\text{Aff}(2, \mathbb{Z})$ which leaves a vertical line (with orientation) invariant. In other words an element of \mathcal{T} is a composition of a vertical translation and an element of $\{T^k, k \in \mathbb{Z}\} \subset \text{GL}(2, \mathbb{Z})$.

Theorem 3.8 *Given any $\vec{\varepsilon} \in \{-1, +1\}^{m_f}$, there exists a homeomorphism f from B to $f(B) \in \mathbb{R}^2$ such that*

1. $f|_{(B \setminus \ell^{\vec{\varepsilon}})}$ is a diffeomorphism (into its image).
2. $f|_{(B_r \setminus \ell^{\vec{\varepsilon}})}$ is affine: it sends the integral affine structure of B_r to the standard structure of $\mathbb{A}_{\mathbb{Z}}^2$.
3. f preserves J : ie $f(x, y) = (x, f^{(2)}(x, y))$.
4. $f|_{(B_r \setminus \ell^{\vec{\varepsilon}})}$ extends to a smooth multi-valued map from B_r to \mathbb{R}^2 and for any $i = 1, \dots, m_f$ and any $c \in \mathring{\ell}_i$ then

$$\lim_{\substack{(x,y) \rightarrow c \\ x < x_i}} df(x, y) = T^{k(c)} \lim_{\substack{(x,y) \rightarrow c \\ x > x_i}} df(x, y), \quad (6)$$

where $k(c)$ is the multiplicity of c .

5. *The image of f is a rational convex polygon.*

Such an f is unique modulo a left composition by a transformation in \mathcal{T} .

Proof. We cannot show separately each point in the theorem. However we shall split the proof into several important steps.

0.— First of all, we use the description of the image of Φ given by theorem 3.4 (and point 4. of proposition 2.9). One can assume $m_f > 0$. The case $m_f = 0$ follows by proposition 2.12 from the standard toric theory (and an argument like paragraph (2.—) below).

1.— For $i = 0, \dots, m_f$, let I_i be the open interval (x_i, x_{i+1}) and (if $I_i \neq \emptyset$) $M_i = J^{-1}(I_i)$, where by convention $x_0 = J_{\min} \in \{-\infty\} \cup \mathbb{R}$ and $x_{m_f+1} = J_{\max} \in \mathbb{R} \cup \{+\infty\}$. Each M_i is an (open) symplectic manifold endowed with the momentum map $\Phi|_{M_i}$, and on which the set of regular values of Φ is connected and simply connected (by theorem 3.4). Moreover, the critical points of $\Phi|_{M_i}$ are non-degenerate and of elliptic type. Thus, as in proposition 2.12, we can define global action coordinates: there exists a smooth map $f_i : M_i \rightarrow \mathbb{R}^2$ which is a diffeomorphism into its image $B_i := \Phi(M_i)$ and such that $f_i \circ \Phi$ is momentum map for a torus action on M_i .

2.— Actually, since $J|_{M_i}$ already defines an S^1 action, there exists an integer $p \neq 0$ such that \mathcal{X}_J/p can be chosen to be the first element of an integral basis of the period lattice defining action variables. In other words, f_i can be chosen of the form $f_i(x, y) = (x/p, f_i^{(2)}(x, y))$. But then one can see that the action of J is effective if and only if $p = \pm 1$ (we leave this to the reader). Therefore one can always chose $f_i(x, y) = (x, f_i^{(2)}(x, y))$.

3.— Assume first for simplicity that all x_i 's are different. Then $B \setminus \ell^{\tilde{e}}$ is connected and simply connected. Since $(f_0)|_{B_0 \cap B_r}$ is a section of the previously introduced sheaf of basic action variables on B_r , f_0 can be uniquely extended to a global section f over $B \setminus \ell^{\tilde{e}}$, and J is always the first action variable.

4.— Remark that $B \setminus \ell^{\tilde{e}}$ can be seen as a fundamental domain for the universal cover of B_r , and f is a developing map for the affine structure. We look now at what happens at the gluing between B_i and B_{i+1} (fix $i = 0$ for notational simplicity). Recall that in a neighbourhood of a focus-focus singularities there is a unique (up to a sign) Hamiltonian vector field \mathcal{X}_1 tangent to the fibres and whose flow is 2π -periodic. And this vector field corresponds to a line through c_1 which is fixed by the affine monodromy (see paragraph ???). In our situation \mathcal{X}_1 must be $\pm \mathcal{X}_J$ and hence the fixed line \mathcal{L} is the vertical line through c_1 . This implies that f is continuous at $\ell_1 \setminus \{c_1\}$, and the characterisation (6) follows from (3).

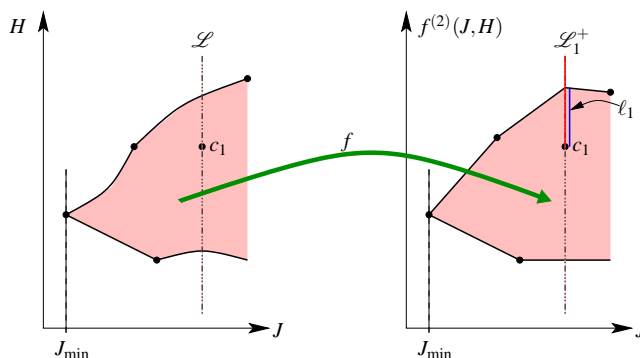


Figure 2: Definition of f at ℓ_1

We prove now that f extends to a continuous map at c_1 . For this one can use the local normal form of [23]. Since $f^{(2)}(J, H)$ is an action variable in $U \setminus \ell_1$, where U is a neighbourhood of c_1 , it follows from [23, remark 3.2] that in coordinates (\tilde{x}, \tilde{y}) of the form $\tilde{x} = x, \tilde{y} = \varphi(x, y)$ for some function $\varphi \in C^\infty(\mathbb{R}^2, 0)$,

$$f^{(2)}(x, y) = \tilde{y} \ln |\tilde{z}| - \tilde{x} \arg \tilde{z} + g(\tilde{x}, \tilde{y}),$$

where $(x, y) \in U \setminus \ell_1$, $\tilde{z} := \tilde{x} + i\tilde{y}$, and g is smooth at the origin. This shows that the function equal to $f^{(2)}$ in $U \setminus \ell_1$ and to $\tilde{y} \ln |\tilde{z}| + g(0, \tilde{y})$ on $U \cap \ell_1$ is continuous in U .

5.— Notice that our construction of f amounts to saying that f_0 on B_0 has been extended to B_1 by following paths in B_r whose rule is to go only *below* c_1 or *above* c_1 (depending on the sign of ε_1). If several x_i 's are equal, one cannot necessarily find a path that goes only below some c_i and above some others (in other words, $B \setminus \ell^{\tilde{\varepsilon}}$ is not necessarily connected). But we shall do the following: chose an arbitrary order i_1, \dots, i_n for the indices i with the same value of x_i . Then there is a unique (up to homotopy) path that connects B_0 and B_1 avoiding the c_i 's such that the whole picture is isotopic to a one where $x_{i_1} < x_{i_2} < \dots < x_{i_n}$ and the path respects the above rule (see fig. 3). Since the monodromy is Abelian [3], the choice of the ordering does not affect the definition of f in B_1 , and the results follow as well.

6.— Since $f \circ \Phi$ is momentum map for a torus action on M_i , the boundary of $f \circ \Phi(M_i)$ corresponding to elliptic singularities is piecewise linear. Thus, the functions $[J_{\min}, J_{\max}] \ni x \rightarrow f^{(2)}(x, H^\pm(x))$ (with the notations of theorem 3.4) are piecewise linear with rational slopes, and we have shown in paragraph (4.—) that they are continuous. It remains to show that the polygon $f \circ \Phi(M)$ is convex, which amounts to prove that $f^{(2)}(x, H^+(x))$ is convex and $f^{(2)}(x, H^-(x))$ is concave. For this it suffices to look at the vertices. At elliptic-elliptic critical values, the result follows from the normal form. The other vertices that can appear are the points $(x_i, f^{(2)}(x_i, H^\pm(x_i)))$. Let us look for instance at the image of $v_1 := (x_1, H^+(x_1))$.

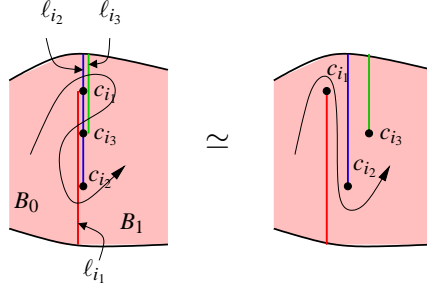


Figure 3: Extension of f_0 to B_1

Up to a change of sign for $f^{(2)}$, one can assume that $f(v_1)$ is still on the top boundary (which says that f preserves the orientation). Let $\alpha = \lim_{\substack{x \rightarrow x_1 \\ x < x_1}} (H^+)'(x_1)$, ie. the slope of the left-hand tangent to the boundary of $\Phi(M)$ at v_1 , and β the slope of the right-hand tangent. If v_1 is not the image of an elliptic-elliptic critical point then $\alpha = \beta$, otherwise $\beta < \alpha$ (the precise relation between α and β is not needed here but will be given in section 5.1 below). Call α' and β' the corresponding slopes for the new “momentum map” $f \circ \Phi$. (In other words they are the slopes of the edges of our moment polygon connecting at $f(v_1)$). Using (6) we compute

$$\begin{aligned} \beta' - \alpha' &= \lim_{\substack{(x,y) \rightarrow v_1 \\ x > x_1}} \left(\frac{\partial f^{(2)}}{\partial x} + \beta \frac{\partial f^{(2)}}{\partial y} \right) - \lim_{\substack{(x,y) \rightarrow v_1 \\ x < x_1}} \left(\frac{\partial f^{(2)}}{\partial x} + \alpha \frac{\partial f^{(2)}}{\partial y} \right) \\ &= -k(v_1) + (\beta - \alpha) \frac{\partial f^{(2)}}{\partial y}(v_1). \end{aligned} \quad (7)$$

Since f is orientation preserving, one has $\partial f^{(2)}/\partial y > 0$, hence

$$\beta' - \alpha' \leq -k(v_1). \quad (8)$$

Since $k(v_1) \geq 0$ (the cuts ℓ_i that can attain v_1 are only those that go up: for which $\varepsilon_i = 1$), the polygon is locally convex at the vertex $f(v_1)$ (or possibly flat if there is no cut and $\alpha = \beta$).

Finally, if v_1 is an elliptic-elliptic vertex then $f \circ \Phi_{M_1}$ extends naturally to a smooth momentum map near $\Phi^{-1}(v_1)$ that gives local action coordinates. Hence we know as before that the slopes of the boundary of the local angular sector obtained by this momentum map are rational. This means that the last term in (7) is rational, and $\beta' - \alpha'$ is thereby always rational.

The other cases are handled in the same way, modulo only some sign changes.

□

Remark 3.9 As I learned afterwards, the use of such branch cuts was also crucial in Symington’s work [18]. They were called “branch curves”; switching from upward to downward or vice-versa is a special case of her “branch moves”. \triangle

4 The group of polygons

Let M be a symplectic 4-manifold equipped with a semi-toric momentum map $\Phi = (J, H)$ with m_f focus-focus critical fibres. For any $\vec{\varepsilon} \in \{-1, +1\}^{m_f}$ theorem 3.8 gives an equivalence class of rational convex polygons that we denote by $\mathfrak{P}_{\vec{\varepsilon}}$, where the equivalence is given by the action of transformations in \mathcal{T} . If one changes $\vec{\varepsilon}$ the class of $\mathfrak{P}_{\vec{\varepsilon}}$ modulo \mathcal{T} might change. We investigate here the relations between all these classes of polygons.

Given an affine vertical line $\mathcal{L} \subset \mathbb{R}^2$ and an integer $n \in \mathbb{Z}$ we define a piecewise affine transformation $t_{\mathcal{L}}^n$ of \mathbb{R}^2 as follows: \mathcal{L} splits \mathbb{R}^2 into two half-spaces. $t_{\mathcal{L}}^n$ acts as the identity on the left half-space, and as the matrix T^n (defined in equation (4)) on the right one, for an origin of the affine plane \mathbb{R}^2 placed arbitrarily in \mathcal{L} (recall that T^n fixes \mathcal{L}).

We consider now the vertical lines \mathcal{L}_i through the focus-focus critical values c_1, \dots, c_{m_f} , and for any $\vec{n} := (n_1, \dots, n_{m_f}) \in \mathbb{Z}^{m_f}$ we construct the piecewise affine transformation of \mathbb{R}^2 $t_{\vec{n}} := t_{\mathcal{L}_1}^{n_1} \circ \dots \circ t_{\mathcal{L}_{m_f}}^{n_{m_f}}$. This defines an Abelian action of \mathbb{Z}^{m_f} on \mathbb{R}^2 . Finally let $G = \{0, 1\}^{m_f}$ (viewed as the Abelian group $(\mathbb{Z}/2\mathbb{Z})^{m_f}$) and let $\vec{k} = (k_1, \dots, k_{m_f})$ where $k_i \in \mathbb{N}^*$ is the number of focus-focus critical points in the fibre $\Phi^{-1}(c_i)$.

Proposition 4.1 *Let G acts transitively on the set*

$$\mathfrak{P} := \{\mathfrak{P}_{\vec{\varepsilon}}, \vec{\varepsilon} \in \{-1, +1\}^{m_f}\}$$

by the formula

$$G \times \mathfrak{P} \ni (\vec{u}, \mathfrak{P}_{\vec{\varepsilon}}) \rightarrow \vec{u} \cdot \mathfrak{P}_{\vec{\varepsilon}} := \mathfrak{P}_{(-2\vec{u}+1) \cdot \vec{\varepsilon}}. \quad (9)$$

Then this action is given by the $t_{\vec{n}}$ transformations as follows:

$$\vec{u} \cdot \mathfrak{P}_{\vec{\varepsilon}} = t_{\vec{u} \cdot \vec{\varepsilon} \cdot \vec{k}} \mathfrak{P}_{\vec{\varepsilon}}, \quad (10)$$

The action is free if and only if the abscissae x_i 's of the focus-focus critical values are pairwise distinct.

In the statement of the proposition, the dot \cdot between $\vec{\varepsilon}$, \vec{u} or \vec{k} means pointwise multiplication in \mathbb{Z}^{m_f} , after the involved quantities $e_i \in \{-1, +1\}$ or $u_i \in \{0, 1\}$ are naturally injected in \mathbb{Z} . Notice that $\vec{u} \rightarrow -2\vec{u} + 1$ is just the standard group isomorphism between $(\{0, 1\}^{m_f}, \oplus)$ and $(\{-1, +1\}^{m_f}, \times)$, where \oplus is the addition modulo 2.

Proof. The action $t_{\vec{n}}$ commutes with \mathcal{T} and therefore induces an action on equivalences classes modulo \mathcal{T} . It is then clear that both formulas (10) and (9) define actions of G on some sets of equivalences classes modulo \mathcal{T} . So to prove that these actions coincide (and thereby that the result of (10) is indeed in \mathfrak{P}) is suffices to look at generators of G .

We use here the notations of the proof of theorem 3.8. Selecting an element of the class of $\mathfrak{P}_{\vec{\varepsilon}} \bmod \mathcal{T}$ amounts to fixing the starting local basis of action

variables f_0 in M_0 . Any other representative of that class can be obtained upon composing f_0 by a transformation in \mathcal{T} . So in what follows we fix f_0 and by the notation $\mathfrak{P}_{\vec{\varepsilon}}$ we always mean the particular representative obtained from f_0 by the process of theorem 3.8.

We assume here that the x_i 's are pairwise distinct. The general case follows, as before, by a splitting argument.

Consider the action of G given by equation (9): a "1" in the i th coefficient of \vec{u} corresponds to a sign change in the i th component of ε , which flips the corresponding half line ℓ_i with respect to the point c_i . As a set of generators of G , we take the elements that have only one non-trivial coefficient. Consider for instance the first one: $u = (1, 0, \dots, 0)$ and let it act on the polytope associated to the identity element $\vec{1}$: $\vec{u} \cdot \mathfrak{P}_{\vec{1}} = \mathfrak{P}_{\vec{\varepsilon}}$, where $\varepsilon = (-1, 1, \dots, 1)$. Let f_1 and \tilde{f}_1 be the local action variables in M_1 obtained for $\mathfrak{P}_{\vec{1}}$ and $\mathfrak{P}_{\vec{\varepsilon}}$, respectively. Let us fix for instance $y > y_1$ (recall that $c_1 = (x_1, y_1)$ is a focus-focus critical value). By (6) one has for $\mathfrak{P}_{\vec{1}}$

$$\lim_{\substack{(x,y) \rightarrow c \\ x < x_i}} df_0(x, y) = T^{k_1} \lim_{\substack{(x,y) \rightarrow c \\ x > x_i}} df_1(x, y),$$

whereas for $\mathfrak{P}_{\vec{\varepsilon}}$ the formula reads

$$\lim_{\substack{(x,y) \rightarrow c \\ x < x_i}} df_0(x, y) = T^0 \lim_{\substack{(x,y) \rightarrow c \\ x > x_i}} d\tilde{f}_1(x, y),$$

entailing

$$\lim_{\substack{(x,y) \rightarrow c \\ x > x_i}} d\tilde{f}_1(x, y) = T^{k_1} \lim_{\substack{(x,y) \rightarrow c \\ x > x_i}} df_1(x, y),$$

and therefore, since in M_1 f_1 and \tilde{f}_1 must differ only by an element of \mathcal{T} ,

$$\tilde{f}_1 = T^{k_1} \circ f_1. \quad (11)$$

Now for $i > 1$ the half lines ℓ_i are identical for $\mathfrak{P}_{\vec{\varepsilon}}$ and $\mathfrak{P}_{\vec{1}}$; this means that both f_1 and \tilde{f}_1 are extended further in the same way, ensuring that for all $i > 1$, $\tilde{f}_i = T^{k_1} \circ f_i$. This in turn says that the polytopes are precisely related by the formula $\mathfrak{P}_{\vec{\varepsilon}} = t_{\vec{u}, \vec{k}} \mathfrak{P}_{\vec{1}}$. Doing this for all generators \vec{u} we have proved that for all $\vec{u} \in G$,

$$\mathfrak{P}_{(-2\vec{u}+1), \vec{1}} = t_{\vec{u}, \vec{k}} \mathfrak{P}_{\vec{1}}.$$

We conclude for a general $\vec{\varepsilon}$ by the following elementary chasing around: let $\varphi : (\{-1, +1\}^{m_f}, \times) \rightarrow (\{0, 1\}^{m_f}, \oplus)$ be the isomorphism used in the statement of the proposition: $\varphi^{-1}(\vec{u}) = -2\vec{u} + 1$. Thus one can write $\mathfrak{P}_{\vec{\varepsilon}} = \mathfrak{P}_{\vec{\varepsilon}, \vec{1}} = t_{\varphi(\vec{\varepsilon}), \vec{k}} \mathfrak{P}_{\vec{1}}$. Therefore $\mathfrak{P}_{\varphi^{-1}(\vec{u}), \vec{\varepsilon}} = \mathfrak{P}_{\varphi^{-1}(\vec{u}), \vec{\varepsilon}, \vec{1}} = \mathfrak{P}_{\varphi^{-1}(\vec{u} \oplus \varphi(\vec{\varepsilon}), \vec{1}} = t_{\vec{u} \oplus \varphi(\vec{\varepsilon}), \vec{k} - \varphi(\vec{\varepsilon}), \vec{k}} \mathfrak{P}_{\vec{\varepsilon}}$. Now it is straightforward to check that $(\vec{u} \oplus \varphi(\vec{\varepsilon}) - \varphi(\vec{\varepsilon})) = \vec{u} \cdot \vec{\varepsilon}$. This shows that the right hand sides of (10) and (9) are indeed equal.

The transitivity of the action is ensured by $\mathfrak{P}_{\vec{\varepsilon}} = \varphi(\vec{\varepsilon}) \cdot \mathfrak{P}_{\vec{1}}$. Finally, the subgroup of affine transformations generated by $t_{\mathcal{L}}$ acts freely on the set of all non-vertical segments starting on the right of \mathcal{L} . Applying this fact to the edges of

the polygons $\mathfrak{P}_{\vec{\varepsilon}}$ one sees that the action of G on \mathfrak{P} is free provided the x_i 's are distinct. Now suppose $x_i = x_{i+1} = \dots = x_{i+j}$. Then the order in which we consider c_i, \dots, c_{i+j} is irrelevant, and the corresponding permutation group in $j+1$ elements acts trivially on \mathfrak{P} . In particular the action of G is not free. \square

Remark 4.2 Let $\tilde{\mathfrak{P}}$ be the set of all possible polygons obtained for a given semitoric momentum map (J, H) . As remarked in the proof, fixing a starting set of action variables f_0 gives a way of selecting a representative in each class $\mathfrak{P}_{\vec{\varepsilon}}$. This says that $\tilde{\mathfrak{P}}$ is in bijection with $\mathfrak{P} \times \mathcal{T}$, acquiring thereby a natural group structure, where the identity element is the representative of the class $\mathfrak{P}_{\vec{1}}$. In other words one has a short exact sequence

$$0 \rightarrow \mathcal{T} \rightarrow \tilde{\mathfrak{P}} \rightarrow \mathfrak{P} \rightarrow 0,$$

which has a cross section given by the choice of f_0 . If all the x_i 's are distinct then $\tilde{\mathfrak{P}}$ is isomorphic to $G \times \mathcal{T}$. \triangle

5 Duistermaat-Heckman measures

5.1 The S^1 action

The polygons introduced in theorem 3.8 are a very efficient tool for recovering the various invariants attached to the momentum map Φ , and in particular to the effective S^1 action defined by J .

We consider here the standard Duistermaat-Heckman measure μ_J for the Hamiltonian J . Recall that by definition $\mu_J([a, b]) = \text{vol}(J^{-1}([a, b]))$, where vol means the symplectic (or Liouville) volume in M . It is known (see [6]) that

$$\mu_J := \rho_J(x) \frac{|dx|}{2\pi},$$

where the density $\rho_J(x)$ (sometimes called the Duistermaat-Heckman function) is a continuous function, equal to the symplectic volume of the reduced orbifold $J^{-1}(x)/S^1$.

Proposition 5.1 *Given any $\vec{\varepsilon} \in \{-1, 1\}^{m_f}$ and any polygon P in $\mathfrak{P}_{\vec{\varepsilon}}$, $\rho_J(x)$ is equal to the length of the vertical segment, intersection of the vertical line through x and the (filled) polygon P . Hence $\rho_J(x)$ is piecewise linear.*

Proof. Of course the fact that $\rho_J(x)$ is piecewise linear also follows from the theorem of Duistermaat and Heckman. It comes very easily here because we are in an integrable situation. Namely let f be the homeomorphism given by theorem 3.8. Then in each ‘‘cell’’ M_i , $\tilde{\Phi} := f(J, H) = (J, f^{(2)}(J, H))$ is a set of smooth action variables. It follows from Liouville-Arnold-Mineur theorem that the Duistermaat-Heckman measure $\mu_{\tilde{\Phi}}$ on \mathbb{R}^2 associated to $\tilde{\Phi}$ has density 1 over $|dx \wedge dy| / (2\pi)^2$. Integrating in the vertical direction one finds the result. \square

Remark 5.2 This shows that the lengths of the vertical segments of the polygons in \mathfrak{P} don't depend either on $\vec{\epsilon}$ or on the particular representative. This, of course, can also be checked directly from the definition of these polygon (the action of \mathcal{T} does not change vertical lengths). \triangle

We calculate now $\rho_J(x)$ in terms of the generalised moment polygons of theorem 3.8. Let $\vec{\epsilon} \in \{-1, +1\}^{m_f}$ and let $f = f_{\vec{\epsilon}}$ be the homeomorphism given by the theorem. As before, c_j 's are the focus-focus critical values and k_j is the number of focus-focus point in the fibre above c_j .

If c is a critical value of maximal corank of Φ , then $\Phi^{-1}(c)$ is either of focus-focus point or an elliptic-elliptic point. In the latter case we call c a ‘‘top vertex’’ if it lies in the graph of H^+ and a ‘‘bottom vertex’’ if it lies in the graph of H^- (in the terminology of theorem 3.4). At such a critical point J can be written in suitable symplectic coordinates under the form $J = a(x^2 + \xi^2)/2 + b(y^2 + \eta^2)/2$ for integer a, b which are called *isotropy weights* of the S^1 action defined by J [6, 10].

Theorem 5.3 *If $\alpha^+(x)$ (resp. $\alpha^-(x)$) denotes the slope of the top (resp. bottom) boundary of the polygon $f \circ \Phi(M)$, then the derivative of the Duistermaat-Heckman function is*

$$\rho'_J(x) = \alpha^+(x) - \alpha^-(x) \quad (12)$$

and is locally constant on $J(M) \setminus \{\pi_x(f(\Sigma_0(\Phi)))\} \in \mathbb{R}$, where $\Sigma_0(\Phi)$ is the set of critical values of Φ of maximal corank and π_x is the projection $(x, y) \rightarrow x$. If $(x, y) \in \Sigma_0(\Phi)$ then

$$\rho'_J(x+0) - \rho'_J(x-0) = - \sum_j k_j - e^+ - e^-, \quad (13)$$

where the sum runs over the set of all indices j such that $\pi_x(c_j) = x$, and e^+ (respectively e^-) is non-zero if and only if an elliptic top vertex (resp. a bottom vertex) projects down onto x . If this occurs then

$$e^\pm = - \frac{1}{a^\pm b^\pm} \geq 0,$$

where a^\pm, b^\pm are the isotropy weights for the S^1 action at the corresponding vertices.

Proof. The first point is obvious in view of proposition 5.1. Notice that in general the discontinuities of ρ'_J occur at the singularities of J . Here these singularities (except possibly for the maxima and minima of J) are exactly critical values of maximal corank of Φ .

The second point is just a small refinement of formula (7). This formula says that

$$\rho'_J(x+0) - \rho'_J(x-0) = - \sum_j k_j + (r^+(x) - r^-(x))$$

where, as explained at the end of the proof of theorem 3.8, $r^\pm(x)$ is computed as follows. The item 4. of theorem 3.8 says that in a small neighbourhood of the point in

the boundary $v^\pm := (x, H^\pm(x))$, f can be smoothly extended (either from the region $\leq x$ or $\geq x$) to a smooth map \tilde{f}^\pm such that $\tilde{f}^\pm \circ \Phi$ is a toric momentum map near $\Phi^{-1}(v^\pm)$. Then the local image of $\tilde{f}^\pm \circ \Phi$ is an angular sector and $r^\pm(x)$ is the difference between the slopes of the right-hand and left-hand edges at the vertex $f(v^\pm)$ of this sector. It does not depend on the way f was extended since it is invariant by a transformation in \mathcal{T} . Precisely, there is a matrix $A^\pm = \begin{pmatrix} a^\pm & b^\pm \\ c^\pm & d^\pm \end{pmatrix} \in SL(2, \mathbb{Z})$ and canonical coordinates (x, y, ξ, η) near the elliptic-elliptic point $\Phi^{-1}(v^\pm)$ such that $\tilde{f}^\pm \circ \Phi = A^\pm \circ (\frac{x^2 + \xi^2}{2}, \frac{y^2 + \eta^2}{2})$. In particular $J = a^\pm (\frac{x^2 + \xi^2}{2}) + b^\pm (\frac{y^2 + \eta^2}{2})$. If x is not an extremal value for J , a^\pm and b^\pm do not vanish and have different signs; for the top vertex $v^+(x)$ one must have $a^+ < 0$. Then

$$r^+(x) = \frac{d^+}{b^+} - \frac{c^+}{a^+} = \frac{1}{a^+ b^+}.$$

At the bottom vertex $v^-(x)$ the coefficient a^- is positive, and $r^-(x) = -\frac{1}{a^- b^-}$. \square

Remark 5.4 Nothing in this theorem is essentially new, apart from the proof (and maybe also the fact that M is not necessarily compact). Compared to the usual theory, our proof follows very easily and elementarily from our moment polygons. For general Hamiltonian torus actions on compact symplectic manifolds, a formula analogous to (13) follows from the Duistermaat-Heckman formula (or the localisation formula of Atiyah-Bott-Berline-Vergne) for the Fourier transform of μ_J , and a Fourier inversion argument as in [8] (see also [10]). The main difference with our formula is that we separate the contribution of focus-focus points from elliptic-elliptic points, which of course is not possible in the context of a general S^1 action. This again is not really new since the link between the monodromy and Duistermaat-Heckman's theory was recently pointed out by Nguyễn Tiên Zung in [25]. However Zung's construction was a local one using integrable surgery, whereas we express it in a global situation. \triangle

This theorem (together with theorem 3.8) has some easy corollaries of topological nature.

Corollary 5.5 *If a symplectic manifold M admits a semi-toric momentum map (J, H) with at least one critical value of maximal corank ($dJ(m) = dH(m) = 0$) then J is bounded from below or from above.*

Proof. By the theorem 5.3, the strict inequality $\rho'_J(x+0) - \rho'_J(x-0) < 0$ holds at at least one point. Hence there is a point x_0 for which $\rho'_J(x_0) \neq 0$. Suppose for instance $\rho'_J(x_0) < 0$. Then the length of the interval $f^{(2)}(x, J^{-1}(x))$ (or the Duistermaat-Heckman measure at x) is bounded from above by $\text{const} + x\rho'_J(x)$ and hence by convexity of the polygon must vanish for a finite value of $x > x_0$. The point for which it vanishes has to be the maximal value of J . \square

Corollary 5.6 *Let a symplectic manifold M admit a semi-toric momentum map $\Phi = (J, H)$ such that J is bounded from below with minimal value J_{\min} . If Φ has more than $\rho'_J(J_{\min} + 0)$ focus-focus points (counted with multiplicity) then M is compact.*

Proof. By the theorem 5.3 if x is greater than the maximum of the abscissae of the focus-focus critical values then $\rho'_J(x) < 0$, and we conclude as above that J has a finite maximal value. Hence M is compact by properness of J . \square

Remark 5.7 In case of a compact M , one can write an explicit upper bound for the symplectic volume of M (the area of the polygon: see next section) in terms of $\rho'_J(J_{\min} + 0)$, the symplectic volume of $J^{-1}(J_{\min})$ (which may be zero), and the abscissae and multiplicities of all focus-focus critical values. We leave this to the reader. \triangle

Corollary 5.8 *If M admits a semi-toric momentum map $\Phi = (J, H)$ with $m_f \geq 2$ focus-focus critical fibres and such that J has a unique minimum (or maximum) then M is compact.*

Proof. If J has a unique minimum its image under Φ is an elliptic-elliptic corner of any associated moment polygon, open in the direction $y \geq 0$. But the edges of an elliptic-elliptic corner are directed along integral vectors (a, c) and (b, d) such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$. Hence $\rho'_J(J_{\min} + 0) = d/b - c/a = 1/ab \leq 1$ and the result follows from the corollary 5.6 above. \square

Remark 5.9 In contrast with the hypothesis of this corollary, $\rho'_J(J_{\min} + 0)$ can take any integral value if J has a non-trivial submanifold of minima (which means that the moment polygons have a vertical edge at J_{\min}). \triangle

Corollary 5.10 *If M admits a semi-toric momentum map then the number m_f of focus-focus critical fibres is finite.*

Proof. If $m_f > 0$ then by corollary 5.5 J is semi-bounded (say for instance from below). Then by corollary 5.6 if m_f is very large M must be compact. Since the c_i 's are isolated m_f has to be finite. \square

Remark 5.11 If one knows the value of ρ_J at some point, then theorem 5.3 gives all one needs to reconstruct ρ_J by integration. Such a formula will be given below for the ‘‘generalised’’ S^1 actions. \triangle

5.2 The generalised S^1 actions

The construction of the polygons in theorem 3.8 leads naturally to considering another type of Duistermaat-Heckman measure, namely the one associated with

horizontal slices of the polygons. In other words, given an $\vec{\epsilon} \in \{-1, 1\}^{m_f}$ and a map f as provided by the mentioned theorem, we consider the push-forward of the Liouville measure by the second “action” variable $K := f^{(2)}(J, H)$. K is continuous, but it is not smooth along the vertical lines through focus-focus points. Where it is smooth, K does define a Hamiltonian S^1 -action. The only problem for defining the Duistermaat-Heckman measure μ_K is that K is not assumed to be proper. Therefore in what follows we either assume M to be compact or we restrict M to the compact symplectic manifold with boundary $J^{-1}([a, b])$, for some bounded interval $[a, b]$.

Then the Duistermaat-Heckman function ρ_K such that $\mu_K = \rho_K(y) |dy|$ can be described exactly as we did for ρ_J . In particular $\rho_K(y)$ is the length of the horizontal slice of the moment polygon with ordinate y . However this definition is not so easy to use here since the order of the vertical projections of the polygon vertices—and hence $\rho_K(y)$ —strongly depends on the $\vec{\epsilon}$ chosen to construct it. It is more adequate to express ρ_K as much as possible in terms of the J -data.

For this purpose we slightly change the notation by letting $x_0 < x_1 < \dots < x_N$ be the abscissæ of all critical values of rank zero of Φ (including focus-focus and elliptic-elliptic points). Let y_i^+ (resp. y_i^-) be the ordinate of the intersection of the vertical line at x_i with the top (resp. bottom) boundary of the polygon. Finally for $i \in [0..N-1]$ let $\alpha_i^\pm = \frac{y_{i+1}^\pm - y_i^\pm}{x_{i+1} - x_i}$ be the slope of the corresponding (top or bottom) edge of the polygon. Contrary to x_i , α_i^\pm and y_i^\pm depend on $\vec{\epsilon}$. One has :

$$y_i^\pm = y_0^\pm + \sum_{j=0}^{i-1} h_j \alpha_j^\pm, \text{ where } h_i = (x_{i+1} - x_i),$$

and $\alpha_{i+1}^\pm - \alpha_i^\pm$ is given by theorem 5.3 in terms of $\vec{\epsilon}$, the monodromy indices, and fixed point data of J .

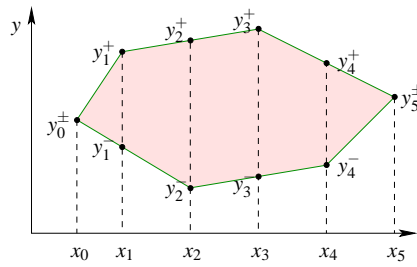


Figure 4: Notations for cutting the polygon

We need some non-standard conventions in order to state the following theorem. If a, b are real numbers, we denote by $[a, b]$ the interval $[\min(a, b), \max(a, b)]$. If I is an interval, χ_I designates the characteristic function of I . If I is a point, by convention $\chi_I = 0$, and for any number $\beta \in \mathbb{R} \cup \{\infty\}$, $\beta \chi_I = 0$.

Theorem 5.12 *With the notation defined above, the Duistermaat-Heckman function ρ_K is the continuous, piecewise linear function given by the following formula :*

$$\rho_K(y) = \sum_{i=0}^{N-1} \left(\frac{1}{|\alpha_i^-|} (y - y_i^-) \chi_{[y_i^-, y_{i+1}^-]} + h_i \chi_{[y_{i+1}^-, y_i^+]} + \frac{1}{|\alpha_i^+|} (y_{i+1}^+ - y) \chi_{[y_i^+, y_{i+1}^+]} \right). \quad (14)$$

In particular the derivative of ρ_K is the piecewise constant function given by

$$\rho'_K(y) = \sum_{i=0}^{N-1} \left(\frac{1}{|\alpha_i^-|} \chi_{[y_i^-, y_{i+1}^-]} - \frac{1}{|\alpha_i^+|} \chi_{[y_i^+, y_{i+1}^+]} \right).$$

Proof. The term in the sum for a fixed i corresponds to the calculation of ρ_K restricted to the elementary cell $J^{-1}([x_i, x_{i+1}])$, which is a simple exercise. \square

6 Examples

6.1 Coupled angular momenta on $S^2 \times S^2$

The first example that motivated this paper (with some others to come), and which I still think is of primary interest, has been described first by Sadovskií and Zhilinskií in [17]. It is the problem of two coupled angular momenta, describing for instance a so-called “spin-orbit coupling”. The momentum map on $S^2 \times S^2$ depends on an additional parameter t as follows : $\Phi_t = (J, H_t)$, where $J = N_z + S_z$ and

$$H_t = \frac{1-t}{|\mathbf{S}|} S_z + \frac{t}{|\mathbf{N}||\mathbf{S}|} \langle \mathbf{N}, \mathbf{S} \rangle, \quad 0 \leq t \leq 1.$$

We have denoted by $\mathbf{S} = (S_x, S_y, S_z)$ et $\mathbf{N} = (N_x, N_y, N_z)$ the angular momentum variables on each S^2 factor. In other words these spheres are standard symplectic spheres but with radius $|\mathbf{S}|$ for the first one and $|\mathbf{N}|$ for the second one.

Then one can show that Φ_t is semi-toric except for two values of t , and *not* of toric type for t in a bounded open interval containing $1/2$, where Φ_t has a focus-focus critical point. For t around $1/2$ the image of the momentum map and the two generalised polygons are depicted in the figure 5. We don't show the details here because they are partly computed in [17] and go along the same lines as the next example.

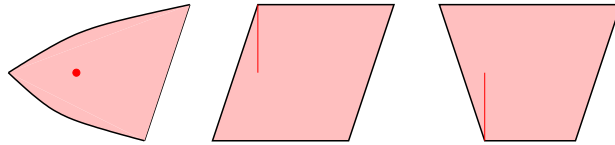


Figure 5: Image of the momentum map (left) and generalised polygons for the coupled angular momenta.

6.2 Coupled spin and oscillator on $S^2 \times \mathbb{R}^2$

Using the previous example by Sadovskii and Zhilinskiĭ, one can construct an example on $S^2 \times \mathbb{R}^2$ with one focus-focus singularity, just by linearising one of the spheres at a pole. In addition to being interesting by its computational simplicity, it provides an example of a non-compact manifold that shows that corollary 5.8 is optimal.

On S^2 one has a natural Hamiltonian S^1 action whose Hamiltonian is the “vertical coordinate” z , where we embed S^2 in \mathbb{R}^3 as $\{x^2 + y^2 + z^2 = 1\}$. The sign of the “standard” symplectic form on S^2 is chosen such that the flow turns around the vertical axis in the direct sense (counterclockwise). The total symplectic volume is chosen such that the flow of z is 2π -periodic.

On $\mathbb{R}^2 = \{(u, v)\}$ with canonical symplectic form our standard S^1 action is the harmonic oscillator $N := (u^2 + v^2)/2$ with 2π -periodic flow.

On $M = S^2 \times \mathbb{R}^2$ we define an S^1 action by the Hamiltonian

$$J := N + z.$$

Using the embedding of S^2 in \mathbb{R}^3 , define the orthogonal projector π_z from S^2 onto \mathbb{R}^2 viewed as the $z = 0$ hyperplane. Let $(m, p) \in S^2 \times \mathbb{R}^2$. Then under the flow of J the points m and p are moving along the flows of z and N , respectively, with the same angular velocity. Therefore the scalar product $\langle \pi_z(m), p \rangle$ is constant. That is,

$$K := (m, p) \rightarrow \langle \pi_z(m), p \rangle = ux + vy$$

commutes with J : $\{K, J\} = 0$. Now we define

$$H_t := (1 - 2t)(N - z) + tK, \quad \text{and} \quad \Phi_t := (J, H_t).$$

When $t = 0$, $\Phi_0 = (N + z, N - z)$ defines an effective \mathbb{T}^2 action and hence is *toric*. The moment polygon is depicted in Fig. 6. Notice that Φ_0 is affinely equivalent to the momentum map (z, N) in which the variables are “separated”, or “uncoupled”. Physically it describes a classical spin and a harmonic oscillator. Hence the name we gave to this example (but it probably deserves a better one). The particular linear scaling $(1 - 2t$ and $t)$ is not important; it is just chosen in such a way that the spectrum of the linearised Hamiltonian at the focus-focus point is very simple.

Proposition 6.1 1. For $t \in \mathbb{R} \setminus \{1/3, 1\}$ the momentum map Φ_t is semi-toric;

2. for $t < 1/3$ and $t > 1$ the momentum map Φ_t is actually of toric type (in the sense of definition 2.1);
3. for $t \in (1/3, 1)$ the momentum map Φ_t is semi-toric with one simple focus-focus point;
4. for $t \in \{1/3, 1\}$ the momentum map Φ_t has a degenerate singularity (and hence is not almost-toric).

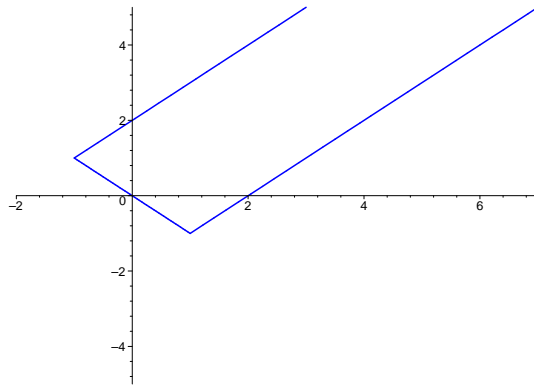


Figure 6: The standard moment polytope at $t = 0$ for example 6.2.

If one needs only an example with one focus-focus point, the simplest of course is to take $t = 1/2$ or $\Phi = (J, K)$.

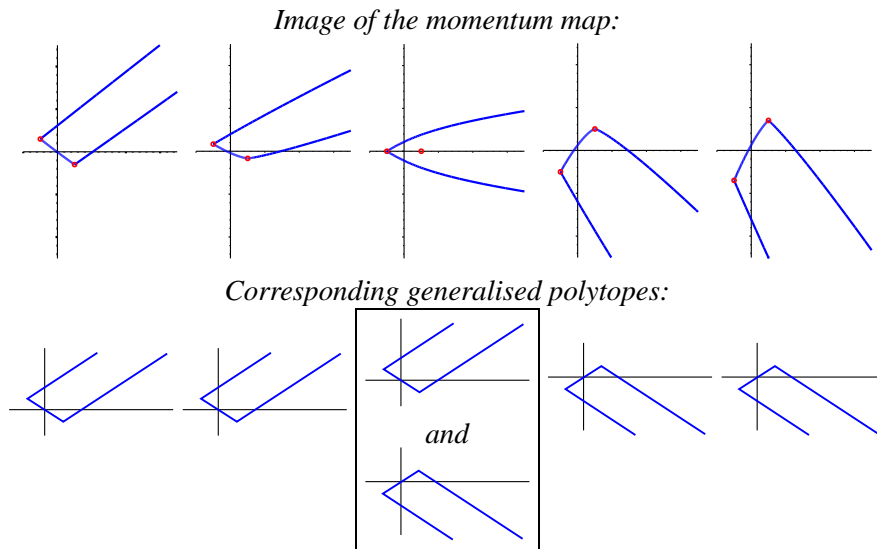


Figure 7: Bifurcation of the image of the momentum map of example 6.2. Here $t = 0, 1/3, 1/2, 1$ and 1.2 .

Proof. It is clear that J defines a proper S^1 action on M . It remains to find the singularities of Φ and compute the spectrum of the linearised Hamiltonians; we leave the details to the reader. For instance, the two critical points of rank zero are $A_t = (-1, 1 - 2t)$ and $B_t = (1, -1 + 2t)$. The spectrum of the linearisation of H_t at A_t is composed of two purely imaginary eigenvalues of multiplicity two $\pm i\sqrt{5t^2 - 4t + 1}$ and hence A_t is always elliptic-elliptic, whereas the spectrum at B_t is composed of two eigenvalues of multiplicity two $\pm\sqrt{-3t^2 + 4t - 1}$, which are

real if and only if $t \in [1/3, 1]$. In each 2-dimensional eigenspace the eigenvalues of J are $\pm i$. Hence B_t is elliptic-elliptic for $t < 1/3$ and $t > 1$ and focus-focus for $t \in (1/3, 1)$. \square

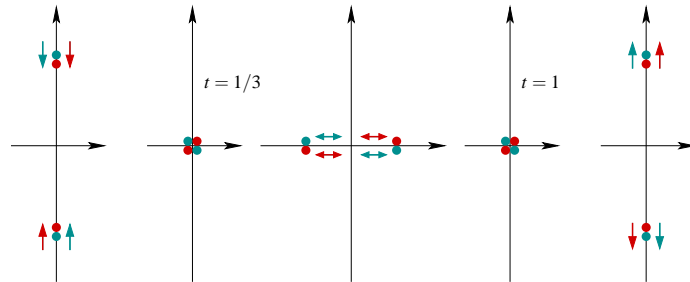


Figure 8: bifurcation of the spectrum of the linearisation of H_t at B_t

For $t \in (1/3, 1)$ we have two generalised polygons. Notice that since Φ_t depends continuously on t while the polygons are rational and hence locally constant, they actually don't depend on $t \in (1/3, 1)$. This of course is also a consequence of the description in terms of fixed point data. At the south pole (elliptic-elliptic point) the isotropy weights for J are $(1, 1)$ and at the north pole (focus-focus point) the isotropy weights are $(1, -1)$. We deduce that the generalised polygons are the one in figure 6 and its mirror image with respect to the horizontal axis.

7 Final remarks

The construction of the moment polygons for semi-toric momentum maps was originally motivated by a question of Zhiilinskiĭ about the redistribution of semi-classical eigenvalues in one-parameter families of quantum Hamiltonian systems. Some hints were given in the very interesting article [17], where the example 6.1 mentioned above was studied from different viewpoints. In an article in preparation [19] I give an answer to Zhiilinskiĭ's question in the semi-toric framework. The moment polygons are a very natural and efficient tool for proving and stating the result. Roughly speaking, it is shown first using a global version of Bohr-Sommerfeld rules that the number of eigenvalues in each "polyad" is given in terms of the Duistermaat-Heckman measure for the Hamiltonian H . Secondly, the bifurcation of the system as the parameter varies is interpreted in terms of an action of the group G on the initial moment polygon, which gives a geometric formula for the variation of the Duistermaat-Heckman measure.

Finally I would like to point out that I did not consider in this article "inverse questions" such as which polygons can show up and to what extent a given class of polygons determines the symplectic manifold with momentum map Φ . I hope to return on these problems in a future article, using the invariants of focus-focus foliations of [23]. However it is easy to see using the classification by Karshon [10]

that in case M is compact, a given polygon uniquely determines M with the S^1 -momentum map J . In particular this shows that M always admit a Kähler structure. But it is not always possible to find a \mathbb{T}^2 momentum map extending J . In view of Zhilinskiĭ's problem this issue is not particularly interesting because the initial Lagrangian foliation would in general be completely different from the toric one that one could possibly construct (focus-focus leaves do not appear in toric foliations). For instance it is true that $S^2 \times S^2$ (in example 6.1) is toric, but this does not help understanding the redistribution problem, whereas the polygons of theorem 3.8 contain all the information we need.

Acknowledgements

This article was originally strongly motivated by lively discussions with Boris Zhilinskiĭ. A first version was developed under the auspices of regretted European network MASIE. Later on I had the chance to get many useful comments by Margaret Symington, which improved a lot the manuscript. I wish therefore to express all my gratitude to Boris and Margaret, and to Mark Robert, responsible for MASIE business.

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