On semi-global invariants for *focus-focus* singularities

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Abstract

This article gives a classification, up to symplectic equivalence, of singular Lagrangian foliations given by a completely integrable system of a 4-dimensional symplectic manifold, in a full neighbourhood of a singular leaf of *focus-focus* type.

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1 Introduction

In the study of completely integrable Hamiltonian systems, and more generally for any dynamical system, finding normal forms is often the easiest way of understanding the behaviour of the trajectories. Normal forms generally deal with a *local* issue. But the locality here depends on one's viewpoint: one can be local near a point, an orbit, or any invariant submanifold. If $F = (H_1, \ldots, H_n)$ is a completely integrable system on a $2n$-symplectic manifold M (meaning that $\{H_j, H_i\} = 0$), several normal forms hold:

- near a point m where $dH_j(m)$, $j = 1, \ldots, n$ are linearly independent, one can construct *Darboux-Carathéodory* coordinates: a neighbourhood of m is symplectomorphic to a neighbourhood of the origin in $\mathbb{R}^{2n}$ with its canonical coordinates (x, ξ), in such a way that $H_j - H_j(m) = \xi_j$.

- if c is a regular value of F, one has near any compact connected component Λ_c of $F^{-1}(c)$ the *Liouville-Arnold* theorem which states that the system is symplectomorphic to a neighbourhood of the zero section of $T^*(\mathbb{T}^n)$ in such a way that there is a change of coordinates Φ in $\mathbb{R}^n$ such that $F \circ \Phi = (\xi_1, \ldots, \xi_n)$. Here $\mathbb{T}^n$ is the torus $\mathbb{R}^n/2\pi\mathbb{Z}^n$ and the cotangent bundle $T^*(\mathbb{T}^n)$ is equipped with canonical coordinates (x, ξ).

The first one is typically a local normal form, while I would refer to the Liouville-Arnold theorem as a *semi-global* result, for it classifies a neighbourhood of a whole invariant Lagrangian leaf Λ_c. These two statements above are now fairly standard. They can be extended in different directions: a) trying to globalise: what can be said at the level of the whole fibration of regular fibres Λ_c ? This of course involves more topological invariants, as described in Duistermaat's paper [4]; b) including critical points, which is the main incentive for this article.

A Morse-Bott like theoretical study of critical point of completely integrable Hamiltonian systems exists, which yields a local symplectic classification of non-degenerate singularities (see Eliasson [5]). These results have been used by Nguyên Tiên Zung [8] (extending previous results by Fomenko) to obtain a topological semi-global classification of the singular foliation. This work does not give the corresponding smooth symplectic classification, where new semi-global invariants show up, as demonstrated in the "1-D" (one degree of freedom, i.e. $n = 1$) case by [3]. The point of our present article is to extend the results of [3] to the 2-D case of *focus-focus* singularities. Note that our arguments could easily be applied in the 1-D case, thus supplying for the lack of proofs in [3].

Between the pure topological classification of the singular foliation and the "exact" symplectic classification, some other interesting notions of equivalence have been introduced (see eg. [1]), which are all weaker that what we shall present here.

The semi-global viewpoint seems to be able to shed some new light in semi-classical mechanics, where a quantum state is associated to a Lagrangian submanifold. Quantum states associated to singular manifolds have a particularly rich structure, strongly linked to the local (for this, see [11]) and semi-global symplectic invariants of the foliation. We expect to return on this in a future paper.

2 Statement of the result

In this article, (M,ω) is a 4-dimensional symplectic manifold, equipped with the symplectic Poisson bracket $\{\cdot,\cdot\}$. Any smooth function H on M gives rise to a Hamiltonian vector field denoted by $\mathscr{X}_H$.

The word smooth always means of C^∞ category and a function f is said *flat* at a point m if f and all its derivatives vanish at m.

Definition 2.1 *A map $F = (H_1,H_2)$ defined on some open subset U of M with values in $\mathbb{R}^2$ is called a **momentum map** if dF is surjective almost everywhere in U and $\{H_1,H_2\} = 0$.*

Definition 2.2 *A **singular Liouville foliation** $\mathscr{F}$ is a disjoint union of connected subsets of M called leaves for which there exists a momentum map F defined in some neighbourhood Ω of $\mathscr{F}$ such that the leaves of $\mathscr{F}$ are the connected components of the level sets $F^{-1}(c)$, for c in some open subset of $\mathbb{R}^2$.*

The total space of the foliation is also denoted by $\mathscr{F}$. The above definition implies that $\mathscr{F}$ is an open subset of M.

Definition 2.3 *Let $m \in \mathscr{F}$. The maximum of the set $\{\mathrm{rank}(dF(m)), F \text{ defining } \mathscr{F}\}$ is called the rank of m. m is called **regular** if its rank is maximal ($= 2$). Otherwise it is called **singular**.*

If m is a regular point, then there is an open neighbourhood of m in which all points are regular, and if F_1 F_2 are associated momentum maps near m, one has $F_1 = \varphi \circ F_2$, for some local diffeomorphism φ of $\mathbb{R}^2$ (these facts come from the local submersion theorem).

Note that the condition $\{H_1,H_2\} = 0$ implies that the leaves are local Lagrangian manifolds near any regular point. However, the foliation near a regular leaf (=a leaf without any singular point) is not the most general Lagrangian foliation (which would be defined as a foliation admitting *locally* associated momentum maps), since the latter does not necessarily admit a global momentum map (see [12]).

In what follows, the word "Liouville" is often omitted. If $m \in \mathscr{F}$, we denote by $\mathscr{F}_m$ the leaf containing m.

Definition 2.4 *A singular Liouville foliation $\mathscr{F}$ is called of* ***simple*** **focus-focus** ***type*** *whenever the following conditions are satisfied:*

1. *$\mathscr{F}$ has a unique singular point m;*
2. *the singularity at m is of* focus-focus *type;*
3. *the leaf $\mathscr{F}_m$ is compact.*

The leaf $\mathscr{F}_m$ is called the focus-focus *leaf.*

Recall that the second condition means that there exists a momentum map $F = (H_1, H_2)$ for the foliation at m such that the Hessians of H_1 and H_2 span a subalgebra of quadratic forms that admits, in some symplectic coordinates (x, y, ξ, η), the following basis:

$$q_1 = x\xi + y\eta, \qquad q_2 = x\eta - y\xi. \tag{1}$$

This implies that *focus-focus* points are isolated, which ensures that the above definition is non-void. Note that *focus-focus* singularities are one of the four types of singularities of Morse-Bott type in dimension 4, in the sense of Eliasson [6].

Definition 2.5 *Two singular foliations $\mathscr{F}$ and $\tilde{\mathscr{F}}$ in the symplectic manifolds (M, ω) and $(\tilde{M}, \tilde{\omega})$ are* ***equivalent*** *is there exists a symplectomorphism $\varphi : \mathscr{F} \to \tilde{\mathscr{F}}$ that sends leaves to leaves.*

Definition 2.6 *Let $\mathscr{F}$ and $\tilde{\mathscr{F}}$ be singular foliations in M, and $m \in \mathscr{F} \cap \tilde{\mathscr{F}}$ such that $\mathscr{F}_m = \tilde{\mathscr{F}}_m$. The* ***germs*** *of $\mathscr{F}$ and $\tilde{\mathscr{F}}$ at $\mathscr{F}_m$ are equal if and only if there exists a saturated neighbourhood Ω of $\mathscr{F}_m$ in $\mathscr{F}$ such that $\mathscr{F} \cap \Omega = \tilde{\mathscr{F}} \cap \Omega$.*

The classification of germs of Liouville foliations near a compact regular leaf is given by the Liouville-Arnold theorem that asserts that they are all equivalent to the horizontal fibration by tori of $T^*\mathbb{T}^n$. The presence of singularities imposes more rigidity, and we have the following theorem (which is natural in view of [3]):

Theorem 2.1 *The set of equivalence classes of germs of singular Liouville foliations of* focus-focus *type at the* focus-focus *leaf is in natural bijection with $\mathbb{R}[[X, Y]]_0$, where $\mathbb{R}[[X, Y]]$ is the algebra of real formal power series in two variables, and $\mathbb{R}[[X, Y]]_0$ is the subspace of such series with vanishing constant term.*

This formal statement does not contain the most interesting part of the result, which is the geometric description of the power series involved (it is essentially the Taylor series of a regularisation of some action integral). The rest of the paper is devoted to this description – which is the "⇒" sense of the theorem, and to the proof of the "⇐" sense, for which we provide a normal form corresponding to any given power series in $\mathbb{R}[[X,Y]]_0$.

The articles ends up with a sketchy argument as to how the result can be extended to handle the case of several *focus-focus* points in the singular leaf.

3 The regularised action

Let $\mathscr{F}$ be a singular foliation of simple *focus-focus* type. Then in some neighbourhood U of the *focus-focus* point m, the following linearisation result holds (Eliasson [5]): there exist symplectic coordinates in U in which the map (q_1,q_2) (defined in (1)) is a momentum map for the foliation. Notice therefore that, contrary to what the picture of Figure 1 may suggest, $\mathscr{F}_m$ is diffeomorphic near m to the union of two 2-dimensional planes transversally intersecting at m. Let A be a point in $\mathscr{F}_m \cap U \setminus \{m\}$, and Σ be a small 2-dimensional surface transversal to the foliation at A, and Ω be the open neighbourhood of $\mathscr{F}_m$ consisting of leaves intersecting Σ. In what follows, we restrict the foliation to Ω.

Let $\tilde{F}$ be a momentum map for the whole foliation $\mathscr{F}$ satisfying the hypothesis of Definition 2.4. In a neighbourhood of Σ, $\tilde{F}$ and $q=(q_1,q_2)$ are regular local momentum maps, hence $q=\varphi\circ\tilde{F}$, for some local diffeomorphism φ of $\mathbb{R}^2$. Now let $F=\varphi\circ\tilde{F}$. It is a global momentum map for $\mathscr{F}$ that extends q. We denote $F=(H_1,H_2)$ and $\Lambda_c=F^{-1}(c)$.

Near m, the Hamiltonian flow of q_2 is 2π-periodic, and – assuming U to be invariant with respect to this flow – the associated S^1-action is free in $U\setminus\{m\}$. Since this action commutes with the flow of H_1, the H_2-orbits must be periodic of primitive period 2π for any point in a (non-trivial) trajectory of $\mathscr{X}_{H_1}$. On the leaf $\mathscr{F}_m=\Lambda_0$, these trajectories are homoclinic orbits for the point m, which implies that the flow of H_2 generates an S^1-action on a whole neighbourhood of $\mathscr{F}_m$ (see [10] for details).

For any point $A\in\Lambda_c$, c a regular value of F, let $\tau_1(c)>0$ be the time of first return for the $\mathscr{X}_{H_1}$-flow to the $\mathscr{X}_{H_2}$-orbit through A, and $\tau_2(c)\in\mathbb{R}/2\pi\mathbb{Z}$ the time it takes to close up this trajectory under the flow of $\mathscr{X}_{H_2}$ (see Fig. 1). These times are independent of the initial point A on Λ_c.

For any regular value c of F, the set of points $(a,b)\in\mathbb{R}^2$ such that $a\mathscr{X}_{H_1}+b\mathscr{X}_{H_2}$ has a 1-periodic flow on Λ_c is a sublattice of $\mathbb{R}^2$ called the *period lattice* [4]. The vector fields $\tau_1\mathscr{X}_{H_1}+\tau_2\mathscr{X}_{H_2}$ and $2\pi\mathscr{X}_{H_2}$ both define 1-periodic flows, hence

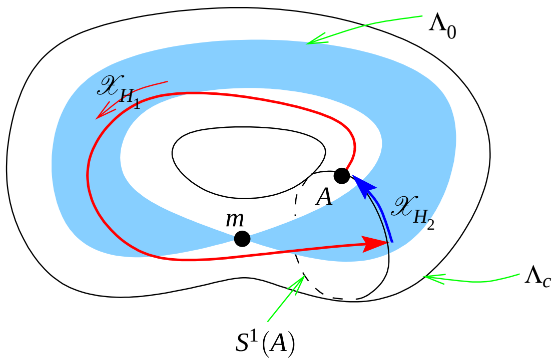

Figure 1: Construction of the "periods" $\tau_j(c)$

(τ_1, τ_2) and $(0, 2\pi)$ form a $\mathbb{Z}$-basis of the period lattice (see Remark 3.3). As we shall see, the classification we are looking for relies on the behaviour of this basis as c tends to 0. One immediate fact is that the cycle associated to $\mathscr{X}_{H_2}$ shrinks to a point (vanishing cycle). On the other hand, the coefficients of the first vector field display a logarithmic divergence, as stated in the following proposition.

Proposition 3.1 *Let* $\ln c$ *be the some determination of the complex logarithm, where* $c = (c_1, c_2)$ *is identified with* $c_1 + ic_2$*. Then the following quantities*

$$\begin{cases} \sigma_1(c) &= \tau_1(c) + \Re(\ln c) \\ \sigma_2(c) &= \tau_2(c) - \Im(\ln c) \end{cases}$$

extend to smooth and single-valued functions in a neighbourhood of 0*. The differential 1-form*

$$\sigma := \sigma_1 dc_1 + \sigma_2 dc_2$$

is closed.

Proof. As before, let U be the neighbourhood of m found using Eliasson's result, with canonical coordinates (x, y, ξ, η). In U, we use the complex coordinates $z = (z_1, z_2)$ with $z_1 = x + iy$ and $z_2 = \xi + i\eta$, so that $q_1(z) + iq_2(z) = \bar{z_1} z_2$. The flow of q_1 is

$$(z_1(t), z_2(t)) = (e^t z_1(0)), e^{-t} z_2(0), \tag{2}$$

while the flow of q_2 is the S^1-action given by

$$(z_1(t), z_2(t)) = e^{it}(z_1(0), z_2(0)). \tag{3}$$

Fix some small $\varepsilon > 0$. Then the local submanifolds $\Sigma_u = \{z_1 = \varepsilon, |z_2| \text{ small}\}$ and $\Sigma_s = \{|z_1| \text{ small}, z_2 = \varepsilon,\}$ are transversal to the foliation $\Lambda_c = \{(z_1, z_2), \quad \bar{z_1} z_2 =$

$c\}$; therefore, the intersections $A(c) := \Sigma_u \cap \Lambda_c$ and $B(c) := \Sigma_s \cap \Lambda_c$ are smooth families of points.

The S^1-orbits of $\Sigma_{u/s}$ form two small hypersurfaces transversal to the flow of q_1; therefore one can uniquely define $\tau_1^{A,B}(c)$ as the time of first hit on Σ_s for the $\mathscr{X}_{H_1}$-flow starting at $A(c)$ (and hence flowing outside of U), and $\tau_2^{A,B}(c)$ as the time it takes to finally reach $B(c)$ under the $\mathscr{X}_{H_2}$-flow. $\tau_1^{A,B}(c)$ and $\tau_2^{A,B}(c)$ are smooth functions of c in a neighbourhood of 0.

Interchanging the roles of A and B – and thus of Σ_u and Σ_s, the times $\tau_j^{B,A}(c)$ for $j = 1,2$ are defined in the same way. But since the corresponding flows now take place inside U, where a singular point occur, $\tau_j^{B,A}(c)$ is not defined for $c = 0$. On the other hand, equations (2) and (3) yield the following explicit formula:

$$\tau_1^{B,A}(c) + i\tau_2^{B,A}(c) = \ln \frac{z_1(A)}{z_1(B)} = \ln z_1(A)\bar{z_2}(B) - \ln \bar{c} = \ln \varepsilon^2 - \ln \bar{c} \qquad (4)$$

Writing now

$$\tau_1(c) + i\tau_2(c) = \left(\tau_1^{A,B}(c) + \tau_1^{B,A}(c)\right) + i\left(\tau_2^{A,B}(c) + \tau_2^{B,A}(c)\right),$$

using (4), and the fact that $\ln \bar{c} = \ln |c| - i \arg c$, we obtain that

$$\sigma_1(c) + i\sigma_2(c) = \tau_1^{A,B}(c) + i\tau_2^{A,B}(c) + \ln \varepsilon^2,$$

which proves the first statement of the proposition.

Let us show now that for regular values of c the 1-form $\tau_1(c)dc_1 + \tau_2(c)dc_2$ is closed. For this we fix a regular value c_0 and introduce the following action integral, for c in a small ball of regular values around c_0:

$$\mathscr{A}(c) := \int_{\gamma_c} \alpha, \qquad (5)$$

where α is any 1-form on some neighbourhood of Λ_c in M such that $d\alpha = \omega$ (which always exists since Λ_c is Lagrangian), and $c \to \gamma_c$ is a smooth family of loops on the torus Λ_c with the same homology class as the trajectory of the joint flow of (H_1, H_2) during the time $(\tau_1(c), \tau_2(c))$. A simple argument (see for instance [10, Lemma 3.6]) shows that $\frac{\partial \mathscr{A}(c)}{\partial c_j} = \int_{\gamma_c} \kappa_j$, where κ_j is the closed 1-form on Λ_c defined by $\iota_{\mathscr{X}_{H_i}} \kappa = \delta_{i,j}$. In other words, the integral of κ_j along a trajectory of the flow of H_j measures the increase of the time t_j along this trajectory. This means that

$$d\mathscr{A}(c) = \tau_1(c)dc_1 + \tau_2(c)dc_2, \qquad (6)$$

and thus proves the closedness of the right-hand side.

Another way of proving this fact would be to apply the Liouville-Arnold theorem, which ensures that any 1-form $adc_1 + bdc_2$, where a, b depend smoothly on c near a regular value, such that (a,b) is in the *period lattice*, is closed (see remark 3.3).

Adding the fact that $\ln(c)dc$ is closed as a holomorphic 1-form, we obtain the closedness of σ at any regular value of c, and hence at $c = 0$ as well. $\square$

Remark 3.1. From this proposition, one easily recovers the result of [9] stating that the monodromy of the Lagrangian fibration around a *focus-focus* fibre is generated by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. $\triangle$

Notice that the function $c \to \sigma_2(c)$ is defined modulo the addition of a fixed constant in $2\pi\mathbb{Z}$. We shall from now on assume that $\sigma_2(0) \in [0, 2\pi[$. This amounts to choosing the determination of the complex logarithm in accordance with the determination of τ_2.

Definition 3.1 *Let S be the unique smooth function defined in some neighbourhood of $0 \in \mathbb{R}^2$ such that $dS = \sigma$ and $S(0) = 0$. The Taylor expansion of S at $c = 0$ is called the* ***symplectic invariant*** *of Theorem 2.1. It is denoted by $(S)^\infty$.*

Remark 3.2. Using equation (6), one can interpret S as a *regularised action integral*:

$$S(c) = \mathscr{A}(c) - \mathscr{A}(0) + \Re(c \ln c - c).$$

$\triangle$

Remark 3.3. The formula (6) defines the 1-form $\tau = \tau_1 dc_1 + \tau_2 dc_2$ independently of the choice of the coordinate system (c_1, c_2). Another (standard) way of viewing this is the following. First let $\mathscr{B}$ be the set of regular leaves of $\mathscr{F}$, and π be the projection (which is a Lagrangian fibration) $\mathscr{F} \xrightarrow{\pi} \mathscr{B}$. The choice of a particular semi-global momentum map $F := (H_1, H_2)$ for the system (near a Lagrangian leaf $\Lambda_c := \pi^{-1}(c)$, for some $c \in \mathscr{B}$) is equivalent to the choice of a *local chart* ϕ for $\mathscr{B}$ near c: $F = \phi \circ \pi$.

Then for each $c \in \mathscr{B}$, $T_c^*\mathscr{B}$ acts naturally on Λ_c by the time-1 flows of the vector fields symplectically dual to the pull backs by π of the 1-forms in $T_c^*\mathscr{B}$. This action extends to a Hamiltonian action in a neighbourhood of Λ_c if and only if we restrict to closed 1-forms on $\mathscr{B}$. (In the local coordinates (c_1, c_2) of $\mathscr{B}$ given by the choice of a momentum map $F = (H_1, H_2)$, the constant 1-forms dc_1, dc_2 act by the flows of $\mathscr{X}_{H_1}$, $\mathscr{X}_{H_2}$, respectively).

The *stabiliser* of this action form a particularly interesting lattice in $T_c^*\mathscr{B}$, which is another representation of the "period lattice" [4]. It is the main point of the Liouville-Arnold theorem to show that, as c varies, the points of this lattice are associated to *closed* 1-forms, called *period 1-forms*. (Indeed, in action-angle coordinates, the period 1-forms have constant coefficients). In our case, the period lattice is computed using a local chart given by Eliasson's theorem. First we see that this lattice has a privileged direction given by the S^1-action of q_2. Then we construct a "minimal" basis of this lattice by choosing the generator of this S^1-action (ie $2\pi dc_2$) together with the smallest transversal vector τ that has positive coefficients on dc_1 and dc_2. This is what we have done in this section. △

4 Uniqueness

In order to show that the above invariant $(S)^\infty$ is indeed symplectic and uniquely defined by the foliation, we need to prove that it does not depend on any choice made to define them. A priori, $(S)^\infty = (S)^\infty(\mathscr{F}, \chi)$ depends on the foliation $\mathscr{F}$ and on the choice of the chart χ that puts a neighbourhood of the *focus-focus* point m into normal form. It follows from the definition that if φ is a symplectomorphism sending $\mathscr{F}$ to $\tilde{\mathscr{F}}$, then $(S)^\infty(\tilde{\mathscr{F}}, \tilde{\chi}) = (S)^\infty(\mathscr{F}, \tilde{\chi} \circ \varphi)$. So $(S)^\infty$ is well-defined as a symplectic invariant of $\mathscr{F}$ if and only if, for any choice of two chart χ and χ' putting a neighbourhood of m into normal form, $(S)^\infty(\mathscr{F}, \chi) = (S)^\infty(\mathscr{F}, \chi')$. This is guaranteed by the following lemma:

Lemma 4.1 *If φ is a local symplectomorphism of $(\mathbb{R}^4, 0)$ preserving the standard* focus-focus *foliation $\{q := (q_1, q_2) = const\}$ near the origin, then there exists a unique germ of diffeomorphism $G : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$q \circ \varphi = G \circ q, \tag{7}$$

and G is of the form $G = (G_1, G_2)$, where $G_2(c_1, c_2) = \varepsilon_2 c_2$ and $G_1(c_1, c_2) - \varepsilon_1 c_1$ is flat at the origin, with $\varepsilon_j = \pm 1$.

Remark 4.1. This uniqueness statement about Eliasson's normal form does not appear in [5]. △

Proof *of the lemma.* The existence of some unique G satisfying (7) is standard (because the leaves of the *focus-focus* foliation are locally connected around the origin). What interests us here are the last properties. As before, we use the complex coordinates $(z_1, z_2) \in \mathbb{C}^2 = \mathbb{R}^4$, and $c = \bar{z_1} z_2 \in \mathbb{C} = \mathbb{R}^2$. Let $\delta > 0$ be such that φ is defined in the box $\mathscr{B} = \{|z_1| \leqslant 2\delta, |z_2| \leqslant 2\delta\}$.

Since the flow of q_2 is 2π-periodic, (7) implies that the Hamiltonian vector field $\partial_1 G_2 \mathscr{X}_{q_1} + \partial_2 G_2 \mathscr{X}_{q_2}$ is also 2π-periodic (with 2π as a primitive period). But on Λ_0 the only linear combinations of $\mathscr{X}_{q_1}$ and $\mathscr{X}_{q_2}$ that are periodic are the integer multiples of $\mathscr{X}_{q_2}$. Hence $\partial_1 G_2(0) = 0$ and $\partial_2 G_2(0) = \pm 1$.

The flow of q_1 on Λ_0 is radial: any line segment $]0,A[$ for some $A \in \Lambda_0$ is a trajectory. Then by (7) it image by φ must be a trajectory of $G_1 \circ q$. Since φ is smooth at the origin, the image of $]0,A[$ for $A \in \mathscr{B}$ close enough to 0 lies in some proper sector of the plane $\Pi \subset \Lambda_0$ containing $\varphi(A)$ (Π is either $\{z_1 = 0\}$ or $\{z_2 = 0\}$). But the only linear combinations of $\mathscr{X}_{q_1}$ and $\mathscr{X}_{q_2}$ which yield trajectories that are confined in a proper sector of Π are the multiples of $\mathscr{X}_{q_1}$. Hence $\partial_2 G_1(0) = 0$. It follows now from the previous paragraph that $\partial_1 G_1(0) \neq 0$ (since G is a local diffeomorphism).

φ preserves the critical set of q; since left composition of φ by the symplectomorphism $(z_1, z_2) \to (-z_2, z_1)$ leaves (7) unchanged (except for the sign of G_1), we may assume that each "axis" ($\{z_2 = 0\}$ and $\{z_1 = 0\}$ respectively) is preserved by φ. But then $\{z_2 = 0\}$ is the local unstable manifold for both q_1 and $G_1(q_1, q_2)$, which says that $\partial_1 G_1(0) > 0$.

Using (2) and (3), it is immediate to check that the joint flow of (q_1, q_2) taken at the joint time $(-\ln|c/\delta|, \arg c)$ sends the point $(\bar{c}, \delta)$ to the point (δ, c), and hence extends to a smooth and single-valued map Φ from a neighbourhood of $(0, \delta)$ to a neighbourhood of $(\delta, 0)$.

$\varphi^{-1} \circ \Phi \circ \varphi$ sends a neighbourhood of $\varphi^{-1}(0, \delta) = (0, a)$ to a neighbourhood of $\varphi^{-1}(\delta, 0) = (b, 0)$ and, because of (7), it is equal – in the complement of the singular leaf Λ_0 – to the joint flow of $G \circ q$ at the joint time $(-\ln|c/\delta|, \arg c)$, which is equal to the joint flow of q at the joint time

$$(-\partial_1 G_1 \ln|c/\delta| + \partial_1 G_2 \arg c, -\partial_2 G_1 \ln|c/\delta| + \partial_2 G_2 \arg c).$$

Since $\varphi^{-1} \circ \Phi \circ \varphi$ is smooth at the origin, we obtain by restricting the first component of this map to the "Poincaré" surface $\{(\bar{c}, a)$ with c near 0 in $\mathbb{C}\}$ that the map:

$$c \to \exp\big((1 - \partial_1 G_1)\ln|c| + \partial_1 G_2 \arg c + i\big((\partial_2 G_2 - 1)\arg c - \partial_2 G_1 \ln|c|\big)\big) \quad (8)$$

is single-valued and smooth at the origin. (We have factored out the terms $\exp(\partial_j G_1 \ln \delta)$, $j = 1, 2$, which are obviously smooth.)

The single-valuedness of (8) implies that $\partial_1 G_2 \equiv 0$ and $\partial_2 G_2 \in \mathbb{Z}$. Hence $\partial_2 G_2 = \pm 1$.

Now the smoothness of (8) says that the following two functions:

$$c \to (1 - \partial_1 G_1)\ln|c| \qquad \text{and} \qquad c \to -\partial_2 G_1 \ln|c|$$

are smooth at the origin, which easily implies that $(1 - \partial_1 G_1)$ and $\partial_2 G_1$ are flat at the origin, yielding the result. □

Suppose we define two semi-global invariants $(S)^\infty(\mathscr{F}, \chi)$ and $(S)^\infty(\mathscr{F}, \tilde{\chi})$ by choosing two different charts χ and $\tilde{\chi}$ which put a neighbourhood of the *focus-focus* point into normal form. As before, one defines the momentum maps F and

$\tilde{F}$, which are the extensions to $\mathscr{F}$ of $q \circ \chi$ and $q \circ \tilde{\chi}$, and computes the corresponding period 1-forms τ and $\tilde{\tau}$. Then we can invoke the lemma to $\varphi = \tilde{\chi} \circ \chi^{-1}$, and the conclusions apply to $G = \tilde{F}F^{-1}$.

Suppose that $\varepsilon_j = 1$, $j = 1, 2$, i.e. G is infinitely tangent to the identity. Then the same type of arguments as above (a logarithm cannot compete against a flat term) shows that, since the vector fields $\mathscr{X}_{H_j}$ and $\mathscr{X}_{\tilde{H}_j}$ are infinitely tangent to each other, τ and $\tilde{\tau}$ must differ by a flat term. Actually, since by remark 3.3 $G^*\tilde{\tau}$ is also a period 1-form associated with the momentum map F, one has $\tau = G^*\tilde{\tau}$. This implies that $\sigma(c) = \tau(c) + \Re(\ln c dc)$ and $\tilde{\sigma} = (G^{-1})^*\sigma$ differ by a flat form, hence $(S)^\infty(\mathscr{F}, \chi) = (S)^\infty(\mathscr{F}, \tilde{\chi})$.

If $\varepsilon_2 = -1$, it suffices to compose with the symplectomorphism $(x, \xi) \to (-x, -\xi)$, which sends (q_1, q_2) to $(q_1, -q_2)$ and leaves σ invariant (both σ_2 and dc_2 change sign). An analogous remark holds with the symplectomorphism $(z_1, z_2) \to (-z_2, z_1)$, which sends (q_1, q_2) to $(-q_1, q_2)$ and leaves σ invariant, while changing the sign of ε_1.

5 Injectivity

Let $\mathscr{F}$ and $\tilde{\mathscr{F}}$ are two singular foliations of simple *focus-focus* type on the symplectic manifolds (M, ω) and $(\tilde{M}, \tilde{\omega})$. Assume that they have the same invariant $(S)^\infty(\mathscr{F}) = (S)^\infty(\tilde{\mathscr{F}}) \in \mathbb{R}[\![X, Y]\!]_0$. We shall prove here that $\mathscr{F}$ and $\tilde{\mathscr{F}}$ are semi-globally equivalent, ie. there exists a foliation preserving symplectomorphism between some neighbourhoods of the *focus-focus* leaves.

For each of the foliations $\mathscr{F}$ and $\tilde{\mathscr{F}}$, we choose a chart of Eliasson's type around the *focus-focus* point, and thus define the period 1-forms τ and $\tilde{\tau}$ on $(\mathbb{R}^2 \setminus \{0\}, 0)$. The hypothesis implies that there is a smooth closed 1-form $\pi = \pi_1 dc_1 + \pi_2 dc_2$ on $(\mathbb{R}^2, 0)$ whose coefficients are flat functions of c at the origin such that

$$\tilde{\tau} = \tau + \pi.$$

Lemma 5.1 *One can chose symplectic charts of Eliasson's type at the* focus-focus *points in such a way that $\pi = 0$, ie:*

$$\tilde{\tau} = \tau.$$

Proof. 1. We first prove that there exists a local diffeomorphism G of $(\mathbb{R}^2, 0)$ isotopic to the identity such that $(G^{-1})^*\tau = \tilde{\tau}$. We wish to realise G as G_1 where G_t is a flow satisfying

$$G_t^*(\tau + t\pi) = \tau.$$

This amounts to finding the associated vector field Y_t which must satisfy

$$d(\iota_{Y_t}(\tau + t\pi)) = -\pi.$$

We can write $\pi = dP$ for some smooth function P which is flat at 0. Assume we look for a field Y_t of the form $Y_t = f_t(c)\frac{\partial}{\partial c_1}$. We obtain the following equation:

$$f_t(c) = \frac{-P(c)}{\tau_1(c) + t\pi_1} = \frac{-P(c)}{\ln|c| - \sigma_1(c) + t\pi_1}.$$

Since P is flat at 0, the right-hand-side is indeed a (flat) smooth function depending smoothly on t, and the result is proved.

2. Notice also that G is infinitely tangent to the identity, and moreover leaves the second variable c_2 unchanged. Now we show that for any diffeomorphism G of $(\mathbb{R}^2, 0)$ sharing these properties (which are those of Lemma 4.1) there exists a symplectomorphism χ near the *focus-focus* point m such that

$$G(q_1, q_2) \circ \chi = (q_1, q_2).$$

Here again we seek χ as the time-1 map of the flow of some vector field X_t. Of course we shall look now for a Hamiltonian vector field $X_t = \mathscr{X}_{f_t}$ to ensure the symplecticity of χ_t. Then the requirement

$$\chi_t^* q_t = q_0,$$

where $q_t = (q_{t,1}, q_{t,2}) \stackrel{\text{def}}{=} tG(q_1, q_2) + (1-t)(q_1, q_2)$, leads to the following system

$$\begin{array}{rcl} \{f_t, q_{t,1}\} & = & g_1 \\ \{f_t, q_{t,2}\} & = & 0, \end{array}$$

with $(g_1, 0) = (q_1, q_2) - G(q_1, q_2)$. By hypothesis g_1 is a flat function at the origin, and the fact that $\{q_{t,1}, q_{t,2}\} \equiv 0$ implies that $\{g_1, q_{t,2}\} = 0$. Moreover the quadratic part of q_t is q_0, so we know (see [5]) that such a system admits a solution f_t.

It remains to put all our remarks together: Point 2) shows that left composition by χ of the Eliasson chart we have chosen at m is again an admissible chart of Eliasson's type, yielding the new momentum map $G(q_1, q_2)$. Using the G obtained at Point 1) and in view of the naturality property (remark 3.3), the new period 1-form (denoted by τ again) satisfies $\tau = \tilde{\tau}$. □

We are now is position to construct the required equivalence. Applying the lemma we get a local symplectomorphism that allows us to identify some neighbourhoods U and $\tilde{U}$ of the *focus-focus* points m and $\tilde{m}$, and two momentum maps F and $\tilde{F}$ (both equal to (q_1, q_2) inside their respective neighbourhoods of the *focus-focus* points) which define the same closed 1-form σ on $(\mathbb{R}^2, 0)$. We denote $\Lambda_c = F^{-1}(c)$ and $\tilde{\Lambda}_c = \tilde{F}^{-1}(c)$.

Let $\mathfrak{U}$ be an open ball strictly contained in U, let $\Sigma_u \subset \mathfrak{U}$ be a transversal section as defined in the proof of Proposition 3.1, and construct in the same way

$\tilde{\Sigma}_u$ for the foliation $\tilde{\mathscr{F}}$ (so that Σ_u and $\tilde{\Sigma}_u$ are identified by the above symplectomorphism). Reduce $\mathscr{F}$ (and $\tilde{\mathscr{F}}$) to the neighbourhoods of the *focus-focus* leaves composed of the leaves intersecting Σ_u (or $\tilde{\Sigma}_u$). We construct our equivalence by extending the identity outside $\mathfrak{U}$. Let $x \in \Lambda_c \setminus \mathfrak{U}$, and define $t(x) \in]0, \tau_1(c)[$ to be the smallest time it takes for the point $\Sigma_u \cap \Lambda_c$ to reach the $\mathscr{X}_{H_2}$-orbit of x. (Recall that H_2 generates an S^1 action.) Now define $s(x) \in \mathbb{R}/2\pi\mathbb{Z}$ as the remaining time to finally reach x under the $\mathscr{X}_{H_2}$-flow. To this x we associate the point $\tilde{x} \in \tilde{\mathscr{F}}$ obtained from the point $\tilde{\Sigma}_u \cap \tilde{\Lambda}_c$ by letting the joint flow of $\tilde{F}$ act during the times $(t(x), s(x))$. This map — let's call it Ψ — is well defined because of the equality $\tau = \tilde{\tau}$. It is a bijection since the inverse is equally well-defined just by interchanging the roles of $\mathscr{F}$ and $\tilde{\mathscr{F}}$. Between U and $\tilde{U}$, Ψ is a symplectomorphism since through Eliasson's charts, it is just the identity. Concerning now the symplecticity of Ψ in the complement of the singular points, one can prove it for $c \neq 0$ (which is sufficient by continuity) by invoking the Liouville-Arnold theorem, which shows that Ψ is symplectically conjugate to a translation in the fibres. Then the symplectic property near the singular points implies that this translation must be symplectic everywhere. A similar argument using the less sophisticated Darboux-Carathéodory theorem could also do. But the simplest is maybe the following. It is clear from the construction that Ψ is equivariant with respect to the joint flows of our Hamiltonian dynamics:

$$\forall (t_1, t_2), \qquad \Psi \circ \varphi_{t_1,t_2} = \tilde{\varphi}_{t_1,t_2} \circ \Psi, \tag{9}$$

where φ_{t_1,t_2} and $\tilde{\varphi}_{t_1,t_2}$ are the joint flows of F and $\tilde{F}$ at the joint time (t_1, t_2). Using (9) together with the fact that $\tilde{\varphi}_{t_1,t_2}$ is symplectic, we see that $\varphi^*_{t_1,t_2}(\Psi^*\tilde{\omega}) = \Psi^*\tilde{\omega}$; in other words, $\Psi^*\tilde{\omega}$ is invariant under the joint flow φ_{t_1,t_2}. Since ω has the same property, so has $\Psi^*\tilde{\omega} - \omega$. Since $\Psi^*\tilde{\omega} - \omega = 0$ near m, it must vanish as well on the whole $\mathscr{F}$.

6 Surjectivity

We prove here that any formal power series $(S)^\infty \in \mathbb{R}[\![X,Y]\!]_0$ is the symplectic invariant — in the sense of Definition 3.1 — of some Liouville foliation of simple *focus-focus* type. More precisely, we construct a foliation $\mathscr{F}$ together with a local chart χ that puts a neighbourhood of the *focus-focus* point into normal form such that (using the notation of Section 4) $(S)^\infty = (S)^\infty(\mathscr{F}, \chi)$. Another proof of this result has been proposed by Castano-Bernard [2].

Using the same notations as before, we let $(q_1, q_2) = \bar{z_1} z_2$ be the standard *focus-focus* fibration $\mathbb{R}^4 \simeq \mathbb{C}^2 \to \mathbb{C} \simeq \mathbb{R}^2$ defined in (1). The joint flow will be denoted by φ_{t_1,t_2}.

Invoking Borel's construction, let $S \in C^\infty(\mathbb{R}^2)$ be a function vanishing at the origin and whose Taylor series is $(S)^\infty$. We shall denote by S_1, S_2 the partial derivatives $\partial_X S$ and $\partial_Y S$, respectively.

Let us define two "Poincaré" surfaces in $\mathbb{C}^2$ by means of the following embeddings of the ball $D_\varepsilon = B(0,\varepsilon) \subset \mathbb{C}$, for some $\varepsilon \in]0,1[$:

$$\begin{aligned} &\Pi_1(c) = (\bar{c}, 1) \\ &\Pi_2(c) = (e^{S_1(c)+iS_2(c)}, ce^{-S_1(c)+iS_2(c)}). \end{aligned}$$

Notice that for each c, the points $\Pi_j(c)$, $j = 1,2$ belong to the (non-compact) Lagrangian submanifold $\Lambda_c := \{\bar{z_1} z_2 = c\}$. $\Pi_j(D_\varepsilon)$, $j = 1,2$ are smooth 2-dimensional manifolds constructed in such a way that for any $c \neq 0$, $\Pi_2(c)$ is the image of $\Pi_1(c)$ by the joint flow of (q_1, q_2) at the time $(S_1(c) - \ln|c|, S_2(c) + \arg(c))$.

Let Φ be this diffeomorphism, defined on all $\Pi_j(D_\varepsilon)$ by the embeddings:

$$\begin{array}{ccc} \Pi_1(D_\varepsilon) & \xrightarrow{\quad\Phi\quad} & \Pi_2(D_\varepsilon) \\ & {\scriptstyle \Pi_1} \nwarrow \quad \nearrow {\scriptstyle \Pi_2} & \\ & D_\varepsilon & \end{array}$$

$\Pi_1(D_\varepsilon)$ and $\Pi_2(D_\varepsilon)$ are transversal to the Lagrangian foliation, and Φ can be extended uniquely to a diffeomorphism between small neighbourhoods of $\Pi_1(D_\varepsilon)$ and $\Pi_2(D_\varepsilon)$ by requiring that it commute with the joint flow:

$$\Phi\left(\varphi_{t_1,t_2}(m)\right) = \varphi_{t_1,t_2}(\Phi(m)). \tag{10}$$

Lemma 6.1 *Φ is a symplectomorphism.*

Proof. One can write Φ in terms of Π_1 and Π_2 and check the result by explicit calculation. However, the reason why it works is the following:

Since we already know that Φ is smooth, it is enough to prove the lemma outside of the singular Lagrangian Λ_0. So fix $c_0 \neq 0$; we can construct a Darboux-Carathéodory chart $(x,\xi) \in \mathbb{R}^4$ in a connected open subset of Λ_{c_0} containing both $\Pi_1(c_0)$ and $\Pi_2(c_0)$. In these coordinates, the momentum map is (ξ_1, ξ_2) and the flow is linear: φ_{t_1,t_2} is the translation by (t_1,t_2) in the x variables.

Through this chart, Φ is by construction a "fibre translation":

$$\Phi(x,\xi) = (x + f(\xi), \xi), \tag{11}$$

where

$$f(\xi) = (S_1(\xi), S_2(\xi)) + (\ln|\xi|, -\arg(\xi)). \tag{12}$$

Now, it is easy to check that (11) defines a symplectomorphism if and only if the 1-form

$$f_1(\xi)d\xi_1 + f_2(\xi)d\xi_2$$

is closed. In our case the closedness is automatic since $S_1 dX + S_2 dY = dS$. $\square$

Let Σ_j, $j = 1,2$ be the S^1-orbit of $\Pi_j(D_\varepsilon)$. Construct a 4-dimensional cylinder $\mathscr{C}$ by letting the q_2-flow take Σ_1 to Σ_2, namely:

$$\mathscr{C} := \overline{\bigsqcup_{c \in D_\varepsilon \setminus \{0\}} \mathscr{C}_c}$$

where $\mathscr{C}_c \subset \Lambda_c$ is the 2-dimensional cylinder spanned by $\varphi_{t_1,t_2}(\Pi_1(c))$, for $(t_1,t_2) \in [0, S_1(c) + \ln|c|] \times [0, 2\pi]$. Finally, let M be the symplectic manifold obtained by

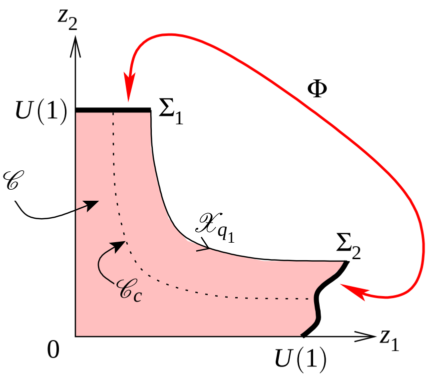

Figure 2: Construction of the symplectic manifold M

gluing the two ends Σ_j of the cylinder $\mathscr{C}$ using the symplectomorphism Φ. Since Φ preserves the momentum map (q_1, q_2), the latter yields a valid momentum map F on M. The corresponding Lagrangian foliation $F^{-1}(c)$ is given by $\mathscr{C}_c$ with its two ends identified by Φ. In particular all leaves are compact and the foliation is of simple *focus-focus* type.

The S^1 action is unchanged, while the transversal period $(\tau_1(c), \tau_2(c))$ on $F^{-1}(c)$ is by construction the time it takes for the joint flow to reach $\Pi_2(c)$ from $\Pi_1(c)$, ie

$$(\tau_1(c), \tau_2(c)) = (S_1(c) - \ln|c|, S_2(c) + \arg(c)).$$

Then by definition 3.1 the symplectic invariant of the foliation is given by the Taylor expansion of the primitive of the 1-form $S_1 dc_1 + S_2 dc_2$ vanishing at 0, ie. $(S)^\infty$.

7 Further remarks

Multiple *focus-focus*. Assume now that the singular fibre Λ_0 carries k *focus-focus* points $m_0, \ldots, m_{k-1}$. Then Λ_0 is a k-times pinched torus, and Theorem 2.1 can be generalised. In this case, the regularisation of the action integral S must take into account all the singular points. In order to do this, one has to consider $k-1$ local invariants, which are also formal power series in $\mathbb{R}[\![X,Y]\!]$, and which measure the obstruction to construct a semi-global momentum map that is in Eliasson normal form simultaneously at two different singular points. Here follows a sketch of the argument.

Let F be a semi-global momentum map. At each point m_j one has a local normal form $F \circ \varphi_j = G^j(q_1, q_2)$. Because of Lemma 4.1, one can extend q_2 to a periodic Hamiltonian on a whole neighbourhood of Λ_0, and one can always assume that φ_j is orientation preserving — that means we fix once and for all the sign of the ε_j. If now F if of the form (H_1, q_2) then G^j takes the form $G^j(q_1, q_2) = (F^j(q_1, q_2), q_2)$. By the implicit function theorem, F^j is locally invertible with respect to the variable q_1. Let $(F^j)^{-1}$ be this inverse, and define $G^{i,j} = (F^i)^{-1} F^j$. Again by Lemma 4.1, the Taylor expansions of $G^{i,j}$ are invariants of the foliation.

Assume the points m_i are ordered according to the flow of H_1, with indices $i \in \mathbb{Z}/k\mathbb{Z}$. Similarly to the case $k = 1$, one can define a regularised period 1-form σ by the following formula:

$$\sigma := \sum_{i=0}^{k-1} (G_0^{-1} G_i)^* \left(\sigma_1^{i,i+1}(c) dc_1 + \sigma_2^{i,i+1}(c) dc_2 \right), \tag{13}$$

with

$$\begin{cases} \sigma_1^{i,i+1}(c) &= \tau_1^{i,i+1}(c) + \Re(\ln c) \\ \sigma_2^{i,i+1}(c) &= \tau_2^{i,i+1}(c) - \Im(\ln c) \end{cases}, \tag{14}$$

where $(\tau_1^{i,i+1}(c), \tau_2^{i,i+1}(c))$ are the smallest positive times needed to reach $A_{i+1}(c)$ from $A_i(c)$ under the flow of $(G^i)^{-1} \circ F$ — which is the momentum map (q_1, q_2) in the normal form coordinates near point A_i. Here we have chosen a point $A_i(c)$ in a Poincaré section of each local stable manifold near m_i. Of course $\sigma_j^{i,i+1}(c)$ depends heavily on the choice of A_i and A_{i+1}, but the sums appearing in (13) does not, and the resulting 1-form σ is closed. Notice that the definition of σ depends on the choice of a start point m_0. Thus we are here classifying a singular foliation with a distinguished *focus-focus* point m_0.

Let $(S)^\infty$ be the Taylor series of the primitive of σ vanishing at the origin. Then $(S)^\infty$ and the $k-1$ ordered invariants $(G^{i,i+1})^\infty$ are independent and entirely classify a neighbourhood of the critical fibre Λ_0 with distinguished point m_0. The arguments of the proof are similar to the ones of the case $k = 1$. An abstract construction of a foliation admitting a given set of invariants is proposed in Figure 4.

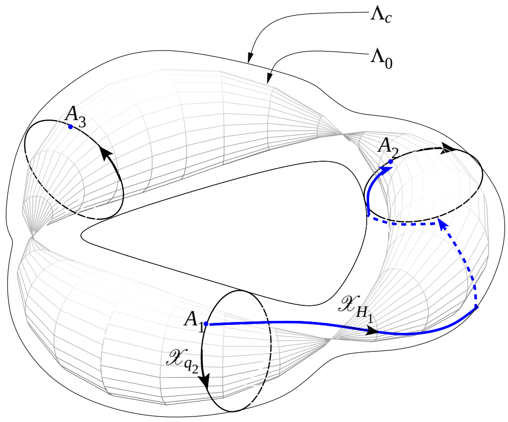

Figure 3: The multi-pinched torus

There the local pictures are described by canonical coordinates respectively given by (q_1,q_2), $(G^{1,2}(q_1,q_2),q_2)$, $(G^{1,2}(G^{2,3}(q_1,q_2),q_2),q_2)$, etc. and the gluing diffeomorphisms $\Phi_{i,i+1}$ are constructed as in section 6 using the following functions, respectively: $S_{0,1} = S_{1,2} = \cdots = S_{k-2,k-1} = 0$ and $S_{0,k-1}$ is a resummation of $(S)^\infty$.

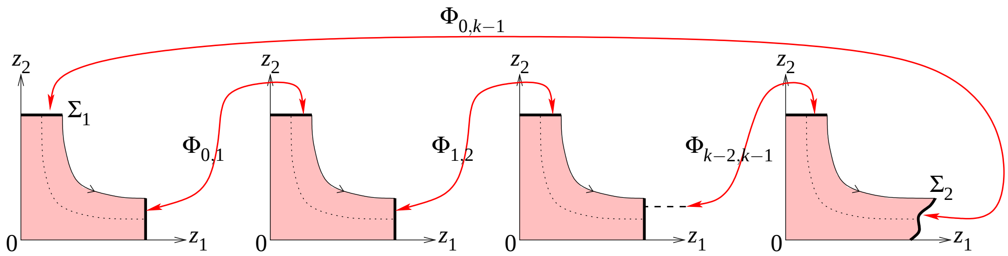

Figure 4: multiple gluing

Remark 7.1. We can regard the reduced space Λ_0/S^1 as a cyclic graph $\mathscr{G}$ whose vertices are the *focus-focus* points m_i, and which is oriented by the flow of H_1. For each edge $[i,i+1]$ one can define a 1-form

$$\sigma^{i,i+1} := (G_0^{-1}G_i)^* \left(\sigma_1^{i,i+1}dc_1 + \sigma_2^{i,i+1}dc_2\right) \in \Omega^1(D)$$

(for some fixed small disc D around the origin in $\mathbb{R}^2$). This defines a 1-cocycle on $\mathscr{G}$ with values in the vector space $\Omega^1(D)$. If one varies the points A_j, this cocycle is easily seen to change by a coboundary; hence the set of $\{\sigma^{i,i+1}\}$ naturally defines a well-defined cohomology class on $\mathscr{G}$. Be the same argument as in the case $k=1$ (ie. essentially Arnold-Liouville's theorem) this class is *closed*, in the sense that the cochain $\{\sigma^{i,i+1}\}$, modulo some coboundary, can be chosen to consist only of closed 1-forms. Hence we end up with a class $[\sigma] \in H^1(\mathscr{G}, H^1(D))$. Since $\mathscr{G}$ is homeomorphic to a circle, $H^1(\mathscr{G}, H^1(D)) \simeq H^1(D)$ and $[\sigma]$ is represented by the de Rham cohomology class of the closed 1-form $\sigma = \sum \sigma^{i,i+1}$ defined in (13).

Now, the functor that produces Taylor series of 1-forms can be applied to the coefficients of this cochain, yielding a cocycle with values in formal closed 1-forms and whose class is represented by the differential of our invariant $(S)^\infty$. △

"Exact" version. If one intends to extend the results to a semiclassical setting, general symplectomorphism do not suffice: one needs to control the action integrals (in the standard semiclassical pseudo-differential theory, a potential α for the symplectic form: $d\alpha = \omega$ is part of the data). In view of Remark 3.2, this is naturally done by including the constant term in the Taylor series of S as being the integral

$$S_0 := \int_{\gamma_0} \alpha,$$

where γ_0 is the generator of $H_1(\Lambda_0)$.

Acknowledgements. This article answers a question that J.J. Duistermaat asked me when I was in Utrecht in 1998. I wrote then a short — and incomplete — draft, and that was it. Two years after I finally read the note by Toulet and al. [3], and a fruitful discussion with Richard Cushman made me realise that I had the result at hand. I wish to thank him for this. I would also like to thank Ricardo Castano-Bernard for interesting discussions, and for showing me an alternative proof of the "surjectivity" part using general arguments developed for mirror symmetry via special Lagrangian fibrations.

After I wrote this article, P. Molino informed me of an unpublished work of his (in collaboration with one of his students [7]) concerning the same problem. They defined a similar invariant, and the classification result was conjectured.

References

[1] A.V. Bolsinov and A.T. Fomenko. Orbital equivalence of integrable hamiltonian systems with two degrees of freedom. a classification theorem I. *Russian Acad. Sci. Sb. Math.*, 81(2):421–465, 1995. (translated from the Russian).

[2] R. Castano-Bernard, 2000. personal communication.

[3] J.-P. Dufour, P. Molino, and A. Toulet. Classification des systèmes intégrables en dimension 2 et invariants des modèles de Fomenko. *C. R. Acad. Sci. Paris Sér. I Math.*, 318:949–952, 1994.

[4] J.J. Duistermaat. On global action-angle variables. *Comm. Pure Appl. Math.*, 33:687–706, 1980.

[5] L.H. Eliasson. *Hamiltonian systems with Poisson commuting integrals*. PhD thesis, University of Stockholm, 1984.

[6] L.H. Eliasson. Normal forms for hamiltonian systems with Poisson commuting integrals – elliptic case. *Comment. Math. Helv.*, 65:4–35, 1990.

[7] D. Grossi. Systèmes intégrables non dégénérés — invariants symplectiques d'une singularité focus-focus simple. Mémoire de DEA, université de Montpellier, 1995.

[8] Z. Nguyên Tiên. A topological classification of integrable hamiltonian systems. In R. Brouzet, editor, *Séminaire Gaston Darboux de géometrie et topologie différentielle*, pages 43–54. Université Montpellier II, 1994-1995.

[9] Z. Nguyên Tiên. A note on focus-focus singularities. *Diff. Geom. Appl.*, 7(2):123–130, 1997.

[10] S. Vũ Ngọc. Bohr-Sommerfeld conditions for integrable systems with critical manifolds of focus-focus type. *Comm. Pure Appl. Math.*, 53(2):143–217, 2000.

[11] S. Vũ Ngọc. Formes normales semi-classiques des systèmes complètement intégrables au voisinage d'un point critique de l'application moment. *Asymptotic Analysis*, 24(3,4):319–342, 2000.

[12] A. Weinstein. Symplectic manifolds and their lagrangian submanifolds. *Adv. in Math.*, 6:329–346, 1971.