GEOMETRY AND SPECTRUM IN 2D MAGNETIC WELLS

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Abstract. This paper is devoted to the classical mechanics and spectral analysis of a pure magnetic Hamiltonian in $\mathbb{R}^2$. It is established that both the dynamics and the semiclassical spectral theory can be treated through a Birkhoff normal form and reduced to the study of a family of one dimensional Hamiltonians. As a corollary, recent results by Helffer-Kordyukov are extended to higher energies.

1. Introduction

We consider in this article a charged particle in $\mathbb{R}^2$ moving under the action of a non-vanishing, time-independent magnetic field which is orthogonal to the plane. We will study both the classical and quantum (non relativistic) cases, in a regime where the energy is low but the magnetic field is strong.

This problem has given rise to many semiclassical investigations in the last fifteen years. Most of them are motivated by the study of the Ginzburg-Landau functional and its third critical field H_{C_3} which can be related to the lowest eigenvalue of the magnetic Laplacian (see [9]). Many cases involving the geometry of the possible boundary and the variations of the magnetic field have been analyzed (see [25, 15, 16, 17, 12, 27, 10, 13, 14]). Due to the initial motivation, most of the papers provide only asymptotic expansions of the lowest eigenvalue and do not provide the corresponding approximation for the eigenfunctions. The only paper which explicitly tackles the approximation of the eigenfunctions and their microlocal properties is [8], where the authors combine pseudo-differential techniques and a Grushin reduction. More recently, the contributions [28, 7, 26] display that the magnetic 2-form and the geometry combine in the semiclassical limit to produce very fine microlocalization properties for the eigenfunctions. In particular, it is shown, in various geometric and magnetic settings, that a normal form procedure can reveal a double scale structure of the magnetic Laplacian, which is reminiscent of the famous Born-Oppenheimer approximation. It also established that an effective electric operator generates asymptotic series for the lowest eigenpairs. Such results suggest the fact that a full Birkhoff normal form analysis in the spirit of [34, 3, 35] could be implemented for the magnetic Laplacian.

This is a remarkable fact that the Birkhoff procedure has never been implemented to enlighten the effect of magnetic fields on the low lying eigenvalues of the magnetic Laplacian. A reason might be that, compared to the case of a Schrödinger operator with an electric potential, the magnetic case presents a major difficulty: the symbol itself is not enough to confine the dynamics in a compact set. Therefore, it is not possible to start with a simple harmonic approximation at the principal level. This difficulty can be seen in the recent papers by Helffer and Kordyukov [13] (dimension two) and [14] (dimension three) which treat cases without boundary. In dimension three they provide accurate constructions of quasimodes, but do not establish the asymptotic

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expansions of the eigenvalues which is still an open problem. In dimension two, they prove that if the magnetic field has a unique and non-degenerate minimum, the j-th eigenvalue admits an expansion in powers of $\hbar^{1/2}$ of the form:

$$\lambda_j(\hbar) \sim \hbar \min_{q\in\mathbb{R}^2} B(q) + \hbar^2(c_1(2j-1)+c_0) + O(\hbar^{5/2}),$$

where c_0 and c_1 are constants depending on the magnetic field. In this paper, we extend their result by obtaining a complete asymptotic expansion — without odd powers of $\hbar^{1/2}$ (see Corollary 1.7)— which actually applies to more general magnetic wells — see for instance Corollary 1.8.

Let us describe now the methods and results of the paper. As we shall recall below, a particle in a magnetic field has a fast rotating motion, coupled to a slow drift. It is of course expected that the long-time behaviour of the particle is governed by this drift. We show in this article that it is indeed the case, and that the drift motion can be obtained by a one degree of freedom Hamiltonian system, both in the classical or the quantum setting. What's more, the effective Hamiltonian is, for small energies, approximated by the magnetic field itself.

In order to achieve this, we obtain a normal form that explicitly reduces the study of the original system to a one degree of freedom Hamiltonian. In the classical case, this gives an approximation of the dynamics for long times, of order $\mathcal{O}(1/E^\infty)$, where E is the energy. In the quantum case, this gives a complete asymptotic expansion of the eigenvalues up to $\mathcal{O}(\hbar^\infty)$, where $\hbar$ is the semiclassical parameter (Planck's constant).

Classical dynamics. Let (e_1, e_2, e_3) be an orthonormal basis of $\mathbb{R}^3$. Our configuration space is $\mathbb{R}^2 = \{q_1e_1 + q_2e_2;\ (q_1, q_2) \in \mathbb{R}^2\}$, and the magnetic field is $\vec{B} = B(q_1, q_2)e_3$. For the moment we only assume that $q := (q_1, q_2)$ belongs to an open set Ω where B does not vanish.

With appropriate constants, Newton's equation for the particle under the action of the Lorentz force writes

$$\ddot{q} = 2\dot{q} \wedge \vec{B}. \tag{1.1}$$

The kinetic energy $E = \frac{1}{4}\|\dot{q}\|^2$ is conserved. If the speed $\dot{q}$ is small, we may linearize the system, which amounts to have a constant magnetic field. Then, as is well known, the integration of Newton's equations gives a circular motion of angular velocity $\dot{\theta} = -2B$ and radius $\|\dot{q}\|/2B$. Thus, even if the norm of the speed is small, the angular velocity may be very important. Now, if B is in fact not constant, then after a while, the particle may leave the region where the linearization is meaningful. This suggests a separation of scales, where the fast circular motion is superposed with a slow motion of the center (Figure 1).

It is known that the system (1.1) is Hamiltonian. In fact, the Hamiltonian is simply the kinetic energy, but the definition of the phase space requires the introduction of a magnetic potential. Let $\mathbf{A} \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ such that

$$\vec{B} = \nabla \wedge \mathbf{A}.$$

We may identify $\mathbf{A} = (A_1, A_2)$ with the 1-form $A = A_1dq_1 + A_2dq_2$. Then, as a differential 2-form, $dA = (\frac{\partial A_2}{\partial q_1} - \frac{\partial A_1}{\partial q_2})dq_1 \wedge dq_2 = Bdq_1 \wedge dq_2$. Thus, by Poincaré lemma we see that, given any smooth magnetic function $B(q_1, q_2)$, such a potential $\mathbf{A}$ always exists.

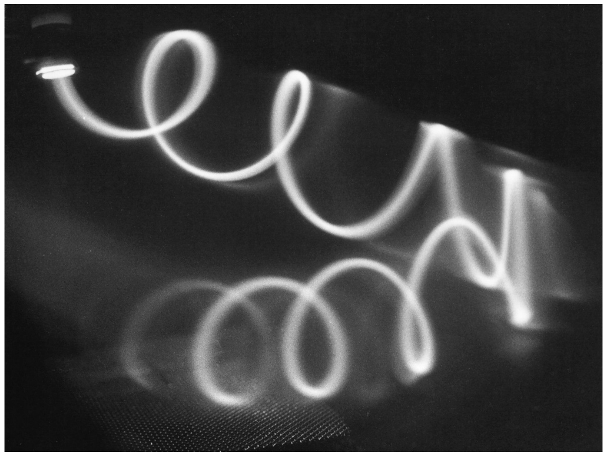

FIGURE 1. This photograph shows the motion of an electron beam in a non-uniform magnetic field. One can clearly see the fast rotation coupled with a drift. In the magnetic literature, the turning point (here on the right), due to the projection of the phase space motion onto the position space, is called a *mirror point*. Credits: Prof. Reiner Stenzel, http://www.physics.ucla.edu/plasma-exp/beam/BeamLoopyMirror.html

In terms of canonical variables $(q,p) \in T^*\mathbb{R}^2 = \mathbb{R}^4$ the Hamiltonian of our system is

$$H(q,p) = \|p - A(q)\|^2 . \tag{1.2}$$

We use here the Euclidean norm on $\mathbb{R}^2$, which allows the identification of $\mathbb{R}^2$ with $(\mathbb{R}^2)^*$ by

$$\forall (v,p) \in \mathbb{R}^2 \times (\mathbb{R}^2)^*, \qquad p(v) = \langle p, v\rangle. \tag{1.3}$$

Thus, the canonical symplectic structure ω on $T^*\mathbb{R}^2$ is given by

$$\omega((Q_1,P_1),(Q_2,P_2)) = \langle P_1, Q_2\rangle - \langle P_2, Q_1\rangle. \tag{1.4}$$

It is easy to check that Hamilton's equations for H imply Newton's equation (1.1). In particular, through the identification (1.3) we have $\dot{q} = 2(p - A)$.

Main results. We can now state our main results. The starting point is to consider large time classical dynamics. Indeed, while it is quite easy (and well known) to find an approximation of the dynamics for finite time, the large time problem has to face the issue that the conservation of the energy H is not enough to confine the trajectories in a compact set: the set $H^{-1}(E)$ is not bounded.

The first result shows the existence of a smooth symplectic diffeomorphism that transforms the initial Hamiltonian into a normal form, up to any order in the distance to the zero energy surface.

Theorem 1.1. *Let*

$$H(q,p) := \|p - \mathbf{A}(q)\|^2 , \quad (q,p) \in T^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2,$$

where the magnetic potential $\mathbf{A} : \mathbb{R}^2 \to \mathbb{R}^2$ *is smooth. Let* $B := \frac{\partial A_2}{\partial q_1} - \frac{\partial A_1}{\partial q_2}$ *be the corresponding magnetic field. Let* $\Omega \subset \mathbb{R}^2$ *be an open set where* B *does not vanish. Then there exists a symplectic diffeomorphism* Φ*, defined in an open set* $\tilde{\Omega} \subset \mathbb{C}_{z_1} \times \mathbb{R}^2_{z_2}$*, with values in* $T^*\mathbb{R}^2$*, which sends the plane* $\{z_1 = 0\}$ *to the surface* $\{H = 0\}$*, and such that*

$$H \circ \Phi = |z_1|^2 f(z_2, |z_1|^2) + \mathcal{O}(|z_1|^\infty), \tag{1.5}$$

where $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ *is smooth. Moreover, the map*

$$\varphi : \Omega \ni q \mapsto \Phi^{-1}(q, \mathbf{A}(q)) \in (\{0\} \times \mathbb{R}^2_{z_2}) \cap \tilde{\Omega} \tag{1.6}$$

is a local diffeomorphism and

$$f \circ (\varphi(q), 0) = |B(q)| .$$

Note that similar results were discovered by Littlejohn [22] (who used non-canonical coordinates but kept track of the modifications of the symplectic form); actually Littlejohn was one of the first to emphasize the role of Hamiltonian techniques for physics calculations.

In the following theorem we denote by $K = |z_1|^2 f(z_2, |z_1|^2) \circ \Phi^{-1}$ the (completely integrable) normal form of H given be Theorem 1.1 above. Let φ^t_H be the Hamiltonian flow of H, and let φ^t_K be the Hamiltonian flow of K. Since K has separated variables, it is easy to compute its flow. The following result ensures that φ^t_K is a very good approximation to φ^t_H for large times.

Theorem 1.2. *Assume that the magnetic field $B > 0$ is confining: there exists $C > 0$ and $M > 0$ such that $B(q) \geqslant C$ if $\|q\| \geqslant M$. Let $C_0 < C$. Then*

1. *The flow φ^t_H is uniformly bounded for all starting points (q, p) such that $B(q) \leqslant C_0$ and $H(q, p) = \mathcal{O}(\epsilon)$ and for times of order $\mathcal{O}(1/\epsilon^N)$, where N is arbitrary.*
2. *Up to a time of order $T_\epsilon = \mathcal{O}(|\ln \epsilon|)$, we have*

$$\left\|\varphi^t_H(q, p) - \varphi^t_K(q, p)\right\| = \mathcal{O}(\epsilon^\infty) \tag{1.7}$$

for all starting points (q, p) such that $B(q) \leqslant C_0$ and $H(q, p) = \mathcal{O}(\epsilon)$.

It is interesting to notice that, if one restricts to regular values of B, one obtains the same control for a much longer time, as stated below.

Theorem 1.3. *Under the same confinement hypothesis as Theorem 1.2, let $J \subset (0, C_0)$ be a closed interval such that dB does not vanish on $B^{-1}(J)$. Then up to a time of order $T = \mathcal{O}(1/\epsilon^N)$, for an arbitrary $N > 0$, we have*

$$\left\|\varphi^t_H(q, p) - \varphi^t_K(q, p)\right\| = \mathcal{O}(\epsilon^\infty)$$

for all starting points (q, p) such that $B(q) \in J$ and $H(q, p) = \mathcal{O}(\epsilon)$.

It is possible that the longer time $T = \mathcal{O}(1/\epsilon^N)$ reached in (1.7) could apply as well for some types of singularities of B; this seems to be an open question at the moment.

We may now describe the magnetic dynamics in terms of a fast rotating motion with a slow drift. In order to do this, we introduce the adiabatic action

$$I := |z_1|^2 = \int_\gamma p dq,$$

where γ is the loop corresponding to the fast motion (which we can obtain by using a local approximation by a constant magnetic field). Since $\{I, K\} = 0$, I is a constant of motion for the flow φ^t_K. Moreover, the Hamiltonian flow of I generates a 2π-periodic S^1 action on the level set $\{I = \text{const}\}$. For $I \neq 0$, the reduced symplectic manifold $\Sigma_I := \{I = \text{const}\}/S^1$ may be identified with $\Sigma := I^{-1}(0) = H^{-1}(0)$, endowed with the symplectic form $d\xi_2 \wedge dx_2$. (As we shall see in Lemma 2.1 below, we may also identify Σ with $\mathbb{R}^2_{(q_1,q_2)}$ endowed with the symplectic form $Bdq_1 \wedge dq_2$.) Then, for each value of I, the function K defines a Hamiltonian h_I on Σ:

$$h_I(z_2) := If(z_2, I).$$

In the next statement, we assume that B is confining and we denote by $T(\epsilon)$ the time given by Theorems 1.2 or 1.3, depending on the initial value of B. In view of the fact that the Hamiltonian vector field of K splits into the sum of commuting vector fields

$$\mathcal{X}_K = f\mathcal{X}_I + I\mathcal{X}_{f(z_2,I)},$$

we immediately obtain the following corollary, which is illustrated by Figure 2.

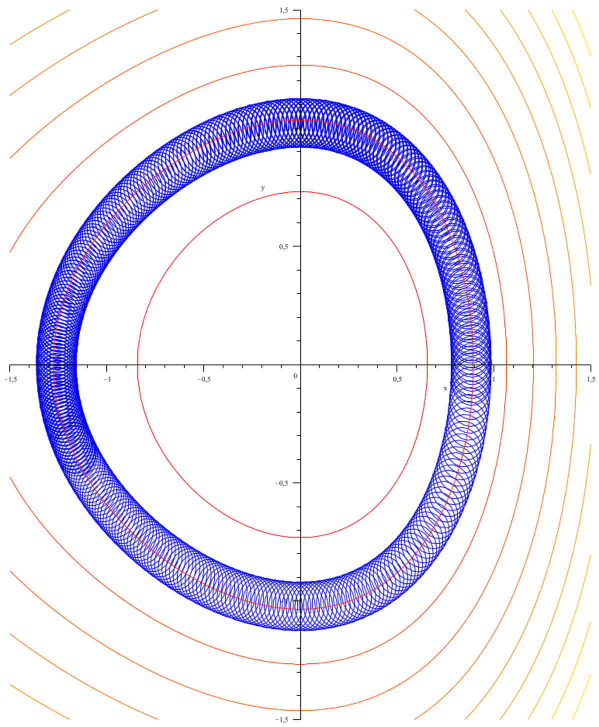

FIGURE 2. Numerical simulation of the flow of H when the magnetic field is given by $B(x,y) = 2 + x^2 + y^2 + \frac{x^3}{3} + \frac{x^4}{20}$, and $\epsilon = 0.05$, $t \in [0, 500]$. The picture also displays in red some level sets of B.

Corollary 1.4 (fast/slow decomposition)**.** *Let $N > 0$. There exists a small energy $E_0 > 0$ such that, for all $E < E_0$, for times $t \leqslant T(E)$, the magnetic flow φ_H^t at kinetic energy $H = E$ is, up to an error of order $\mathcal{O}(E^\infty)$, the Abelian composition of two motions:*

- [fast rotating motion] *a periodic flow around the S^1-orbits, with frequency $\frac{1}{2\pi}\frac{\partial K}{\partial I}$;*
- [slow drift] *the Hamiltonian flow of h_I on $\Sigma \simeq \Sigma_I$.*

Thus, we can informally describe the motion as a coupling between a fast rotating motion around a center $c(t) \in H^{-1}(0)$ and a slow drift of the point $c(t)$. The rotating motion depends smoothly on E; in terms of the original variables (q_1, q_2), it has a small radius

$$r = \frac{\sqrt{E}}{B(q)} + \mathcal{O}(E^{3/2})$$

and a fast angular velocity

$$\dot{\theta} = -2B(q) + \mathcal{O}(E).$$

The motion of $c(t)$, up to an error of order $\mathcal{O}(E^\infty)$, is given by the effective 1D Hamiltonian h_I, depending smoothly on the adiabatic action I, of the form

$$h_I(x_2,\xi_2) = IB(q) + \mathcal{O}(I^2),$$

where q and $z_2 = (x_2, \xi_2)$ are related by (1.6). Notice that, at first order, the flow of h_I is given by the flow of IB; thus, modulo an error of order E^2, the trajectories follow the level sets of the magnetic field; Figure 2 gives a striking numerical evidence of this.

Under additional hypothesis on h_I, one can of course say much more. For instance, if h_I has no critical points at a given energy (as in Theorem 1.3), then the trajectories are diffeomorphic to circles; then we can introduce a second adiabatic invariant. In this case, it could be interesting to improve the estimates using KAM/Nekhoroshev methods.

We turn now to the quantum counterpart of these results. Let $\mathcal{H}_{\hbar,\mathbf{A}} = (-i\hbar\nabla - \mathbf{A})^2$ be the magnetic Laplacian on $\mathbb{R}^2$, where the potential $\mathbf{A} : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth, and such that $\mathcal{H}_{\hbar,\mathbf{A}} \in S(m)$ for some order function m on $\mathbb{R}^4$ (see [6, Chapter 7]). We will work with the Weyl quantization; for a classical symbol $a = a(x,\xi) \in S(m)$, it is defined as:

$$\mathsf{Op}_\hbar^w a\, \psi(x) = \frac{1}{(2\pi\hbar)^2} \int\int e^{i(x-y)\cdot\xi/\hbar} a\left(\frac{x+y}{2}, \xi\right) \psi(y)\,\mathrm{d}y\,\mathrm{d}\xi, \quad \forall\psi \in \mathcal{S}(\mathbb{R}^2).$$

The first result shows that the spectral theory of $\mathcal{H}_{\hbar,\mathbf{A}}$ is governed at first order by the magnetic field itself, viewed as a symbol.

Theorem 1.5. *Assume that the magnetic field B is non vanishing ($\Omega = \mathbb{R}^2$). Let $\mathcal{H}_\hbar^0 = \mathsf{Op}_\hbar^w(H^0)$, where $H^0 = B(\varphi^{-1}(z_2))|z_1|^2$ and the diffeomorphism φ is defined in (1.6). Then there exists a bounded classical pseudo-differential operator $Q_\hbar$, such that*

- *$Q_\hbar$ commutes with $\mathsf{Op}_\hbar^w(|z_1|^2)$;*
- *$Q_\hbar$ is relatively bounded with respect to $\mathcal{H}_\hbar^0$ with an arbitrarily small relative bound;*
- *its Weyl symbol is $O_{z_2}(\hbar^2 + \hbar\,|z_1|^2 + |z_1|^4)$,*

so that the following holds. Assume that the magnetic field is confining: there exist constants $\tilde{C}_1 > 0$, $M_0 > 0$ such that

$$B(q) \geqslant \tilde{C}_1 \quad \text{for} \quad |q| \geqslant M_0. \tag{1.8}$$

Let $0 < C_1 < \tilde{C}_1$. Then the spectra of $\mathcal{H}_{\hbar,\mathbf{A}}$ and $\mathcal{N}_\hbar := \mathcal{H}_\hbar^0 + Q_\hbar$ in $(-\infty, C_1\hbar]$ are discrete. We denote by $0 < \lambda_1(\hbar) \leqslant \lambda_2(\hbar) \leqslant \cdots$ the eigenvalues of $\mathcal{H}_{\hbar,\mathbf{A}}$ and by $0 < \mu_1(\hbar) \leqslant \mu_2(\hbar) \leqslant \cdots$ the eigenvalues of $\mathcal{N}_\hbar$. Then for all $j \in \mathbb{N}^*$ such that $\lambda_j(\hbar) \leqslant C_1\hbar$ and $\mu_j(\hbar) \leqslant C_1\hbar$, we have

$$|\lambda_j(\hbar) - \mu_j(\hbar)| = O(\hbar^\infty).$$

The proof of Theorem 1.5 relies on the following theorem, which provides in particular an accurate description of $Q_\hbar$. In the statement, we use the notation of Theorem 1.1; we recall that Σ is the zero set of the classical Hamiltonian H.

Theorem 1.6. *For $\hbar$ small enough there exists a Fourier Integral Operator $U_\hbar$ such that*

$$U_\hbar^* U_h = I + Z_\hbar, \qquad U_\hbar U_\hbar^* = I + Z'_\hbar,$$

where $Z_\hbar, Z'_\hbar$ are pseudo-differential operators that microlocally vanish in a neighborhood of $\tilde{\Omega} \cap \Sigma$, and

$$U_\hbar^* \mathcal{H}_{\hbar,\mathbf{A}} U_\hbar = \mathcal{I}_\hbar F_\hbar + R_\hbar, \tag{1.9}$$

where

(1) $\mathcal{I}_\hbar := -\hbar^2 \frac{\partial^2}{\partial x_1^2} + x_1^2$;

(2) $F_\hbar$ *is a classical pseudo-differential operator in* $S(m)$ *that commutes with* $\mathcal{I}_\hbar$;

(3) *For any Hermite function* $h_n(x_1)$ *such that* $\mathcal{I}_\hbar h_n = \hbar(2n-1)h_n$, *the operator* $F_\hbar^{(n)}$ *acting on* $L^2(\mathbb{R}_{x_2})$ *by*

$$h_n \otimes F_\hbar^{(n)}(u) = F_\hbar(h_n \otimes u)$$

is a classical pseudo-differential operator in $S_{\mathbb{R}^2}(m)$ *with principal symbol*

$$F^{(n)}(x_2, \xi_2) = B(q),$$

where $(0, x_2 + i\xi_2) = \varphi(q)$ *as in* (1.6);

(4) *Given any classical pseudo-differential operator* $D_\hbar$ *with principal symbol* d_0 *such that* $d_0(z_1, z_2) = c(z_2)|z_1|^2 + O(|z_1|^3)$, *and any* $N \geqslant 1$, *there exist classical pseudo-differential operators* $S_{\hbar,N}$ *and* K_N *such that:*

$$R_\hbar = S_{\hbar,N}(D_\hbar)^N + K_N + O(\hbar^\infty), \tag{1.10}$$

with K_N *compactly supported away from a fixed neighborhood of* $|z_1| = 0$.

(5) $\mathcal{I}_\hbar F_\hbar = \mathcal{N}_\hbar = \mathcal{H}_\hbar^0 + Q_\hbar$, *where* $\mathcal{H}_\hbar^0 = \mathsf{Op}_\hbar^w(H^0)$, $H^0 = B(\varphi^{-1}(z_2))|z_1|^2$, *and the operator* $Q_\hbar$ *is relatively bounded with respect to* $\mathcal{H}_\hbar^0$ *with an arbitrarily small relative bound.*

We recover the result of [13], adding the fact that no odd power of $\hbar^{1/2}$ can show up in the asymptotic expansion.

Corollary 1.7 (Low lying eigenvalues)**.** *Assume that* B *has a unique non-degenerate minimum. Then there exists a constant* c_0 *such that for any* j, *the eigenvalue* $\lambda_j(\hbar)$ *has a full asymptotic expansion in integral powers of* $\hbar$ *whose first terms have the following form:*

$$\lambda_j(\hbar) \sim \hbar \min B + \hbar^2(c_1(2j-1) + c_0) + O(\hbar^3),$$

with $c_1 = \frac{\sqrt{\det(B" \circ \varphi^{-1}(0))}}{2B \circ \varphi^{-1}(0)}$, *where the minimum of* B *is reached at* $\varphi^{-1}(0)$.

Proof. The first eigenvalues of $\mathcal{H}_{\hbar,A}$ are equal to $\hbar$ times the eigenvalues of $F_\hbar^{(1)}$ (in point (3) of Theorem 1.6). Since B has a non-degenerate minimum, the symbol of $F_\hbar^{(1)}$ has a non-degenerate minimum, and the spectral asymptotics of the low-lying eigenvalues for such a 1D pseudo-differential operator are well known. We get

$$\lambda_j(\hbar) \sim \hbar \min B + \hbar^2(c_1(2j-1) + c_0) + O(\hbar^3),$$

with $c_1 = \sqrt{\det(B \circ \varphi^{-1})"(0)/2}$. One can easily compute

$$c_1 = \frac{\sqrt{\det(B" \circ \varphi^{-1}(0))}}{2\,|\det(D\varphi^{-1}(0))|} = \frac{\sqrt{\det(B" \circ \varphi^{-1}(0))}}{2B \circ \varphi^{-1}(0)}.$$

□

Under reasonable assumptions on B, Theorems 1.6 and 1.5 should yield precise asymptotic expansions even in the regime of energies larger than $c\hbar$, where $c > \min B$. For instance, we obtain the following result.

Corollary 1.8 (Magnetic excited states)**.** *Let* $c < \tilde{C}_1$ *be a regular value of* B, *and assume that the level set* $B^{-1}(c)$ *is connected. Then there exists* $\epsilon > 0$ *such that the eigenvalues of the magnetic Laplacian lying in the interval* $[\hbar(c-\epsilon), \hbar(c+\epsilon)]$ *have the form*

$$\lambda_j(\hbar) = (2n-1)\hbar f_\hbar(\hbar n(j), \hbar k(j)) + O(\hbar^\infty), \quad (n(j), k(j)) \in \mathbb{Z}^2,$$

where $f_\hbar = f_0 + \hbar f_1 + \cdots$ *admits an asymptotic expansion in powers of* $\hbar$ *with smooth coefficients* $f_i \in C^\infty(\mathbb{R}^2;\mathbb{R})$ *and* $\partial_1 f_0 = 0$, $\partial_2 f_0 \neq 0$. *Moreover, the corresponding eigenfunctions are microlocalized in the annulus* $B^{-1}([c-\epsilon, c+\epsilon])$.

In particular, if $n = 1$ *and* $c \in (\min B, 3\min B)$*, the eigenvalues of the magnetic Laplacian in the interval* $[\hbar(c-\epsilon), \hbar(c+\epsilon)]$ *have gaps of order* $O(\hbar^2)$.

Proof. As before, the spectrum of $\mathcal{H}_{\hbar,A}$ below $C_1\hbar$ is the union of the eigenvalues below $C_1\hbar$ of $(2n-1)\hbar F_\hbar^{(n)}$, $n \in \mathbb{N}^*$. For each n, the usual Bohr-Sommerfeld rules for 1D semiclassical pseudo-differential operators (see for instance [33] and the references therein) state that the eigenvalues of $F_\hbar^{(n)}$ in the interval $[c-\epsilon, c+\epsilon]$ admit a complete asymptotic expansion of the form

$$f_0^{(n)}(\hbar j) + \hbar f_1^{(n)}(\hbar j) + \cdots,$$

where $f_0^{(n)}, f_1^{(n)}, \dots,$ are smooth functions and $f_0^{(n)} = f_0$ does not depend on n and satisfies $(f_0^{(n)})' \neq 0$ (precisely, $2\pi f_0^{-1}(c)$ is the area of the curve $B^{-1}(c)$ viewed in Σ, up to a constant). □

• *Comments on Theorem 1.6.* When finishing to write this paper, we discovered that Theorem 1.6 appears in a close form in [19, Theorem 6.2.7]. However, several differences have to be mentioned. Our proof uses a deformation argument *à la* Moser which relies on a global symplectic parameterization of Σ and an intrinsic description of the symplectic normal bundle $N\Sigma$. Both the classical and quantum Birkhoff normal forms are obtained simultaneously by endowing the space of formal series with the semiclassical Weyl product, instead of the usual product. Actually, the particular grading in $(z_1, \hbar)$ that we use is tightly linked to the physical nature of the problem. The result itself is different since we obtain a uniform remainder $R_\hbar$ which vanishes to any order in that grading.

• *Higher dimensions.* In [14], the asymptotic expansion of the eigenvalues is not proved. We believe that the methods presented in our paper are likely to apply in their context and should help prove their conjecture.

Organization of the paper. The paper is organized as follows. Section 2 is devoted to the proof of Theorems 1.1 and 1.6. Then, we prove Theorems 1.2 and 1.3 in Section 3. Finally in Section 4 we provide the proof of Theorem 1.5.

2. Magnetic Birkhoff normal form

In this section we prove Theorem 1.1.

2.1. Symplectic normal bundle of Σ.

We introduce the submanifold of all particles at rest ($\dot q = 0$):

$$\Sigma := H^{-1}(0) = \{(q,p); \qquad p = A(q)\}.$$

Since it is a graph, it is an embedded submanifold of $\mathbb{R}^4$, parameterized by $q \in \mathbb{R}^2$.

Lemma 2.1. Σ *is a symplectic submanifold of* $\mathbb{R}^4$. *In fact,*

$$j^*\omega_{\restriction\Sigma} = dA \simeq B,$$

where $j : \mathbb{R}^2 \to \Sigma$ *is the embedding* $j(q) = (q, A(q))$.

Proof. We compute $j^*\omega = j^*(dp_1 \wedge dq_1 + dp_2 \wedge dq_2) = (-\frac{\partial A_1}{\partial q_2} + \frac{\partial A_2}{\partial q_1})dq_1 \wedge dq_2 \neq 0.$ □

Since we are interested in the low energy regime, we wish to describe a small neighborhood of Σ in $\mathbb{R}^4$, which amounts to understanding the normal symplectic bundle of Σ. For any $q \in \Omega$, we denote by $T_q\mathbf{A} : \mathbb{R}^2 \to \mathbb{R}^2$ the tangent map of $\mathbf{A}$. Then of course the vectors $(Q, T_q\mathbf{A}(Q))$, with $Q \in T_q\Omega = \mathbb{R}^2$, span the tangent space $T_{j(q)}\Sigma$. It is interesting to notice that the symplectic orthogonal $T_{j(q)}\Sigma^{\perp}$ is very easy to describe as well.

Lemma 2.2. *For any $q \in \Omega$, the vectors*

$$u_1 := \frac{1}{\sqrt{|B|}}(e_1, {}^tT_q\mathbf{A}(e_1)); \quad v_1 := \frac{\sqrt{|B|}}{B}(e_2, {}^tT_q\mathbf{A}(e_2))$$

form a symplectic basis of $T_{j(q)}\Sigma^{\perp}$.

Proof. Let $(Q_1, P_1) \in T_{j(q)}\Sigma$ and (Q_2, P_2) with $P_2 = {}^tT_q\mathbf{A}(Q_2)$. Then from (1.4) we get

$$\begin{aligned}\omega((Q_1, P_1), (Q_2, P_2)) &= \langle T_q\mathbf{A}(Q_1), Q_2\rangle - \langle {}^tT_q\mathbf{A}(Q_2), Q_1\rangle \\ &= 0.\end{aligned}$$

This shows that u_1 and v_1 belong to $T_{j(q)}\Sigma^{\perp}$. Finally

$$\begin{aligned}\omega(u_1, v_1) &= \frac{1}{B}\left(\langle {}^tT_q\mathbf{A}(e_1), e_2\rangle - \langle {}^tT_q\mathbf{A}(e_2), e_1\rangle\right) \\ &= \frac{1}{B}\langle e_1, (T_q\mathbf{A} - {}^tT_q\mathbf{A})(e_2)\rangle \\ &= \frac{1}{B}\langle e_1, \vec{B} \wedge e_2\rangle = -\frac{B}{B}\langle e_1, e_1\rangle = -1.\end{aligned}$$

□

Thanks to this lemma, we are able to give a simple formula for the transversal Hessian of H, which governs the linearized (fast) motion:

Lemma 2.3. *The transversal Hessian of H, as a quadratic form on $T_{j(q)}\Sigma^{\perp}$, is given by*

$$\forall q \in \Omega, \forall (Q, P) \in T_{j(q)}\Sigma^{\perp}, \quad d_q^2 H((Q, P)^2) = 2\|Q \wedge \vec{B}\|^2.$$

Proof. Let $(q, p) = j(q)$. From (1.2) we get

$$dH = 2\langle p - A, dp - T_q\mathbf{A} \circ dq\rangle.$$

Thus

$$d^2H((Q, P)^2) = 2\|(dp - T_q\mathbf{A} \circ dq)(Q, P)\|^2 + \langle p - A, M((Q, P)^2)\rangle,$$

and it is not necessary to compute the quadratic form M, since $p - A = 0$. We obtain

$$\begin{aligned}d^2H((Q, P)^2) &= 2\|P - T_q\mathbf{A}(Q)\|^2 \\ &= 2\|({}^tT_q\mathbf{A} - T_q\mathbf{A})(Q)\|^2 = 2\|Q \wedge \vec{B}\|^2.\end{aligned}$$

□

We may express this Hessian in the symplectic basis (u_1, v_1) given by Lemma 2.2:

$$d^2 H_{\restriction T_{j(q)}\Sigma^\perp} = \begin{pmatrix} 2\,|B| & 0 \\ 0 & 2\,|B| \end{pmatrix}. \tag{2.1}$$

Indeed, $\|e_1 \wedge \vec{B}\|^2 = B^2$, and the off-diagonal term is $\frac{1}{B}\langle e_1 \wedge \vec{B}, e_2 \wedge \vec{B}\rangle = 0$.

2.2. **Proof of Theorem 1.1.** We use the notation of the previous section. We endow $\mathbb{C}_{z_1} \times \mathbb{R}^2_{z_2}$ with canonical variables $z_1 = x_1 + i\xi_1$, $z_2 = (x_2, \xi_2)$, and symplectic form $\omega_0 := d\xi_1 \wedge dx_1 + d\xi_2 \wedge dx_2$. By Darboux's theorem, there exists a diffeomorphism $g : \Omega \to g(\Omega) \subset \mathbb{R}^2_{z_2}$ such that $g(q_0) = 0$ and

$$g^*(d\xi_2 \wedge dx_2) = j^*\omega.$$

(We identify g with φ in the statement of the theorem.) In other words, the new embedding $\tilde{\jmath} := j \circ g^{-1} : \mathbb{R}^2 \to \Sigma$ is symplectic. In fact we can give an explicit choice for g by introducing the global change of variables:

$$x_2 = q_1, \quad \xi_2 = \int_0^{q_2} B(q_1, s)\,\mathrm{d}s.$$

Consider the following map $\tilde{\Phi}$:

$$\mathbb{C} \times \Omega \xrightarrow{\tilde{\Phi}} N\Sigma \tag{2.2}$$

$$(x_1 + i\xi_1, z_2) \mapsto x_1 u_1(z_2) + \xi_1 v_1(z_2), \tag{2.3}$$

where $u_1(z_2)$ and $v_1(z_2)$ are the vectors defined in Lemma 2.2 with $q = g^{-1}(z_2)$. This is an isomorphism between the normal symplectic bundle of $\{0\} \times \Omega$ and $N\Sigma$, the normal symplectic bundle of Σ: indeed, Lemma 2.2 says that for fixed z_2, the map $z_1 \mapsto \tilde{\Phi}(z_1, z_2)$ is a linear symplectic map. This implies, by a general result of Weinstein [36], that there exists a symplectomorphism Φ from a neighborhood of $\{0\} \times \Omega$ to a neighborhood of $\tilde{\jmath}(\Omega) \subset \Sigma$ whose differential at $\{0\} \times \Omega$ is equal to $\tilde{\Phi}$. Let us recall how to prove this.

First, we may identify $\tilde{\Phi}$ with a map into $\mathbb{R}^4$ by

$$\tilde{\Phi}(z_1, z_2) = \tilde{\jmath}(z_2) + x_1 u_1(z_2) + \xi_1 v_1(z_2).$$

Its Jacobian at $z_1 = 0$ in the canonical basis of $T_{z_1}\mathbb{C} \times T_{z_2}\Omega = \mathbb{R}^4$ is a matrix with column vectors $[u_1, v_1, T_{z_2}\tilde{\jmath}(e_1), T_{z_2}\tilde{\jmath}(e_2)]$, which by Lemma 2.2 is a basis of $\mathbb{R}^4$: thus $\tilde{\Phi}$ is a local diffeomorphism at every $(0, z_2)$. Therefore if $\epsilon > 0$ is small enough, $\tilde{\Phi}$ is a diffeomorphism of $B(\epsilon) \times \Omega$ into its image.

($B(\epsilon) \subset \mathbb{C}$ is the open ball of radius ϵ).

Since $\tilde{\jmath}$ is symplectic, Lemma 2.2 implies that the basis $[u_1, v_1, T_{z_2}\tilde{\jmath}(e_1), T_{z_2}\tilde{\jmath}(e_2)]$ is symplectic in $\mathbb{R}^4$; thus the Jacobian of $\tilde{\Phi}$ on $\{0\} \times \Omega$ is symplectic. This in turn can be expressed by saying that the 2-form

$$\omega_0 - \tilde{\Phi}^*\omega_0$$

vanishes on $\{0\} \times \Omega$.

Lemma 2.4. *Let us consider ω_0 and ω_1 two 2-forms on $\mathbb{R}^4$ which are closed and non degenerate. Let us assume that $\omega_{1|\hat{x}_1=0} = \omega_{0|\hat{x}_1=0}$. There exist a neighborhood of $(0,0,0,0)$ and a change of coordinates ψ_1 such that:*

$$\psi_1^*\omega_1 = \omega_0 \quad \text{and} \quad \psi_1 = \mathsf{Id} + O(\hat{x}_1^2).$$

Proof. The proof of this relative Darboux lemma is standard but we recall it for completeness (see [24, p. 92]).

• *Relative Poincaré Lemma.* Let us begin to recall how we can find a 1-form σ defined in a neighborhood of $\hat{z}_1 = 0$ such that:

$$\tau := \omega_1 - \omega_0 = d\sigma \quad \text{and} \quad \sigma = O(\hat{x}_1^2).$$

We introduce the family of diffeomorphisms $(\phi_t)_{0<t\leqslant 1}$ defined by:

$$\phi_t(\hat{x}_1, \hat{x}_2, \hat{\xi}_1, \hat{\xi}_2) = (t\hat{x}_1, \hat{x}_2, \hat{\xi}_1, \hat{\xi}_2)$$

and we let:

$$\phi_0(\hat{x}_1, \hat{x}_2, \hat{\xi}_1, \hat{\xi}_2) = (0, \hat{x}_2, \hat{\xi}_1, \hat{\xi}_2).$$

We have:

$$\phi_0^*\tau = 0 \quad \text{and} \quad \phi_1^*\tau = \tau; \tag{2.4}$$

Let us denote by X_t the vector field associated with ϕ_t:

$$X_t = \frac{d\phi_t}{dt}(\phi_t^{-1}) = (t^{-1}\hat{x}_1, 0, 0, 0) = t^{-1}\hat{x}_1 e_1,$$

with $e_1 := (1, 0, 0, 0)$. Let us compute the Lie derivative of τ along X_t: $\frac{d}{dt}\phi_t^*\tau = \phi_t^*\mathcal{L}_{X_t}\tau$. From the Cartan formula, we have: $\mathcal{L}_{X_t} = \iota(X_t)d\tau + d(\iota(X_t)\tau)$. Since τ is closed on $\mathbb{R}^4$, we have $d\tau = 0$. Therefore it follows:

$$\frac{d}{dt}\phi_t^*\tau = d(\phi_t^*\iota(X_t)\tau). \tag{2.5}$$

We consider the 1-form $\sigma_t := \phi_t^*\iota(X_t)\tau = \hat{x}_1\tau_{\phi_t(\hat{x}_1,\hat{x}_2,\hat{\xi}_1,\hat{\xi}_2)}(e_1, \nabla\phi_t(\cdot)) = O(\hat{x}_1^2)$. We see from (2.5) that the map $t \mapsto \phi_t^*\tau$ is smooth on $[0,1]$. To conclude, let $\sigma = \int_0^1 \sigma_t\, dt$; it follows from (2.4) and (2.5) that:

$$\frac{d}{dt}\phi_t^*\tau = d\sigma_t \quad \text{and} \quad \tau = d\sigma.$$

• *Conclusion.* We use a standard deformation argument due to Moser. For $t \in [0,1]$, we let: $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$. The 2-form ω_t is closed and non degenerate (up to choosing a neighborhood of $\hat{z}_1 = 0$ small enough). We look for ψ_t such that:

$$\psi_t^*\omega_t = \omega_0.$$

For that purpose, let us determine a vector field Y_t such that:

$$\frac{d}{dt}\psi_t = Y_t(\psi_t).$$

By using again the Cartan formula, we get:

$$0 = \frac{d}{dt}\psi_t^*\omega_t = \psi_t^*\left(\frac{d}{dt}\omega_t + \iota(Y_t)d\omega_t + d(\iota(Y_t)\omega_t)\right).$$

This becomes:

$$\omega_0 - \omega_1 = d(\iota(Y_t)\omega_t).$$

We are led to solve:

$$\iota(Y_t)\omega_t = -\sigma.$$

By non degeneracy of ω_t, this determines Y_t. Choosing a neighborhood of $(0,0,0,0)$ small enough, we infer that ψ_t exists until the time $t = 1$ and that it satisfies $\psi_t^*\omega_t = \omega_0$. Since $\sigma = O(\hat{x}_1^2)$, we get $\psi_1 = \mathsf{Id} + O(\hat{x}_1^2)$. □

Lemma 2.5. *There exists a map $S : B(\epsilon)\times\Omega \to B(\epsilon)\times\Omega$, which is tangent to the identity along $\{0\}\times\Omega$, such that*

$$S^*\tilde{\Phi}^*\omega = \omega_0.$$

Proof. It is sufficient to apply Lemma 2.4 to $\omega_1 = \tilde{\Phi}^*\omega_0$. □

We let $\Phi := \tilde{\Phi}\circ S$; this is the claimed symplectic map.

Let us now analyze how the Hamiltonian H is transformed under Φ. The zero-set $\Sigma = H^{-1}(0)$ is now $\{0\}\times\Omega$, and the symplectic orthogonal $T_{\tilde{\jmath}(0,\hat{z}_2)}\Sigma^\perp$ is canonically equal to $\mathbb{C}\times\{\hat{z}_2\}$. By (2.1), the matrix of the transversal Hessian of $H\circ\Phi$ in the canonical basis of $\mathbb{C}$ is simply

$$d^2(H\circ\Phi)_{\upharpoonright\mathbb{C}\times\{\hat{z}_2\}} = d^2_{\Phi(0,\hat{z}_2)}H\circ(d\Phi)^2 = \begin{pmatrix} 2\left|B(g^{-1}(\hat{z}_2))\right| & 0 \\ 0 & 2\left|B(g^{-1}(\hat{z}_2))\right| \end{pmatrix}. \tag{2.6}$$

Therefore, by Taylor's formula in the $\hat{z}_1$ variable (locally uniformly with respect to the $\hat{z}_2$ variable seen as a parameter), we get

$$\begin{aligned} H\circ\Phi(\hat{z}_1,\hat{z}_2) &= H\circ\Phi_{\upharpoonright\hat{z}_1=0} + dH\circ\Phi_{\upharpoonright\hat{z}_1=0}(\hat{z}_1) + \frac{1}{2}d^2(H\circ\Phi)_{\upharpoonright\hat{z}_1=0}(\hat{z}_1^2) + \mathcal{O}(|\hat{z}_1|^3) \\ &= 0 + 0 + \left|B(g^{-1}(\hat{z}_2))\right||\hat{z}_1|^2 + \mathcal{O}(|\hat{z}_1|^3). \end{aligned}$$

In order to obtain the result claimed in the theorem, it remains to apply a formal Birkhoff normal form in the $\hat{z}_1$ variable, to simplify the remainder $\mathcal{O}(\hat{z}_1^3)$. This classical normal form is a particular case of the semiclassical normal form that we prove below (Proposition 2.7); therefore we simply refer to this proposition, and this finishes the proof of the theorem, where, for simplicity of notation, the variables (z_1,z_2) actually refer to $(\hat{z}_1,\hat{z}_2)$.

2.3. **Semiclassical Birkhoff normal form.** We follow the spirit of [3, 35]. In the coordinates $\hat{x}_1,\hat{\xi}_1,\hat{x}_2,\hat{\xi}_2$, the Hamiltonian takes the form:

$$\hat{H}(\hat{z}_1,\hat{z}_2) = H^0 + O(|\hat{z}_1|^3), \qquad \text{where } H^0 = B(g^{-1}(\hat{z}_2))|\hat{z}_1|^2. \tag{2.7}$$

Let us now consider the space of the formal power series in $\hat{x}_1,\hat{\xi}_1,\hbar$ with coefficients smoothly depending on $(\hat{x}_2,\hat{\xi}_2)$: $\mathcal{E} = \mathcal{C}^\infty_{\hat{x}_2,\hat{\xi}_2}[\hat{x}_1,\hat{\xi}_1,\hbar]$. We endow $\mathcal{E}$ with the Moyal product (compatible with the Weyl quantization) denoted by $\star$ and the commutator of two series κ_1 and κ_2 is defined as:

$$[\kappa_1,\kappa_2] = \kappa_1\star\kappa_2 - \kappa_2\star\kappa_1.$$

Notation 2.6. The degree of $\hat{x}_1^\alpha\hat{\xi}_1^\beta\hbar^l$ is $\alpha+\beta+2l$. $\mathcal{D}_N$ denotes the space of the monomials of degree N. $\mathcal{O}_N$ is the space of formal series with valuation at least N.

Proposition 2.7. *Given $\gamma\in\mathcal{O}_3$, there exist formal power series $\tau,\kappa\in\mathcal{O}_3$ such that:*

$$e^{i\hbar^{-1}\mathsf{ad}_\tau}(H^0+\gamma) = H^0+\kappa,$$

with: $[\kappa, H^0] = 0$.

Proof. Let $N\geqslant 1$. Assume that we have, for $N\geqslant 1$ and $\tau_N\in\mathcal{O}_3$:

$$e^{ih^{-1}\mathsf{ad}_{\tau_N}}(H^0+\gamma) = H^0 + K_3 + \cdots + K_{N+1} + R_{N+2} + \mathcal{O}_{N+3},$$

where $K_i\in\mathcal{D}_i$ commutes with $|\hat{z}_1|^2$ and where $R_{N+2}\in\mathcal{D}_{N+2}$.

Let $\tau' \in \mathcal{D}_{N+2}$. A computation provides:

$$e^{ih^{-1}\mathsf{ad}_{\tau_N+\tau'}}(H^0+\gamma) = H^0 + K_3 + \cdots + K_{N+1} + K_{N+2} + \mathcal{O}_{N+3},$$

with:

$$K_{N+2} = R_{N+2} + B(g^{-1}(\hat{z}_2))i\hbar^{-1}\mathsf{ad}_{\tau'}|\hat{z}_1|^2 = R_{N+2} - B(g^{-1}(\hat{z}_2))i\hbar^{-1}\mathsf{ad}_{|\hat{z}_1|^2}\tau'.$$

We can write:

$$R_{N+2} = K_{N+2} + B(g^{-1}(\hat{z}_2))i\hbar^{-1}\mathsf{ad}_{|\hat{z}_1|^2}\tau'.$$

Since $B(g^{-1}(\hat{z}_2)) \neq 0$, we deduce the existence of τ' and K_{N+2} such that K_{N+2} commutes with $|\hat{z}_1|^2$. Note that $i\hbar^{-1}\mathsf{ad}_{|\hat{z}_1|^2} = \{|\hat{z}_1|^2, \cdot\}$. □

2.4. **Proof of Theorem 1.6.** Since the formal series κ given by Proposition 2.7 commutes with H^0, we can write it as a polynomial in $|\hat{z}_1|^2$:

$$\kappa = \sum_{k\geqslant 0}\sum_{2l+m=k} \hbar^l c_{l,m}(\hat{z}_2)|\hat{z}_1|^{2m}.$$

This formal series can be reordered by using the monomials $(|\hat{z}_1|^2)^{\star m}$ for the product law $\star$:

$$\kappa = \sum_{k\geqslant 0}\sum_{2l+m=k} \hbar^l c^{\star}_{l,m}(\hat{z}_2)(|\hat{z}_1|^2)^{\star m}.$$

Thanks to the Borel lemma, there exists a smooth function with compact support $f^\star(\hbar, |\hat{z}_1|^2, \hat{z}_2)$ such that the Taylor expansion with respect to $(\hbar, |\hat{z}_1|^2)$ of $f^\star(\hbar, |\hat{z}_1|^2, \hat{z}_2)$ is given by κ and:

$$\sigma^{\mathsf{T},w}\left(\mathsf{Op}^w_\hbar\left(f^\star(\hbar,\mathcal{I},z_2)\right)\right) = \kappa, \tag{2.8}$$

where $\sigma^{\mathsf{T},w}$ means that we consider the formal Taylor series of the Weyl symbol with respect to $(\hbar, \hat{z}_1)$. The operator $\mathsf{Op}^w_\hbar\left(f^\star(\hbar,\mathcal{I},z_2)\right)$ has to be understood as the Weyl quantization with respect to $\hat{z}_2$ of an operator valued symbol. We can write it in the form:

$$\mathsf{Op}^w_\hbar f^\star(\hbar,\mathcal{I}_\hbar,\hat{z}_2) = \mathcal{I}_\hbar \mathsf{Op}^w_\hbar \tilde{f}^\star(\hbar,\mathcal{I}_\hbar,\hat{z}_2)$$

so that, up to choosing the support of $f^\star$ small enough, there exists η_0 such that for $\eta \in (0,\eta_0)$, we have, for all $\psi \in C^\infty_0(\mathbb{R}^2)$,

$$|\langle \mathsf{Op}^w_\hbar f^\star(\hbar,\mathcal{I}_\hbar,\hat{z}_2)\psi,\psi\rangle| \leqslant \eta\|\mathcal{I}_\hbar^{1/2}\psi\|^2. \tag{2.9}$$

Moreover we can also introduce a smooth symbol $a_\hbar$ with compact support such that $\sigma^{\mathsf{T},w}(a_\hbar) = \tau$. Using (2.7) and applying the Egorov theorem (see [23, Theorems 5.5.5 and 5.5.9], [30] or [37]), we can find a microlocally unitary Fourier Integral Operator $V_\hbar$ such that:

$$V_\hbar^*\mathcal{H}_{\hbar,\mathbf{A}}V_\hbar = C_0\hbar + \mathcal{H}^0 + \mathsf{Op}^w_\hbar(r_\hbar), \quad \text{with } \mathcal{H}^0 = \mathsf{Op}^w_\hbar(H^0)$$

so that $\sigma^{\mathsf{T},w}\left(\mathsf{Op}^w_\hbar(r_\hbar)\right) = \gamma \in \mathcal{O}_3$. In fact, one can choose $V_\hbar$ such that the subprincipal symbol is preserved by conjugation (see for instance [18, Appendix A]), which implies that $C_0 = 0$[1]. It remains to use Proposition 2.7 and again the Egorov theorem to notice that $e^{i\hbar^{-1}\mathsf{Op}^w_\hbar(a_\hbar)}\mathsf{Op}^w_\hbar(r_\hbar)e^{-i\hbar^{-1}\mathsf{Op}^w_\hbar(a_\hbar)}$ is a pseudo-differential operator such that the formal Taylor series of its symbol is κ. Therefore, recalling (2.8), we have found a microlocally unitary Fourier Integral Operator $U_\hbar$ such that:

$$U_\hbar^*\mathcal{H}_{\hbar,\mathbf{A}}U_\hbar = \mathcal{H}^0 + \mathsf{Op}^w_\hbar\left(f^\star(\hbar,\mathcal{I},z_2)\right) + R_\hbar, \tag{2.10}$$

[1]We give another proof of this fact in Remark 2.8 below.

where $R_\hbar$ is a pseudo-differential operator such that $\sigma^{\mathsf{T},w}(R_\hbar) = 0$. It remains to prove the division property expressed in the last statement of item (4) of Theorem 1.6. By the Morse Lemma, there exists in a fixed neighborhood of $z_1 = 0$ in $\mathbb{R}^4$ a (non symplectic) change of coordinates $\tilde{z}_1$ such that $d_0 = c(z_2)\,|\tilde{z}_1|^2$. It is enough to prove the result in this microlocal neighborhood. Now, for any $N \geqslant 1$, we proceed by induction. We assume that we can write $R_\hbar$ in the form:

$$R_\hbar = \mathsf{Op}_\hbar^w\left(s_0 + \hbar s_1 + \cdots + \hbar^k s_k\right) D_\hbar^N + O(\hbar^{k+1}),$$

with symbols s_j which vanish at infinite order with respect to $\hat{z}_1$. We look for s_{k+1} such that:

$$R_\hbar = \mathsf{Op}_\hbar^w\left(s_0 + \hbar s_1 + \cdots + \hbar^k s_k + \hbar^{k+1} s_{k+1}\right) D_\hbar^N + O(\hbar^{k+2})\tilde{R}_{\hbar,k}.$$

We are reduced to find s_{k+1} such that:

$$\tilde{r}_{0,k} = d_0^N s_{k+1}.$$

Since $\tilde{r}_{0,k}$ vanishes at any order at zero we can find a smooth function ϕ_k such that:

$$\tilde{r}_{0,k} = |\tilde{z}_1|^{2N}\phi.$$

We have $s_{k+1}(\tilde{z}_1, z_2) = \frac{\phi_k(\tilde{z}_1,z_2)}{c(z_2)^N}$.

This ends the proof of Theorem 1.6.

Remark 2.8. It is well known that (see [16, Theorem 1.1]), when $B > 0$, the smallest eigenvalue $\lambda_1(\hbar)$ of $\mathcal{H}_{\hbar,A}$ has the following asymptotics

$$\lambda_1(\hbar) \sim \hbar \min_{q\in\mathbb{R}^2} B(q).$$

We will see in Section 4.1 that the corresponding eigenfunctions are microlocalized on Σ at the minima of B. Therefore the normal form would imply, by a variational argument, that

$$\lambda_1(\hbar) \geqslant C_0\hbar + \mu_1(\hbar) + o(\hbar), \tag{2.11}$$

where $\mu_1(\hbar)$ is the smallest eigenvalue of $\mathcal{N}_\hbar := \mathcal{H}^0 + \mathsf{Op}_\hbar^w\left(f^\star(\hbar, \mathcal{I}, z_2)\right)$. Similarly, we will see in 4.2 that the lowest eigenfunctions of $\mathcal{N}_\hbar$ are also microlocalized in $\hat{z_1}$ and $\hat{z_2}$, and therefore

$$\lambda_1(\hbar) \sim C_0\hbar + \mu_1(\hbar).$$

By Gårding's inequality and point (5) of Theorem 1.6, $\mu_1(\hbar) \sim \hbar \min B$. Comparing with (2.11), we see that $C_0 = 0$.

3. Long time dynamics at low energy

The goal of this section is to prove Theorems 1.2 and 1.3. We shall rely on the following localization lemma.

We work in the open set Ω equipped with the coordinates (z_1, z_2) given by the normal form of Theorem 1.1; thus, we may write $H(z_1, z_2) = K + R$, where $K = |z_1|^2 f(z_2, |z_1|^2)$ and the Taylor series of R with respect to z_1 vanishes for all z_2. On Ω, the magnetic field B has a fixed sign. For notational simplicity we may assume that $B > 0$. We denote by φ_H^t the Hamiltonian flow of H, $I = |z_1|^2$, and $I(t) := I \circ \varphi_H^t$. We also denote $z_2(t) := z_2 \circ \varphi_H^t$.

Lemma 3.1. *Let $C_f > 0$, $M > 0$ be such that*

$$f(z_2, 0) > C_f, \qquad \forall |z_2| > M. \tag{3.1}$$

Let $0 < \tilde{c}_0 < c_0 < C_0 < \tilde{C}_0 < C_f$. For any $\epsilon > 0$ we define the bounded open set

$$U_\epsilon := \left\{(z_1, z_2); \qquad |z_1|^2 < \frac{\epsilon}{2}, \quad c_0 < f(z_2, 0) < C_0\right\}, \tag{3.2}$$

which is contained in the compact set

$$V_\epsilon := \left\{(z_1, z_2); \qquad |z_1|^2 \leqslant \epsilon, \quad \tilde{c}_0 \leqslant f(z_2, 0) \leqslant \tilde{C}_0\right\}. \tag{3.3}$$

Let

$T_\epsilon := \sup\{T > 0; \quad \forall t \in [-T, T], \varphi_H^t$ *exists and* $(z_1(t), z_2(t)) \in V_\varepsilon$ *for any starting point in* $U_\epsilon\}$.

Then for any $N > 0$, there exists $\epsilon_0 > 0$ and a constant $C > 0$ such that

$$\forall \epsilon \leqslant \epsilon_0, \qquad T_\epsilon \geqslant \frac{C}{\epsilon^N}.$$

Proof. Let $N > 1/2$. Since V_ϵ is compact, we have $T_\varepsilon > 0$; moreover, there exists ϵ_0 such that $U_\epsilon \subset \Omega$ for all $\epsilon \leqslant \epsilon_0$. Since the z_1-Taylor series of R vanishes, we can write $R = I^N R_N$, where R_N is smooth. Thus

$$\{H, I\} = I^N \{R_N, I\},$$

which implies

$$|\{H, I\}| \leqslant 2I^{N+1/2} \|\nabla R_N\|.$$

Therefore, we get, on U_ϵ,

$$\forall |t| < T_\epsilon, \quad \left|\frac{d}{dt} I(t)\right| = \left|\{H, I\} \circ \varphi_H^t\right| \leqslant 2C_N I(t)^{N+1/2},$$

where $C_N := \sup_{V_{\epsilon_0}} \|\nabla R_N\|$. By integration, we get

$$\forall |t| < T, \qquad |I(t) - I(0)| \leqslant 2C_N |t| \epsilon^{N+1/2}. \tag{3.4}$$

We apply a similar argument for $K(t) := K \circ \varphi_H^t$. We have $\{H, K\} = \{I^N R_N, K\} = I^N \{R_N, K\}$. Thus we get, on U_ϵ,

$$\left|\frac{d}{dt} K(t)\right| \leqslant I^{N+1/2} C'_N,$$

with $C'_N := 3 \sup_{V_{\epsilon_0}} |\{R_N, K/I\}|$. Therefore $|K(t) - K(0)| \leqslant C'_N I^{N+1/2} |t|$, which implies, since $K = I f(z_2, I)$,

$$|f(z_2(t), I(t)) - f(z_2(0), I(0))| \leqslant C'_N I^{N-1/2} |t| \leqslant C'_N \epsilon^{N-1/2} |t|. \tag{3.5}$$

We may write $f(z_2, I) = f(z_2, 0) + I\tilde{f}$, for some smooth function $\tilde{f}$.

Let us fix $\ell > 0$ such that $C_0 + \frac{C_f - C_0}{\ell} < \tilde{C}_0$ and $c_0 - \frac{C_f - C_0}{\ell} > \tilde{c}_0$. Assume that ϵ_0 is small enough so that

$$\sup_{V_{\epsilon_0}} \tilde{f} \leqslant (C_f - C_0)/(2\ell\epsilon_0). \tag{3.6}$$

Assume by contradiction that there exists $\epsilon \leqslant \epsilon_0$ such that

$$C'_N \epsilon^{N-1/2} T_\epsilon \leqslant (C_f - C_0)/(2\ell), \tag{3.7}$$

and

$$2C_N\epsilon^{N+1/2}T_\epsilon \leqslant \epsilon/3. \tag{3.8}$$

By (3.2), Equations (3.8) and (3.4) imply $I(t) \leqslant \epsilon/2 + \epsilon/3 = 5\epsilon/6$. Equations (3.7) and (3.5) imply $f(z_2(t), I(t)) \leqslant C_0 + (C_f - C_0)/(2\ell)$, and hence, by (3.6),

$$f(z_2(t), 0) \leqslant \tilde{C}_0 - \left(\tilde{C}_0 - C_0 - \frac{C_f - C_0}{\ell}\right) = \tilde{C}_1 < \tilde{C}_0 < C_f.$$

In the same way we find

$$f(z_2(t), 0) \geqslant c_0 - \frac{C_f - C_0}{\ell} = \tilde{c}_1 > \tilde{c}_0.$$

Now we remark that, by definition of T_ϵ, the flow φ_H^t is uniformly bounded for all $|t| < T_\epsilon$; therefore, there exists $T' > T_\epsilon$ for which the flow φ_H^t is defined for all $|t| < T'$. Since $I(t) \leqslant 5\epsilon/6$ and $\tilde{c}_1 \leqslant f(z_2(t), 0) \leqslant \tilde{C}_1$ for all $t < T_\epsilon$, we can find $T' > T_\epsilon$ such that $z(t) \in V_\varepsilon$ which contradicts the definition of T_ϵ.

Therefore one of (3.7) or (3.8) must be false. In both cases, we find a constant $C > 0$ such that

$$\forall \epsilon < \epsilon_0, \qquad T_\epsilon \geqslant \frac{C}{\epsilon^{N-1/2}},$$

which gives the result. □

3.1. **Proof of Theorems 1.2 and 1.3.** The confining assumption on B implies (3.1) — with different constants. Hence, we may apply Lemma 3.1 to H and K which implies that the flows φ_H^t and φ_K^t remain in the region V_ε for times of order ϵ^{-N}, and starting position in U_ϵ. This proves the first point of Theorem 1.2.

Now, let $N' > N$. Writing $H = K + I^{N'}R_{N'}$, we see that the Hamiltonian vector fields $\mathcal{X}_H$ and $\mathcal{X}_K$ differ by $\mathcal{O}(\epsilon^{N'-1/2})$. Let $\mathcal{F}(t) = \varphi_H^t - \varphi_K^t$; $d\mathcal{F}/dt = \mathcal{X}_H \circ \varphi_H^t - \mathcal{X}_K \circ \varphi_K^t$. By Taylor, we get

$$\frac{d\mathcal{F}}{dt} = \mathcal{X}_{H-K} \circ \varphi_H^t + \mathcal{O}(\varphi_K^t - \varphi_H^t),$$

where the $\mathcal{O}$ is given by the derivatives of $\mathcal{X}_K$ and thus is uniform for $|t| < T_\epsilon$. Thus there exist constants C_1, C_2 such that

$$\left\|\frac{d\mathcal{F}}{dt}\right\| \leqslant C_1\epsilon^{N'-1/2} + C_2 \left\|\mathcal{F}(t)\right\|.$$

Here we use $\|\cdot\|$ for the Euclidean norm in $\mathbb{R}^4$. Therefore, since $\mathcal{F}(0) = 0$, the Gronwall lemma provides

$$\|\mathcal{F}(t)\| \leqslant \frac{C_1\epsilon^{N'-1/2}}{C_2}(e^{C_2|t|} - 1).$$

Thus, if $|t| \leqslant C_3 |\ln\epsilon|$ we get $\|\mathcal{F}(t)\| \leqslant C_1C_2^{-1}\epsilon^{N'-1/2-C_2C_3}$, which proves (1.7) since N' is arbitrary, thereby establishing Theorem 1.2.

The naive estimate used above in the proof of Theorem 1.2 cannot yield the stronger conclusion of Theorem 1.3, because it does not take into account the commutation $\{H, K\} = 0$. For this we consider the composition $\varphi_K^t \circ \varphi_H^{-t}$. Notice that, thanks to Lemma 3.1, φ_H^{-t} sends U_ϵ into V_ϵ for times of order $\mathcal{O}(\epsilon^{-N})$, and that V_ϵ is globally invariant by φ_K^t for all times. Thus, the composition $\varphi_K^t \circ \varphi_H^{-t}$ is well defined on U_ϵ and takes values in V_ϵ, for times of order $\mathcal{O}(\epsilon^{-N})$.

Let us fix an arbitrary smooth function $z : \mathbb{R}^4 \to \mathbb{R}$ and introduce, on U_ϵ, the family of functions

$$\mathcal{D}(t) := z \circ \varphi_K^t \circ \varphi_H^{-t}.$$

Using, among others, the equivariance of the Poisson bracket under symplectomorphism, we get

$$\frac{d\mathcal{D}(t)}{dt} = -\{H, \mathcal{D}\} + \{K \circ \varphi_H^{-t}, \mathcal{D}\} = \{-R \circ \varphi_H^{-t}, \mathcal{D}\} = -\{R, z \circ \varphi_K^t\} \circ \varphi_H^{-t}.$$

The goal is now to estimate $\{R, z \circ \varphi_K^t\}$ on V_ϵ. We have

$$\mathcal{X}_K = f\mathcal{X}_I + I\mathcal{X}_{f(z_2,I)},$$

and since $\{I, f(z_2, I)\} = 0$, the flow of K can be written as

$$\varphi_K^t = \varphi_I^{tf} \circ \varphi_{f(z_2,I)}^{It},$$

and I is constant along this flow. We use now the assumptions of Theorem 1.3; thus, $d_{z_2} f(z_2, 0)$ does not vanish on the annulus $c_0 \leqslant f(z_2, 0) \leqslant C_0$, where $J = [c_0, C_0]$. This implies that the same holds for $d_{z_2} f(z_2, I)$, when $I < \epsilon_0$ is small enough. Therefore, for each value of I one can apply the action-angle theorem to the Hamiltonian $z_2 \mapsto f(z_2, I)$: there exists a symplectic change of coordinates $(r, \theta) = \psi_I(z_2)$, with $(r, \theta) \in \mathbb{R} \times S^1$, such that

$$\varphi_{f(z_2,I)}^t(r, \theta) = (r, \theta + tg(I, f)),$$

where g is smooth. This yields the following formula for the flow of K in the variables (z_1, r, θ):

$$\varphi_K^t(z_1, r, \theta) = (e^{-2itf} z_1, r, \theta + tIg(I, f)).$$

From this we obtain the estimate for the spacial derivative:

$$\left\| d\varphi_K^t \right\| \leqslant C(|t| + 1) \qquad \text{on } V_\epsilon,$$

for some constant $C > 0$ (involving the C^1-norms of f and g on V_ϵ), and for any $t \in \mathbb{R}$. Now, as above, we write $R = I^{N'} R_{N'}$ and get

$$\{R, z \circ \varphi_K^t\} = I^{N'} \{R_{N'}, z \circ \varphi_K^t\} + N' R_{N'} I^{N'-1} \{I, z \circ \varphi_K^t\},$$

hence

$$\left|\{R, z \circ \varphi_K^t\}\right| \leqslant C_{N'} I^{N'-1/2} \left\| d\varphi_K^t \right\| \leqslant \tilde{C}_{N'} I^{N'-1/2} (1 + |t|).$$

Thus, if $|t| \leqslant T_\epsilon = \mathcal{O}(\epsilon^{-N})$, we obtain, on U_ϵ,

$$|\mathcal{D}(t) - \mathcal{D}(0)| \leqslant \hat{C}_{N,N'} I^{N'-N-1/2}.$$

Taking z to be any coordinate function, we get, for $m \in U_\epsilon$,

$$\left\| \varphi_K^t \circ \varphi_H^{-t}(m) - m \right\| \leqslant C_{N,N'} \epsilon^{N'-N-1/2}.$$

Notice that an estimate of the same kind is also valid for $m \in \overset{\circ}{V}_\varepsilon$. For any $m' \in U_\epsilon$ we may let $m = \varphi_H^t(m')$, which yields

$$\left\| \varphi_K^t(m') - \varphi_H^t(m') \right\| \leqslant C_{N,N'} \epsilon^{N'-N-1/2}.$$

This gives the conclusion of Theorem 1.3 by choosing N' large enough.

4. Spectral theory

This section is devoted to the proof of Theorem 1.5. The main idea is to use the eigenfunctions of $\mathcal{H}_{\hbar,\mathbf{A}}$ and $\mathcal{N}_\hbar$ as test functions in the pseudo-differential identity (1.9) given in Theorem 1.6 and to apply the variational characterization of the eigenvalues given by the min-max principle. In order to control the remainders we shall prove the microlocalization of the eigenfunctions of $\mathcal{H}_{\hbar,\mathbf{A}}$ and $\mathcal{N}_\hbar$ thanks to the confinement assumption (1.8).

4.1. **Localization and microlocalization of the eigenfunctions of $\mathcal{H}_{\hbar,\mathbf{A}}$.** The space localization of the eigenfunctions of $\mathcal{H}_{\hbar,\mathbf{A}}$, which is the quantum analog of Theorem 1.2, is a consequence of the so-called Agmon estimates.

Proposition 4.1. *Let us assume (1.8). Let us fix $0 < C_1 < \tilde{C}_1$ and $\alpha \in (0, \frac{1}{2})$. There exist $C, \hbar_0, \varepsilon_0 > 0$ such that for all $0 < \hbar \leqslant \hbar_0$ and for all eigenpair (λ, ψ) of $\mathcal{H}_{\hbar,\mathbf{A}}$ such that $\lambda \leqslant C_1\hbar$, we have:*

$$\int |e^{\chi(q)\hbar^{-\alpha}|q|}\psi|^2 \, \mathrm{d}q \leqslant C\|\psi\|^2,$$

where χ is zero for $|q| \leqslant M_0$ and 1 for $|q| \geqslant M_0 + \varepsilon_0$. Moreover, we also have the weighted H^1 estimate:

$$\int |e^{\chi(q)\hbar^{-\alpha}|q|} h\nabla\psi|^2 \, \mathrm{d}q \leqslant C\hbar\|\psi\|^2.$$

Proof. Let us denote by $q_{\hbar,\mathbf{A}}$ the quadratic form associated with the magnetic Laplacian $\mathcal{H}_{\hbar,\mathbf{A}}$. We write the Agmon formula (see [1, 2]) for the eigenpair (λ, ψ) with $\lambda \leqslant C_1\hbar$:

$$q_{\hbar,\mathbf{A}}(e^\Phi\psi) = \lambda\|e^\Phi\psi\| + \hbar^2\|\nabla\Phi e^\Phi\|^2.$$

We recall that:

$$q_{\hbar,\mathbf{A}}(e^\Phi\psi) \geqslant \int \hbar B(q)|e^\Phi\psi|^2 \, \mathrm{d}q$$

so that:

$$\int \left(\hbar B(q) - C_1\hbar - \hbar^2\|\nabla\Phi\|^2\right) |e^\Phi\psi|^2 \, \mathrm{d}q \leqslant 0.$$

We split this integral into two parts:

$$\int_{|q|\geqslant M_0} \left(\hbar B(q) - C_1\hbar - \hbar^2\|\nabla\Phi\|^2\right) |e^\Phi\psi|^2 \, \mathrm{d}q \leqslant \int_{|q|\leqslant M_0} \left(-\hbar B(q) + C_1\hbar + \hbar^2\|\nabla\Phi\|^2\right) |e^\Phi\psi|^2 \, \mathrm{d}q.$$

Let us choose Φ:

$$\Phi = \chi(q)\hbar^{-\alpha}|q|.$$

We get:

$$\int_{|q|\geqslant M_0} \left(\hbar B(q) - C_1\hbar - \hbar^2\|\nabla\Phi\|^2\right) |e^\Phi\psi|^2 \, \mathrm{d}q \leqslant Ch\|\psi\|^2.$$

Then we have:

$$\int_{|q|\geqslant M_0} \left(\hbar C_1 - C_1\hbar - \tilde{C}\hbar^{2-2\alpha}\right) |e^\Phi\psi|^2 \, \mathrm{d}q \leqslant Ch\|\psi\|^2.$$

We infer that:

$$\int_{|q|\geqslant M_0} |e^\Phi\psi|^2 \, \mathrm{d}q \leqslant C\|\psi\|^2, \quad \int |e^{\chi(q)\hbar^{-\alpha}|q|}\psi|^2 \, \mathrm{d}q \leqslant C\|\psi\|^2$$

and then:

$$q_{h,\mathbf{A}}(e^\Phi\psi) \leqslant C\hbar\|\psi\|^2.$$

□

Remark 4.2. This estimate is interesting when $|x| \geqslant M_0 + \varepsilon_0$. In this region, we deduce by standard elliptic estimates that $\psi = O(h^\infty)$ in suitable norms (see for instance [11, Proposition 3.3.4] or more recently [29, Proposition 2.6]). Therefore, the eigenfunctions are localized in space in the ball of center $(0,0)$ and radius $M_0 + \varepsilon_0$.

We shall now prove the microlocalization of the eigenfunctions near the zero set of the magnetic Hamiltonian Σ.

Proposition 4.3. *Let us assume* (1.8)*. Let us fix* $0 < C_1 < \tilde{C}_1$ *and consider* $\delta \in \left(0, \frac{1}{2}\right)$*. Let* (λ, ψ) *be an eigenpair of* $\mathcal{H}_{\hbar,\mathbf{A}}$ *with* $\lambda \leqslant C_1\hbar$*. Then, we have:*

$$\psi = \chi_1\left(\hbar^{-2\delta}\mathcal{H}_{\hbar,\mathbf{A}}\right)\chi_0(q)\psi + O(\hbar^\infty),$$

where χ_0 *is smooth cutoff function supported in a compact set in the ball of center* $(0,0)$ *and radius* $M_0 + \varepsilon_0$ *and where* χ_1 *a smooth cutoff function being* 1 *near* 0*.*

Proof. In view of Proposition 4.1, it is enough to prove that

$$\left(1 - \chi_1\left(\hbar^{-2\delta}\mathcal{H}_{\hbar,\mathbf{A}}\right)\right)(\chi_0(q)\psi) = O(h^\infty). \tag{4.1}$$

By the space localization, we have:

$$\mathcal{H}_{\hbar,\mathbf{A}}(\chi_0(q)\psi) = \lambda\chi_0(q)\psi + O(\hbar^\infty).$$

Then, we get:

$$\left(1 - \chi_1\left(\hbar^{-2\delta}\mathcal{H}_{\hbar,\mathbf{A}}\right)\right)\mathcal{H}_{\hbar,\mathbf{A}}(\chi_0(q)\psi) = \lambda\left(1 - \chi_1\left(\hbar^{-2\delta}\mathcal{H}_{\hbar,\mathbf{A}}\right)\right)(\chi_0(x)\psi) + O(\hbar^\infty).$$

This implies:

$$\begin{aligned}\hbar^{2\delta}\|\left(1 - \chi_1\left(\hbar^{-2\delta}\mathcal{H}_{\hbar,\mathbf{A}}\right)\right)(\chi_0(q)\psi)\|^2 &\leqslant q_{\hbar\,A}\left(\left(1 - \chi_1\left(\hbar^{-2\delta}\mathcal{H}_{\hbar,\mathbf{A}}\right)\right)\mathcal{H}_{\hbar,\mathbf{A}}(\chi_0(q)\psi)\right)\\ &\leqslant C_1\hbar\|\left(1 - \chi_1\left(\hbar^{-2\delta}\mathcal{H}_{\hbar,\mathbf{A}}\right)\right)(\chi_0(q)\psi)\|^2 + O(\hbar^\infty)\|\psi\|^2.\end{aligned}$$

Since $\delta \in \left(0, \frac{1}{2}\right)$, we deduce (4.1). □

4.2. Microlocalization of the eigenfunctions of $\mathcal{N}_\hbar$.

The next two propositions state the microlocalization of the eigenfunctions of the normal form $\mathcal{N}_\hbar$.

Proposition 4.4. *Let us consider the pseudo-differential operator:*

$$\mathcal{N}_\hbar = \mathcal{H}_\hbar^0 + \mathsf{Op}_\hbar^w f^\star(\hbar, \mathcal{I}_\hbar, \hat{z}_2).$$

We assume the confinement assumption (1.8)*. We can consider* $\tilde{M}_0 > 0$ *such that* $B \circ \varphi^{-1}(\hat{z}_2) \geqslant \tilde{C}_1$ *for* $|\hat{z}_2| \geqslant \tilde{M}_0$*. Let us consider* $C_1 < \tilde{C}_1$ *and an eigenpair* (λ, ψ) *of* $\mathcal{N}_\hbar$ *such that* $\lambda \leqslant C_1\hbar$*. Then, for all* $\varepsilon_0 > 0$ *and for all smooth cutoff function* χ *supported in* $|\hat{z}_2| \geqslant \tilde{M}_0 + \varepsilon_0$*, we have:*

$$\mathsf{Op}_\hbar^w\left(\chi(\hat{z}_2)\right)\psi = O(\hbar^\infty).$$

Proof. We notice that:

$$\mathcal{N}_\hbar\mathsf{Op}_\hbar^w\left(\chi(\hat{z}_2)\right)\psi = \lambda\mathsf{Op}_\hbar^w\left(\chi(\hat{z}_2)\right)\psi + \hbar\mathcal{R}_\hbar\psi,$$

where the symbol of $\mathcal{R}_\hbar$ is supported in compact slightly smaller than the support of χ. We may consider a cutoff function $\underline{\chi}$ which is 1 on a small neighborhood of this support. We get:

$$\langle \mathcal{N}_\hbar \mathsf{Op}_\hbar^w\left(\chi(\hat{z}_2)\right)\psi, \mathsf{Op}_\hbar^w\left(\chi(\hat{z}_2)\right)\psi\rangle \leqslant \lambda\|\mathsf{Op}_\hbar^w\left(\chi(\hat{z}_2)\right)\psi\|^2 + C\hbar\|\mathsf{Op}_\hbar^w\left(\underline{\chi}(\hat{z}_2)\right)\psi\|\|\mathsf{Op}_\hbar^w\left(\chi(\hat{z}_2)\right)\psi\|$$

Thanks to the Gårding inequality, we have:

$$\begin{aligned}\langle \mathcal{H}_\hbar^0 \mathsf{Op}_\hbar^w\left(\chi(\hat{z}_2)\right)\psi, \mathsf{Op}_\hbar^w\left(\chi(\hat{z}_2)\right)\psi\rangle &\geqslant (\tilde{C}_1 - C\hbar)\|\mathsf{Op}_\hbar^w\left(\chi(\hat{z}_2)\right)\mathcal{I}_\hbar^{1/2}\psi\|^2\\ &\geqslant (\tilde{C}_1 - C\hbar)\hbar\|\mathsf{Op}_\hbar^w\left(\chi(\hat{z}_2)\right)\psi\|^2.\end{aligned}$$

We can consider $\mathsf{Op}_\hbar^w f^\star(\hbar, \mathcal{I}_\hbar, \hat{z}_2)$ as a perturbation of $\mathcal{H}_\hbar^0$ (see (2.9)). Since $C_1 < \tilde{C}_1$ we infer that:

$$\|\mathsf{Op}_\hbar^w\left(\chi(\hat{z}_2)\right)\psi\| \leqslant C\hbar\|\mathsf{Op}_\hbar^w\left(\underline{\chi}(\hat{z}_2)\right)\psi\|.$$

Then a standard iteration argument provides $\mathsf{Op}_\hbar^w\left(\chi(\hat{z}_2)\right)\psi = O(\hbar^\infty)$. □

Proposition 4.5. *Keeping the assumptions and the notation of Proposition 4.4, we consider $\delta \in \left(0, \frac{1}{2}\right)$ and an eigenpair (λ, ψ) of $\mathcal{N}_\hbar$ with $\lambda \leqslant C_1\hbar$. Then, we have:*

$$\psi = \chi_1\left(\hbar^{-2\delta}\mathcal{I}_\hbar\right)\mathsf{Op}_\hbar^w\left(\chi_0(\hat{z}_2)\right)\psi + O(\hbar^\infty),$$

for all smooth cutoff function χ_1 supported in a neighborhood of zero and all smooth cutoff function χ_0 being 1 near zero and supported in the ball of center 0 and radius $\tilde{M}_0 + \varepsilon_0$.

Proof. The proof follows the same lines as for Proposition 4.4 and Proposition 4.3. □

4.2.1. *Proof of Theorem 1.5.* As we proved in the last section, each eigenfunction of $\mathcal{H}_{\hbar,\mathbf{A}}$ or $\mathcal{N}_\hbar$ is microlocalized. Nevertheless we do not know yet if all the functions in the range of the spectral projection below $C_1 h$ are microlocalized. This depends on the rank of the spectral projection. The next two lemmas imply that this rank does not increase more than polynomially in $\hbar^{-1}$ (so that the functions lying in the range of the spectral projection are microlocalized). We will denote by $N(\mathcal{M}, \lambda)$ the number of eigenvalues of $\mathcal{M}$ less than or equal to λ.

Lemma 4.6. *There exists $C > 0$ such that for all $\hbar > 0$, we have:*

$$N(\mathcal{H}_{\hbar,\mathbf{A}}, C_1\hbar) \leqslant C\hbar^{-1}.$$

Proof. We notice that:

$$N(\mathcal{H}_{\hbar,\mathbf{A}}, C_1\hbar) = N(\mathcal{H}_{1,\hbar^{-1}A}, C_1\hbar^{-1})$$

and that, for all $\varepsilon \in (0,1)$:

$$q_{1,\hbar^{-1}\mathbf{A}}(\psi) \geqslant (1-\varepsilon)q_{1,\hbar^{-1}\mathbf{A}}(\psi) + \varepsilon\int_{\mathbb{R}^2}\frac{B(x)}{\hbar}|\psi(x)|^2\,\mathrm{d}x$$

so that we infer:

$$N(\mathcal{H}_{\hbar,\mathbf{A}}, C_1\hbar) \leqslant N(\mathcal{H}_{1,\hbar^{-1}\mathbf{A}} + \varepsilon(1-\varepsilon)^{-1}\hbar^{-1}B, (1-\varepsilon)^{-1}C_1\hbar^{-1}).$$

Then, the diamagnetic inequality [2] jointly with a Lieb-Thirring estimate (see the original paper [21]) provides for all $\gamma > 0$ the existence of $L_{\gamma,2} > 0$ such that for all $\hbar > 0$ and $\lambda > 0$:

$$\sum_{j=1}^{N(\mathcal{H}_{1,\hbar^{-1}\mathbf{A}}+\varepsilon(1-\varepsilon)^{-1}\hbar^{-1},\lambda)} \left|\tilde{\lambda}_j(\hbar) - \lambda\right|^\gamma \leqslant L_{\gamma,2}\int_{\mathbb{R}^2} (\varepsilon(1-\varepsilon)^{-1}\hbar^{-1}B(x) - \lambda)_-^{1+\gamma}\,\mathrm{d}x.$$

We apply this inequality with $\lambda = (1+\eta)(1-\varepsilon)^{-1}C_1\hbar^{-1}$, for some $\eta > 0$. This implies that:

$$\sum_{j=1}^{N_{\epsilon,\hbar,\eta}} \left|\tilde{\lambda}_j(\hbar) - \lambda\right|^\gamma \leqslant L_{\gamma,2}\int_{B(x)\leqslant(1+\eta)C_1/\epsilon} (\lambda - \varepsilon(1-\varepsilon)^{-1}\hbar^{-1}B(x))^{1+\gamma}\,\mathrm{d}x$$

with $N_{\epsilon,\hbar,\eta} := N(\mathcal{H}_{1,\hbar^{-1}\mathbf{A}} + \varepsilon(1-\varepsilon)^{-1}\hbar^{-1}B, (1-\varepsilon)^{-1}C_1\hbar^{-1})$, so that:

$$(\eta(1-\varepsilon)^{-1}C_1\hbar^{-1})^\gamma N_{\epsilon,\hbar,\eta} \leqslant L_{\gamma,2}(\hbar(1-\varepsilon))^{-1-\gamma}\int_{B(x)\leqslant\frac{(1+\eta)C_1}{\varepsilon}} ((1+\eta)C_1 - \varepsilon B(x))^{1+\gamma}\,\mathrm{d}x.$$

For η small enough and ε is close to 1, we have $(1+\eta)\varepsilon^{-1}C_1 < \tilde{C}_1$ so that the integral is finite, which gives the required estimate. □

Lemma 4.7. *There exists $C > 0$ and $\hbar_0 > 0$ such that for all $\hbar \in (0, \hbar_0)$, we have:*

$$N(\mathcal{N}_\hbar, C_1\hbar) \leqslant C\hbar^{-1}.$$

Proof. Let $\epsilon \in (0,1)$. By point (5) of Theorem 1.6, it is enough to prove that $N(\mathcal{H}_\hbar^0, \frac{C_1\hbar}{1-\epsilon}) \leqslant Ch^{-1}$. The eigenvalues and eigenfunctions of $\mathcal{H}_\hbar^0$ can be found by separation of variables: $\mathcal{H}_\hbar^0 = \mathcal{I}_\hbar \otimes \mathsf{Op}_\hbar^w(B\circ\varphi^{-1})$, where $\mathcal{I}_\hbar$ acts on $L^2(\mathbb{R}_{x_1})$ and $\hat{B}_\hbar := \mathsf{Op}_\hbar^w(B\circ\varphi^{-1})$ acts on $L^2(\mathbb{R}_{x_2})$. Thus,

$$N(\mathcal{H}_\hbar^0, \hbar C_{1,\epsilon}) = \#\{(n,m)\in(\mathbb{N}^*)^2;\quad (2n-1)\hbar\gamma_m(\hbar) \leqslant \hbar C_{1,\epsilon}\},$$

where $C_{1,\epsilon} := \frac{C_1}{1-\epsilon}$, and $\gamma_1(\hbar) \leqslant \gamma_2(\hbar) \leqslant \cdots$ are the eigenvalues of $\hat{B}_\hbar$. A simple estimate gives

$$N(\mathcal{H}_\hbar^0, C_{1,\epsilon}) \leqslant \left(1 + \left\lfloor \frac{1}{2} + \frac{C_{1,\epsilon}}{2\gamma_1(\hbar)}\right\rfloor\right)\cdot\#\{m\in\mathbb{N}^*;\quad \gamma_m(\hbar)\leqslant C_{1,\epsilon}\}.$$

If ϵ is small enough, $C_{1,\epsilon} < \tilde{C}_1$, and then Weyl asymptotics (see for instance [6, Chapter 9]) for $\hat{B}_\hbar$ gives

$$N(\hat{B}_\hbar, C_{1,\epsilon}) \sim \frac{1}{2\pi\hbar}\mathrm{vol}\{B\circ\varphi^{-1}\leqslant C_{1,\epsilon}\},$$

and Gårding's inequality implies $\gamma_1(\hbar) \geqslant \min_{q\in\mathbb{R}^2} B - O(\hbar)$, which finishes the proof. □

Remark 4.8. With additional hypotheses on the magnetic field, it has been proved that the $O(\hbar^{-1})$ estimate is in fact optimal: see for instance [4] and [32, Remark 1]. Actually, it would likely follow from Theorem 1.5 and Theorem 1.6 that these Weyl asymptotics hold in general under the confinement assumption.

Let us now consider $\lambda_1(\hbar), \cdots, \lambda_{N(\mathcal{H}_{\hbar,\mathbf{A}},C_1\hbar)}(\hbar)$ the eigenvalues of $\mathcal{H}_{\hbar,\mathbf{A}}$ below $C_1\hbar$. We can consider corresponding normalized eigenfunctions ψ_j such that : $\langle\psi_j,\psi_k\rangle = \delta_{kj}$. We introduce the N-dimensional space:

$$V = \chi_1\left(\hbar^{-2\delta}\mathcal{H}_{\hbar,\mathbf{A}}\right)\chi_0(q)\operatorname*{\mathsf{span}}_{1\leqslant j\leqslant N}\psi_j.$$

[2]See [5, Theorem 1.13] and the link with the control of the resolvent kernel in [20, 31].

Let us bound from above the quadratic form of $\mathcal{N}_\hbar$ denoted by $\mathcal{Q}_\hbar$. For $\psi \in \underset{1\leqslant j\leqslant N}{\mathsf{span}}\, \psi_j$, we let:

$$\tilde{\psi} = \chi_1\left(\hbar^{-2\delta}\mathcal{H}_{\hbar,\mathbf{A}}\right)\chi_0(q)\psi$$

and we can write:

$$\mathcal{Q}_\hbar(U_\hbar^*\tilde{\psi}) = \langle U_\hbar \mathcal{N}_\hbar U_\hbar^*\tilde{\psi}, \tilde{\psi}\rangle = \langle U_\hbar U_\hbar^* \mathcal{H}_{\hbar,\mathbf{A}} U_\hbar U_\hbar^*\tilde{\psi}, \tilde{\psi}\rangle - \langle U_\hbar R_\hbar U_\hbar^*\tilde{\psi}, \tilde{\psi}\rangle.$$

Since $U_\hbar$ is microlocally unitary, the elementary properties of the pseudo-differential calculus yield:

$$\langle U_\hbar \mathcal{N}_\hbar U_\hbar^*\tilde{\psi}, \tilde{\psi}\rangle = \langle \mathcal{H}_{\hbar,\mathbf{A}}\tilde{\psi}, \tilde{\psi}\rangle - \langle U_\hbar R_\hbar U_\hbar^*\tilde{\psi}, \tilde{\psi}\rangle + O(\hbar^\infty)\|\tilde{\psi}\|^2.$$

Then, thanks to Proposition 4.3 and Lemma 4.6 we may replace $\tilde{\psi}$ by ψ up to a remainder of order $O(\hbar^\infty)\|\tilde{\psi}\|$:

$$\langle U_\hbar \mathcal{N}_\hbar U_\hbar^*\tilde{\psi}, \tilde{\psi}\rangle = \langle \mathcal{H}_{\hbar,\mathbf{A}}\psi, \psi\rangle - \langle U_\hbar R_\hbar U_\hbar^*\tilde{\psi}, \tilde{\psi}\rangle + O(\hbar^\infty)\|\tilde{\psi}\|^2$$

so that:

$$\langle U_\hbar \mathcal{N}_\hbar U_\hbar^*\tilde{\psi}, \tilde{\psi}\rangle \leqslant \lambda_N(\hbar)\|\psi\|^2 + |\langle U_\hbar R_\hbar U_\hbar^*\tilde{\psi}, \tilde{\psi}\rangle| + O(\hbar^\infty)\|\tilde{\psi}\|^2$$

and:

$$\langle U_\hbar \mathcal{N}_\hbar U_\hbar^*\tilde{\psi}, \tilde{\psi}\rangle \leqslant \lambda_N(\hbar)\|U_\hbar^*\tilde{\psi}\|^2 + |\langle U_\hbar R_\hbar U_\hbar^*\tilde{\psi}, \tilde{\psi}\rangle| + O(\hbar^\infty)\|U_\hbar^*\tilde{\psi}\|^2.$$

Let us now estimate the remainder term $U_\hbar R_\hbar U_\hbar^*\tilde{\psi}$. We have:

$$U_\hbar R_\hbar U_\hbar^*\tilde{\psi} = U_\hbar R_\hbar U_\hbar^*\underline{\chi_1}\left(\hbar^{-2\delta}\mathcal{H}_{\hbar,\mathbf{A}}\right)\tilde{\psi} = U_\hbar R_\hbar U_\hbar^*\underline{\chi_1}\left(\hbar^{-2\delta}\mathcal{H}_{\hbar,\mathbf{A}}\right)(U_\hbar^*)^{-1}U_\hbar^*\tilde{\psi} + O(\hbar^\infty)\|U_\hbar^*\tilde{\psi}\|,$$

where $\underline{\chi_1}$ has a support slightly bigger then the one of χ_1. We notice that

$$U_\hbar^*\underline{\chi_1}\left(\hbar^{-2\delta}\mathcal{H}_{\hbar,\mathbf{A}}\right)(U_\hbar^*)^{-1} = \underline{\chi_1}\left(\hbar^{-2\delta}U_\hbar^*\mathcal{H}_{\hbar,\mathbf{A}}(U_\hbar^*)^{-1}\right).$$

Let us now apply (1.10) with $D_\hbar = U_\hbar^*\mathcal{H}_{\hbar,\mathbf{A}}(U_\hbar^*)^{-1}$ to get:

$$R_\hbar = S_{\hbar,M}(U_\hbar^*\mathcal{H}_{\hbar,\mathbf{A}}(U_\hbar^*)^{-1})^M + K_N + O(\hbar^\infty)$$

so that:

$$\|U_\hbar R_\hbar U_\hbar^*\underline{\chi_1}\left(\hbar^{-2\delta}\mathcal{H}_{\hbar,\mathbf{A}}\right)\tilde{\psi}\| = O(\hbar^{2M\delta})\|U_\hbar^*\tilde{\psi}\|^2.$$

We infer that:

$$\mathcal{Q}_\hbar(U_\hbar^*\tilde{\psi}) \leqslant \lambda_N(\hbar)\|U_\hbar^*\tilde{\psi}\|^2 + O(\hbar^{2M\delta})\|U_\hbar^*\tilde{\psi}\|^2.$$

From the min-max principle, it follows that:

$$\mu_N(\hbar) \leqslant \lambda_N(\hbar) + O(\hbar^{2M\delta}).$$

The converse inequality follows from a similar proof, using Proposition 4.5 and Lemma 4.7. This ends the proof of Theorem 1.5.

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