Diophantine tori and spectral asymptotics for non-selfadjoint operators

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Abstract

We study spectral asymptotics for small non-selfadjoint perturbations of selfadjoint h-pseudodifferential operators in dimension 2, assuming that the classical flow of the unperturbed part possesses several invariant Lagrangian tori enjoying a Diophantine property. We get complete asymptotic expansions for all eigenvalues in certain rectangles in the complex plane in two different cases: in the first case, we assume that the strength ϵ of the perturbation is $\mathcal{O}(h^\delta)$ for some $\delta > 0$ and is bounded from below by a fixed positive power of h. In the second case, ϵ is assumed to be sufficiently small but independent of h, and we describe the eigenvalues completely in a fixed h-independent domain in the complex spectral plane.

Keywords and Phrases: Non-selfadjoint, eigenvalue, spectral asymptotics, Lagrangian torus, Diophantine condition, completely integrable, KAM

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1 Introduction and statement of main results

Recently there has been a large number of new developments for non-selfadjoint problems. These include semiclassical spectral asymptotics for non-selfadjoint operators in low dimensions [22], [29], [28], [40], [35], resolvent estimates and pseudospectral behavior [15], [13], [6], spectral instability questions [19], [34], and evolution problems and decay to equilibrium for the Fokker-Planck operator [20]. The purpose of this work is to continue a line of development initiated in [28], which opened up the possibility of carrying out a spectral analysis for non-selfadjoint operators in dimension two, that is as precise as the corresponding analysis for selfadjoint problems in dimension one. In [28], it was established that for a wide and stable class of non-selfadjoint operators in dimension two, it is possible to describe all eigenvalues individually in a fixed domain in the complex plane, by means of a Bohr-Sommerfeld quantization condition. The underlying reason for this result is a version of the KAM theorem without small divisors, in a complex domain.

The work [28] has been continued in a series of papers [41], [23], [24], [25], all of them done in the context of small non-selfadjoint perturbations of selfadjoint operators, with the important additional assumption that the classical flow of the leading symbol of the unperturbed part should be periodic in some energy shell. While the case of a periodic classical flow is very special indeed, in the aforementioned works, we have already given some applications of the general results to spectral asymptotics for damped wave equations on analytic Zoll surfaces [39], [21], while barrier top resonances for semiclassical Schrödinger operators have been treated in [26].

Now a classical Hamiltonian with a periodic flow can be naturally viewed as a degenerate case of a completely integrable symbol, and an even more general and much more interesting dynamical situation occurs when considering a symbol that is merely close to a completely integrable one. Continuing our previous works, in this case it seems to be of interest to study the spectrum of non-selfadjoint operators that are small perturbations of a selfadjoint operator, whose classical flow is close to a completely integrable one. The present work is the first one where we begin to study this problem, and when doing so, as our starting point, we shall take a general assumption that the real energy surface of the unperturbed leading symbol contains several flow invariant Lagrangian tori satisfying a Diophantine condition.

According to a classical theorem of Kolmogorov [3], the existence of such invariant tori is guaranteed when the unperturbed symbol in question is close to a completely integrable non-degenerate one.

We shall begin by describing the general assumptions on our operators, which will be the same as in [23], [24], and [25]. Let M denote $\mathbf{R}^2$ or a compact real-analytic manifold of dimension 2. We shall let $\widetilde{M}$ stand for a complexification of M, so that $M = \mathbf{C}^2$ in the Euclidean case, and in the manifold case, $\widetilde{M}$ is a Grauert tube of M.

When $M = \mathbf{R}^2$, let

$$P_\epsilon = P(x, hD_x, \epsilon; h) \tag{1.1}$$

be the Weyl quantization on $\mathbf{R}^2$ of a symbol $P(x, \xi, \epsilon; h)$ depending smoothly on $\epsilon \in \mathrm{neigh}\,(0, \mathbf{R})$ with values in the space of holomorphic functions of (x, ξ) in a tubular neighborhood of $\mathbf{R}^4$ in $\mathbf{C}^4$, with

$$|P(x, \xi, \epsilon; h)| \leq \mathcal{O}(1) m(\mathrm{Re}\,(x, \xi)) \tag{1.2}$$

there. Here m is assumed to be an order function on $\mathbf{R}^4$, in the sense that $m > 0$ and

$$m(X) \leq C_0 \langle X - Y \rangle^{N_0} m(Y),\ X, Y \in \mathbf{R}^4, \quad C_0,\ N_0 > 0. \tag{1.3}$$

We also assume that

$$m \geq 1. \tag{1.4}$$

We further assume that

$$P(x, \xi, \epsilon; h) \sim \sum_{j=0}^{\infty} p_{j,\epsilon}(x, \xi) h^j,\ h \to 0, \tag{1.5}$$

in the space of such functions. We make the ellipticity assumption

$$|p_{0,\epsilon}(x, \xi)| \geq \frac{1}{C} m(\mathrm{Re}\,(x, \xi)),\ |(x, \xi)| \geq C, \tag{1.6}$$

for some $C > 0$.

When M is a compact manifold, we let P_ϵ be a differential operator on M, such that for every choice of local coordinates, centered at some point of M, it takes the form

$$P_\epsilon = \sum_{|\alpha| \leq m} a_{\alpha,\epsilon}(x; h)(hD_x)^\alpha, \tag{1.7}$$

where $a_{\alpha,\epsilon}(x;h)$ is a smooth function of ϵ with values in the space of bounded holomorphic functions in a complex neighborhood of $x = 0$. We further assume that

$$a_{\alpha,\epsilon}(x;h) \sim \sum_{j=0}^{\infty} a_{\alpha,\epsilon,j}(x)h^j, \;\; h \to 0, \tag{1.8}$$

in the space of such functions. The semi-classical principal symbol $p_{0,\epsilon}$, defined on T^*M, takes the form

$$p_{0,\epsilon}(x,\xi) = \sum a_{\alpha,\epsilon,0}(x)\xi^\alpha, \tag{1.9}$$

if (x, ξ) are canonical coordinates on T^*M, and we make the ellipticity assumption

$$|p_{0,\epsilon}(x,\xi)| \geq \frac{1}{C}\langle\xi\rangle^m, \;\; (x,\xi) \in T^*M, \; |\xi| \geq C, \tag{1.10}$$

for some large $C > 0$. (Here we assume that M has been equipped with some Riemannian metric, so that $|\xi|$ and $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$ are well-defined.)

Sometimes, we write p_ϵ for $p_{0,\epsilon}$ and simply p for $p_{0,0}$. Assume

$$P_{\epsilon=0} \text{ is formally selfadjoint.} \tag{1.11}$$

In the case when M is compact, we let the underlying Hilbert space be $L^2(M, \mu(dx))$ for some positive real-analytic density $\mu(dx)$ on M.

Under these assumptions, P_ϵ will have discrete spectrum in some fixed neighborhood of $0 \in \mathbf{C}$, when $h > 0, \epsilon \geq 0$ are sufficiently small, and the spectrum in this region will be contained in a band $|\operatorname{Im} z| \leq \mathcal{O}(\epsilon)$.

Assume for simplicity that (with $p = p_{\epsilon=0}$)

$$p^{-1}(0) \cap T^*M \text{ is connected,} \tag{1.12}$$

and let us also assume that the energy level $E = 0$ is non-critical, so that $dp \neq 0$ along $p^{-1}(0) \cap T^*M$.

Let $H_p = p'_\xi \cdot \frac{\partial}{\partial x} - p'_x \cdot \frac{\partial}{\partial \xi}$ be the Hamilton field of p. We introduce the following hypothesis, assumed to hold throughout this work:

> The set $p^{-1}(0) \cap T^*M$ contains finitely many analytic H_p-invariant Lagrangian tori Λ_j, $1 \leq j \leq L$, such that each Λ_j carries real analytic coordinates x_1, x_2 identifying Λ_j with $\mathbf{T}^2$ so that along Λ_j, we have, (1.13)
>
> $$H_p = a_1\partial_{x_1} + a_2\partial_{x_2}, \tag{1.14}$$

where a_1, $a_2 \in \mathbf{R}$ satisfy the Diophantine condition,

$$|a \cdot k| \geq \frac{1}{C_0 \, |k|^{N_0}}, \quad 0 \neq k \in \mathbf{Z}^2, \tag{1.15}$$

for some fixed C_0, $N_0 > 0$. Here $\mathbf{T}^2 = \mathbf{R}^2/2\pi\mathbf{Z}^2$ is the standard 2-torus.

We write out the first few terms in a Taylor expansion of p_ϵ in a neighborhood of $p^{-1}(0) \cap T^*M$,

$$p_\epsilon = p + i\epsilon q + \mathcal{O}(\epsilon^2). \tag{1.16}$$

When $0 \leq K \in C_0^\infty(\mathbf{R})$ is such that $\int K(t)\, dt = 1$ and $T > 0$, we introduce a "smoothed out" flow average of q,

$$\langle q \rangle_{T,K,p} = \langle q \rangle_{T,K} = \int K_T(-t) q \circ \exp(tH_p)\, dt, \quad K_T(t) = \frac{1}{T} K\left(\frac{t}{T}\right), \tag{1.17}$$

defined near $p^{-1}(0) \cap T^*M$. The standard flow average corresponds to taking $K = 1_{[-1,0]}$, and we shall then write $\langle q \rangle_{T,K} = \langle q \rangle_T$.

Let G_T be an analytic function, defined in a neighborhood of $p^{-1}(0) \cap T^*M$, such that

$$H_p G_T = q - \langle q \rangle_{T,K}.$$

This is a convolution equation along the H_p–trajectories, and as in [39] and [24], we solve it by setting

$$G_T = \int T J_T(-t) q \circ \exp(tH_p)\, dt, \quad J_T(t) = \frac{1}{T} J\left(\frac{t}{T}\right), \tag{1.18}$$

where the function J is compactly supported, smooth away from 0, with

$$J'(t) = \delta(t) - K(t).$$

Composing the principal symbol (1.16) with the holomorphic canonical transformation $\exp(i\epsilon H_{G_T})$, and conjugating the operator P_ϵ by means of the corresponding Fourier integral operator $U_\epsilon = e^{\frac{\epsilon}{h} G_T(x, hD_x)}$, defined microlocally near $p^{-1}(0) \cap T^*M$, we may reduce our operator to a new one, still denoted by P_ϵ, which has the principal symbol

$$p_\epsilon \circ \exp(i\epsilon H_{G_T}) = p + i\epsilon \langle q \rangle_{T,K} + \mathcal{O}_T(\epsilon^2).$$

Moreover, it is still true that $P_{\epsilon=0}$ is the original selfadjoint operator. Repeating an argument, explained for example in [39], which makes use of the sharp Gårding inequality, we obtain a first localization of the spectrum of P_ϵ: if $z \in \mathbf{C}$ in the spectrum of P_ϵ is such that $|\mathrm{Re}\, z| \leq \delta$, then as $\epsilon, \delta, h \to 0$,

$$\frac{\mathrm{Im}\, z}{\epsilon} \in \left[\lim_{T\to\infty} \inf_{p^{-1}(0)} \mathrm{Re}\, \langle q \rangle_{T,K} - o(1), \lim_{T\to\infty} \sup_{p^{-1}(0)} \mathrm{Re}\, \langle q \rangle_{T,K} + o(1) \right]. \tag{1.19}$$

This estimate remains valid for $K = 1_{[-1,0]}$. Let us also notice that along the diophantine torus Λ_j, $1 \leq j \leq L$, we have uniformly, as $T \to \infty$,

$$\langle q \rangle_T = F_j + \mathcal{O}\left(\frac{1}{T}\right). \tag{1.20}$$

Here F_j is the mean value of q over Λ_j, computed with respect to the natural smooth flow-invariant measure on Λ_j, with respect to which the H_p-flow on Λ_j is ergodic. In the case when $K \in C_0^\infty(\mathbf{R})$, using the rapid decay of $\widehat{K}$, it is easy to see that (1.20) improves to

$$\langle q \rangle_{T,K} = F_j + \mathcal{O}\left(\frac{1}{T^\infty}\right).$$

We shall assume from now on that

$$F_1 = F_2 = \ldots = F_L \quad \text{is independent of } j, \tag{1.21}$$

and we write then $F_j = F$, $1 \leq j \leq L$.

As will be explained in section 2, for each j, there exists a smooth canonical transformation

$$\kappa_{\infty,j} : \mathrm{neigh}(\Lambda_j, T^*M) \to \mathrm{neigh}(\xi = 0, T^*\mathbf{T}^2),$$

mapping Λ_j to $\xi = 0$, such that

$$p_\epsilon \circ \kappa_{\infty,j}^{-1} = p_{\infty,j}(\xi) + i\epsilon q_j(x,\xi) + \mathcal{O}(\epsilon^2) + \mathcal{O}(\xi^\infty), \tag{1.22}$$

and $p_{\infty,j}(\xi) = a \cdot \xi + \mathcal{O}(\xi^2)$, with a satisfying (1.15). We furthermore may assume that the energy surface $p_{\infty,j}^{-1}(0)$ has the form $\xi_2 = f_j(\xi_1)$, for some smooth function f_j, with $f_j(0) = 0$, $f_j'(0) \neq 0$. Using the coordinates ξ_1, ξ_2, we define

$$\langle q_j \rangle(\xi) = \frac{1}{(2\pi)^2} \int q_j(x,\xi)\, dx. \tag{1.23}$$

For each small $\delta > 0$, we use the coordinate functions $\xi_1 \circ \kappa_{\infty,j}$ and $\xi_2 \circ \kappa_{\infty,j}$ near Λ_j to decompose the real energy surface as follows,

$$p^{-1}(0) = \Omega_-(\delta) \cup \Lambda_\delta \cup \Omega_+(\delta),$$

where

$$\Lambda_\delta = p^{-1}(0) \cap \bigcup_{j=1}^{L} (\xi_1 \circ \kappa_{\infty,j})^{-1}((-\delta, \delta)).$$

Here the sets $\Omega_\pm(\delta)$ are disjoint, compact, with finitely many connected components, while in general they are not invariant under the H_p–flow.

Recall that F stands for the common value of the average of q over the tori Λ_j, $1 \leq j \leq L$. We introduce the following global assumption:

There exist N_1, $N_2 \in \mathbf{N}\backslash\{0\}$ and a sequence $\delta = \delta_j \to 0$ such that (1.24)

$$\inf_{\Omega_+(\delta)} \left(\langle \mathrm{Re}\, q\rangle_{\delta^{-N_1},K} - \mathrm{Re}\, F\right) \geq \delta^{N_2}, \quad \sup_{\Omega_-(\delta)} \left(\langle \mathrm{Re}\, q\rangle_{\delta^{-N_1},K} - \mathrm{Re}\, F\right) \leq -\delta^{N_2}.$$

Here $0 \leq K \in C_0^\infty$ is as in (1.17), and we adopt the convention that when $N_2 = 1$, $\pm\delta^{N_2}$ in the right hand side of (1.24) should be replaced by $\pm\delta/C_1$ for some $C_1 > 0$.

Theorem 1.1 *Let $\alpha_{1,j}$ and $\alpha_{2,j}$ be the fundamental cycles in Λ_j, $1 \leq j \leq L$, defined by*

$$\kappa_{\infty,j}(\alpha_{k,j}) = \{x \in \mathbf{T}^2; x_k = 0\}, \quad k = 1, 2.$$

We write then $S_j = (S_{1,j}, S_{2,j})$ and $k_j = (k(\alpha_{1,j}), k(\alpha_{2,j}))$ for the values of the actions and the Maslov indices of the cycles, respectively. Let us make the global dynamical assumption (1.24), *and assume that the differentials of the functions $p_{\infty,j}$ and* $\mathrm{Re}\,\langle q_j\rangle$, *defined in* (1.22) *and* (1.23) *are linearly independent when $\xi = 0$, $1 \leq j \leq L$. Assume furthermore that $\epsilon = \mathcal{O}(h^\delta)$, $\delta > 0$, satisfies $\epsilon \geq h^K$, for some K fixed but arbitrarily large. Let $C > 0$ be sufficiently large. Then the eigenvalues of P_ϵ in the rectangle*

$$|\mathrm{Re}\, z| < \frac{h^\delta}{C}, \quad |\mathrm{Im}\, z - \epsilon \mathrm{Re}\, F| < \frac{\epsilon h^\delta}{C} \tag{1.25}$$

are given by

$$P_j^{(\infty)}\left(h\left(k - \frac{k_j}{4}\right) - \frac{S_j}{2\pi}, \epsilon; h\right) + \mathcal{O}(h^\infty), \quad k \in \mathbf{Z}^2, \quad 1 \leq j \leq L.$$

Here $P_j^{(\infty)}(\xi, \epsilon; h)$ is smooth in $\xi \in \mathrm{neigh}(0, \mathbf{R}^2)$ and $\epsilon \in \mathrm{neigh}(0, \mathbf{R})$, real-valued for $\epsilon = 0$, and has an asymptotic expansion in the space of such functions,

$$P_j^{(\infty)}(\xi, \epsilon; h) \sim \sum_{l=0}^\infty h^l p_{j,l}^{(\infty)}(\xi, \epsilon), \quad 1 \leq j \leq L.$$

We have

$$p_{j,0}^{(\infty)}(\xi, \epsilon) = p_{\infty,j}(\xi) + i\epsilon\langle q_j\rangle(\xi) + \mathcal{O}(\epsilon^2).$$

Our next result treats the case when the strength of the perturbation ϵ is sufficiently small but independent of h. In this case, we obtain a complete spectral result in a fixed h-independent domain.

Theorem 1.2 *Let us continue to write S_j and k_j for the actions and Maslov indices of the fundamental cycles in Λ_j, $1 \leq j \leq L$. Assume that $h^{1/3-\delta} < \epsilon \leq \epsilon_0 \ll 1$, for some $\delta > 0$. As in Theorem 1.1, we make the assumption (1.24) and assume that the differentials of $p_{\infty,j}(\xi)$ and $\mathrm{Re}\,\langle q_j\rangle(\xi)$ are linearly independent for $\xi = 0$, $1 \leq j \leq L$. Let $C > 0$ be large enough. Then the eigenvalues of P_ϵ in*

$$|\mathrm{Re}\, z| \leq \frac{\epsilon^{1/\widetilde{N}}}{C}, \quad \left|\frac{\mathrm{Im}\, z}{\epsilon} - \mathrm{Re}\, F\right| \leq \frac{\epsilon^{1/\widetilde{N}}}{C} \tag{1.26}$$

are given by

$$z(j,k) \sim \sum_{n=0}^{\infty} h^n \widetilde{p}_{j,n}^{(\infty)} \left(h\left(k - \frac{k_j}{4}\right) - \frac{S_j}{2\pi}, \epsilon \right), \quad k \in \mathbf{Z}^2,\ 1 \leq j \leq L,$$

with

$$\widetilde{p}_{j,n}^{(\infty)}(\xi,\epsilon) = \mathcal{O}(\epsilon^{-2(n-1)_+ - n/\widetilde{N}}),\ n = 0,1,2,\dots, \quad 1 \leq j \leq L,$$

holomorphic for $\xi = \mathcal{O}(\epsilon^{1/\widetilde{N}})$, and

$$\widetilde{p}_{j,0}^{(\infty)}(\xi,\epsilon) = p_j(\xi) + i\epsilon q_j(\xi,\epsilon) + \mathcal{O}\left(\epsilon^{\frac{N}{\widetilde{N}}-1}\right).$$

Here p_j is real on the real domain and the differentials of $p_j(\xi)$ and $\mathrm{Re}\, q_j(\xi,\epsilon)$ are linearly independent when $\xi = \epsilon = 0$, $1 \leq j \leq L$. We have

$$p_j(\xi) + i\epsilon q_j(\xi,\epsilon) = a \cdot \xi + i\epsilon F + \mathcal{O}((\xi,\epsilon)^2), \quad a = a_j.$$

The parameters $\widetilde{N}$ and $N/\widetilde{N}$ can be taken arbitrarily large.

Remark. As we shall see in section 2, the assumption (1.24) implies the existence of a suitable weight function, which allows us to microlocalize the spectral problem for P_ϵ in a rectangle of the form (1.25) or (1.26) to a small neighborhood of the union of the tori Λ_j, $1 \leq j \leq L$—see Proposition 2.3. Indeed, it is the existence of the weight function that allows us to carry out the complete spectral analysis in such rectangles. The purpose of the condition (1.24) is to provide an explicit criterion of a purely dynamical nature, which suffices for the construction of the global weight. As will also be seen in section 2, in the case when the H_p-flow is completely integrable, the condition (1.24) can be replaced by a slightly different, although equivalent one, having the advantage of being easier to verify in practice—see (2.53) below, Proposition 2.5, and the discussion in section 7. In particular, in section 7, we show that in the completely integrable case, the set of values F in (1.21) to which Theorem 1.1 and Theorem 1.2 apply, covers the entire region (1.19), apart from a subset of a suitably small measure. Such a result is stated in Theorem 7.6, where, using the isoenergetic KAM theorem, we also extend it to the perturbed situation, when the completely integrable symbol p is replaced by $p + \mathcal{O}(\lambda)$, with $\lambda > 0$ small enough.

The construction of quasimodes associated to invariant Diophantine Lagrangian tori and to Cantor families of such tori has a long tradition in the selfadjoint case, and we refer to [18], [10], [27], and [32] for the results in this direction. Especially relevant in the present two-dimensional context is the work of Shnirelman [36] in the selfadjoint case, which contains the idea of using invariant Lagrangian tori to split up the real energy surface into different invariant regions. See also [11]. In [36], the main focus is on constructing quasimodes associated with the gaps between the invariant tori in a perturbative situation. In our non-selfadjoint case, the idea of using the invariant tori as separatrices becomes more efficient than in the standard selfadjoint setting, and in a future work we plan to use it to study the global distribution of eigenvalues inside the entire band corresponding to different invariant tori.

In the present work, we essentially exploit only the invariant tori (1.13), corresponding to a given value of the average of the leading perturbation, and obtain complete spectral results in the spirit of our previous works [41], [23], and [24]. In particular, the work [24] introduced and exploited a dynamical condition somewhat reminiscent of (1.24). Technically speaking, many important differences occur, however, due to the more general behavior of the classical flow away from the union of the invariant tori (1.13).

The plan of the paper is as follows.

In section 2, we use the global dynamical condition (1.24) to construct a globally defined compactly supported weight function, which allows us to microlocalize the spectral problem for P_ϵ to a small neighborhood of the union of the Λ_j's, $1 \leq j \leq L$. We also give a discussion of a modified version of (1.24), when the H_p-flow is completely integrable.

In section 3, we carry out a construction of a quantum Birkhoff normal form for P_ϵ, valid to an arbitrarily high order along the invariant torus, and in (ϵ, h).

In section 4, we solve an appropriate Hamilton-Jacobi equation, to be used in the spectral analysis of P_ϵ in the case when ϵ is sufficiently small but independent of h. The ideas used here are similar to [28] and [41].

In section 5, we justify the eigenvalue computation based on the Birkhoff normal form construction from section 3 and prove Theorem 1.1, by solving a suitable global Grushin problem.

In section 6, we use the results of section 4 to modify the Grushin analysis from the previous section, in order to establish Theorem 1.2.

In section 7, we first study the completely integrable case and verify the global dynamical condition (1.24), or rather (2.53) below, under some general assumptions. This is then applied to the case of convex analytic surfaces of revolution, and complex perturbations, close to rotationally symmetric ones. We then exploit the isoenergetic KAM theorem and study the dynamical condition in the case when the unperturbed symbol is close to a completely integrable one. The results of this section are summarized in Theorem 7.6, which together with Theorem 1.1 and Theorem 1.2, can be considered as the final result of the present work.

In section 8, we give an application of Theorem 1.2 to the barrier top resonances for the semiclassical Schrödinger operator and obtain an extension of the result of [26] to an h-independent region in the complex plane.

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2 Construction of the global weight

In [23] and [24], the basic weight function was coming from an averaging procedure along the H_p–flow, and was given by G_T, defined in (1.18), with T being a common period for the flow. This weight was employed in a full neighborhood of $p^{-1}(0) \cap T^*M$. As we shall see in section 3, in the present case, the Birkhoff normal form construction will in general be valid only to an infinite order along an invariant torus, which will force us to work in small h-dependent neighborhoods of the tori. We shall therefore find it convenient to reduce q to the flow average $\langle q \rangle_{T,K}$ by means of G_T when away from a small but fixed neighborhood of the union of Λ_j, $1 \leq j \leq L$, while near each Λ_j, we shall use a somewhat different sort of average, which is close to $\langle q \rangle_{T,K}$, when T is large.

We keep all the assumptions of the introduction, and consider an operator P_ϵ

with the leading symbol

$$p + i\epsilon q + \mathcal{O}(\epsilon^2), \tag{2.1}$$

microlocally in a neighborhood of $p^{-1}(0) \cap T^*M$.

Let us say that a smooth multi-valued function defined on or near a torus is grad-periodic if its gradient is single-valued. In what follows, we shall work microlocally near a fixed invariant torus, say Λ_1. From the assumption (1.13), we recall that there exist analytic and grad-periodic coordinates on Λ_1, x_1 and x_2, which induce an identification between Λ_1 and $\mathbf{T}^2$. Applying Weinstein's tubular neighborhood theorem, see [9], or simply following the argument, described in the beginning of section 1 of [28], we see that we can extend x_1 and x_2 to a tubular neighborhood of Λ_1 and complete them with analytic functions ξ_1 and ξ_2, defined near Λ_1 and vanishing on this set, so that we get a real analytic canonical transformation,

$$\kappa_1 : \mathrm{neigh}(\Lambda_1, T^*M) \to \mathrm{neigh}(\xi = 0, T^*\mathbf{T}^2), \tag{2.2}$$

which maps Λ_1 to the zero section in $T^*\mathbf{T}^2$, and such that when expressed in terms of the coordinates x and ξ, the unperturbed leading symbol p of $P_{\epsilon=0}$ becomes

$$p(x,\xi) = a \cdot \xi + \mathcal{O}(\xi^2), \tag{2.3}$$

with a satisfying (1.15). It will now be convenient to perform an additional real canonical transformation on $T^*\mathbf{T}^2$, in order to make p independent of x to a high order in ξ. To this end, we observe that a straightforward Birkhoff normal form construction, very similar to the one described in detail in section 3, shows that there exists a sequence of real-valued functions $G_1, G_2 \ldots$, with G_j being homogeneous of degree $j+1$ in ξ and depending analytically on x, such that if

$$G \sim G_1 + G_2 + \ldots\,,$$

then at the level of formal Taylor expansions in ξ, we have

$$p \circ \exp(H_G) = p_\infty(\xi) + \mathcal{O}(\xi^\infty),$$

where $p_\infty(\xi) = a \cdot \xi + \mathcal{O}(\xi^2)$ depends on ξ only. See also (1.22). Since we work in the analytic category and do not wish to consider convergence questions for the Birkhoff normal forms and associated canonical transformations (in this connection, see [30]), we truncate the series at some fixed but arbitrarily large order N, and write

$$p \circ \exp(H_{G_{(N)}}) = p_N(\xi) + \mathcal{O}(\xi^{N+1}),$$

with $G_{(N)} = G_1 + G_2 + \ldots + G_{N-1}$, and $p_N(\xi) = a \cdot \xi + \mathcal{O}(\xi^2)$ independent of x. Notice that since $G_{(N)}(\xi) = \mathcal{O}(\xi^2)$, the analytic canonical transformation

$$\kappa^{(N)} := \exp\left(H_{G_{(N)}}\right) \tag{2.4}$$

maps the zero section $\xi = 0$ to itself and it also preserves the action integrals along closed loops. The composition of the transforms, $\kappa^{(N)} \circ \kappa_1$, with κ_1 defined in (2.2), maps symplectically a neighborhood of Λ_1 in T^*M onto a neighborhood of $\xi = 0$ in $T^*\mathbf{T}^2$, in such a way that Λ_1 is mapped onto the zero section $\xi = 0$. Implementing

$$\kappa_{N,1} = \kappa^{(N)} \circ \kappa_1 \tag{2.5}$$

by means of a microlocally unitary Fourier integral operator with a real phase, U, and conjugating P_ϵ by means of U, we obtain a new operator, still denoted by P_ϵ, which is microlocally defined near the zero section in $T^*\mathbf{T}^2$, and has the leading symbol

$$p_0(x, \xi, \epsilon) = p(x, \xi) + i\epsilon q(x, \xi) + \mathcal{O}(\epsilon^2), \tag{2.6}$$

with

$$p(x, \xi) = p_N(\xi) + \mathcal{O}(\xi^{N+1}), \quad p_N(\xi) = a \cdot \xi + \mathcal{O}(\xi^2). \tag{2.7}$$

Here the operator $P_{\epsilon=0}$ is selfadjoint. Moreover, the complete symbol of P_ϵ is a holomorphic function in a fixed complex neighborhood of $\xi = 0$, and it depends smoothly on $\epsilon \in \mathrm{neigh}(0, \mathbf{R})$. On the operator level, P_ϵ acts in the space of microlocally defined Floquet periodic functions on $\mathbf{T}^2$, $L^2_\theta(\mathbf{T}^2) \subset L^2_{\mathrm{loc}}(\mathbf{R}^2)$, elements u of which satisfy

$$u(x - \nu) = e^{i\theta\cdot\nu} u(x), \quad \theta = \frac{S}{2\pi h} + \frac{k_0}{4}, \quad \nu \in 2\pi\mathbf{Z}^2. \tag{2.8}$$

Here $S = (S_1, S_2)$ are the classical actions,

$$S_j = \int_{\alpha_j} \xi \, dx, \quad j = 1, 2,$$

and α_j form a system of fundamental cycles in Λ_1, defined by

$$\kappa_1(\alpha_j) = \beta_j, \quad j = 1, 2, \quad \beta_j = \{x \in \mathbf{T}^2; x_j = 0\}.$$

The tuple $k_0 = (k_0(\alpha_1), k_0(\alpha_2))$ stands for the Maslov indices of the cycles α_j, $j = 1, 2$.

In what follows we shall work with the operator P_ϵ, microlocally defined near $\xi = 0$ in $T^*\mathbf{T}^2$, which has the principal symbol (2.6), (2.7). For future reference, let us also remark that an application of the implicit function theorem shows that we may assume that the level set $p_N = 0$ is of the form

$$\xi_2 = f(\xi_1), \tag{2.9}$$

for some analytic function f with $f(0) = 0$, $f'(0) \neq 0$.

We shall now discuss the problem of solving the equation

$$H_p G = q - r, \tag{2.10}$$

considered near $\xi = 0$ in $T^*\mathbf{T}^2$, for a suitable remainder r. The torus average of the left hand side of (2.10) is $\mathcal{O}(\xi^N)$, and we shall try to make r independent of x at least to that order in ξ. Also, the function G should be analytic near $\xi = 0$. When solving (2.10), we first consider the problem of solving

$$H_{p_N} G = q - r. \tag{2.11}$$

We analyze this problem on the level of formal Taylor series in ξ, and look for G in terms of a formal expansion,

$$G \sim \sum_{k=0}^{\infty} G_k,$$

where G_k is homogeneous of degree k in ξ. Taking also a (finite) Taylor expansion of p_N with respect to ξ, we easily see that

$$H_{p_N} G \sim \sum_{n=0}^{\infty} f_n,$$

where f_n is homogeneous of degree n in ξ and is given by

$$f_n = \sum_{k+l-1=n} \{p_{N,l}, G_k\},$$

with $p_{N,l}$ homogeneous of degree $l \geq 1$ in ξ. Introducing also a Taylor expansion of q with respect to ξ, $q \sim q_0 + q_1 + \dots$, and using the fact that the operator $a \cdot \partial_x$ is globally analytic hypoelliptic, we determine successively analytic functions $G_0, G_1, \dots$, with G_k homogeneous of degree k in ξ, such that

$$q_n - f_n = \langle q_n \rangle, \quad n = 0, 1, \dots .$$

Here, as in (1.23), we write

$$\langle f\rangle(\xi) = \frac{1}{(2\pi)^2}\int f(x,\xi)\,dx,$$

for the torus average of a smooth function f defined near $\xi = 0$ in $T^*\mathbf{T}^2$. Taking the finite sum $G_0 + G_1 + \ldots + G_N$, we get an analytic function G such that

$$H_{p_N} G = q - r,$$

where

$$r(x,\xi) = \langle q\rangle(\xi) + \mathcal{O}(\xi^{N+1}).$$

To solve (2.10), we just use that

$$H_p G = H_{p_N} G + \mathcal{O}(\xi^N).$$

We conclude therefore that for any fixed $N \in \mathbf{N}$ as in (2.7), we can find an analytic function G defined in a fixed N-independent neighborhood of $\xi = 0$ and solving (2.10), with the remainder r satisfying

$$r(x,\xi) = \langle q\rangle(\xi) + \mathcal{O}(\xi^N). \tag{2.12}$$

Composing G with the inverse of $\kappa_{N,1}$, defined in (2.5), we get an analytic function

$$G_1 = G \circ \kappa_{N,1}^{-1}, \tag{2.13}$$

defined in a small but fixed neighborhood of $\Lambda_1 \subset T^*M$.

Coming back to the torus model, let us recall that we are working in real symplectic coordinates (x,ξ) for which (2.7) holds. We shall consider the behavior of the H_p-flow near $\xi = 0$ for large, but finite times. With $(x(t),\xi(t)) = \exp(tH_p)(x(0),\xi(0))$ we have by the Hamilton equations,

$$\dot{\xi}(t) = \mathcal{O}(|\xi(t)|^{N+1}),$$

and a standard "continuous induction" argument shows that for $|t| \leq \mathcal{O}_N(1)\,|\xi(0)|^{-N}$, we have

$$\xi(t) = \mathcal{O}(|\xi(0)|),$$

provided that $|\xi(0)|$ is small enough. We get, for these times,

$$\xi(t) = \xi(0) + t\mathcal{O}(|\xi(0)|^{N+1}). \tag{2.14}$$

Similarly, we have

$$\dot{x}(t) = \frac{\partial p_N}{\partial \xi} + \mathcal{O}(|\xi|^N), \tag{2.15}$$

and with $(x_0(t), \xi(0)) = \exp(tH_{p_N})(x(0), \xi(0))$, we get for $|t| \leq \mathcal{O}_N(1)\,|\xi(0)|^{-N}$, and writing (x, ξ) rather than $(x(0), \xi(0))$,

$$\xi(t) = \xi + t\mathcal{O}(|\xi|^{N+1}), \quad x(t) = x_0(t) + t\mathcal{O}(|\xi|^N). \tag{2.16}$$

For future reference, we shall also investigate the behavior of $\langle q\rangle_{T,K,p}$ near $\xi = 0$, as $T \to \infty$. In doing so, we notice that (2.16) gives

$$\langle q\rangle_{T,K,p}(x,\xi) = \langle q\rangle_{T,K,p_N}(x,\xi) + \mathcal{O}(T\,|\xi|^N), \quad T \leq \mathcal{O}_N(1)\,|\xi|^{-N}\,. \tag{2.17}$$

where we may also remark that the $\mathcal{O}$-term in (2.17) depends only on the Lipschitz norm of q. When analyzing $\langle q\rangle_{T,K,p_N}$, we write

$$\langle q\rangle_{T,K,p_N}(x,\xi) = \frac{1}{T}\int K\left(\frac{-t}{T}\right) q(x + tp_N'(\xi), \xi)\, dt,$$

and expanding $q(\cdot, \xi)$ in a Fourier series, we get

$$\langle q\rangle_{T,K,p_N}(x,\xi) = \langle q\rangle(\xi) + \sum_{0\neq k\in \mathbf{Z^2}} e^{ikx}\widehat{q}(k,\xi)\widehat{K}(Tp_N'(\xi)\cdot k). \tag{2.18}$$

Here $\widehat{q}(k,\xi)$ are the Fourier coefficients of $q(x,\xi)$ and $\widehat{K}(\xi) = \int e^{-it\xi}K(t)\,dt$ is the Fourier transform of K. Now the Diophantine condition (1.15) gives

$$|p_N'(\xi)\cdot k| \geq \frac{1}{C_0\,|k|^{N_0}} - C_1\,|\xi|\,|k|\,, \quad k \neq 0, \quad C_1 > 0,$$

and therefore

$$|p_N'(\xi)\cdot k| \geq \frac{1}{2C_0\,|k|^{N_0}}, \text{ if } C_1\,|\xi|\,|k| \leq \frac{1}{2C_0\,|k|^{N_0}}, \ k \neq 0.$$

Let $0 \leq \chi \in C_0^\infty((-1,1))$ be equal to 1 on $(-1/2, 1/2)$, and let us decompose the sum in (2.18) as follows:

$$\langle q\rangle_{T,K,p_N}(x,\xi) - \langle q\rangle(\xi) = \sum_{0\neq k\in\mathbf{Z^2}} \chi\left(2C_0C_1\,|\xi|\,|k|^{N_0+1}\right) e^{ix\cdot k}\widehat{q}(k,\xi)\widehat{K}(Tp_N'(\xi)\cdot k)$$

$$+ \sum_{0\neq k\in\mathbf{Z^2}} (1-\chi)\left(2C_0C_1\,|\xi|\,|k|^{N_0+1}\right) e^{ix\cdot k}\widehat{q}(k,\xi)\widehat{K}(Tp_N'(\xi)\cdot k) = \mathrm{I} + \mathrm{II},$$

with the natural definitions of I and II. We get for each $M \in \mathbf{N}$,

$$|I| \leq \mathcal{O}_M(1) \sum_{k\neq 0} \frac{\chi\left(2C_0C_1\,|\xi|\,|k|^{N_0+1}\right)}{|Tp_N'(\xi)\cdot k|^M}\,|\widehat{q}(k,\xi)| \leq \mathcal{O}_M(1)T^{-M}\sum_{k\neq 0}|k|^{MN_0}\,|\widehat{q}(k,\xi)|\,,$$

and therefore,

$$I = \mathcal{O}(T^{-\infty}).$$

When treating II, we only use that $\widehat{K}$ is bounded and because of the presence of $1-\chi$, we have $|k| \geq \mathcal{O}(1)^{-1}\,|\xi|^{-1/(N_0+1)}$ in the sum, so that the contribution coming from II is $\mathcal{O}(|\xi|^{\infty})$. We conclude that

$$\langle q\rangle_{T,K,p_N}(x,\xi) = \langle q\rangle(\xi) + \mathcal{O}(\xi^{\infty} + T^{-\infty}). \tag{2.19}$$

The discussion above is summarized in the following proposition.

Proposition 2.1 *Let (x,ξ) be real symplectic coordinates so that (2.7) is true. Here N is fixed but arbitrarily large. We then have for ξ small enough,*

$$\langle q\rangle_{T,K,p}(x,\xi) = \langle q\rangle(\xi) + \mathcal{O}\left(\xi^{\infty} + T^{-\infty}\right) + \mathcal{O}(T\,|\xi|^N), \quad T \leq \mathcal{O}_N(1)\,|\xi|^{-N}. \tag{2.20}$$

For future reference, we shall pause here to recall some general estimates for convolutions of the form $K_T * g$, where $g \in L^{\infty}(\mathbf{R})$. The following discussion is motivated by the fact that the flow average of q,

$$\langle q\rangle_{T,K}(\rho) = \int K_T(-t)q(\exp(tH_p)(\rho))\,dt$$

is a convolution in the time variable along the H_p-trajectory passing through ρ. The starting point is that

$$K = K_{\epsilon} * K + r_{\epsilon}, \quad ||\,r_{\epsilon}\,||_{L^1} = \mathcal{O}(\epsilon), \tag{2.21}$$

for $0 < \epsilon \leq 1$. It follows that for $0 < S \leq T$,

$$K_T = K_S * K_T + r_{\frac{S}{T},T}, \tag{2.22}$$

where $r_{\frac{S}{T},T}(t) = \frac{1}{T}r_{\frac{S}{T}}(\frac{t}{T})$, so that

$$||\,r_{\frac{S}{T},T}\,||_{L^1} = ||\,r_{\frac{S}{T}}\,||_{L^1} = \mathcal{O}\left(\frac{S}{T}\right).$$

If $g \in L^\infty(\mathbf{R})$ is real-valued, we have

$$K_T * g = K_T * K_S * g + r_{\frac{S}{T},T} * g, \tag{2.23}$$

with

$$|| \, r_{\frac{S}{T},T} * g \, ||_{L^\infty} \leq \mathcal{O}\left(\frac{S}{T}\right) || \, g \, ||_{L^\infty}. \tag{2.24}$$

In particular, when $|| \, g \, ||_{L^\infty} = \mathcal{O}(1)$, we get

$$\inf_{\mathbf{R}}(K_T * g) \geq \inf_{\mathbf{R}}(K_S * g) - \mathcal{O}\left(\frac{S}{T}\right), \tag{2.25}$$

and

$$\sup_{\mathbf{R}}(K_T * g) \leq \sup_{\mathbf{R}}(K_S * g) + \mathcal{O}\left(\frac{S}{T}\right). \tag{2.26}$$

Here we can be a little more precise about where to take the sup and the inf. If $I \subset \mathbf{R}$ is an interval, then

$$\inf_{I}(K_T * g) \geq \inf_{I - T\mathrm{supp}K}(K_S * g) - \mathcal{O}\left(\frac{S}{T}\right), \tag{2.27}$$

$$\sup_{I}(K_T * g) \leq \sup_{I - T\mathrm{supp}K}(K_S * g) + \mathcal{O}\left(\frac{S}{T}\right). \tag{2.28}$$

We also notice that if $\widetilde{K}$ has the same properties as K, then (2.21) can be generalized to

$$K - K * \widetilde{K}_\epsilon = \mathcal{O}(\epsilon) \quad \text{in } L^1,$$

leading to the possibility of replacing K_S by $\widetilde{K}_S$ in (2.22), (2.23), and (2.25)–(2.28).

Let us return now to the operator P_ϵ with the leading symbol (2.6), (2.7). We shall assume from now on that

$$dp_N(0) = a \text{ and } d\mathrm{Re}\,\langle q \rangle(0) \text{ are linearly independent.} \tag{2.29}$$

Notice that this condition is independent of N. Then, possibly after changing the sign of the ξ_1-coordinate and using (2.9), we get

$$\frac{d}{d\xi_1}\Big|_{\xi_1=0} \langle \mathrm{Re}\, q \rangle(\xi_1, f(\xi_1)) > 0. \tag{2.30}$$

Lemma 2.2 *Assume that* (2.29) *holds true, and let* N *in* (2.7) *be sufficiently large. There exists a constant* $C_0 > 0$ *such that if* (1.24) *holds for some* $1 \leq N_1, N_2$*, and* $\widetilde{N}_1 \in \mathbf{N}\backslash\{0\}$*, then there exists* $\delta_0 > 0$ *such that*

$$\inf_{\Omega_+(\delta)} \left(\langle \operatorname{Re} q \rangle_{\delta^{-\widetilde{N}_1},K} - \operatorname{Re} F \right) \geq \frac{\delta}{C_0}, \quad 0 < \delta \leq \delta_0, \tag{2.31}$$

and

$$\sup_{\Omega_-(\delta)} \left(\langle \operatorname{Re} q \rangle_{\delta^{-\widetilde{N}_1},K} - \operatorname{Re} F \right) \leq -\frac{\delta}{C_0}, \quad 0 < \delta \leq \delta_0. \tag{2.32}$$

Proof: It suffices to prove only (2.31), and we may also assume that $F = 0$. Moreover, as will be clear, the following argument will not depend on the choice of N in (2.7), provided that it is large enough, and to simplify the discussion notationally, we shall take $N = \infty$. We shall work near one of the tori Λ_j. From Proposition 2.1 we then know that for every fixed $N_1 \in \mathbf{N}\backslash\{0\}$, we have

$$\langle q \rangle_{\delta^{-N_1},K}(x,\xi) = \langle q \rangle(\xi) + \mathcal{O}(\delta^\infty), \quad |\xi| < \delta,$$

if $0 < \delta \leq \delta(N_1) > 0$. Take δ as in (1.24) and let $\widetilde{\delta} \in [\delta^{M_1}, \delta^{M_0}]$, where $1 < M_0 < M_1$ will be fixed later. Then

$$\inf_{\Omega_+(\widetilde{\delta})} \langle \operatorname{Re} q \rangle_{\widetilde{\delta}^{-\widetilde{N}_1},K} \geq \min \left(\inf_{\Omega_+(\widetilde{\delta})\backslash\Omega_+(2\delta)} \langle \operatorname{Re} q \rangle_{\widetilde{\delta}^{-\widetilde{N}_1},K}, \inf_{\Omega_+(2\delta)} \langle \operatorname{Re} q \rangle_{\widetilde{\delta}^{-\widetilde{N}_1},K} \right), \tag{2.33}$$

and we also get

$$\inf_{\Omega_+(\widetilde{\delta})\backslash\Omega_+(2\delta)} \langle \operatorname{Re} q \rangle_{\widetilde{\delta}^{-\widetilde{N}_1},K} \geq \frac{\widetilde{\delta}}{C_0}, \tag{2.34}$$

for some $C_0 > 0$, when δ is small enough. We may assume that $C_0 \geq 2$.

If we choose M_0 with $M_0\widetilde{N}_1 \geq N_1$, then $\widetilde{\delta}^{-\widetilde{N}_1} \geq \delta^{-N_1}$, and using the general estimate (2.27) in the form

$$\langle q \rangle_{T,K}(\rho) \geq \inf_{t\in[0,T]} \langle q \rangle_{S,K}(\exp{(tH_p)}(\rho)) - \mathcal{O}\left(\frac{S}{T}\right),$$

and the fact that in the region $|\xi| < \delta$, the H_p-trajectories are approximately confined to the tori $\xi = \text{Const}$, for times bounded by any fixed inverse power of $|\xi|$, we get

$$\inf_{\Omega_+(2\delta)} \langle \operatorname{Re} q \rangle_{\widetilde{\delta}^{-\widetilde{N}_1},K} \geq \inf_{\Omega_+(\delta)} \langle \operatorname{Re} q \rangle_{\delta^{-N_1},K} - \mathcal{O}(\delta^{-N_1}\widetilde{\delta}^{\widetilde{N}_1}) \geq \delta^{N_2} - \mathcal{O}(\delta^{-N_1}\widetilde{\delta}^{\widetilde{N}_1}). \tag{2.35}$$

Here we should replace δ^{N_2} by δ/C_1 if $N_2 = 1$. Now assume that $M_0 \geq N_2$, $M_0\widetilde{N}_1 - N_1 > N_2$, so that $\delta^{N_2} \geq \widetilde{\delta}$, $\delta^{-N_1}\widetilde{\delta}^{\widetilde{N}_1} \ll \delta^{N_2}$. Then (2.35) implies that

$$\inf_{\Omega_+(2\delta)} \langle \operatorname{Re} q \rangle_{\delta^{-\tilde{N}_1},K} \geq \frac{\widetilde{\delta}}{2}.$$

Combining this with (2.33) and (2.34), we get

$$\inf_{\Omega_+(\widetilde{\delta})} \langle \operatorname{Re} q \rangle_{\widetilde{\delta}^{-\tilde{N}_1},K} \geq \frac{\widetilde{\delta}}{C_0},$$

for $\delta > 0$ sufficiently small and for $\widetilde{\delta} \in [\delta^{M_1}, \delta^{M_0}]$, provided that

$$M_1 > M_0 \geq \max\left(N_2, \frac{N_1 + N_2}{\widetilde{N}_1}\right),$$

and $\delta > 0$ is small enough depending on the exponents and such that the estimate (1.24) holds true.

We conclude that we have the estimate in (1.24) with $N_1 = \widetilde{N}_1$ and $N_2 = 1$ for δ replaced by $\widetilde{\delta} \in [\delta^{M_1}, \delta^{M_0}]$. A special case of this is when we start with δ with $N_1 = \widetilde{N}_1$, $N_2 = 1$, and then conclude that (1.24) holds with δ replaced by $\widetilde{\delta}$ in $[\delta^{M_1}, \delta^{M_0}]$, when $M_1 > M_0 \geq (\widetilde{N}_1 + 1)/\widetilde{N}_1$. Choose $M_1 = M_0^2$. Then the argument can be iterated and we get (1.24) with $N_1 = \widetilde{N}_1$, $N_2 = 1$, for δ replaced by any $\widetilde{\delta}$ in $[\delta^{M_0^2}, \delta^{M_0}] \cup [\delta^{M_0^3}, \delta^{M_0^2}] \cup \ldots = (0, \delta^{M_0}]$, and the lemma follows. $\Box$

Lemma 2.2 shows that (1.24) is independent of the choice of N_1 and N_2. The proof also shows that it is independent of $K \in C_0^\infty(\mathbf{R})$. We also notice that the proof shows that the validity of (1.24) is stable under small perturbations of p that preserve the invariant tori Λ_j and the Diophantine condition (1.15).

We shall now construct a global weight function $\widetilde{G}$. When doing so, let us recall from (2.13) the analytic functions $G_j = G \circ \kappa_{N,j}^{-1}$, defined in small neighborhoods of Λ_j, $1 \leq j \leq L$. Here $\kappa_{N,j}$ is defined exactly as $\kappa_{N,1}$ in (2.5). When constructing the global weight $\widetilde{G}$, we shall glue together the analytic functions G_T, defined in (1.18) and G_j. In doing so, it will be sufficient to work in a fixed sufficiently small neighborhood of $\xi = 0$ in $T^*\mathbf{T}^2$. Let $0 \leq \chi \leq 1$, $\chi \in C_0^\infty(\mathbf{R}^2)$, be supported in $|\xi| < 2$, with $\chi = 1$ on $|\xi| < 1$. With $0 < \mu \ll 1$ to be determined, we put

$$\widetilde{G} = \left(1 - \sum_{j=1}^{L} \chi\left(\frac{\xi \circ \kappa_{N,j}}{\mu}\right)\right) G_T + \sum_{j=1}^{L} \chi\left(\frac{\xi \circ \kappa_{N,j}}{\mu}\right) G_j. \tag{2.36}$$

In the following calculation, in order to ease the notation, we shall take $L = 1$, in which case we may omit $\kappa_{N,1}$ from the notation and write

$$\widetilde{G} = (1 - \chi_\mu) G_T + \chi_\mu G, \tag{2.37}$$

where $\chi_\mu(\xi) = \chi(\xi/\mu)$. We shall compute $H_p \widetilde{G}$, and in doing so we notice first that

$$H_p \chi_\mu = \mathcal{O}(\xi^N),$$

uniformly in $\mu > 0$. We get

$$H_p \widetilde{G} = (1 - \chi_\mu)\left(q - \langle q \rangle_{T,K,p}\right) + \chi_\mu \left(q - \langle q \rangle\right) + \mathcal{O}(\xi^N)\chi_\mu + H_p \chi_\mu \left(G - G_T\right),$$

which we rewrite as

$$H_p \widetilde{G} = q - \langle q \rangle - (1 - \chi_\mu)(\langle q \rangle_{T,K,p} - \langle q \rangle) + \mathcal{O}(\xi^N)\chi_\mu + H_p \chi_\mu (G - G_T). \tag{2.38}$$

Now $\chi_\mu = \mathcal{O}(1)$ and (1.18) gives $G - G_T = \mathcal{O}(T)$, so that

$$H_p \widetilde{G} = q - \langle q \rangle - (1 - \chi_\mu)(\langle q \rangle_{T,K,p} - \langle q \rangle) + \mathcal{O}(T\xi^N), \quad T \geq 1.$$

Proposition 2.1 gives then that uniformly in $\mu > 0$ we have

$$H_p \widetilde{G} = q - \langle q \rangle - (1 - \chi_\mu)\left(\mathcal{O}(T\xi^N) + \mathcal{O}(\xi^\infty + T^{-\infty})\right) + \mathcal{O}(T\xi^N), \quad T \leq \mathcal{O}_N(1)|\xi|^{-N}. \tag{2.39}$$

We let first N be sufficiently large depending on N_1, and take then $\delta > 0$ small enough but fixed. Take then $\mu \sim \delta$, and put

$$T = \mu^{-N_1},$$

In the region $|\xi| < \mu$, we have $\widetilde{G} = G$, while $\widetilde{G} = G_T$ for $|\xi| > 2\mu$. In the intermediate region where $0 < \chi_\mu < 1$, we have $|\xi| \sim \mu$, and it follows that in this region

$$q - H_p \widetilde{G} = \langle q \rangle(\xi) + \mathcal{O}(\xi^{N - N_1}).$$

The discussion above can be summarized in the following proposition.

Proposition 2.3 *Assume* (2.29) *for each* $1 \leq j \leq L$, *and* (1.24) *for some* N_1 *and* $N_2 \in \mathbf{N}\backslash\{0\}$. *When* $1 \leq j \leq L$, *let* G_j *be an analytic solution near* Λ_j *of the equation* (2.10), *with* r *satisfying* (2.12) *for some* N *large enough. Let* W_j *be a sufficiently small neighborhood of* Λ_j, $W_j = (\xi_1 \circ \kappa_{N,j})^{-1}((-\delta_0, \delta_0))$, $\delta_0 > 0$. *Then there exists a smooth compactly supported function* $\widetilde{G}$ *on all of* T^*M, *such that* $\widetilde{G} = G_j$ *in* W_j,

$1 \leq j \leq L$, and if $W = \cup_{j=1}^{L} W_j$, $\delta(W) > 0$ is sufficiently small, and $\widetilde{\Omega}_{\pm}(\delta(W))$ stand for the components of $\Omega_{\pm}(\delta_0)$ in $p^{-1}\left((-\delta(W), \delta(W))\right) \backslash W$, we have

$$\inf_{\widetilde{\Omega}_{+}(\delta(W))} \left(\mathrm{Re}\,(q - H_p \widetilde{G} - F)\right) \geq \frac{1}{C(W)},$$

and

$$\sup_{\widetilde{\Omega}_{-}(\delta(W))} \left(\mathrm{Re}\,(q - H_p \widetilde{G} - F)\right) \leq \frac{-1}{C(W)}.$$

Here F is the average of q over Λ_j, $1 \leq j \leq L$, and $C(W) > 0$.

We shall now discuss a modified version of the global assumption (1.24), available if we assume that the H_p-flow is completely integrable. In particular, the results of the following discussion will be applied in section 7, where, as an example, we consider p corresponding to the geodesic flow on an analytic surface of revolution.

Assuming the complete integrability for the flow, we know that there exists an analytic real-valued function f, which Poisson commutes with p, and such that each Λ_j, $1 \leq j \leq L$, is a level set for the associated mapping,

$$(p, f) : T^*M \to \mathbf{R}^2, \tag{2.40}$$

corresponding to a regular value. Then each torus Λ_j is embedded in a Lagrangian foliation of H_p-invariant tori, given by

$$\Lambda_{E,F} : p = E, \quad f = F,$$

for $(E, F) \in \mathrm{neigh}((0, F_j), \mathbf{R}^2)$, where $(0, F_j)$ is a regular value of (2.40). We may then introduce the action-angle coordinates, given by a real analytic canonical transformation,

$$\kappa_j : \mathrm{neigh}(\Lambda_j, T^*M) \to \mathrm{neigh}(\xi = 0, T^*\mathbf{T}^2), \quad 1 \leq j \leq L,$$

such that Λ_j is mapped to the zero section in $T^*\mathbf{T}^2$, and such that when expressed in terms of the coordinates x and ξ, the unperturbed leading symbol p becomes a function of ξ only, with

$$p(\xi) = a \cdot \xi + \mathcal{O}(\xi^2), \quad a = a_j, \tag{2.41}$$

As in (2.9), we may assume that the energy surface $p^{-1}(0)$ takes the form $\xi_2 = f_j(\xi_1)$, where f_j is analytic with $f_j(0) = 0$. The tori $\Lambda_{0,F}$, for $F \in \mathrm{neigh}(F_j, \mathbf{R})$

then take the form $\xi_1 = \mu$, $\xi_2 = f_j(\mu)$, for $|\mu| < b$, with $0 < b \ll 1$. Consider a flow-invariant neighborhood of the union of the Λ_j of the form

$$\bigcup_{|\mu|<b} \bigcup_{j=1}^{L} (\xi_1 \circ \kappa_j)^{-1}(\mu) \cap p^{-1}(0) = \bigcup_{|\mu|<b} \Lambda_\mu.$$

Here Λ_μ is a disjoint union of L flow-invariant tori. We decompose the real energy surface as follows,

$$p^{-1}(0) = \Lambda_{-b} \cup \bigcup_{|\mu|<b} \Lambda_\mu \cup \Lambda_b,$$

where $\Lambda_{\pm b}$ are two disjoint compact flow-invariant domains with at most finitely many connected components. This gives a decomposition of the energy surface

$$p^{-1}(0) = \bigcup_{\mu \in M} \Lambda_\mu, \tag{2.42}$$

where $M = [-b, b]$ and each Λ_μ is a compact H_p-invariant set, with finitely many connected components $\Lambda_{\mu,j}$. Assume also the continuity property:

For any $\mu_0 \in M, \epsilon > 0$ there exists $\delta > 0$, such that for (2.43)
$\mathrm{dist}(\mu, \mu_0) < \delta$ we have $\Lambda_\mu \subset \Lambda_{\mu_0,\epsilon} := \{\rho \in p^{-1}(0); \mathrm{dist}(\rho, \Lambda_{\mu_0}) < \epsilon.\}$

As in (1.17), we put

$$\langle q \rangle_{T,K}(\rho) = \int q(\exp(tH_p)(\rho)) K_T(-t)\, dt. \tag{2.44}$$

Let $Q_T(\mu, j)$ and $Q_T(\mu)$ be the ranges of $\mathrm{Re}\, \langle q \rangle_{T,K}$ restricted to $\Lambda_{\mu,j}$ and Λ_μ, respectively. Then $Q_T(\mu)$ is a finite union of the closed intervals $Q_T(\mu, j)$, and according to (2.27) and (2.28), we have

$$Q_T(\mu) \subset Q_S(\mu) + \mathcal{O}(1) \| \, q|_{\Lambda_\mu} \|_{L^\infty} \left[-\frac{S}{T}, \frac{S}{T} \right], \tag{2.45}$$

uniformly for $0 < S \le T$, $\mu \in M$. Let $Q_\infty(\mu, j)$ be the non-empty intersection of all the $Q_T(\mu, j)$ for $T \ge 1$, and define $Q_\infty(\mu)$ to be the union of all the $Q_\infty(\mu, j)$, so that

$$Q_\infty(\mu) = \bigcup_j \left[\lim_{T\to\infty} \inf_{\Lambda_{\mu,j}} \mathrm{Re}\, \langle q \rangle_{T,K}, \lim_{T\to\infty} \sup_{\Lambda_{\mu,j}} \mathrm{Re}\, \langle q \rangle_{T,K} \right]. \tag{2.46}$$

We may also remark that according to (2.45), $Q_\infty(\mu) \subset Q_T(\mu)$, for all $T \ge 1$. It is also easy to see that $Q_\infty(\mu)$ does not depend on the choice of K. Put

$$\mathcal{Q}_T = \{(\mu, E); \mu \in M, E \in Q_T(\mu)\}, \quad 0 < T \le \infty. \tag{2.47}$$

Lemma 2.4 *We have*

1. *$\mathcal{Q}_T$ is closed.*
2. *For every neighborhood $\mathcal{U}$ of $\mathcal{Q}_\infty$ there exists a $T_0 \in (0, \infty)$ such that $\mathcal{Q}_T \subset \mathcal{U}$ for $T \geq T_0$.*

Proof: We prove (1) first. It is clear that $\mathcal{Q}_T$ is closed for each finite T. Let $(\mu_j, E_j) \in \mathcal{Q}_\infty$ be a convergent sequence so that $\mu_j \to \mu_0 \in M$, $E_j \to E_0 \in \mathbf{R}$. Let $\epsilon > 0$. Then there exists $T_0 \in (0, \infty)$ such that $Q_{T_0}(\mu_0) \subset Q_\infty(\mu_0) + [-\epsilon, \epsilon]$. We fix such a number. If $\delta > 0$ is small enough, we have

$$Q_\infty(\mu) \subset Q_{T_0}(\mu) \subset Q_{T_0}(\mu_0) + [-\epsilon, \epsilon] \subset Q_\infty(\mu_0) + [-2\epsilon, 2\epsilon], \tag{2.48}$$

for $\mathrm{dist}(\mu, \mu_0) < \delta$. In particular, $E_0 \in Q_\infty(\mu_0) + [-2\epsilon, 2\epsilon]$, and letting $\epsilon \to 0$, we get $E_0 \in Q_\infty(\mu_0)$, proving the closedness of $\mathcal{Q}_\infty$.

We now establish (2). If $T = kT_0$, $k \geq 1$, we get from (2.45) and (2.48) that for $\mathrm{dist}(\mu, \mu_0) < \delta$,

$$Q_T(\mu) \subset Q_\infty(\mu_0) + [-2\epsilon, 2\epsilon] + \left[-\frac{\mathcal{O}(1)}{k}, \frac{\mathcal{O}(1)}{k}\right],$$

and hence

$$Q_T(\mu) \subset Q_\infty(\mu_0) + [-3\epsilon, 3\epsilon],$$

for T large enough. This means that $\{(\mu, E) \in \mathcal{Q}_T; \mathrm{dist}(\mu, \mu_0) < \delta\}$ is within a distance $3\epsilon + \delta$ from $\mathcal{Q}_\infty$. The statement (2) now follows by a covering argument. $\Box$

If $|\mu| < b$, then each connected component $\Lambda_{\mu,j}$, $1 \leq j \leq L$, of Λ_μ is diffeomorphic to $\mathbf{T}^2$ in such a way that $H_p|_{\Lambda_{\mu,j}}$ becomes $a_1 \partial_{x_1} + a_2 \partial_{x_2}$, $a_j \in \mathbf{R}$, and we have

$$\langle q \rangle_{T,K}(x) = \sum_{k \in \mathbf{Z}^2} \widehat{K}(Ta \cdot k) \widehat{q}(k, \mu) e^{ix \cdot k}, \tag{2.49}$$

which, as $T \to \infty$, converges uniformly in x (using now also the smoothness of q), to

$$\langle q \rangle_{\infty,\mu}(x) = \sum_{a \cdot k = 0} \widehat{q}(k, \mu) e^{ix \cdot k}. \tag{2.50}$$

Notice that $Q_\infty(\mu, j)$ is the range of this function.

When $|\mu| < b$ and $1 \leq j \leq L$, we shall now study the compact interval $Q_\infty(\mu, j)$ in more detail. In doing so, we remark that if we have an equation of the form (2.10) on $\cup_{|\mu|<b}\Lambda_{\mu,j}$, with G smooth and bounded, then $Q_\infty(\mu,j) \subset \operatorname{Re} r(\Lambda_\mu)$. In fact, with $K = 1_{[-1,0]}$, we have for $\rho \in \Lambda_{\mu,j}$,

$$\langle r\rangle_T(\rho) = \langle q\rangle_T(\rho) - \frac{1}{T}\left(G(\exp(tH_p)(\rho)) - G(\rho)\right) = \langle q\rangle_T(\rho) + \mathcal{O}\left(\frac{1}{T}\right),$$

and it suffices to let $T \to \infty$, and use that $\operatorname{Re}\langle r\rangle_T(\Lambda_\mu) \subset \operatorname{Re} r(\Lambda_\mu)$.

Recall now that we have seen that we can solve (2.10) with r satisfying (2.12), and we conclude that

$$Q_\infty(\mu,j) \subset \operatorname{Re}\langle q_j\rangle(\mu, f(\mu)) + \mathcal{O}(\mu^N)[-1,1], \quad |\mu| < b, \quad q_j = q. \tag{2.51}$$

As in the general case, from now on we introduce the assumption (2.29) for each $1 \le j \le L$. In view of (2.51), this implies (possibly after shrinking b again) that for $0 < \widetilde{b} < b$,

$$\inf \bigcup_{\mu\in[\widetilde{b},b)} Q_\infty(\mu) > \operatorname{Re} F_0 + \frac{\widetilde{b}}{C}, \quad \sup \bigcup_{\mu\in(-b,-\widetilde{b}]} Q_\infty(\mu) < \operatorname{Re} F_0 - \frac{\widetilde{b}}{C}, \tag{2.52}$$

for some constant $C > 0$. The global assumption in the completely integrable case is then the following one: For any $\widetilde{b} \in (0,b)$ there exists $C(\widetilde{b}) > 0$ such that

$$\inf_{\mu\in[\widetilde{b},b]} Q_\infty(\mu) > \operatorname{Re} F_0 + \frac{1}{C(\widetilde{b})}, \quad \sup_{\mu\in[-b,-\widetilde{b}]} Q_\infty(\mu) < \operatorname{Re} F_0 - \frac{1}{C(\widetilde{b})}. \tag{2.53}$$

Using (2.51) together with Proposition 2.1 and Lemma 2.4 it is easy to see that in the completely integrable case, the condition (2.53) is equivalent to (1.24).

Proposition 2.5 *Assume that the H_p-flow is completely integrable, so that* (2.41) *holds true. Assume furthermore that* (2.29) *and* (2.53) *are valid. Then we have the same conclusion as in Proposition* 2.3.

In what follows, to ease the notation, we shall drop the tilde and write G instead of $\widetilde{G}$, defined in Proposition 2.3. Associated with $G \in C_0^\infty(T^*M)$, there is a globally defined IR-manifold

$$\Lambda_{\epsilon G} = \{\rho + i\epsilon H_G(\rho);\ \rho \in T^*M\} \subset T^*\widetilde{M}, \tag{2.54}$$

and when acting on the Hilbert space $H(\Lambda_{\epsilon G})$, associated to $\Lambda_{\epsilon G}$ by means of the FBI-Bargmann transform

$$Tu(x) = Ch^{-3/2}\int e^{\frac{i\varphi(x,y)}{h}} u(y)\,dy, \quad \varphi(x,y) = \frac{i}{2}(x-y)^2,$$

the operator P_ϵ gets the leading symbol

$$\big(p + i\epsilon q + \mathcal{O}(\epsilon^2)\big)\,(\rho + i\epsilon H_G \rho) = p(\rho) + i\epsilon\,(q - H_p G)\,(\rho) + \mathcal{O}(\epsilon^2). \tag{2.55}$$

Here we are tacitly assuming that $M = \mathbf{R}^2$ so that T^*M is a linear space. In the manifold case, using that G is analytic near Λ_j, $1 \leq j \leq L$, we define $\Lambda_{\epsilon G} = \exp(i\epsilon H_G)(T^*M)$ in a complex neighborhood of Λ_j, and elsewhere we let $\widetilde{G} \in C_0^\infty(T^*\widetilde{M})$ stand for an almost holomorphic extension of G, and $\Lambda_{\epsilon G}$ is then defined as

$$\Lambda_{\epsilon G} = \exp(\epsilon H^{\mathrm{Im}\sigma}_{\mathrm{Re}\widetilde{G}})(T^*M).$$

Here the Hamilton vector field of $\mathrm{Re}\,\widetilde{G}$ is computed with respect to the imaginary part $\mathrm{Im}\,\sigma$ of the complex symplectic form σ on $T^*\widetilde{M}$.

It follows from Proposition 2.3 that when $m \in \Lambda_{\epsilon G}$ is away from a small but fixed neighborhood of the union of the Lagrangian tori $\widetilde{\Lambda}_j$, such that each $\widetilde{\Lambda}_j$ is the image of Λ_j in $\Lambda_{\epsilon G}$ under the map $T^*M \ni \rho \mapsto m = \rho + i\epsilon H_G(\rho)$, it is true that

$$|\mathrm{Im}\, P_\epsilon(m) - \epsilon \mathrm{Re}\, F_0| \geq \frac{\epsilon}{\mathcal{O}(1)},$$

provided that $|\mathrm{Re}\, P_\epsilon(m)| \leq 1/C$ for a sufficiently large $C > 0$. Moreover, microlocally near $p^{-1}(0)$, the operator P_ϵ acting on $H(\Lambda_{\epsilon G})$ is unitarily equivalent to an operator acting on $L^2(M)$, which has the leading symbol (2.55).

Summing up the discussion of this section, we have achieved that microlocally near each $\widetilde{\Lambda}_j \subset \Lambda_{\epsilon G}$, $1 \leq j \leq L$, the operator

$$P_\epsilon : H(\Lambda_{\epsilon G}) \to H(\Lambda_{\epsilon G})$$

is unitarily equivalent to an operator $\widetilde{P}_\epsilon$, acting on $L^2_\theta(\mathbf{T}^2)$ and defined microlocally near $\xi = 0$ in $T^*\mathbf{T}^2$. Here $\widetilde{P}_\epsilon$ is such that

$$\widetilde{P}_\epsilon \sim \sum_{\nu=0}^{\infty} h^\nu \widetilde{p}_\nu(x, \xi, \epsilon), \tag{2.56}$$

with $\widetilde{p}_\nu$ holomorphic in a fixed complex neighborhood of $\xi = 0$ and

$$\widetilde{p}_0 = p_N(\xi) + i\epsilon\langle q\rangle(\xi) + \mathcal{O}(\epsilon^2) + \mathcal{O}(\xi^{N+1}) + \epsilon\mathcal{O}(\xi^N), \tag{2.57}$$

with $p_N(\xi) = a \cdot \xi + \mathcal{O}(\xi^2)$, and where the term $\mathcal{O}(\xi^{N+1})$ is real on the real domain. Here N is fixed but arbitrarily large. In what follows, we shall drop the tildes and write P_ϵ and p_ν, $\nu \geq 0$, instead of $\widetilde{P}_\epsilon$ and $\widetilde{p}_\nu$, $\nu \geq 0$, respectively.

3 The normal form construction

We recall that we have reduced the analysis to the operator P_ϵ with a complete symbol (2.56) and a leading symbol (2.57), and recall also the assumption (2.29). Our goal in this section is to construct a quantum Birkhoff normal form for P_ϵ—see also [10], [32], and [37]. We may write

$$p_0(x,\xi,\epsilon) = p_{0,1}(x,\xi,\epsilon) + p_{0,2}(x,\xi,\epsilon) + \ldots + p_{0,N}(x,\xi,\epsilon) + \mathcal{O}((\epsilon,\xi)^{N+1}), \quad (3.1)$$

where $p_{0,j}$ is homogeneous of degree j in (ξ,ϵ), so that in particular

$$p_{0,1}(x,\xi,\epsilon) = a\cdot\xi + i\epsilon\langle q\rangle(0)$$

is independent of x.

We shall now remove the x-dependence also in the terms $p_{0,j}$, $2\le j\le N$. In doing so, we consider the formal power series

$$G = \sum_{j=1}^{\infty} G_j(x,\xi,\epsilon), \quad (3.2)$$

where $G_j = \mathcal{O}((\xi,\epsilon)^{j+1})$ depends analytically on x and is a homogeneous polynomial of degree $j+1$ in (ξ,ϵ). In the sense of formal Taylor expansions, we then have

$$p_0\circ\exp(H_G) = p_0 + \sum_{k=1}^{\infty}\frac{1}{k!}H_G^k p_0 \;=\; p_0 + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{k!}H_G^k p_{0,l}$$

$$= p_0 + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{j_1=1}^{\infty}\ldots\sum_{j_k=1}^{\infty}\frac{1}{k!}H_{G_{j_1}}\ldots H_{G_{j_k}}p_{0,l} \;=\; \sum_{n=1}^{\infty} q_n.$$

Here q_n is a homogeneous polynomial of degree n in (ξ,ϵ). We also notice that in the sum above $H_{G_m}p_{0,l}$ is homogeneous of degree $m+l$ in (ξ,ϵ). It follows that $q_1 = p_{0,1}$ is independent of x, $q_2 = p_{0,2} + H_{G_1}p_{0,1}$, and $q_{n+1} = p_{0,n+1} + H_{G_n}p_{0,1} + \widetilde{q}_{n+1}$, $n\ge 2$, where $\widetilde{q}_{n+1}$ depends only on $G_1,\ldots G_{n-1}$. We are therefore able to determine $G_1, G_2\ldots G_{N-1}$ successively by solving the cohomological equations

$$H_{p_{0,1}}G_n = a\cdot\partial_x G_n = p_{0,n+1} + \widetilde{q}_{n+1} - \langle p_{0,n+1} + \widetilde{q}_{n+1}\rangle, \quad 1\le n\le N-1. \quad (3.3)$$

In this way we achieve that all the q_n, $n\le N$, are independent of x.

Remark. It follows from the construction that the function G_1 is independent of ξ and is of the form $G_1(x,\xi,\epsilon) = \epsilon^2 h(x)$ for some analytic function h. Furthermore, it is clear that

$$G_j = \mathcal{O}_j(\epsilon^2), \quad 1 \le j \le N-1.$$

Remark. In the construction above, we could also have used a slightly different argument and looked for G as a formal power series in ϵ alone,

$$G \sim G_1(x,\xi) + \epsilon G_2(x,\xi) + \dots$$

Then composing the symbol p_0 with the canonical transformation $\exp(\epsilon^2 H_G)$ and repeating the previous arguments, we would have been able to determine each G_j modulo $\mathcal{O}(\xi^{N+1})$, and the final result would have been the same.

Summarizing the discussion so far, we get the following result.

Proposition 3.1 *Let* $p_0(x,\xi,\epsilon) = p_N(\xi) + i\epsilon\langle q\rangle(\xi) + \mathcal{O}(\epsilon^2) + \mathcal{O}(\xi^{N+1}) + \epsilon\mathcal{O}(\xi^N)$ *be an analytic function defined near* $\xi = 0$ *in* $T^*\mathbf{T}^2$*, depending smoothly on* $\epsilon \in$ neigh$(0,\mathbf{R})$*. Here* $N \in \mathbf{N}$ *is fixed but can be taken arbitrarily large. Assume that*

$$p_N(\xi) = a\cdot\xi + \mathcal{O}(\xi^2),$$

where a *satisfies* (1.15)*. Then we can find analytic functions* $G_1, G_2, \dots, G_{N-1}$*, with* $G_j(x,\xi,\epsilon)$ *being a homogeneous polynomial of degree* $j+1$ *in* (ξ,ϵ)*, such that* $G_j = \mathcal{O}_j(\epsilon^2)$*,* $1 \le j \le N-1$*,* G_1 *is independent of* ξ*, and such that if*

$$G^{(N)} = G_1 + G_2 + \dots + G_{N-1}, \tag{3.4}$$

then

$$p_0 \circ \exp(H_{G^{(N)}}) = p^{(N)}(\xi,\epsilon) + r_{N+1}(x,\xi,\epsilon).$$

Here

$$p^{(N)}(\xi,\epsilon) = a\cdot\xi + i\epsilon\langle q\rangle(0) + \mathcal{O}((\xi,\epsilon)^2)$$

is independent of x *and* $r_{N+1}(x,\xi,\epsilon) = \mathcal{O}((\xi,\epsilon)^{N+1})$*. Writing* $p^{(N)}(\xi,\epsilon) = p(\xi) + i\epsilon q(\xi,\epsilon)$*, where* p *is real, we have*

$$d_\xi p(0) \;\; \mathrm{Re}\, d_\xi q(0,0) \;\; \textit{are linearly independent.}$$

In section 5, we shall see that we can quantize the holomorphic canonical transformation

$$\widehat{\kappa} := \exp(H_{G^{(N)}}) \tag{3.5}$$

by means of an analytic elliptic Fourier integral operator in the complex domain. In this section we shall proceed somewhat formally, and carrying out the corresponding conjugation of P_ϵ, we may assume from now on that we are given an h-pseudodifferential operator, still denoted by P_ϵ, defined microlocally near $\xi = 0$ in $T^*\mathbf{T}^2$, whose full symbol has a complete asymptotic expansion,

$$P_\epsilon(x, \xi, \epsilon; h) = p_0 + hp_1 + h^2 p_2 + \dots, \tag{3.6}$$

with all $p_j = p_j(x, \xi, \epsilon)$ holomorphic in a fixed complex neighborhood of $\xi = 0$, depending smoothly on $\epsilon \in \mathrm{neigh}(0, \mathbf{R})$, and such that

$$p_0(x, \xi, \epsilon) = p^{(N)}(\xi, \epsilon) + \mathcal{O}((\xi, \epsilon)^{N+1}), \quad p^{(N)}(\xi, \epsilon) = a \cdot \xi + i\epsilon \langle q \rangle(0) + \mathcal{O}((\xi, \epsilon)^2). \tag{3.7}$$

Our goal now is to make the lower order terms p_j, $j \geq 1$, in (3.6) independent of x, to a high order in ξ and ϵ. This will be achieved by means of the usual conjugation of P_ϵ by an elliptic pseudodifferential operator of the form $\exp(Q(x, hD_x, \epsilon; h))$, where Q is of order 0 in h. Write $Q(x, hD_x, \epsilon; h) \sim q_0 + hq_1 + \dots$. Then on the operator level, we have

$$\begin{aligned} e^Q P_\epsilon e^{-Q} &= P_\epsilon + \sum_{k=1}^\infty \sum_{l=0}^\infty \frac{1}{k!} h^l \left(\mathrm{ad} Q\right)^k p_l \\ = \quad & P_\epsilon + \sum_{k=1}^\infty \sum_{l=0}^\infty \sum_{j_1=0}^\infty \dots \sum_{j_k=0}^\infty \frac{1}{k!} h^{l+j_1+\dots+j_k+k} \left(\frac{1}{h}\mathrm{ad}(q_{j_1})\right) \dots \left(\frac{1}{h}\mathrm{ad}(q_{j_k})\right) p_l, \end{aligned}$$

and on the symbol level we get

$$e^Q P_\epsilon e^{-Q} \sim \sum_{n=0}^\infty h^n s_n. \tag{3.8}$$

Here $s_0 = p_0$, $s_1 = p_1 + iH_{p_0} q_0$, and for $n \geq 1$, the term s_{n+1} has the form $s_{n+1} = p_{n+1} + iH_{p_0} q_n + \widetilde{s}_{n+1}$, where $\widetilde{s}_{n+1}$ depends only on $q_0, \dots q_{n-1}$. We wish to make the terms s_n, $n \leq N$, independent of x, modulo $\mathcal{O}((\xi, \epsilon)^{N+1})$. Taking a Taylor expansion of q_0 in (ξ, ϵ), and solving the cohomological equations as before, we see that we can choose q_0 so that s_1 becomes independent of x modulo $\mathcal{O}((\xi, \epsilon)^{N+1})$. Repeating this argument, we determine successively $q_1, q_2, \dots$ so that

$$s_n = s_n^{(N)} + \mathcal{O}((\xi, \epsilon)^{N+1}), \quad n = 0, 1, 2, \dots,$$

with $s_n^{(N)} = s_n^{(N)}(\xi, \epsilon)$ independent of x.

With $Q^{(N)} = \sum_{n=0}^{N-1} h^n q_n$, we get

$$e^{Q^{(N)}} P_\epsilon e^{-Q^{(N)}} = P^{(N)}(hD_x, \epsilon; h) + R_{N+1}(x, hD_x, \epsilon; h), \tag{3.9}$$

where the full symbol of $P^{(N)}$ is independent of x and the symbol of R_{N+1} is $\mathcal{O}((h, \xi, \epsilon)^{N+1})$. Moreover, on the symbol level we have

$$P^{(N+1)}(\xi, \epsilon; h) - P^{(N)}(\xi, \epsilon; h) = \mathcal{O}((\xi, \epsilon, h)^{N+1}). \tag{3.10}$$

Using this compatibility property, we introduce a limiting operator

$$P^{(\infty)} = P^{(\infty)}(hD_x, \epsilon; h), \tag{3.11}$$

such that

$$P^{(\infty)}(\xi, \epsilon; h) = P^{(N)}(\xi, \epsilon; h) + \mathcal{O}((\xi, \epsilon, h)^{N+1}),$$

for all N. Then $P^{(\infty)}(\xi, \epsilon; h)$ is well-defined modulo $\mathcal{O}((\xi, \epsilon, h)^{\infty})$.

We summarize the discussion of this section in the following proposition.

Proposition 3.2 *Let*

$$P_\epsilon \sim p_0 + hp_1 + \dots, \quad |\xi| \le \frac{1}{\mathcal{O}(1)},$$

be such that

$$p_0 = p_N(\xi) + i\epsilon\langle q\rangle(\xi) + \mathcal{O}(\epsilon^2) + \mathcal{O}(\xi^{N+1}) + \epsilon\mathcal{O}(\xi^N),$$

for some fixed integer N that can be taken arbitrarily large. Then there exist

$$G^{(N)}(x, \xi, \epsilon) = \sum_{j=1}^{N-1} G_j(x, \xi, \epsilon), \tag{3.12}$$

with $G_j = \mathcal{O}_j(\epsilon^2)$, $1 \le j \le N-1$, being homogeneous of degree $j+1$ in (ξ, ϵ) and depending analytically on x, and

$$Q^{(N)}(x, \xi, \epsilon; h) = \sum_{j=0}^{N-1} h^j q_j(x, \xi, \epsilon), \tag{3.13}$$

where q_j are analytic, such that on the operator level, $e^{\mathrm{ad} Q^{(N)}} e^{\frac{i}{h}\mathrm{ad} G^{(N)}} P_\epsilon$ is of the form

$$P^{(N)}(hD_x, \epsilon; h) + R_{N+1}(x, hD_x, \epsilon; h), \tag{3.14}$$

where the full symbol of $P^{(N)}(hD_x, \epsilon; h)$ is independent of x and

$$R_{N+1}(x, \xi, \epsilon; h) = \mathcal{O}((h, \xi, \epsilon)^{N+1}).$$

The leading symbol of $P^{(N)}(hD_x, \epsilon; h)$ is $p_0^{(N)}(\xi, \epsilon) = p^{(N)}(\xi, \epsilon) = a \cdot \xi + i\epsilon\langle q\rangle(0) + \mathcal{O}((\xi, \epsilon)^2) = p(\xi) + i\epsilon q(\xi, \epsilon)$, where

$$d_\xi p(0),\ \operatorname{Re} d_\xi q(0,0) \quad \textit{are linearly independent.} \tag{3.15}$$

Remark. Recalling the operator $P^{\infty}(hD_x, \epsilon; h)$ from (3.11), we notice that formally, Proposition 3.2 leads to the quasi-eigenvalues

$$P^{(\infty)}\left(h\left(k - \frac{k_0}{4}\right) - \frac{S}{2\pi}, \epsilon; h\right) + \mathcal{O}(h^\infty), \quad k \in \mathbf{Z}^2, \tag{3.16}$$

provided that $\epsilon = \mathcal{O}(h^\delta)$ and $kh = \mathcal{O}(h^\delta)$, for some fixed $\delta > 0$. Here we have also used that the space $L^2_\theta(\mathbf{T}^2)$, equipped with the L^2-norm over a fundamental domain of $\mathbf{T}^2$ has an orthonormal basis given by

$$e_k(x) = e^{\frac{i}{h}x\cdot\left(h(k-\frac{k_0}{4})-\frac{S}{2\pi}\right)}, \quad k \in \mathbf{Z}^2. \tag{3.17}$$

Remark. The pseudodifferential Birkhoff normal form construction carried out in Proposition 3.1 and Proposition 3.2 could also have been done in one stroke, by expanding the full symbol of P_ϵ in a sum of homogeneous terms in all the variables (h, ξ, ϵ), and making a similar decomposition for the operator $G^{(N)}$. We hope to be able to return to this idea in a future paper.

4 The Hamilton-Jacobi equation

The results of this section will be instrumental in carrying out a complete spectral analysis of the operator P_ϵ, in the case when ϵ is sufficiently small but fixed. Here we shall do the work that will allow us to construct an additional complex canonical transformation which, together with the normal form construction of section 3, will allow us to reduce the leading symbol of P_ϵ to a function on $T^*\mathbf{T}^2$, which is independent of the spatial variables. The construction will be somewhat similar to the corresponding constructions in [28] and [41], and as there, the basic idea is to work with cohomological equations of $\overline{\partial}$–type, liberating ourselves from Diophantine conditions.

Let us recall from Proposition 3.1 that on the principal symbol level, we can reduce ourselves to the case of a symbol

$$p(x,\xi,\epsilon) = p^{(N-1)}(\xi,\epsilon) + r_N(x,\xi,\epsilon) \tag{4.1}$$

in a complex neighborhood of $\xi = 0$ in $T^*\mathbf{T}^2$. Here $p^{(N-1)}(\xi,\epsilon)$ is a polynomial of degree at most $N-1$ in (ξ,ϵ), and $r_N(x,\xi,\epsilon) = \mathcal{O}((\xi,\epsilon)^N)$. Assume more generally that $r = r_N = \mathcal{O}\left(\epsilon^M + |\xi|^N\right)$, for $2 \leq M \leq N$. Here $M, N \in \mathbf{N}$ can be arbitrarily large but fixed.

Write

$$p^{(N-1)}(\xi,\epsilon) = p(\xi) + i\epsilon q(\xi,\epsilon), \tag{4.2}$$

and recall from Proposition 3.1 that (p is real and that)

$$d_\xi p(0),\ d_\xi \mathrm{Re}\, q(0,0) \quad \text{are linearly independent.} \tag{4.3}$$

Let $\widetilde{\epsilon} > 0$ be a small parameter and assume that $\xi = \mathcal{O}(\widetilde{\epsilon})$. We put

$$z(\xi,\epsilon) = p(\xi) + i\epsilon q(\xi,\epsilon). \tag{4.4}$$

We shall try to find a grad-periodic function $\psi(x) = \psi(x,\xi)$ with $\psi'(x) = \mathcal{O}(\widetilde{\epsilon})$, so that $\varphi(x,\xi) = x\cdot\xi + \psi(x)$ solves the Hamilton-Jacobi equation

$$p(x,\varphi'(x),\epsilon) - z(\xi,\epsilon) = 0, \tag{4.5}$$

so that

$$p(\xi + \psi'_x(x)) + i\epsilon q(\xi + \psi'_x(x),\epsilon) + r_N(x,\xi+\psi'_x(x),\epsilon) = z(\xi,\epsilon). \tag{4.6}$$

Here we "linearize" the first two terms and get

$$\left(p'(\xi) + i\epsilon q'_\xi(\xi,\epsilon)\right)\cdot\partial_x\psi + r_N(x,\xi+\psi'_x(x),\epsilon) + A(\xi,\psi'_x(x),\epsilon)\psi'_x(x)\cdot\psi'_x(x) = 0, \tag{4.7}$$

where

$$A(\xi,\eta,\epsilon) = \int_0^1 (1-t)\,(p+i\epsilon q)''\,(\xi+t\eta)\,dt. \tag{4.8}$$

In order to apply some standard results about non-linear functions of Sobolev class functions, we write

$$\psi = \widetilde{\epsilon}\rho, \tag{4.9}$$

and (4.7) becomes

$$\left(p'(\xi) + i\epsilon q'_\xi(\xi;\epsilon)\right)\cdot\partial_x\rho + G(x,\xi,\rho'_x,\epsilon,\widetilde{\epsilon}) = 0, \tag{4.10}$$

where

$$G(x,\xi,\eta,\epsilon,\widetilde{\epsilon}) = \frac{1}{\widetilde{\epsilon}} r_N(x,\xi+\widetilde{\epsilon}\eta;\epsilon) + \widetilde{\epsilon} A(\xi,\widetilde{\epsilon}\eta,\epsilon)\eta\cdot\eta. \tag{4.11}$$

Let $y_j(x) = y_j(x,\xi;\epsilon)$, $j=1,2$, be linear functions of $x \in \mathbf{R}^2$, with

$$l_j(\xi,\epsilon)\cdot\partial_x y_k = \delta_{j,k}, \tag{4.12}$$

where $l_1 = p'_\xi(\xi)$ and $l_2 = q'_\xi(\xi,\epsilon)$. Put

$$w^* = \frac{1}{2}\left(\epsilon y_1 + \frac{1}{i} y_2\right), \tag{4.13}$$

and

$$w = \frac{1}{2}\left(\epsilon y_1 - \frac{1}{i} y_2\right), \tag{4.14}$$

so that

$$\begin{aligned} \left(p'(\xi) + i\epsilon q'_\xi(\xi,\epsilon)\right)\cdot\partial_x w^* &= \epsilon, \\ \left(p'(\xi) + i\epsilon q'_\xi(\xi,\epsilon)\right)\cdot\partial_x w &= 0. \end{aligned} \tag{4.15}$$

We now look for grad-periodic solutions to (4.10) of the form

$$\rho = \rho_{\mathrm{per}}(x,\xi,\epsilon,\widetilde{\epsilon}) + b(\xi,\epsilon,\widetilde{\epsilon}) w^*(x,\xi,\epsilon), \tag{4.16}$$

where ρ_{per} is single-valued, and try to find ρ as a limit $\rho = \lim_{j\to\infty}\rho_j$, where

$$\rho_0 = 0, \tag{4.17}$$

$$\left(p'_\xi(\xi) + i\epsilon q'_\xi(\xi,\epsilon)\right)\cdot\partial_x\rho_j + G\left(x,\xi,(\rho_{j-1})'_x,\epsilon,\widetilde{\epsilon}\right) = 0, \quad j \geq 1. \tag{4.18}$$

We shall estimate the gradient of the solutions to (4.18) in the standard H^s-norms on $\mathbf{T}^2$ for $s > 1$, and observe that

$$||\,\rho'_x\,||_{H^s} \sim ||\,(\rho_{\mathrm{per}})'_x\,||_{H^s} + |b|\,, \tag{4.19}$$

uniformly with respect to ϵ, for ρ of the form (4.16). We shall also use the fact that if

$$\left(p'_\xi + i\epsilon q'_\xi\right)\cdot\partial_x u = v,$$

with u, v periodic, then

$$||\,u'_x\,||_{H^s} \leq \frac{\mathcal{O}(1)}{\epsilon}||\,v\,||_{H^s}.$$

We notice now that for a given function ρ_{j-1} of the form (4.16), (4.18) is solvable with a solution of the same form, unique up to a constant, and that this equation can be decomposed into the following two equations

$$\epsilon b_j + \mathcal{F}\left(G(\cdot,\xi,(\rho_{j-1})'_x,\epsilon,\widetilde{\epsilon})\right)(0) = 0, \tag{4.20}$$

and

$$\left(p'(\xi) + i\epsilon q'_\xi(\xi,\epsilon)\right)\cdot\partial_x(\rho_{\mathrm{per},j}) + G(x,\xi,(\rho_{j-1})'_x,\epsilon,\widetilde{\epsilon}) - \mathcal{F}\left(G(\cdot,\xi,(\rho_{j-1})'_x,\epsilon,\widetilde{\epsilon})\right)(0) = 0. \tag{4.21}$$

Here we let $\mathcal{F}u(k)$ stand for the Fourier coefficient at $k \in \mathbf{Z}^2$ of the $(2\pi\mathbf{Z})^2$-periodic function u.

In order to treat (4.18) with $j = 1$, we notice that

$$||\, G(\cdot,\xi,0,\epsilon,\widetilde{\epsilon})\,||_{H^s} = \mathcal{O}(1)\frac{1}{\widetilde{\epsilon}}\left(\epsilon^M + \widetilde{\epsilon}^N\right). \tag{4.22}$$

For ρ_1, we then get

$$||\, \rho_1'\,||_{H^s} \leq \mathcal{O}(1)\frac{\epsilon^M + \widetilde{\epsilon}^N}{\epsilon\widetilde{\epsilon}}. \tag{4.23}$$

We may remark that for some component of the gradient, we get a better estimate:

$$||\, p'(\xi)\cdot\partial_x\rho_1\,||_{H^s} \leq \mathcal{O}(1)\frac{\epsilon^M + \widetilde{\epsilon}^N}{\widetilde{\epsilon}}.$$

For $j \geq 2$, we consider in general for u'_x, $v'_x = \mathcal{O}(1)$ in H^s, for some $s > 1$,

$$\begin{aligned} & G(x,\xi,u'_x,\epsilon,\widetilde{\epsilon}) - G(x,\xi,v'_x,\epsilon,\widetilde{\epsilon}) \\ = \; & \frac{1}{\widetilde{\epsilon}}\left(r_N(x,\xi+\widetilde{\epsilon}u'_x,\epsilon) - r_N(x,\xi+\widetilde{\epsilon}v'_x,\epsilon)\right) + \left[\widetilde{\epsilon}A(\xi,\widetilde{\epsilon}\eta,\epsilon)\eta\cdot\eta\right]_{\eta=u'_x}^{v'_x} = \mathrm{I} + \mathrm{II}, \end{aligned}$$

with the natural definitions of I and II. Writing

$$\mathrm{I} = \int_0^1 (\partial_\eta r_N)(x,\xi+\widetilde{\epsilon}(tu'_x + (1-t)v'_x),\epsilon)\,dt\cdot(u'_x - v'_x),$$

we get

$$||\,\mathrm{I}\,||_{H^s} \leq \mathcal{O}(1)\frac{1}{\widetilde{\epsilon}}(\epsilon^M + \widetilde{\epsilon}^N)||\, u'_x - v'_x\,||_{H^s},$$

since by Proposition 2.2 of Chapter 2 in [1], which extends to the case of vector-valued functions,

$$||\,\partial_\eta r_N(x,\xi+\widetilde{\epsilon}(tu'_x + (1-t)v'_x),\epsilon)\,||_{H^s} \leq \mathcal{O}(1)\frac{1}{\widetilde{\epsilon}}(\epsilon^M + \widetilde{\epsilon}^N).$$

Similarly, still assuming that u'_x, $v'_x = \mathcal{O}(1)$ in H^s, we get

$$|| \, \mathrm{II} \, ||_{H^s} \leq \mathcal{O}(1)\widetilde{\epsilon}(|| \, u'_x \, ||_{H^s} + || \, v'_x \, ||_{H^s})|| \, u'_x - v'_x \, ||_{H^s}.$$

Consequently,

$$\begin{aligned} & || \, G(\cdot, \xi, u'_x, \epsilon, \widetilde{\epsilon}) - G(\cdot, \xi, v'_x, \epsilon, \widetilde{\epsilon}) \, ||_{H^s} \\ \leq \quad & \mathcal{O}(1) \left(\frac{1}{\widetilde{\epsilon}}(\epsilon^M + \widetilde{\epsilon}^N) + \widetilde{\epsilon}(|| \, u'_x \, ||_{H^s} + || \, v'_x \, ||_{H^s}) \right) || \, u'_x - v'_x \, ||_{H^s}. \end{aligned} \tag{4.24}$$

We now make the a priori assumption

$$|| \, \rho'_j \, ||_{H^s} = \mathcal{O}(1), \quad j \geq 0, \tag{4.25}$$

uniformly in j, ϵ, and notice that it is valid for $j = 0, 1$, by (4.23), if $\epsilon^{M-1}/\widetilde{\epsilon}$, $\widetilde{\epsilon}^{N-1}/\epsilon \leq \mathcal{O}(1)$. Take the difference between the equations (4.18) for $j+1$ and j:

$$(p'(\xi) + i\epsilon q'_\xi(\xi;\epsilon)) \cdot \partial_x(\rho_{j+1} - \rho_j) + (G(x, \xi, \rho'_j, \epsilon, \widetilde{\epsilon}) - G(x, \xi, \rho'_{j-1}, \epsilon, \widetilde{\epsilon})) = 0. \tag{4.26}$$

In view of (4.24) we get

$$|| \, \rho'_{j+1} - \rho'_j \, ||_{H^s} \leq \mathcal{O}(1) \left(\frac{\epsilon^M + \widetilde{\epsilon}^N}{\epsilon\widetilde{\epsilon}} + \frac{\widetilde{\epsilon}}{\epsilon}(|| \, \rho'_j \, ||_{H^s} + || \, \rho'_{j-1} \, ||_{H^s}) \right) || \, \rho'_j - \rho'_{j-1} \, ||_{H^s}. \tag{4.27}$$

Choose $\widetilde{\epsilon}$ with

$$\frac{\epsilon^M + \widetilde{\epsilon}^N}{\epsilon \min(\epsilon, \widetilde{\epsilon})} \ll 1, \tag{4.28}$$

and notice that this requires that $M > 2$. Then (4.27) implies that

$$|| \, \rho'_{j+1} - \rho'_j \, ||_{H^s} \leq \frac{1}{2} || \, \rho'_j - \rho'_{j-1} \, ||_{H^s}, \tag{4.29}$$

as long as

$$|| \, \rho'_j \, ||, \; || \, \rho'_{j-1} \, || \leq \mathcal{O}(1) \frac{\epsilon^M + \widetilde{\epsilon}^N}{\widetilde{\epsilon}\epsilon},$$

uniformly in j. This holds for $j = 1$, and as long as we have (4.29), we can extend it to the next j-value. Then (4.27) implies that we have a convergent sequence:

$$|| \, \rho'_{j+1} - \rho'_j \, ||_{H^s} \leq \left(\mathcal{O}(1) \frac{\epsilon^M + \widetilde{\epsilon}^N}{\epsilon \min(\epsilon, \widetilde{\epsilon})} \right)^{j-1} \frac{\epsilon^M + \widetilde{\epsilon}^N}{\epsilon\widetilde{\epsilon}}. \tag{4.30}$$

Summing up, we have

Proposition 4.1 *Assume that $\epsilon > 0$, $\widetilde{\epsilon} > 0$ small, are such that*

$$\frac{\epsilon^M + \widetilde{\epsilon}^N}{\epsilon \min(\epsilon, \widetilde{\epsilon})} \ll 1. \tag{4.31}$$

Then for $\xi = \mathcal{O}(\widetilde{\epsilon})$, the Hamilton-Jacobi equation (4.5) *has a solution $\varphi(x,\xi) = x \cdot \xi + \widetilde{\epsilon}\rho(x,\xi)$, with ρ of the form* (4.16), *and*

$$|| \, \rho'_x \, ||_{H^s} = \mathcal{O}((\epsilon^M + \widetilde{\epsilon}^N)/(\epsilon\widetilde{\epsilon})), \quad s > 1,$$

Essentially, the same iteration scheme as above shows that the solution to (4.5) is unique up to a constant, if we require that $\frac{\widetilde{\epsilon}}{\epsilon} || \, \rho'_x \, ||_{H^s} \ll 1$ in addition to (4.31).

5 Global Grushin problem for $\epsilon = \mathcal{O}(h^\delta)$

Throughout this section, it will be assumed that $\epsilon = \mathcal{O}(h^\delta)$ for some $\delta > 0$, fixed, but arbitrarily small. Our goal is to prove Theorem 1.1 and show that the quasi-eigenvalues introduced in (3.16) give all the eigenvalues of P_ϵ, modulo $\mathcal{O}(h^\infty)$, in a rectangle (1.25). In doing so, we shall only rely on the results of sections 2 and 3.

Our starting point in this section is the operator P_ϵ with a leading symbol (2.57). We shall first discuss how to implement the conjugation of P_ϵ by the analytic Fourier integral operator $e^{-iG^{(N)}/h}$, introduced in Proposition 3.2, which quantizes the complex canonical transformation $\widehat{\kappa} = \exp{(H_{G^{(N)}})}$, defined in (3.4), (3.5). In doing so, let us consider the IR-manifold, defined in a complex neighborhood of $\xi = 0$ and given by $\exp{(H_{G^{(N)}})}(T^*\mathbf{T}^2) \subset \widetilde{\mathbf{T}}^2 \times \mathbf{C}^2$. Here $\widetilde{\mathbf{T}}^2 = \mathbf{T}^2 + i\mathbf{R}^2$ stands for the standard complexification of $\mathbf{T}^2$. It follows from Proposition 3.2 that along this manifold we have

$$\operatorname{Im} x = \mathcal{O}(\epsilon^2), \ \operatorname{Im} \xi = \mathcal{O}(\epsilon^2). \tag{5.1}$$

Let us introduce the usual FBI-Bargmann transform

$$Tu(x) = Ch^{-3/2} \int e^{i\varphi(x,y)/h} u(y)\, dy, \quad C > 0,$$

acting on $L^2_\theta(\mathbf{T}^2)$. Here the integration is performed over the whole of $\mathbf{R}^2$, with u being Floquet periodic—see also the discussion in section 3 in [28]. The phase function $\varphi(x,y) = \frac{i}{2}(x-y)^2$ is defined for $x \in \mathbf{T}^2 + i\mathbf{R}^2$, and the associated canonical transformation κ_T is given by

$$T^*\widetilde{\mathbf{T}}^2 \ni (y,\eta) \mapsto (x,\xi) = (y - i\eta, \eta) \in T^*\widetilde{\mathbf{T}}^2. \tag{5.2}$$

The transform κ_T maps the real phase space $T^*\mathbf{T}^2$ to the IR-manifold

$$\Lambda_{\Phi_0} : \ \xi = \frac{2}{i}\frac{\partial \Phi_0}{\partial x} = -\mathrm{Im}\, x, \quad \Phi_0(x) = \frac{1}{2}\left(\mathrm{Im}\, x\right)^2,$$

contained inside $T^*\widetilde{\mathbf{T}}^2$. Here the zero section in $T^*\mathbf{T}^2$ corresponds to $\mathrm{Im}\, x = 0$. On the transform side, the manifold $\exp\left(H_{G^{(N)}}\right)\left(T^*\mathbf{T}^2\right)$ is represented by

$$\xi = \frac{2}{i}\frac{\partial \Phi_\epsilon}{\partial x}, \quad |\mathrm{Im}\, x| \leq \frac{1}{\mathcal{O}(1)},$$

where Φ_ϵ is a smooth strictly plurisubharmonic function, such that

$$\nabla\left(\Phi_\epsilon - \Phi_0\right) = \mathcal{O}(\epsilon^2).$$

By choosing the undetermined constant in Φ_ϵ suitably, we can even arrange that

$$\Phi_\epsilon - \Phi_0 = \mathcal{O}(\epsilon^2). \tag{5.3}$$

Let now $\chi = \chi(\mathrm{Im}\, x) \in C_0^\infty$, $0 \leq \chi \leq 1$, be a standard cutoff function in a neighborhood of 0, and consider the IR-manifold given by

$$\Lambda_{\widetilde{\Phi}_\epsilon} : \quad \xi = \frac{2}{i}\frac{\partial \widetilde{\Phi}_\epsilon}{\partial x},$$

where

$$\widetilde{\Phi}_\epsilon(x) = \chi(\mathrm{Im}\, x)\Phi_\epsilon(x) + \left(1 - \chi(\mathrm{Im}\, x)\right)\Phi_0(x).$$

Here $\widetilde{\Phi}_\epsilon$ is strictly plurisubharmonic, and if we introduce the globally defined IR-manifold

$$\Lambda_\epsilon := \kappa_T^{-1}(\Lambda_{\widetilde{\Phi}_\epsilon}) \subset \widetilde{\mathbf{T}}^2 \times \mathbf{C}^2, \tag{5.4}$$

then Λ_ϵ agrees with $\exp\left(H_{G^{(N)}}\right)(T^*\mathbf{T}^2)$ in a complex neighborhood of the zero section and it is equal to $T^*\mathbf{T}^2$ further away from this set. Moreover, since along $\Lambda_{\widetilde{\Phi}_\epsilon}$,

$$\mathrm{Im}\, \xi = -\frac{\partial \widetilde{\Phi}_\epsilon}{\partial \mathrm{Re} x} = \mathcal{O}(\epsilon^2), \quad \mathrm{Re}\, \xi = -\frac{\partial \widetilde{\Phi}_\epsilon}{\partial \mathrm{Im} x} = -\mathrm{Im}\, x + \mathcal{O}(\epsilon^2),$$

we conclude that along Λ_ϵ, we have

$$\mathrm{Im}\, \xi = \mathcal{O}\left(\epsilon^2\right), \quad \mathrm{Im}\, x = \mathcal{O}(\epsilon^2). \tag{5.5}$$

Now the Fourier integral operator

$$e^{-\frac{i}{h}G^{(N)}} = \mathcal{O}(1) : L^2_\theta(\mathbf{T}^2) \to H_\theta(\Lambda_\epsilon)$$

is such that the action of P_ϵ on the space $H(\Lambda_\epsilon)$, associated to Λ_ϵ by means of the FBI transform T, is microlocally near $\exp(H_{G^{(N)}})(\mathbf{T}^2 \times \{\xi = 0\})$, unitarily equivalent to the operator

$$e^{\frac{i}{h}G^{(N)}} P_\epsilon e^{-\frac{i}{h}G^{(N)}} : L^2_\theta(\mathbf{T}^2) \to L^2_\theta(\mathbf{T}^2). \tag{5.6}$$

From Proposition 3.1 we know that the leading symbol of (5.6) is independent of x, modulo $\mathcal{O}((\xi,\epsilon)^{N+1})$.

Let $0 < \widetilde{\epsilon} \ll 1$ be an additional small parameter such that $\widetilde{\epsilon} \gg \max(\epsilon, h)$. It is then clear, in view of (3.15), that along $T^*\mathbf{T}^2$, in a region where $\widetilde{\epsilon} \leq |\xi| \leq 1/\mathcal{O}(1)$, we have

$$|\mathrm{Re}\, P_\epsilon| \geq \frac{\widetilde{\epsilon}}{\mathcal{O}(1)} \quad \text{or} \quad |\mathrm{Im}\, P_\epsilon - \epsilon \mathrm{Re}\, F_0| \geq \frac{\epsilon\widetilde{\epsilon}}{\mathcal{O}(1)}. \tag{5.7}$$

We now notice that the fact that $P_{\epsilon=0}$ is selfadjoint in $L^2_\theta(\mathbf{T^2})$ implies that the symbol of

$$\mathrm{Im}\, P_\epsilon = \frac{P_\epsilon - P_\epsilon^*}{2i},$$

taken in the operator sense on $H(\Lambda_\epsilon)$, is $\mathcal{O}(\epsilon) + \mathcal{O}(\epsilon h)$. It follows then from the property (5.5) of the IR-deformation given by Λ_ϵ that along Λ_ϵ, outside any $\widetilde{\epsilon}$-neighborhood of the Lagrangian torus $\exp(H_{G_N})(\mathbf{T}^2 \times \{0\})$, the estimates (5.7) still hold true.

Corresponding to the manifold Λ_ϵ on the torus side, we get a globally defined IR-manifold $\Lambda \subset T^*\widetilde{M}$, which is ϵ-close to T^*M everywhere, agrees with that set near infinity, and in a complex neighborhood of Λ_1, it is obtained by replacing $\exp(i\epsilon H_G) \circ \kappa_1^{-1} \circ (\kappa^{(N)})^{-1}(T^*\mathbf{T}^2)$ there by

$$\exp(i\epsilon H_G) \circ \kappa_1^{-1} \circ (\kappa^{(N)})^{-1} \circ \exp(H_{G^{(N)}})(T^*\mathbf{T}^2) = \exp(i\epsilon H_G) \circ \kappa_1^{-1} \circ \kappa_0^{-1}(\Lambda_\epsilon). \tag{5.8}$$

Here we recall that the real analytic canonical transformations κ_1 and $\kappa^{(N)}$ have been defined in (2.2) and (2.4), respectively. (We have also written here $\exp(i\epsilon H_G)$ for the complex canonical transformation identifying $\Lambda_{\epsilon G}$ and T^*M in a neighborhood of Λ_1.)

Using Propositions 2.3 and 2.5, and taking into account also the conjugation by the analytic pseudo-differential operator $\exp(Q^N(x, hD_x, \epsilon; h))$, with Q^N defined in (3.13), we arrive at the following result.

Proposition 5.1 *Keep all the general assumptions from the introduction, and in particular,* (1.13). *We write F to denote the mean value of q over the Diophantine tori Λ_j, $1 \leq j \leq L$, and let us make the global dynamical assumption* (1.24). *In the case when the H_p-flow is completely integrable, we make the assumption* (2.53).

Recall also the real analytic canonical transformations κ_j, $1 \leq j \leq L$, *and* $\kappa^{(N)}$, *where* κ_j *is defined in* (2.2) *for* $j = 1$, *and* $\kappa^{(N)}$ *is defined in* (2.4). *Then the composed transform*

$$\kappa^{(N)} \circ \kappa_j : \mathrm{neigh}(\Lambda_j, T^*M) \to \mathrm{neigh}(\xi = 0, T^*\mathbf{T}^2)$$

maps Λ_j *to* $\xi = 0$, *and has the property that when expressed in terms of the coordinates* x *and* ξ *on the torus side, the leading symbol* p_ϵ *of* P_ϵ *becomes*

$$p_{N,j}(\xi) + i\epsilon q_j(x,\xi) + \mathcal{O}(\epsilon^2) + \mathcal{O}(\xi^{N+1}),$$

with $p_{N,j}(\xi) = a \cdot \xi + \mathcal{O}(\xi^2)$. *Here* $N \geq 1$ *is a fixed positive integer that can be taken arbitrarily large. Using these* (x, ξ)*-coordinates, we then define for* ξ *small,*

$$\langle q_j \rangle(\xi) = \frac{1}{(2\pi)^2} \int q_j(x,\xi)\, dx,$$

and assume that $dp_{N,j}(0) = a_j = a$ *and* $\mathrm{Re}\, d\langle q_j \rangle(0)$ *are linearly independent,* $1 \leq j \leq L$. *Let* $0 < \widetilde{\epsilon} \ll 1$ *be such that* $\widetilde{\epsilon} \gg \max(\epsilon, h)$. *Then there exists a globally defined IR-manifold* $\Lambda \subset T^*\widetilde{M}$ *and* L *smooth Lagrangian tori* $\widehat{\Lambda}_1, \ldots, \widehat{\Lambda}_L \subset \Lambda$, *such that when* $\rho \in \Lambda$ *is away from any* $\widetilde{\epsilon}$*-neighborhood of* $\cup_{j=1}^L \widehat{\Lambda}_j$ *in* Λ, *we have*

$$|\mathrm{Re}\, P_\epsilon(\rho)| \geq \frac{\widetilde{\epsilon}}{\mathcal{O}(1)} \quad \textit{or} \quad |\mathrm{Im}\, P_\epsilon - \epsilon \mathrm{Re}\, F| \geq \frac{\epsilon\widetilde{\epsilon}}{\mathcal{O}(1)}.$$

The manifold Λ *is* ϵ*-close to* T^*M *and agrees with it outside a compact set. We have*

$$P_\epsilon = \mathcal{O}(1) : H(\Lambda, m) \to H(\Lambda),$$

For each j *with* $1 \leq j \leq L$, *there exists an elliptic Fourier integral operator*

$$U_j = \mathcal{O}(1) : H(\Lambda) \to L^2_\theta(\mathbf{T}^2),$$

such that microlocally near $\widehat{\Lambda}_j$, *we have*

$$UP_\epsilon = \left(P_j^{(N)}(hD_x, \epsilon; h) + R_{N+1,j}(x, hD_x, \epsilon; h)\right) U.$$

Here $P_j^{(N)}(hD_x, \epsilon; h) + R_{N+1,j}(x, hD_x, \epsilon; h)$ *is defined microlocally near* $\xi = 0$ *in* $T^*\mathbf{T}^2$, *the full symbol of* $P_j^{(N)}$ *is independent of* x, *and*

$$R_{N+1,j}(x, \xi, \epsilon; h) = \mathcal{O}((h, \xi, \epsilon)^{N+1}).$$

The leading symbol of $P_j^{(N)}(hD_x, \epsilon; h)$ *has the form*

$$a \cdot \xi + i\epsilon F + \mathcal{O}((\epsilon, \xi)^2) = p_j(\xi) + i\epsilon q_j(\xi, \epsilon),$$

where, as before, p_j *is real on the real domain, and* $d_\xi p_j(0)$ *and* $\operatorname{Re} d_\xi q_j(0,0)$ *are linearly independent. We have*

$$\operatorname{Im} P_j^{(N)}(\xi, \epsilon; h) = \epsilon \operatorname{Re} q_j(\xi, \epsilon) + \mathcal{O}(\epsilon h).$$

In what follows we shall choose $\widetilde{\epsilon} \gg \epsilon$ so that

$$h^{1/2-\delta} < \widetilde{\epsilon} = \mathcal{O}(h^\delta). \tag{5.9}$$

As we shall see, later the lower bound on $\widetilde{\epsilon}$ will have to be strengthened.

Our goal is to describe the spectrum of P_ϵ in a rectangle of the form

$$|\operatorname{Re} z| < \frac{\widetilde{\epsilon}}{C}, \quad |\operatorname{Im} z - \epsilon \operatorname{Re} F| < \frac{\epsilon\widetilde{\epsilon}}{C}, \tag{5.10}$$

when $C > 0$ is a sufficiently large but fixed constant. To this end, let us introduce the quasi-eigenvalues coming from Proposition 5.1—see also (3.16),

$$z(j,k) := P_j^{(N)}\left(h\left(k - \frac{k_j}{4}\right) - \frac{S_j}{2\pi}, \epsilon; h\right) + \mathcal{O}(h^{\delta(N+1)}), \quad 1 \le j \le L, \quad k \in \mathbf{Z}^2, \tag{5.11}$$

with $h(k-k_j/4)-S_j/2\pi = \mathcal{O}(\widetilde{\epsilon})$. In what follows we shall work under the assumption that ϵ is bounded from below by an arbitrary but fixed positive power of h, $\epsilon \ge h^K$, $K \gg 1$ is fixed. Let us also remark that it follows from Proposition 5.1 that the distance between two neighboring quasi-eigenvalues is $\ge \epsilon h/\mathcal{O}(1)$, provided that N is large enough.

When $z \in \mathbf{C}$ is in the rectangle (5.10), let us consider the equation

$$(P_\epsilon - z)u = v, \quad u \in H(\Lambda). \tag{5.12}$$

At first, we shall derive an a priori bound for the part of u concentrated away from the set $\cup_{j=1}^L \widehat{\Lambda}_j$.

In what follows we shall write that a function $a \in C^\infty(\Lambda)$ (also depending on h) is in the symbol class $S_{\widetilde{\epsilon}}(1)$ if uniformly on Λ,

$$\nabla^m a = \mathcal{O}_\alpha(\widetilde{\epsilon}^{-m}), \quad m \ge 0.$$

We introduce then a smooth partition of unity on the manifold Λ,

$$1=\sum_{j=1}^{L}\chi_j+\psi_{1,+}+\psi_{1,-}+\psi_{2,+}+\psi_{2,-}. \tag{5.13}$$

Here $0\leq\chi_j\in C_0^\infty(\Lambda)\cap S_{\widetilde{\epsilon}}(1)$ is a cut-off function to an $\widetilde{\epsilon}$-neighborhood of $\widehat{\Lambda}_j$, $1\leq j\leq L$, such that on the operator level we have,

$$[P_\epsilon,\chi_j]=\mathcal{O}(h^{(N+1)\delta}):H(\Lambda)\rightarrow H(\Lambda). \tag{5.14}$$

To construct χ_j satisfying (5.14), we use Proposition 5.1 to pass to the torus model and take there a symbol of the form $\chi_0(\xi/\widetilde{\epsilon})$, where $\chi_0\in C_0^\infty(T^*\mathbf{T}^2)$ is supported in a small neighborhood of $\xi=0$ in $T^*\mathbf{T}^2$, and $\chi_0=1$ near $\xi=0$. Conjugating $\chi_0(hD_x/\widetilde{\epsilon})$ by means of the microlocal inverse of the operator U_j of Proposition 5.1, we obtain the cut-off function χ_j with the required properties.

The functions $0\leq\psi_{1,\pm}\in S_{\widetilde{\epsilon}}(1)$ in (5.13) are chosen so that $\pm\mathrm{Re}\,P_\epsilon\geq\widetilde{\epsilon}/\mathcal{O}(1)$ in the support of $\psi_{1,\pm}$, respectively. Finally, we have $\pm\left(\mathrm{Im}\,P_\epsilon-\epsilon\mathrm{Re}\,F\right)\geq\epsilon\widetilde{\epsilon}/\mathcal{O}(1)$ in the support of $\psi_{2,\pm}\in C_0^\infty(\Lambda)\cap S_{\widetilde{\epsilon}}(1)$, and as in the case of χ_j, we arrange so that in the operator norm,

$$A[P_\epsilon,\psi_{2,\pm}]=\mathcal{O}(h^{(N+1)\delta}), \tag{5.15}$$

where A is a microlocal cut-off to a region where $|\mathrm{Re}\,P_\epsilon|<\widetilde{\epsilon}/\mathcal{O}(1)$.

When u satisfies (5.12), we shall prove that

$$||\left(1-\sum_{j=1}^{L}\chi_j\right)u\,||\leq\frac{\mathcal{O}(1)}{\epsilon\widetilde{\epsilon}}||\,v\,||+\mathcal{O}(h^{(N+1)\delta-K-1})||\,u\,||, \tag{5.16}$$

where we shall choose N such that $(N+1)\delta-K-1\gg 1$, and provided that the lower bound on $\widetilde{\epsilon}$ in (5.9) is suitably strengthened. Here the norms are taken in $H(\Lambda)$. When establishing (5.16), we shall first prove that

$$||\,\psi_{1,+}u\,||\leq\frac{\mathcal{O}(1)}{\widetilde{\epsilon}}||\,v\,||+\mathcal{O}(h^\infty)||\,u\,||, \tag{5.17}$$

which is essentially an elliptic estimate. To prove (5.17), we may follow the method of [23]. (See also [37].) When $M\in\mathbf{N}$, we let

$$\psi_{1,+}=:\psi_0\prec\psi_1\prec\ldots\prec\psi_M$$

be a nested sequence of functions in $S_{\widetilde{\epsilon}}(1)$, supported in a region where $\mathrm{Re}\,P_\epsilon\geq\widetilde{\epsilon}/\mathcal{O}(1)$. Near the support of ψ_j we have $\mathrm{Re}\,(P_\epsilon-z)\geq\widetilde{\epsilon}/\mathcal{O}(1)$. Using $h/\widetilde{\epsilon}^2$ as a new semiclassical parameter that appears naturally on the operator level when reducing $S_{\widetilde{\epsilon}}(1)$ to $S_1(1)$ by dilation, and applying the sharp Gårding inequality we get

$$\begin{aligned} \mathrm{Re}\,((P_\epsilon - z)\psi_j u|\psi_j u) &\geq \left(\frac{\widetilde{\epsilon}}{\mathcal{O}(1)} - \mathcal{O}(1)\frac{h}{\widetilde{\epsilon}^2}\right) ||\,\psi_j u\,||^2 - \mathcal{O}(h^\infty)||\,u\,||^2 \\ &\geq \frac{\widetilde{\epsilon}}{\mathcal{O}(1)}||\,\psi_j u\,||^2 - \mathcal{O}(h^\infty)||\,u\,||^2, \quad j = 0, \dots, M. \end{aligned} \tag{5.18}$$

provided that $h/\widetilde{\epsilon}^3 \ll 1$, and we shall even strengthen this further by assuming that

$$\frac{h}{\widetilde{\epsilon}^3} \leq h^\delta. \tag{5.19}$$

Now using that $\psi_j(1-\psi_{j+1}) = \mathcal{O}(h^\infty)$ in the operator norm, we see that the absolute value of the left hand side in (5.18) does not exceed

$$\mathcal{O}(1)||\,v\,||||\,\psi_j u\,|| + \mathcal{O}\left(\frac{h}{\widetilde{\epsilon}^2}\right)||\,\psi_{j+1}u\,||^2 + \mathcal{O}(h^\infty)||\,u\,||^2.$$

Here we have also used that $[P_\epsilon, \psi_j] = \mathcal{O}(h/\widetilde{\epsilon}^2)$. We get

$$\frac{\widetilde{\epsilon}}{\mathcal{O}(1)}||\,\psi_j u\,||^2 \leq \mathcal{O}(1)||\,v\,||||\,\psi_j u\,|| + \mathcal{O}\left(\frac{h}{\widetilde{\epsilon}^2}\right)||\,\psi_{j+1}u\,||^2 + \mathcal{O}(h^\infty)||\,u\,||^2,$$

and therefore,

$$||\,\psi_j u\,||^2 \leq \frac{\mathcal{O}(1)}{\widetilde{\epsilon}^2}||\,v\,||^2 + \mathcal{O}\left(\frac{h}{\widetilde{\epsilon}^3}\right)||\,\psi_{j+1}u\,||^2 + \mathcal{O}(h^\infty)||\,u\,||^2.$$

Taking into account (5.19), we obtain

$$||\,\psi_j u\,||^2 \leq \frac{\mathcal{O}(1)}{\widetilde{\epsilon}^2}||\,v\,||^2 + \mathcal{O}(h^\delta)||\,\psi_{j+1}u\,||^2 + \mathcal{O}(h^\infty)||\,u\,||^2,$$

and combining these estimates for $j = 0, 1, \dots, M$, we get

$$||\,\psi_{1,+}u\,||^2 \leq \frac{\mathcal{O}(1)}{\widetilde{\epsilon}^2}||\,v\,||^2 + \mathcal{O}_M(h^{M\delta})||\,\psi_M u\,||^2 + \mathcal{O}(h^\infty)||\,u\,||^2.$$

The estimate (5.17) follows, and by repeating this argument, we get the same estimate also for $\psi_{1,-}u$. We next notice that $\mathrm{Im}\,P_\epsilon = \mathcal{O}(\epsilon)$ on Λ, and near $\mathrm{supp}\,\psi_{2,+}$ we have $\mathrm{Im}\, P_\epsilon - \epsilon \mathrm{Re}\, F \geq \epsilon\widetilde{\epsilon}/\mathcal{O}(1)$. An application of the sharp Gårding inequality as in (5.18) gives then for z in (5.10),

$$\mathrm{Im}\, ((P_\epsilon - z)\psi_{2,+}u|\psi_{2,+}u) \geq \frac{\epsilon\widetilde{\epsilon}}{\mathcal{O}(1)}||\, \psi_{2,+}u\, ||^2 - \mathcal{O}(h^\infty)||\, u\, ||^2. \tag{5.20}$$

The left hand side of (5.20) does not exceed

$$\mathcal{O}(1)||\, v\, ||\, ||\, \psi_{2,+}u\, || + ||\, [P_\epsilon, \psi_{2,+}]u\, ||\, ||\, \psi_{2,+}u\, ||,$$

and combining this with (5.20) we get

$$||\, \psi_{2,+}u\, || \leq \frac{\mathcal{O}(1)}{\widetilde{\epsilon}\epsilon}||\, v\, || + \frac{\mathcal{O}(1)}{\widetilde{\epsilon}\epsilon}||\, [P_\epsilon, \psi_{2,+}]u\, || + \mathcal{O}(h^\infty)||\, u\, ||. \tag{5.21}$$

Now using (5.15) together with (5.17) and (5.19) we get

$$||\, [P_\epsilon, \psi_{2,+}]u\, || \leq \mathcal{O}(h^\delta)||\, v\, || + \mathcal{O}(h^{(N+1)\delta})||\, u\, ||,$$

and we conclude from (5.21) that

$$||\, \psi_{2,\pm}u\, || \leq \frac{\mathcal{O}(1)}{\epsilon\widetilde{\epsilon}}||\, v\, || + \mathcal{O}(h^{(N+1)\delta - K - 1})||\, u\, ||.$$

The estimate (5.16) follows.

In the spirit of [23], as a warm-up exercise, we shall now prove that z in the rectangle (5.10) avoids the union of $\epsilon h/\mathcal{O}(1)$-neighborhoods of the quasi-eigenvalues (5.11), then $P_\epsilon - z : H(\Lambda, m) \rightarrow H(\Lambda)$ is invertible. When doing so, we write, for $1 \leq j \leq L$,

$$(P_\epsilon - z)\chi_j u = \chi_j v + [P_\epsilon, \chi_j]u,$$

where the $H(\Lambda)$-norm of the commutator term in the right hand side is $\mathcal{O}(h^{(N+1)\delta})||\, u\, ||$. Applying the operator U_j of Proposition 5.1 we get

$$(P_j^{(N)} + R_{N+1,j} - z)U_j\chi_j u = U_j\chi_j v + T_{N,j}u, \tag{5.22}$$

where

$$T_{N,j} = \mathcal{O}(h^{(N+1)\delta}) : H(\Lambda) \rightarrow L^2_\theta(\mathbf{T}^2).$$

Using the fact that $R_{N+1,j}(x, \xi, \epsilon; h) = \mathcal{O}((h, \epsilon, \xi)^{N+1})$ together with the localization properties of $U_j\chi_j$, we can rewrite (5.22) as

$$(P_j^{(N)} - z)U_j\chi_j u = U_j\chi_j v + T_{1,N,j}u,$$

where $T_{1,N,j}$ has the same bound as $T_{N,j}$. Taking an expansion in Fourier series,

$$v(x) = \sum_{k \in \mathbf{Z}^2} \widehat{v}(k-\theta) e_k(x), \quad v \in L^2_\theta,$$

where e_l are defined in (3.17), we next see that the operator $P_j^{(N)} - z$ acting on $L^2_\theta(\mathbf{T}^2)$ is invertible, microlocally near $\xi = 0$, with a microlocal inverse of the norm $\mathcal{O}(1/\epsilon h)$, provided that z in (5.10) avoids the union of the $\epsilon h/\mathcal{O}(1)$-neighborhoods of the quasi-eigenvalues (5.11). It follows that for $1 \leq j \leq N$,

$$||\, \chi_j u \,|| \leq \frac{\mathcal{O}(1)}{\epsilon h} ||\, v \,|| + \mathcal{O}(h^{(N+1)\delta - K - 1}) ||\, u \,||, \;\; \epsilon \geq h^K, \;\; (N+1)\delta - K - 1 \gg 1.$$

Combining this estimate with (5.16), we see that the operator $P_\epsilon - z : H(\Lambda, m) \rightarrow H(\Lambda)$ is injective, hence bijective, since general arguments based on the ellipticity at infinity (1.6), (1.10) show that it is a Fredholm operator of index zero.

We shall now discuss the setup of the global Grushin problem for $P_\epsilon - z$, in the Hilbert space $H(\Lambda)$, which will be well-posed for z varying in the rectangle (5.10). When doing so, we will continue to assume that $\epsilon \geq h^K$, when $K \gg 1$ is fixed.

With $z(j,k)$, being defined in (5.11), let us introduce for $1 \leq j \leq L$,

$$M_j = \# \left\{ z(j,k); |\mathrm{Re}\, z(k)| < \frac{\widetilde{\epsilon}}{C}, \quad |\mathrm{Im}\, z - \epsilon \mathrm{Re}\, F_0| < \frac{\widetilde{\epsilon}\epsilon}{C} \right\},$$

where C is sufficiently large. Then $M_j = \mathcal{O}(\widetilde{\epsilon}^2 h^{-2})$, and we let $k(j,1), \dots k(j, M_j) \in \mathbf{Z}^2$ be the corresponding lattice points, so that

$$h \left(k(j,l) - \frac{k_0}{4} \right) - \frac{S}{2\pi} = \mathcal{O}(\widetilde{\epsilon}), \quad 1 \leq l \leq M_j, \; 1 \leq j \leq L.$$

We introduce the operator

$$R_+ : H(\Lambda) \rightarrow \mathbf{C}^{M_1} \times \cdots \times \mathbf{C}^{M_L},$$

given by

$$R_+ u(j)(l) = (U_j \chi_j u | e_{k(j,l)}), \quad 1 \leq j \leq L, \quad 1 \leq l \leq M_j.$$

Here $e_{k(j,l)}$ is as in (3.17) and the scalar product in the definition of R_+ is taken in $L^2_\theta(\mathbf{T}^2)$. Define next

$$R_- : \mathbf{C}^{M_1} \times \cdots \times \mathbf{C}^{M_L} \rightarrow H(\Lambda)$$

by

$$R_-u_- = \sum_{j=1}^{L}\sum_{l=1}^{M_j} u_-(j)(l)U_j^{-1}e_{k(j,l)}.$$

Here U_j^{-1} is a microlocal inverse of U_j. Let us recall that $e_k(x)$ is microlocally concentrated in the region of $T^*\mathbf{T}^2$ where $\xi \sim h(k-\frac{k_0}{4}) - \frac{S}{2\pi}$. It follows therefore that for $1 \leq j \leq L$,

$$\chi_j R_- = \sum_{l=1}^{M_j} u_-(j)(l)U_j^{-1}e_{k(j,l)} + \mathcal{O}(h^\infty) : \mathbf{C}^{M_1}\times\cdots\times\mathbf{C}^{M_L} \to H(\Lambda). \tag{5.23}$$

We shall next check that when $z \in \mathbf{C}$ varies in the rectangle (5.10), with an increased value of C, the Grushin problem

$$\begin{cases} (P_\epsilon - z)u + R_-u_- = v, \\ R_+u = v_+ \end{cases} \tag{5.24}$$

has a unique solution $(u,u_-) \in H(\Lambda,m)\times\left(\mathbf{C}^{M_1}\times\cdots\times\mathbf{C}^{M_L}\right)$ for every $(v,v_+) \in H(\Lambda)\times\left(\mathbf{C}^{M_1}\times\cdots\times\mathbf{C}^{M_L}\right)$. Moreover, we have an a priori estimate

$$||\, u\, || + ||\, u_-\, || \leq \frac{\mathcal{O}(1)}{\epsilon\widetilde{\epsilon}}\left(||\, v\, || + ||\, v_+\, ||\right), \tag{5.25}$$

where the norms of u and v are taken in $H(\Lambda)$, and the norms of u_- and v_+ are taken in $\mathbf{C}^{M_1}\times\cdots\times\mathbf{C}^{M_L}$. Indeed, we first notice that in view of (5.16) and (5.23),

$$||\left(1-\sum_{j=1}^{L}\chi_j\right)u\, || \leq \frac{\mathcal{O}(1)}{\epsilon\widetilde{\epsilon}}||\, v\, || + \mathcal{O}(h^{(N+1)\delta-K-1})\left(||\, u\, || + ||\, u_-\, ||\right). \tag{5.26}$$

On the other hand, applying χ_j and then U_j, $1 \leq j \leq L$, to the first equation of (5.24), we obtain

$$\begin{cases} (P_j^{(N)} - z)U_j\chi_j u + \sum_{l=1}^{M_j} u_-(j)(l)e_{k(j,l)} = U_j\chi_j v + w_j, \\ (U_j\chi_j u|e_{k(j,l)}) = v_+(j)(l), \quad 1 \leq l \leq M_j. \end{cases} \tag{5.27}$$

Here the $L^2_\theta(\mathbf{T}^2)$-norm of w_j is $\mathcal{O}(h^{(N+1)\delta})\left(||\, u\, || + ||\, u_-\, ||\right)$. It follows that for $1 \leq j \leq L$,

$$||\, \chi_j u\, || + ||\, u_-(j)\, || \leq \frac{\mathcal{O}(1)}{\epsilon\widetilde{\epsilon}}\left(||\, v\, || + ||\, v_+(j)\, ||\right) + \frac{\mathcal{O}(h^{(N+1)\delta})}{\epsilon\widetilde{\epsilon}}\left(||\, u\, || + ||\, u_-\, ||\right).$$

Here, as in (5.26), we shall choose N so large that $\epsilon\widetilde{\epsilon} \geq \epsilon h \geq h^{K+1} \gg h^{(N+1)\delta}$, which together with (5.26) implies the injectivity, and hence the well-posedness of the Grushin problem (5.24). The bound (5.25) follows.

The solution to (5.24) is given by

$$u = Ev + E_+v_+, \quad u_- = E_-v + E_{-+}v_+,$$

and we recall (see [42] and further references given there) that the eigenvalues of P_ϵ in the rectangle (5.10) are precisely the values z for which the matrix $E_{-+} \in \mathcal{L}(\mathbf{C}^{M_1} \times \cdots \times \mathbf{C}^{M_L}, \mathbf{C}^{M_1} \times \cdots \times \mathbf{C}^{M_L})$ is non-invertible. Arguing precisely as in [28] and [23], we find that modulo an error term $\mathcal{O}(h^{(N+1)\delta})$, E_{-+} is a block-diagonal matrix with the blocks $E_{-+}(z)(j) \in \mathcal{L}(\mathbf{C}^{M_j}, \mathbf{C}^{M_j})$, $1 \leq j \leq L$, given by

$$E_{-+}(z)(j)(m,n) = (z - z(j, k(j,m)))\delta_{mn}, \ 1 \leq m \leq n \leq M_j.$$

The discussion above is summarized in the following theorem.

Theorem 5.2 *Let F stand for the mean value of q along the Diophantine tori Λ_j, $1 \leq j \leq L$. When $\alpha_{1,j}$ and $\alpha_{2,j}$ are the fundamental cycles in Λ_j, $1 \leq j \leq L$, we write $S_j = (S_{1,j}, S_{2,j})$ and $k_j = (k(\alpha_{1,j}), k(\alpha_{2,j}))$ for the actions and the Maslov indices of the cycles, respectively. Assume furthermore that $\epsilon = \mathcal{O}(h^\delta)$, $\delta > 0$ satisfies $\epsilon \geq h^K$, for some K fixed but arbitrarily large. Let $\widetilde{\epsilon} > 0$ be an additional small parameter such that $\widetilde{\epsilon} \gg \epsilon$ and*

$$h^{1/3-\delta} < \widetilde{\epsilon} = \mathcal{O}(h^\delta).$$

We next make the global dynamical assumption (1.24)*, and assume that the differentials of the functions $p_{N,j}(\xi)$ and $\mathrm{Re}\,\langle q_j\rangle(\xi)$, defined in Proposition* 5.1*, are linearly independent, when $\xi = 0$, $1 \leq j \leq L$. In the case when the H_p-flow is completely integrable, instead of* (1.24) *we assume* (2.53)*. Let $C > 0$ be sufficiently large. Then the eigenvalues of P_ϵ in the rectangle*

$$|\mathrm{Re}\, z| < \frac{\widetilde{\epsilon}}{C}, \quad |\mathrm{Im}\, z - \epsilon \mathrm{Re}\, F_0| < \frac{\widetilde{\epsilon}\epsilon}{C} \tag{5.28}$$

are given by

$$P_j^{(N)}\left(h\left(k - \frac{k_j}{4}\right) - \frac{S_j}{2\pi}, \epsilon; h\right) + \mathcal{O}(h^{(N+1)\delta}), \quad k \in \mathbf{Z}^2, \quad 1 \leq j \leq L.$$

Here $N \in \mathbf{N}$ is such that $(N+1)\delta - K - 1 \gg 1$, and

$$P_j^{(N)}(\xi, \epsilon; h) = p_{N,j}(\xi) + i\epsilon\langle q_j\rangle(\xi) + \mathcal{O}(\epsilon^2) + \mathcal{O}(h).$$

Let us finally recall from (3.11) the limiting operator $P_j^{(\infty)}(hD_x, \epsilon; h)$, $1 \leq j \leq L$, well-defined modulo $\mathcal{O}((h, \epsilon, \xi)^\infty)$ and such that for each $N \in \mathbf{N}$,

$$P_j^{(\infty)}(\xi, \epsilon; h) = P_j^{(N)}(\xi, \epsilon; h) + \mathcal{O}((h, \epsilon, \xi)^{N+1}).$$

Then it follows from Theorem 5.2 that the eigenvalues of P_ϵ in the domain (5.28) are of the form

$$P_j^{(\infty)}\left(h\left(k - \frac{k_j}{4}\right) - \frac{S_j}{2\pi}, \epsilon; h\right) + \mathcal{O}(h^\infty), \quad k \in \mathbf{Z}^2, \quad 1 \leq j \leq L.$$

This completes the proof of Theorem 1.1.

Remark. In the end of this section, we would like to mention that the method of Grushin reduction, exploited in this section, has a very long tradition and is closely related and essentially equivalent to the so-called Feshbach projection method—see [16] for a recent use of the latter in the context of spectral analysis of the Pauli-Fierz Hamiltonian in quantum field theory. We also refer to [42] for a systematic presentation of the Grushin method and of some of its applications. In particular, in [42], it is explained how the Feshbach method fits into the framework of Grushin problems.

6 Asymptotic expansion of eigenvalues for large ϵ

In this section we let ϵ be sufficiently small but independent of h and the purpose is to compute all the eigenvalues of the operator P_ϵ in a domain independent of h, thereby proving Theorem 1.2. In doing so, we shall use the results of section 4, and we shall also see that a suitable quantum Birkhoff normal form construction will allow us to extend the result to cover a certain h-dependent range of values of ϵ.

Let us recall from Proposition 3.1 that the leading symbol (2.57) of the operator (2.56) can be reduced to the normal form

$$p(x, \xi, \epsilon) = p_{N-1}(\xi, \epsilon) + r_N(x, \xi, \epsilon), \tag{6.1}$$

with $r_N(x, \xi, \epsilon) = \mathcal{O}((\epsilon, \xi)^N)$, and $p_{N-1}(\xi, \epsilon) = p(\xi) + i\epsilon q(\xi, \epsilon)$, with $p(\xi)$ real, and $d_\xi p(0)$, $d_\xi \mathrm{Re}\, q(0, 0)$ linearly independent. Then from Proposition 4.1 we recall that the Hamilton-Jacobi equation

$$p(x, \varphi'_x, \epsilon) - p_{N-1}(\xi, \epsilon) = 0 \tag{6.2}$$

has the solution

$$\varphi(x, \xi, \epsilon) = x \cdot \xi + \widetilde{\epsilon}\rho(x, \xi, \epsilon, \widetilde{\epsilon}), \tag{6.3}$$

for $\xi = \mathcal{O}(\widetilde{\epsilon})$, $\epsilon \leq \widetilde{\epsilon}$, $\widetilde{\epsilon}^N/\epsilon^2 \ll 1$. Here $||\, \rho_x' \,||_{H^s} = \mathcal{O}(\widetilde{\epsilon}^{N-1}/\epsilon)$, $s > 1$. The construction in section 4 clearly works the same way for x complex with $|\mathrm{Im}\, x| < 1/\mathcal{O}(1)$ and for ξ complex with $\xi = \mathcal{O}(\widetilde{\epsilon})$, if we work in H^s-spaces on each torus $\mathrm{Im}\, x = \mathrm{Const}$. If we normalize the choice of the grad-periodic function ρ by putting $\rho(0, \xi) = 0$, then $\rho(x, \xi)$ becomes holomorphic in x and ξ, for $|\mathrm{Im}\, x| < 1/\mathcal{O}(1)$, $\xi = \mathcal{O}(\widetilde{\epsilon})$. Write

$$\varphi(x, \xi, \epsilon) = x \cdot \xi + \psi(x, \xi, \epsilon), \tag{6.4}$$

so that

$$|\psi(x, \xi, \epsilon)| = \mathcal{O}\left(\frac{\widetilde{\epsilon}^N}{\epsilon}\right), \tag{6.5}$$

for $|\mathrm{Im}\, x| \leq 1/\mathcal{O}(1)$, $\xi = \mathcal{O}(\widetilde{\epsilon})$. From the Cauchy inequalities we get

$$\partial_x^\alpha \partial_\xi^\beta \psi = \mathcal{O}\left(\frac{\widetilde{\epsilon}^{N-|\beta|}}{\epsilon}\right) \tag{6.6}$$

in the same region.

Consider the actions (independent of x):

$$\begin{aligned} \eta_j &= \frac{1}{2\pi}(\varphi(x + 2\pi e_j, \xi, \epsilon) - \varphi(x, \xi, \epsilon)) \\ &= \xi_j + \frac{1}{2\pi}(\psi(x + 2\pi e_j, \xi, \epsilon) - \psi(x, \xi, \epsilon)) = \xi_j + \mathcal{O}\left(\frac{\widetilde{\epsilon}^N}{\epsilon}\right), \end{aligned} \tag{6.7}$$

where $e_1 = (1, 0)$, $e_2 = (0, 1)$. By the implicit function theorem and the Cauchy inequalities, we see that the map $\xi \mapsto \eta(\xi)$, defined for $\xi = \mathcal{O}(\widetilde{\epsilon})$, is bijective with an inverse $\xi(\eta)$, and

$$\eta(\xi) = \xi + \mathcal{O}\left(\frac{\widetilde{\epsilon}^N}{\epsilon}\right), \quad \xi(\eta) = \eta + \mathcal{O}\left(\frac{\widetilde{\epsilon}^N}{\epsilon}\right). \tag{6.8}$$

Again derivatives of these maps can be estimated by means of the Cauchy inequalities. Now write $\varphi(x, \eta, \epsilon)$ instead of $\varphi(x, \xi, \epsilon)$. We have

$$\varphi(x, \eta, \epsilon) = x \cdot \eta + \mathcal{O}\left(\frac{\widetilde{\epsilon}^N}{\epsilon}\right), \quad |\mathrm{Im}\, x| < 1/\mathcal{O}(1), \ |\mathrm{Re}\, x| = \mathcal{O}(1), \ |\eta| < \mathcal{O}(\widetilde{\epsilon}), \tag{6.9}$$

and $\varphi(x, \eta, \epsilon) - x \cdot \eta$ is single-valued.

Consider the canonical transformation

$$\kappa = \kappa_\epsilon : (\partial_\eta \varphi(x, \eta, \epsilon), \eta) \mapsto (x, \partial_x \varphi(x, \eta, \epsilon)). \tag{6.10}$$

Differentiating the identity (6.7), we get

$$\partial_{\eta_k}\varphi(2\pi e_j + x, \eta, \epsilon) = \partial_{\eta_k}\varphi(x, \eta, \epsilon) + 2\pi\delta_{j,k},$$

so it is clear that κ_ϵ is a map from a complex neighborhood of the form $|\mathrm{Im}\, y| < 1/\mathcal{O}(1)$, $\eta = \mathcal{O}(\widetilde{\epsilon})$ of the zero section in $T^*\mathbf{T}^2$ onto another neighborhood which contains a neighborhood of the same form.

By construction, $p \circ \kappa_\epsilon(y, \eta) = p_{N-1}(\xi, \epsilon)$, and since $\xi = \eta + \mathcal{O}(\widetilde{\epsilon}^N/\epsilon)$, we get, expanding the right hand side as a function of (ϵ, η),

$$p \circ \kappa_\epsilon(y, \eta) = p_{N-1}(\eta, \epsilon) + \mathcal{O}\left(\frac{\widetilde{\epsilon}^N}{\epsilon}\right) = p(\eta) + i\epsilon q(\eta, \epsilon) + \mathcal{O}\left(\frac{\widetilde{\epsilon}^N}{\epsilon}\right), \tag{6.11}$$

and this is a function of (η, ϵ) only.

We next want to implement κ by a Fourier integral operator. For that, it will be convenient to work in a fixed neighborhood of the zero section. Let $P = P(x, hD_x, \epsilon; h)$ be an h-pseudodifferential operator with the leading symbol $p(x, \xi, \epsilon)$ as in (6.1) and assume, as we may, for simplicity, that $p(0) = 0$, $q(0, \epsilon) = 0$. Put $\widetilde{h} = h/\widetilde{\epsilon}$, so that

$$\frac{1}{\widetilde{\epsilon}}P(x, hD_x, \epsilon; h) = \frac{1}{\widetilde{\epsilon}}P(x, \widetilde{\epsilon}\widetilde{h}D_x, \epsilon; h). \tag{6.12}$$

As an $\widetilde{h}$-pseudodifferential operator, $\widetilde{\epsilon}^{-1}P$ has a well-defined symbol in a fixed neighborhood of $\xi = 0$, and the leading symbol will be

$$\frac{1}{\widetilde{\epsilon}}p(x, \widetilde{\epsilon}\xi, \epsilon) = \frac{1}{\widetilde{\epsilon}}p(\widetilde{\epsilon}\xi) + i\epsilon\frac{1}{\widetilde{\epsilon}}q(\widetilde{\epsilon}\xi, \epsilon) + \mathcal{O}(\widetilde{\epsilon}^{N-1}). \tag{6.13}$$

Notice that $\widetilde{\epsilon}^{-1}p(\widetilde{\epsilon}\xi)$, $\widetilde{\epsilon}^{-1}q(\widetilde{\epsilon}\xi, \epsilon)$ are uniformly bounded in a fixed complex domain when $\widetilde{\epsilon} \to 0$, and that

$$d_\xi\frac{1}{\widetilde{\epsilon}}p(\widetilde{\epsilon}\xi),\ d_\xi\frac{1}{\widetilde{\epsilon}}q(\widetilde{\epsilon}\xi, 0) \quad \text{are linearly independent for } \xi = 0, \tag{6.14}$$

uniformly when $\widetilde{\epsilon} \to 0$.

Let

$$Uu(x) = h^{-2}\iint e^{\frac{i}{h}(\varphi(x,\eta,\epsilon) - y\cdot\eta)}a(x, \eta; h)u(y)\, dy\, d\eta, \tag{6.15}$$

be an elliptic h-Fourier integral operator, associated to κ. The change of variables, $\widetilde{h} = h/\widetilde{\epsilon}$, $\eta = \widetilde{\epsilon}\widetilde{\eta}$, gives

$$Uu(x) = \widetilde{h}^{-2}\iint e^{\frac{i}{\widetilde{h}}(\frac{1}{\widetilde{\epsilon}}\varphi(x,\widetilde{\epsilon}\widetilde{\eta},\epsilon) - y\cdot\widetilde{\eta})}a(x, \widetilde{\epsilon}\widetilde{\eta}; h)u(y)\, dy\, d\widetilde{\eta}. \tag{6.16}$$

From (6.9) we get

$$\frac{1}{\widetilde{\epsilon}}\varphi(x,\widetilde{\epsilon}\widetilde{\eta},\epsilon) = x\cdot\widetilde{\eta} + \mathcal{O}\left(\frac{\widetilde{\epsilon}^{N-1}}{\epsilon}\right),\ |\mathrm{Im}\,x| < \frac{1}{\mathcal{O}(1)},\quad |\widetilde{\eta}| = \mathcal{O}(1). \tag{6.17}$$

The phase is therefore uniformly non-degenerate and the corresponding canonical transformation is

$$\widetilde{\kappa} : (\varphi'_\eta(x,\widetilde{\epsilon}\widetilde{\eta},\epsilon),\widetilde{\eta}) \to (x, \frac{1}{\widetilde{\epsilon}}\varphi'_x(x,\widetilde{\epsilon}\widetilde{\eta},\epsilon)). \tag{6.18}$$

The conjugated operator $\widetilde{P} = U^{-1}\frac{1}{\widetilde{\epsilon}}PU$ is a uniformly well-behaved $\widetilde{h}$-pseudodifferential operator with leading symbol obtained from (6.11) by division by $\widetilde{\epsilon}$ and substitution $\eta = \widetilde{\epsilon}\widetilde{\eta}$:

$$\frac{1}{\widetilde{\epsilon}}p(\widetilde{\epsilon}\widetilde{\eta}) + i\epsilon\frac{1}{\widetilde{\epsilon}}q(\widetilde{\epsilon}\widetilde{\eta},\epsilon) + \mathcal{O}\left(\frac{\widetilde{\epsilon}^{N-1}}{\epsilon}\right) =: \widetilde{p}(\widetilde{\eta},\widetilde{\epsilon},\epsilon), \tag{6.19}$$

and this is independent of y.

We now want to further simplify $\widetilde{P}$ by conjugation by an elliptic $\widetilde{h}$-pseudodifferential operator, e^A, where A is an $\widetilde{h}$-pseudodifferential operator of order 0. The construction will be uniform in $\widetilde{\epsilon}$ with a power degeneration in ϵ, that we shall control. First we recall that $e^A\widetilde{P}e^{-A} = e^{\mathrm{ad}_A}\widetilde{P} = \sum\frac{1}{k!}\mathrm{ad}_A^k\widetilde{P}$. Let the full symbol of A be $\sum_{k=0}^\infty \widetilde{h}^k a_k$. Then

$$\begin{aligned} e^A\widetilde{P}e^{-A} &= \sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{j_1=0}^\infty\cdots\sum_{j_k=0}^\infty \frac{1}{k!}\widetilde{h}^{j_1+\ldots+j_k+l+k}\left(\frac{1}{\widetilde{h}}\mathrm{ad}_{a_{j_1}}\right)\cdots\left(\frac{1}{\widetilde{h}}\mathrm{ad}_{a_{j_k}}\right)(\widetilde{p}_l) \\ &= \sum_{n=0}^\infty \widetilde{h}^n s_n, \end{aligned}$$

with $s_0 = \widetilde{p}_0$, $s_1 = \frac{1}{i}H_{a_0}\widetilde{p}_0 + \widetilde{p}_1 = iH_{\widetilde{p}_0}a_0 + \widetilde{p}_1, \ldots, s_{n+1} = iH_{\widetilde{p}_0}a_n + \widetilde{s}_{n+1}$, where $\widetilde{s}_{n+1}$ only depends on $a_0,\ldots,a_{n-1}$ and is a sum of coefficients for $\widetilde{h}^{n+1}$ for terms

$$\frac{1}{k!}\widetilde{h}^{j_1+\ldots+j_k+l+k}\left(\frac{1}{\widetilde{h}}\mathrm{ad}_{a_{j_1}}\right)\cdots\left(\frac{1}{\widetilde{h}}\mathrm{ad}_{a_{j_k}}\right)(\widetilde{p}_l),$$

with $j_1+\ldots+j_k+l+k \le n+1$, $j_1,\ldots j_k < n$. By solving ϵ-degenerated Cauchy-Riemann equations, we see that the a_j can be successively chosen so that s_j are independent of y.

Assume by induction that $\nabla a_j = \mathcal{O}(\epsilon^{-1-2j})$, for $j < n-1$ (in a complex domain, so that we have the same estimates for the derivatives of ∇a_j). Then the general term in $\widetilde{s}_{n+1}$ is

$$\mathcal{O}(1)\epsilon^{-1-2j_1}\dots\epsilon^{-1-2j_k}=\mathcal{O}(1)\left(\frac{1}{\epsilon}\right)^{2(j_1+\ldots+j_k)+k}. \tag{6.20}$$

Here

$$2(j_1+\ldots+j_k)+k=2(j_1+\ldots+j_k+k)-k\le 2(n+1-l)-k=2n+2-2l-k,$$

and so the quantity (6.20) is $\mathcal{O}(1)\epsilon^{-2n}$ except possibly when $2l+k<2$, i.e. when $k=l=0$ or when $k=1$, $l=0$. In the first case we get the coefficient for $\widetilde{h}^{n+1}$ in $\widetilde{p}_0$ which is 0. In the second case, we get the coefficient for $\widetilde{h}^{n+1}$ in $\widetilde{h}^{j_1+1}(\frac{1}{\widetilde{h}}\mathrm{ad}_{a_{j_1}})(\widetilde{p}_0)$ with $j_1<n$, which is $\mathcal{O}(1)\epsilon^{-1-2j_1}$. Here $1+2j_1\le 2n$. Thus $\widetilde{s}_{n+1}=\mathcal{O}(\epsilon^{-2n})$, in a complex domain. We can choose a_n grad-periodic, with $iH_{\widetilde{p}_0}a_n=-\widetilde{s}_{n+1}+\langle s_{n+1}(\cdot,\widetilde{\eta})\rangle$ and with $\nabla a_n=\mathcal{O}(\epsilon^{-1-2n})$. This completes the induction step and we conclude that we can find a_k with $\nabla a_k=\mathcal{O}(\epsilon^{-1-2k})$ in a fixed complex neighborhood of $\widetilde{\eta}=0$ such that if

$$A^{(N_1)}=\sum_{k=0}^{N_1-1}\widetilde{h}^k a_k,$$

then

$$\widetilde{P}^{(N_1)}:=e^{A^{(N_1)}}U^{-1}\frac{1}{\widetilde{\epsilon}}PUe^{-A^{(N_1)}}=\sum_{n=0}^{\infty}\widetilde{h}^n\widetilde{p}_n^{(N_1)}, \tag{6.21}$$

where

$$\widetilde{p}_n^{(N_1)}=\mathcal{O}(\epsilon^{-2(n-1)_+}), \tag{6.22}$$

$\widetilde{p}_0^{(N_1)}=\widetilde{p}(\widetilde{\eta},\widetilde{\epsilon},\epsilon)$ in (6.19) and $\widetilde{p}_n^{(N_1)}=\widetilde{p}_n^{(\infty)}$ is independent of y and N_1, for $n\le N_1$. Here N_1 can be taken arbitrarily large.

We shall next look at the quasi-modes. In view of the estimates (6.22) on $\widetilde{p}_n^{(N_1)}$, we first assume that

$$\frac{\widetilde{h}}{\epsilon^2}\ll 1. \tag{6.23}$$

This gives us the quasi-eigenvalues of $\widetilde{\epsilon}^{-1}P$,

$$\sum_{n=0}^{N_1}\widetilde{h}^n\widetilde{p}_n^{(\infty)}\left(\widetilde{h}\left(k-\frac{k_0}{4}\right)-\frac{S}{2\pi\widetilde{\epsilon}},\widetilde{\epsilon},\epsilon\right)+\mathcal{O}\left(\frac{\widetilde{h}^{N_1+1}}{\epsilon^{2(N_1+1)}}\right), \tag{6.24}$$

for $k\in\mathbf{Z}^2$ with $\widetilde{h}\left(k-\frac{k_0}{4}\right)-\frac{S}{2\pi\widetilde{\epsilon}}=\mathcal{O}(1)$. From (6.24) we get the "leading" values

$$\widetilde{p}_0^{(\infty)}\left(\widetilde{h}\left(k-\frac{k_0}{4}\right)-\frac{S}{2\pi\widetilde{\epsilon}},\widetilde{\epsilon},\epsilon\right)=\widetilde{p}\left(\widetilde{h}\left(k-\frac{k_0}{4}\right)-\frac{S}{2\pi\widetilde{\epsilon}},\widetilde{\epsilon},\epsilon\right),$$

and from (6.19) we infer that the distance between the neighboring "leading" values is $\geq \epsilon\widetilde{h}/\mathcal{O}(1)$. The $\widetilde{\eta}$-gradient of $\widetilde{h}^n\widetilde{p}_n^{(\infty)}(\widetilde{\eta},\widetilde{\epsilon},\epsilon)$ is $\mathcal{O}(1)\widetilde{h}^n\epsilon^{-2(n-1)}$ for $1 \leq n \leq N_1$, and this is $\ll \epsilon$, since by (6.23), $\widetilde{h}^n\epsilon^{-2(n-1)} = (\frac{\widetilde{h}}{\epsilon^2})^n\epsilon^2 \ll \epsilon^2$. It is therefore clear that the distance between neighboring values of

$$\sum_{n=0}^{N_1} \widetilde{h}^n \widetilde{p}_n^{(\infty)}\left(\widetilde{h}\left(k-\frac{k_0}{4}\right)-\frac{S}{2\pi\widetilde{\epsilon}},\widetilde{\epsilon},\epsilon\right)$$

is $\geq \epsilon\widetilde{h}/\mathcal{O}(1)$, and (6.24) gives "distinct" quasi-eigenvalues if this minimal separation is $\gg \widetilde{h}^{N_1+1}/\epsilon^{2(N_1+1)}$. We therefore must strengthen (6.23) to $\widetilde{h}^{N_1+1}/\epsilon^{2(N_1+1)} \ll \epsilon\widetilde{h}$, i.e.:

$$\left(\frac{\widetilde{h}}{\epsilon^2}\right)^{N_1+1} \ll \epsilon\widetilde{h}, \tag{6.25}$$

or

$$\left(\frac{h}{\widetilde{\epsilon}\epsilon^2}\right)^{N_1+1} \ll \frac{\epsilon h}{\widetilde{\epsilon}}. \tag{6.26}$$

This will be satisfied if we choose $\epsilon \geq h^\delta$ for some fixed $\delta \in (0, 1/3)$, $\widetilde{\epsilon} \geq \epsilon$, and finally N_1 large enough.

Now in (6.12) we make the substitution $\widetilde{\epsilon} \to \mu\widetilde{\epsilon}$, $\widetilde{h} \to \widetilde{h}/\mu$, $\mu \sim 1$, and obtain the isospectral operator

$$\frac{1}{\widetilde{\epsilon}}P(x,hD_x) = \frac{1}{\widetilde{\epsilon}}P(x,\widetilde{\epsilon}\widetilde{h}D_x) = \mu\frac{1}{\mu\widetilde{\epsilon}}P\left(x,\mu\widetilde{\epsilon}\frac{\widetilde{h}}{\mu}D_x\right),$$

with the quasi-eigenvalues

$$\mu\sum_{n=0}^{N_1}\left(\frac{\widetilde{h}}{\mu}\right)^n \widetilde{p}_n^{(\infty)}\left(\frac{\widetilde{h}}{\mu}\left(k-\frac{k_0}{4}\right)-\frac{S}{2\pi\widetilde{\epsilon}\mu},\mu\widetilde{\epsilon},\epsilon\right)+\mathcal{O}\left(\frac{\widetilde{h}^{N_1+1}}{\mu^{N_1}\epsilon^{2(N_1+1)}}\right).$$

We deduce as in [28] that for each $n = 1, \dots, N_1$,

$$\mu^{1-n}\widetilde{p}_n^{(\infty)}\left(\frac{\eta}{\mu},\mu\widetilde{\epsilon},\epsilon\right) = \widetilde{p}_n^{(\infty)}(\eta,\widetilde{\epsilon},\epsilon). \tag{6.27}$$

We use this to define $\widetilde{p}_n^{(\infty)}(\eta,1,\epsilon)$ by

$$\widetilde{p}_n^{(\infty)}(\eta,1,\epsilon) = \widetilde{\epsilon}^{1-n}\widetilde{p}_n^{\infty}\left(\frac{\eta}{\widetilde{\epsilon}},\widetilde{\epsilon},\epsilon\right). \tag{6.28}$$

Now recall from Proposition 4.1 that we have the conditions $\widetilde{\epsilon} \geq \epsilon$, $\widetilde{\epsilon}^N/\epsilon^2 \ll 1$:

$$\epsilon \leq \widetilde{\epsilon} \ll \epsilon^{2/N}. \tag{6.29}$$

Then $\widetilde{p}_n^{(\infty)}(\eta, \widetilde{\epsilon}, \epsilon)$ is a well-defined analytic function for $|\eta| < \mathcal{O}(1)$ and by choosing μ to be of the order of magnitude $\epsilon^{2/N}$, we see that $\widetilde{p}_n^{(\infty)}(\eta, 1, \epsilon)$ is well-defined and holomorphic for $|\eta| \ll \epsilon^{2/N}$.

Now restrict ϵ by imposing

$$\epsilon \geq h^{1/3-\delta}, \quad \text{for some } \delta > 0. \tag{6.30}$$

Then

$$\frac{h}{\epsilon^2 \widetilde{\epsilon}} \leq \frac{h}{\epsilon^3} \leq h^{3\delta},$$

and the remainder in (6.24) can be bounded by

$$\mathcal{O}(1) \left(\frac{h}{\widetilde{\epsilon}\epsilon^2} \right)^{N_1+1} \leq \mathcal{O}(1) h^{3\delta(N_1+1)}.$$

Combining (6.24) with (6.28) we get that the quasi-eigenvalues of P_ϵ take the form

$$\sum_{n=0}^{N_1} h^n \widetilde{p}_n^{(\infty)} \left(h\left(k - \frac{k_0}{4}\right) - \frac{S}{2\pi}, 1, \epsilon \right) + \mathcal{O}(h^{3\delta(N_1+1)}), \tag{6.31}$$

for $k \in \mathbf{Z}^2$ with $\left| h(k - \frac{k_0}{4}) - \frac{S}{2\pi} \right| \ll \epsilon^{2/N}$.

Theorem 6.1 *Assume that $h^{1/3-\delta} < \epsilon \leq \epsilon_0 \ll 1$, for some $\delta > 0$, and let $\widetilde{\epsilon}$ be an additional small parameter such that $\widetilde{\epsilon} \geq \epsilon$ and*

$$\frac{\widetilde{\epsilon}^{N-3}}{\epsilon^2} \ll 1. \tag{6.32}$$

Recall that F stands for the mean value of q over the invariant tori Λ_j, $1 \leq j \leq L$, and that $S_j \in \mathbf{R}^2$ and $k_j \in \mathbf{Z}^2$ are the actions and Maslov indices of the fundamental cycles in Λ_j, $1 \leq j \leq L$, respectively. In the general case we make the assumption (1.24) and when the H_p-flow is completely integrable, we assume (2.53). Assume finally that the differentials of the functions $p_{N,j}(\xi)$ and $\mathrm{Re}\,\langle q_j \rangle(\xi)$, defined in Proposition 5.1, are linearly independent when $\xi = 0$. Let $C > 0$ be large enough. Then the eigenvalues of P_ϵ in

$$|\mathrm{Re}\, z| \leq \frac{\widetilde{\epsilon}}{C}, \quad |\mathrm{Im}\, z - \epsilon F| \leq \frac{\epsilon\widetilde{\epsilon}}{C}$$

are given by

$$z(j,k)=\sum_{n=0}^{N_1} h^n \widetilde{p}_{j,n}^{(\infty)}\left(h\left(k-\frac{k_j}{4}\right)-\frac{S_j}{2\pi},1,\epsilon\right)+\mathcal{O}(h^{3\delta(N_1+1)}),\quad k\in\mathbf{Z}^2,\quad 1\le j\le L, \tag{6.33}$$

with $\widetilde{p}_{j,n}^{(\infty)}(\xi,1,\epsilon)=\mathcal{O}(\epsilon^{-2(n-1)_+})\widetilde{\epsilon}^{-n}$, $1\le j\le L$, *holomorphic for* $\xi=\mathcal{O}(\widetilde{\epsilon})$, *and*

$$\widetilde{p}_{j,0}^{(\infty)}(\xi,\epsilon)=p_j(\xi)+i\epsilon q_j(\xi,\epsilon)+\mathcal{O}\left(\frac{\widetilde{\epsilon}^N}{\epsilon}\right).$$

Here p_j *is real on the real domain, and the differentials of* $p_j(\xi)$ *and* $\operatorname{Re} q_j(\xi,\epsilon)$ *are linearly independent when* $\xi=\epsilon=0$. *The integers* N *and* N_1 *in* (6.32) *and* (6.33) *can be taken arbitrarily large.*

When proving Theorem 6.1, we assume for simplicity that $F=0$ and recall the leading symbol of P_ϵ from (2.57),

$$p_{N-1}(\xi)+i\epsilon\langle q\rangle(\xi)+\mathcal{O}(\epsilon^2)+\epsilon\mathcal{O}(\xi^{N-1})+\mathcal{O}(\xi^N). \tag{6.34}$$

(Here we have replaced N by $N-1$.) Recall also the canonical transformation $\widehat{\kappa}$ defined in (3.4) and (3.5), and that we have a second canonical transformation given by (6.10),

$$\kappa_\epsilon : (\partial_\eta\varphi(x,\eta,\epsilon),\eta)\to(x,\partial_x\varphi(x,\eta,\epsilon))$$

defined in a complex domain, with the restriction that $|\eta|=\mathcal{O}(\widetilde{\epsilon})$. Moreover, $\varphi(x,\eta,\epsilon)=x\cdot\eta+\mathcal{O}(\widetilde{\epsilon}^N/\epsilon)$, and we conclude that $\kappa_\epsilon=1+\mathcal{O}(\widetilde{\epsilon}^{N-1}/\epsilon)$, so that the differential of κ_ϵ is equal to $\mathcal{O}(\widetilde{\epsilon}^{N-2}/\epsilon)$, while the higher order differentials satisfy $\nabla^k\kappa_\epsilon=\mathcal{O}(\widetilde{\epsilon}^{N-1-k}/\epsilon)$, $k\ge 2$.

As in section 5, we can represent the real phase space $L_0=T^*\mathbf{T}^2$ by the IR-manifold $\xi=\frac{2}{i}\frac{\partial\Phi_0}{\partial x}(x)$, $\Phi_0(x)=\frac{1}{2}(\operatorname{Im}x)^2$, with $\xi=0$ corresponding to $\operatorname{Im}x=0$, and x varying in a neighborhood of $\{x;\operatorname{Im}x=0\}$ in $\mathbf{T}^2+i\mathbf{R}^2$. Then $L_\epsilon=\kappa_\epsilon(L_0)$ is represented by

$$\xi=\frac{2}{i}\frac{\partial\Phi_\epsilon}{\partial x},\quad |\operatorname{Im}x|<\mathcal{O}(\widetilde{\epsilon}),$$

with $\Phi_\epsilon-\Phi_0$, $\nabla\Phi_\epsilon-\nabla\Phi_0=\mathcal{O}(\widetilde{\epsilon}^{N-1}/\epsilon)$, $\nabla^k(\Phi_\epsilon-\Phi_0)=\mathcal{O}(\widetilde{\epsilon}^{N-k}/\epsilon)$, $k\ge 1$.

Consider the IR-manifold $\widetilde{L}_\epsilon$ represented by

$$\xi=\frac{2}{i}\frac{\partial\widetilde{\Phi}_\epsilon}{\partial x}(x),\quad \widetilde{\Phi}_\epsilon(x)=\chi\left(\frac{\operatorname{Im}x}{\widetilde{\epsilon}}\right)\Phi_\epsilon(x)+\left(1-\chi\left(\frac{\operatorname{Im}x}{\widetilde{\epsilon}}\right)\right)\Phi_0(x),$$

where $\chi \in C_0^\infty(\mathbf{R}^2)$ is a standard cut-off to a neighborhood of 0, so that $\widetilde{L}_\epsilon$ coincides with L_ϵ near $\xi = 0$ and with L_0 further away from this set. Moreover,

$$\nabla(\widetilde{\Phi}_\epsilon - \Phi_0), \quad \nabla(\widetilde{\Phi}_\epsilon - \Phi_\epsilon) = \mathcal{O}\left(\frac{\widetilde{\epsilon}^{N-2}}{\epsilon}\right). \tag{6.35}$$

On the other hand, from section 5 we know that $|\operatorname{Im} P_\epsilon \circ \widehat{\kappa}|$ along L_0 is of the order of magnitude $\epsilon\widetilde{\epsilon}$ in the region where $|\xi| \sim \widetilde{\epsilon}$ and $|\operatorname{Re} P_\epsilon| / \widetilde{\epsilon}$ is small. In view of (6.35) we then have the same conclusion for $\operatorname{Im} P_\epsilon \circ \widehat{\kappa}$ along $\widetilde{L}_\epsilon$, provided that $\widetilde{\epsilon}^{N-2}/\epsilon \ll \epsilon\widetilde{\epsilon}$, i.e. for

$$\frac{\widetilde{\epsilon}^{N-3}}{\epsilon^2} \ll 1,$$

which is precisely (6.32).

The final IR-manifold $\widehat{\Lambda}_\epsilon$ that we choose is then given by

$$\exp(i\epsilon H_G) \circ (\kappa_1^{-1}) \circ (\kappa^{(N)})^{-1} \circ \widehat{\kappa}(\widetilde{L}_\epsilon)$$

in a complex neighborhood of Λ_1, and we do a similar modification near the other tori Λ_j, $2 \leq j \leq L$—see also (5.8). Here we are also using that $\widehat{\kappa}(\widetilde{L}_\epsilon)$ coincides with $\widehat{\kappa}(L_0)$ in the region corresponding to $\xi \notin \operatorname{supp}\chi\left(\frac{\cdot}{\widetilde{\epsilon}}\right)$, $\xi = \mathcal{O}(\widetilde{\epsilon})$. Then, along $\widehat{\Lambda}_\epsilon$, $|\operatorname{Im} P_\epsilon| \geq \epsilon\widetilde{\epsilon}/\mathcal{O}(1)$ outside any $\widetilde{\epsilon}$-neighborhood of the union of the Λ_j's, $1 \leq j \leq L$, in the energy slice $|\operatorname{Re} P_\epsilon| \leq \widetilde{\epsilon}/\mathcal{O}(1)$, and moreover the holomorphic canonical transformation $\widehat{\kappa} \circ \kappa_\epsilon$ maps a region $|\xi| < \mathcal{O}(\widetilde{\epsilon})$ in $T^*\mathbf{T}^2$ onto the intersection of $\widehat{\Lambda}_\epsilon$ with a similar neighborhood of Λ_1.

Using these facts and the normal form (6.31), we get Theorem 6.1 by solving a globally well-posed Grushin problem as in section 5. We then take $N_1 \to \infty$, and put $\widetilde{\epsilon} = \epsilon^{1/\widetilde{N}}$, where $\widetilde{N}$ can be arbitrarily large. This completes the proof of Theorem 1.2.

7 The dynamical condition and the isoenergetic KAM theorem

7.1 The completely integrable case.

In the beginning of this section, we shall discuss the case when the H_p-flow is completely integrable. To be precise, as in section 2, we shall assume that there exists an analytic real valued function f on T^*M such that $H_p f = 0$. The sets

$\Lambda_a = p^{-1}(0) \cap f^{-1}(a)$, $a \in J$, then form a singular foliation of $p^{-1}(0)$, and we get a corresponding decomposition of the energy surface,

$$p^{-1}(0) = \bigcup_{a \in J} \Lambda_a. \tag{7.1}$$

Here $J \subset \mathbf{R}$ is a compact interval and we know that for each $a \in J$, Λ_a is a compact H_p-invariant set, such that we have the continuity property (2.43). We now introduce a global assumption that for each $a \in J$, except for a finite set $S \subset J$, the set Λ_a is a finite disjoint union of analytic flow-invariant Lagrangian tori, depending analytically on a, $\Lambda_a = \cup_{j=1}^{N(a)} \Lambda_{a,j}$. Here $N(a)$ is a locally constant bounded function on $J \backslash S$. As we shall see, this assumption is satisfied in the case when M is a convex analytic surface of revolution, with the exceptional set S in that case corresponding to the endpoints of J, and with $N(a) = 1$. In order to simplify the discussion notationally, in what follows we shall assume that each Λ_a, $a \in J \backslash S$, is connected.

Remark. In some situations, it turns out to be more appropriate to assume that J in (7.1) is a circle, or, more generally, a connected graph with finitely many vertices and edges, with S then being the set of vertices—see also [12],[43]. Assuming that J is a graph, the following discussion goes through with minor modifications—see also the end of this section.

It is now classical (see [2]) that locally, near any regular torus Λ_a in T^*M, we may introduce analytic action-angle coordinates $(x_1, x_2; \xi_1, \xi_2)$ with $x_j \in S^1$ and $\xi_j \in \text{neigh}(0, \mathbf{R})$, $j = 1, 2$, so that when expressed in terms of these coordinates, p becomes a function of ξ_1 and ξ_2, $p = p(\xi_1, \xi_2)$. The rotation number of Λ_a, $\omega(a)$, is then defined as the ratio of the frequencies

$$[\partial_{\xi_2} p : \partial_{\xi_1} p], \tag{7.2}$$

viewed as an element of the real projective line. We shall assume that $\omega(a)$ depends analytically on $a \in J \backslash S$ and that $a \mapsto \omega(a)$ is not identically constant on any connected component of $J \backslash S$. Following [2], we recall then that the torus $\Lambda_a \subset p^{-1}(0)$ satisfies the isoenergetic condition when the map $a \mapsto \omega(a)$ is a local diffeomorphism.

When q is a bounded holomorphic function in a tubular neighborhood of T^*M, that is for simplicity assumed to be real on the real domain, we shall consider the limit of

$$\langle q \rangle_T = \frac{1}{T} \int_0^T q \circ \exp(tH_p)\, dt, \quad T \to \infty, \tag{7.3}$$

taken along an invariant torus Λ_a, $a \in J \backslash S$. All the results below will be valid if $\langle q \rangle_T$ is replaced by $\langle q \rangle_{T,K}$, with $K \in C_0^\infty$.

When analyzing (7.3), we switch to the action-angle variables, so that Λ_a becomes the standard torus $\mathbf{T}^2$, H_p along Λ_a becomes $b_1\partial_{x_1} + b_2\partial_{x_2}$, b_1, $b_2 \in \mathbf{R}$, and

$$q = q(x_1, x_2).$$

Then as in (2.49), we get

$$\langle q \rangle_T(x) = \frac{1}{T}\int_0^T q(x+tb)\,dt = \sum_{k\in\mathbf{Z}^2} \widehat{q}(k)e^{ikx}\widehat{1_{[0,1]}}(-Tb\cdot k). \tag{7.4}$$

It follows as in (2.50) that the limit of $\langle q\rangle_T$ along Λ_a is equal to

$$\widehat{q}(0) + \sum_{0\neq k,\, b\cdot k=0} \widehat{q}(k)e^{ikx}. \tag{7.5}$$

We conclude that when Λ_a is irrational (in the sense that its rotation number $\omega(a) = b_2/b_1$ is irrational), the limit is equal to the average of q over the torus, $\langle q\rangle_{\Lambda_a}$, computed with respect to the natural absolutely continuous measure on Λ_a, with respect to which the H_p-flow is ergodic. In the rational case, we write $\omega(a) = m/n$, where $m \in \mathbf{Z}$, $n \in \mathbf{N}$, are relatively prime, and introduce

$$k(\omega(a)) := |m| + |n|\,.$$

Using the smoothness of q, it follows then from (7.5) that the limit of (7.3) along a rational torus is equal to

$$\langle q\rangle_{\Lambda_a} + \mathcal{O}\left(\frac{1}{k(\omega(a))^\infty}\right).$$

Here the integration measure used in the definition of the torus average is absolutely continuous and comes from the diffeomorphism between Λ_a and $\mathbf{T}^2$, given by the action-angle coordinates.

Let us summarize the discussion above in the following proposition.

Proposition 7.1 *Assume that the H_p-flow on T^*M is completely integrable and, more precisely, that* (7.1) *holds true, with Λ_a being an analytic invariant Lagrangian torus when $a \in J\backslash S$, for a finite set S. When q is a real-valued bounded analytic function on T^*M and $a \in J\backslash S$, we define the closed interval $Q_\infty(a) \subset \mathbf{R}$ to be the range of the limit of $\langle q\rangle_T$, as $T \to \infty$, restricted to the invariant torus $\Lambda_a \subset p^{-1}(0)$. We then have*

- *When the rotation number $\omega(a) \notin \mathbf{Q}$, then $Q_\infty(a) = \{\langle q\rangle_{\Lambda_a}\}$.*

- *When* $\omega(a) = m/n$, *with* $m \in \mathbf{Z}$, $n \in \mathbf{N}$ *being relatively prime, then*

$$Q_\infty(a) \subset \langle q \rangle_{\Lambda_a} + \mathcal{O}\left(\frac{1}{k(\omega(a))^\infty}\right)[-1,1], \quad k(\omega(a)) := |m| + |n|\,.$$

In particular,

$$\sum_{a;\,\omega(a)\in\mathbf{Q}} |Q_\infty(a)| < \infty.$$

Here $|I|$ *stands for the length of an interval* $I \subset \mathbf{R}$.

When $a \in S$, we also introduce the compact interval $Q_\infty(a) \subset \mathbf{R}$, defined as in (2.46). We shall assume from now on that the function $\langle q \rangle_{\Lambda_a}$ depends analytically on $a \in J \backslash S$ and that it extends to a continuous function on the whole interval J. We shall also assume that $a \mapsto \langle q \rangle_{\Lambda_a}$ is not identically constant on any connected component of $J \backslash S$.

Let us next consider the behavior of the interval $Q_\infty(a)$ corresponding to a rational torus Λ_a in a neighborhood of a Diophantine torus. In doing so, we assume that Λ_{a_0}, $a_0 \in J \backslash S$, is such that $\omega_0 := \omega(a_0)$ is (α, d)–Diophantine for some $\alpha > 0$ and $d > 0$, i.e. it satisfies the usual Diophantine condition—cf. (1.15),

$$\left|\omega_0 - \frac{p}{q}\right| \geq \frac{\alpha}{q^{2+d}}, \quad p \in \mathbf{Z},\ q \in \mathbf{N}. \tag{7.6}$$

Let $\omega = \omega(a)$, $a \in \mathrm{neigh}(a_0, \mathbf{R})$, be rational, and write $\omega = m/n$ where $m \in \mathbf{Z}$ and $n \in \mathbf{N}$ are relatively prime. Then

$$0 = \omega - \frac{m}{n} = \omega - \omega_0 + \omega_0 - \frac{m}{n},$$

and applying (7.6) with $\frac{p}{q} = \frac{m}{n}$, we get

$$|\omega - \omega_0| \geq \frac{\alpha}{n^{2+d}}.$$

Therefore,

$$\frac{1}{k(\omega)} \leq \mathcal{O}(1)\,|\omega - \omega_0|^{1/(2+d)}\,,$$

and an application of Proposition 7.1 shows that $Q_\infty(a)$ is a closed interval such that

$$Q_\infty(a) \subset \langle q \rangle_{\Lambda_a} + \mathcal{O}\left(|\omega(a) - \omega_0|^\infty\right)[-1,1]. \tag{7.7}$$

This result should be compared with (2.51).

We shall next consider the set of Lagrangian tori in $p^{-1}(0)$, whose rotation numbers are uniformly Diophantine and that satisfy a uniform isoenergetic condition. Put for $\alpha > 0$ and $d > 0$ fixed,

$$\Omega_{\alpha,d} = \left\{ a \in J;\ \mathrm{dist}(a,S) \geq \alpha,\ |\omega'(a)| \geq \alpha,\ \left|\omega(a) - \frac{p}{q}\right| \geq \frac{\alpha}{q^{2+d}},\ p \in \mathbf{Z},\ q \in \mathbf{N} \right\}. \tag{7.8}$$

In what follows, the dependence on d will not be indicated explicitly, and we shall write $\Omega_\alpha = \Omega_{\alpha,d}$. Introduce next

$$\widetilde{\Omega}_\alpha := \Omega_\alpha \cap \{a \in J \backslash S, |d_a \langle q \rangle_{\Lambda_a}| \geq \alpha\}, \tag{7.9}$$

and notice that the measure of the complement of $\widetilde{\Omega}_\alpha$ in the interval J, $\complement\widetilde{\Omega}_\alpha$, is small, together with α, when $d > 0$ is kept fixed. We then define the set of good values $\mathcal{G}_\alpha$ contained inside the closed interval $\langle q \rangle_{\Lambda_a}(J)$, in the following way,

$$\mathcal{G}_\alpha = \complement\left(\langle q \rangle_{\Lambda_a}\left(\complement\widetilde{\Omega}_\alpha\right)\right). \tag{7.10}$$

It follows that the complement of $\mathcal{G}_\alpha$ in $\langle q \rangle_{\Lambda_a}(J)$ has a small measure, when α is small and $d > 0$ is kept fixed. We notice also that it follows from the construction that when $F_0 \in \mathcal{G}_\alpha$, the pre-image

$$\langle q \rangle_{\Lambda_a}^{-1}(F_0)$$

is a finite set,

$$\langle q \rangle_{\Lambda_a}^{-1}(F_0) = \left\{ a_1, \ldots, a_L;\ a_j \in \widetilde{\Omega}_\alpha \right\},$$

and an application of (7.7) shows that

$$\mathrm{dist}\left(Q_\infty(a), F_0\right) \geq \frac{|a - a_j|}{\mathcal{O}(1)}, \quad a \in \mathrm{neigh}(a_j, \mathbf{R}),$$

for any $j = 1, \ldots L$.

When considering the intervals $Q_\infty(a)$ for $a \in J$ away from the a_j, $j = 1, \ldots L$, we notice that an application of Lemma 2.4 shows that the set

$$\bigcup_{a \notin \mathrm{neigh}(\{a_1, \ldots a_L\})} Q_\infty(a)$$

is closed. Our final assumption now is that

$$\sup_{a\in J} |Q_\infty(a)| \text{ is sufficiently small depending on } \alpha \text{ and } d. \tag{7.11}$$

When a is away from the a_j, $j = 1, \dots L$, using (7.11), we get

$$\mathrm{dist}\,(Q_\infty(a), F_0) \geq \frac{1}{\mathcal{O}(1)}.$$

We summarize the discussion above in the following theorem.

Theorem 7.2 *Assume that the H_p-flow on T^*M is completely integrable and assume that the rotation number $\omega(a)$ of the invariant tori Λ_a depends analytically on a and is not identically constant on any connected component of $J\backslash S$. When q is a real-valued bounded analytic function on T^*M, define $\langle q\rangle_{\Lambda_a}$, $a \in J\backslash S$, to be the torus average of q with respect to the natural smooth measure on Λ_a, and assume that $a \mapsto \langle q\rangle_{\Lambda_a}$ is an analytic function on $J\backslash S$, which extends continuously to J and which is not identically constant on any connected component of $J\backslash S$. When $\alpha > 0$, $d > 0$, let us define next the set $\mathcal{G}_\alpha \subset \langle q\rangle_{\Lambda_a}(J)$ according to* (7.10), (7.9)*. Then the measure of the complement of $\mathcal{G}_\alpha$ in $\langle q\rangle_{\Lambda_a}(J)$ is small when α is sufficiently small and d is kept fixed. Assume that*

$$\sup_{a\in J} |Q_\infty(a)|$$

is small enough depending on α, d. When $F_0 \in \mathcal{G}_\alpha$, we have that for any neighborhood W of the finite set $\langle q\rangle_{\Lambda_a}^{-1}(F_0)$ there exists a constant $C(W) > 0$ such that

$$\mathrm{dist}\,(Q_\infty(a), F_0) \geq \frac{1}{C(W)}, \quad a \in \complement W.$$

Here $Q_\infty(a)$, $a \in J$, is defined as in (2.46)*, so that it is equal to the range of the limit of $\langle q\rangle_T$, as $T \to \infty$, along Λ_a, when $a \in J\backslash S$.*

7.2 Surfaces of revolution.

We shall now illustrate Theorem 7.2 in the case when M is an analytic surface of revolution in $\mathbf{R}^3$. In doing so, we normalize M so that the x_3-axis is its axis of revolution, and we parametrize it by the cylinder $[0, L] \times S^1$,

$$[0, L] \times S^1 \ni (s, \theta) \mapsto (f(s)\cos\theta, f(s)\sin\theta, h(s)),$$

assuming, as we may, that the parameter s is the arc-length along the meridians, so that $(f'(s))^2 + (h'(s))^2 = 1$. A simple calculation then shows that in the coordinates (s, θ), the metric on M takes the form

$$g = ds^2 + f^2(s)d\theta^2. \tag{7.12}$$

The functions f and h are assumed to be real analytic on $[0, L]$, and from [4] we recall that the regularity of M at the poles is guaranteed by requiring that for each $k \in \mathbf{N}$,

$$f^{(2k)}(0) = f^{(2k)}(L) = 0,$$

and that $f'(0) = 1$, $f'(L) = -1$, which we shall assume from now on.

Throughout the following discussion, we shall assume furthermore that M is a simple surface of revolution, in the sense that $0 \leq f(s)$ has precisely one critical point $s_0 \in (0, L)$, and that this critical point is a non-degenerate maximum, $f''(s_0) < 0$. To fix the ideas, we shall also assume that $f(s_0) = 1$. For future reference, we notice that s_0 corresponds to the equatorial geodesic $\gamma_E \subset M$ given by $s = s_0$, $\theta \in S^1$. This is an elliptic orbit.

We now come to analyze the geodesic flow on M, which we shall view as the Hamilton flow on T^*M of the dual form to the metric g. In doing so, we write

$$T^* \left(M \backslash \{(0, 0, g(0)), (0, 0, g(L))\} \right) \simeq T^* \left((0, L) \times S^1 \right),$$

and using (7.12) we immediately see that the dual form to g is given by

$$p(s, \theta, \sigma, \theta^*) = \sigma^2 + \frac{(\theta^*)^2}{f^2(s)}. \tag{7.13}$$

Here σ and θ^* are the dual variables to s and θ, respectively. Writing out the Hamilton equations for the H_p–flow, we see next that the functions p and θ^* are in involution, and we recover the well-known fact that the geodesic flow on M is completely integrable.

When $E > 0$ and $|F| < E^{1/2}$, $F \neq 0$, it is true that the set defined by

$$\Lambda_{E,F} : p = E, \ \theta^* = F,$$

is an analytic Lagrangian torus sitting inside the energy surface $p^{-1}(E)$. Geometrically, the torus $\Lambda_{E,F}$ consists of geodesics contained between and intersecting tangentially the parallels $s_\pm(E, F)$ on M defined by the equation

$$f(s_\pm(E, F)) = \frac{|F|}{E^{1/2}}.$$

For $F = 0$, the parallels reduce to the two poles and we obtain a torus consisting of a family of meridians. The case $|F| = E^{1/2}$ is degenerate and corresponds to the equator $s = s_0$, traversed with the two different orientations.

The principal actions of the torus $\Lambda_{E,F}$ are given by

$$I_1(E,F) = \int_{\sigma^2 + \frac{F^2}{f^2(s)} = E} \sigma\, ds = 2 \int_{s_-(E,F)}^{s_+(E,F)} \left(E - \frac{F^2}{f^2(s)} \right)^{1/2} ds, \tag{7.14}$$

and

$$I_2(E,F) = 2\pi F. \tag{7.15}$$

The functions I_1, I_2 depend analytically on E, $F \neq 0$.

Let us now restrict the further discussion to the energy surface $p^{-1}(1)$. We shall derive an explicit expression for the rotation number $\omega(a)$ of the torus $\Lambda_a := \Lambda_{1,a}$, $0 \neq a \in (-1, 1)$. In doing so, we introduce the action coordinates $\eta_1 = I_1/2\pi$, $\eta_2 = I_2/2\pi$, and notice that it follows from (7.14) and (7.15) that along $p^{-1}(1)$, η_1 becomes an analytic function of η_2,

$$\eta_1 = \varphi(\eta_2) := \frac{1}{\pi} \int_{s_-(a)}^{s_+(a)} \left(1 - \frac{\eta_2^2}{f^2(s)} \right)^{1/2} ds, \quad \eta_2 = a, \quad s_\pm(a) := s_\pm(1,a).$$

Differentiating next the relation $p(\varphi(\eta_2), \eta_2) = 1$, we get that the rotation number $\omega(a)$ is given by $-\varphi'(a)$, so that

$$\omega(a) = \frac{a}{\pi} \int_{s_-(a)}^{s_+(a)} \frac{1}{f^2(s)} \left(1 - \frac{a^2}{f^2(s)} \right)^{-1/2} ds. \tag{7.16}$$

We recall that the torus $\Lambda_a \subset p^{-1}(1)$ satisfies the isoenergetic condition precisely when the expression $\omega'(a)$, given by

$$\frac{1}{\pi} \int_{s_(a)}^{s_+(a)} \frac{1}{f^2(s)} \left(1 - \frac{a^2}{f^2(s)} \right)^{-1/2} ds + \frac{a}{\pi} \frac{\partial}{\partial a} \left(\int_{s_-(a)}^{s_+(a)} \frac{1}{f^2(s)} \left(1 - \frac{a^2}{f^2(s)} \right)^{-1/2} ds \right) \tag{7.17}$$

is non-vanishing.

From now on, we shall assume that the surface of revolution M is such that $\omega(a)$ in (7.17) is not identically constant. As explained in [44], examples of such surfaces are provided by ellipsoids of revolution, with $\omega'(a) < 0$ for oblong ellipsoids, and $\omega'(a) > 0$ for oblate ones. The separating case of the sphere is of course degenerate with $\omega'(a) \equiv 0$. More generally, we may also remark that the isoenergetic condition rules out the case of analytic Zoll surfaces of revolution.

Let now q be an analytic function on M, which we shall view as a function on T^*M. We now come to discuss the long time averages of q along the H_p-flow, $\langle q \rangle_T$, $T \gg 1$. In doing so, we shall first consider the case when $q = q_0 = q_0(s)$ is a function of s only. It follows from (7.13) that along an H_p-orbit in Λ_a, $0 \neq a \in (-1,1)$, we have

$$\dot{s}(t) = \pm 2 \left(1 - \frac{a^2}{f^2(s(t))}\right)^{1/2}, \tag{7.18}$$

and then using the rotational invariance of q_0, we get

$$\langle q_0 \rangle_T(m) = \frac{1}{T} \int_0^T q_0(s(t))\, dt = \langle q_0 \rangle_{\Lambda_a} + \mathcal{O}\left(\frac{1}{T}\right), \quad m \in \Lambda_a, \quad T \to \infty.$$

Here, as we immediately compute from (7.18),

$$\langle q_0 \rangle_{\Lambda_a} = \frac{J(q_0, a)}{J(1, a)}, \tag{7.19}$$

where, in general, for an analytic function ψ we write

$$J(\psi, a) = \int_{s_-(a)}^{s_+(a)} \psi(s) \frac{f(s)}{(f^2(s) - a^2)^{1/2}}\, ds, \quad f(s_\pm(a)) = |a|\,. \tag{7.20}$$

We remark that (7.19), (7.20) provide an explicit description of $\langle q_0 \rangle_{\Lambda_a}$ which, as before, is defined as the average of q_0 over the invariant torus Λ_a.

The function $\langle q_0 \rangle_{\Lambda_a}$ depends analytically on $a \in (-1,1)$, and it extends to a continuous function on the whole interval $[-1,1]$. In what follows we shall assume that the function $a \mapsto \langle q_0 \rangle_{\Lambda_a}$ is not identically constant.

Example. Let

$$q_0(s) = (f'(s))^2 = (f''(s_0))^2 (s - s_0)^2 + \mathcal{O}((s - s_0)^3), \quad s \to s_0.$$

Since q_0 is minimal along the equator $s = s_0$, it is clear that $\langle q_0 \rangle_{\Lambda_a}$ is not identically constant and achieves its minimum for $a = \pm 1$. Here we shall compute a leading term in the asymptotic expansion of this function, as $|a| \to 1$. In doing so, following [4] and [44], it will be convenient to make a change of variables on the surface M. We introduce

$$r = \begin{cases} \arcsin f(s), & s \in (0, s_0) \\ \pi - \arcsin f(s), & s \in (s_0, L). \end{cases}$$

It follows then that $s = c(\cos r)$, where $c(x)$, $x \in [-1,1]$, is given by

$$c(x) = \begin{cases} (f|_{[0,s_0]})^{-1}(\sqrt{1-x^2}), & x \in [0,1], \\ (f|_{[s_0,\pi]})^{-1}(\sqrt{1-x^2}), & x \in [-1,0]. \end{cases}$$

Therefore, $\sin r = f(s)$, $f'(c(\cos r))ds = \cos r dr$, and making the change of variables in (7.20), we get

$$J(\psi, a) = \int_{\arcsin|a|}^{\pi - \arcsin|a|} \psi(c(\cos r)) \frac{\sin r h(\cos r)}{(\sin^2 r - a^2)^{1/2}}\, dr = \int_{-\sqrt{1-a^2}}^{\sqrt{1-a^2}} \frac{\psi(c(x))h(x)}{(1-x^2-a^2)^{1/2}}\, dx. \tag{7.21}$$

Here $h(x) = x/f'(c(x))$, $x \neq 0$, $h(0) = 1/|f''(s_0)|^{1/2} \neq 0$.

Taylor expanding h, $h(x) = h_0 + h_1 x + h_2 x^2 + \ldots$, $x \to 0$, we get after a simple computation,

$$J(1,a) = \pi h_0 + \frac{\pi}{2} h_2 (1-a^2) + \mathcal{O}((1-a^2)^2), \quad |a| \to 1.$$

Now

$$q_0(c(x)) = (f'(c(x)))^2 = \frac{x^2}{h(x)^2} = \frac{1}{h_0^2} x^2 + \mathcal{O}(x^3),$$

and using (7.21), we get

$$J(q_0, a) = \frac{\pi}{2h_0}(1-a^2) + \mathcal{O}((1-a^2)^2).$$

It follows that

$$\langle q_0 \rangle_{\Lambda_a} = \frac{J(q_0,a)}{J(1,a)} = \frac{1}{2}(1-a^2) + \mathcal{O}((1-a^2)^2) \tag{7.22}$$

is not identically constant.

Remark. The preceding example is closely related to the computations in Appendix C in [38].

We shall now introduce a weak angular dependence in the perturbation q, and when doing so, for an analytic function q_1 on M, we set

$$q_\eta(s,\theta) = q_0(s) + \eta q_1(s,\theta), \quad 0 < \eta \ll 1. \tag{7.23}$$

As before, we are then interested in the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T q_\eta \circ \exp(tH_p)\, dt, \tag{7.24}$$

taken along the invariant torus Λ_a. When analyzing (7.24), we again switch to the action-angle variables, so that Λ_a becomes the standard torus $\mathbf{T}^2$, H_p along Λ_a becomes $b_1\partial_{x_1} + b_2\partial_{x_2}$, $b_j \in \mathbf{R}$, $b_1 \neq 0$, and

$$q_\eta(x_1, x_2) = q_1(x_1) + \eta q_2(x_1, x_2).$$

It follows from (7.5) that (7.24) is equal to

$$\widehat{q}_1(0) + \eta \sum_{b\cdot k=0} \widehat{q}_2(k) e^{ix\cdot k}.$$

Let us summarize this discussion in the following proposition, which is just a specialization of Proposition 7.1 to the case of surfaces of revolution, with a perturbation of the form (7.23).

Proposition 7.3 *On the surface of revolution M, let us consider an analytic function $q_\eta(s,\theta) = q_0(s) + \eta q_1(s,\theta)$, $0 < \eta \ll 1$. When $a \in (-1,1)$, we define the closed interval $Q_{\infty,\eta}(a) \subset \mathbf{R}$ to be the range of the limit of $\langle q_\eta \rangle_T$, as $T \to \infty$, restricted to the invariant torus $\Lambda_a \subset p^{-1}(1)$. We then have*

- *When the rotation number $\omega(a) \notin \mathbf{Q}$, then $Q_{\infty,\eta}(a) = \{\langle q_\eta \rangle_{\Lambda_a}\}$.*
- *When $\omega(a) = m/n$, with $m \in \mathbf{Z}$, $n \in \mathbf{N}$ being relatively prime, then*

$$Q_{\infty,\eta}(a) \subset \langle q_\eta \rangle_{\Lambda_a} + \eta \mathcal{O}\left(\frac{1}{k(\omega(a))^\infty}\right)[-1,1], \quad k(\omega(a)) := |m| + |n|.$$

Let us now define the set Ω_α as in (7.8), with $J = [-1,1]$ and $S = \{\pm 1\}$, and define following (7.9),

$$\widetilde{\Omega}_{0,\alpha} := \Omega_\alpha \cap \{a \in (-1,1)\backslash\{0\}; |d_a \langle q_0 \rangle_{\Lambda_a}| \geq \alpha\}. \tag{7.25}$$

We then define the η-dependent set of good values $\mathcal{G}_{\alpha,\eta}$ contained inside the closed interval $\langle q_\eta \rangle_{\Lambda_a}([-1,1])$, as in (7.10),

$$\mathcal{G}_{\alpha,\eta} = \complement\left((\langle q_0 \rangle_{\Lambda_a} + \eta \langle q_1 \rangle_{\Lambda_a})\left(\complement \widetilde{\Omega}_{0,\alpha}\right)\right). \tag{7.26}$$

The complement of $\mathcal{G}_{\alpha,\eta}$ has a small measure, when α is small, uniformly in η small enough, and when $d > 0$ is kept fixed.

The general assumptions of Theorem 7.2 are verified in this case, and we get the following result.

Proposition 7.4 *Assume that M is a simple analytic surface of revolution for which the rotation number defined in* (7.16) *is not identically constant. When $q_0 = q_0(s)$ is an analytic function on M, let us define $\langle q_0 \rangle_{\Lambda_a}$ as in* (7.19), (7.20), *and assume that $a \mapsto \langle q_0 \rangle_{\Lambda_a}$ is not identically constant. Set when $0 \leq \eta \ll 1$,*

$$q_\eta(s, \theta) = q_0(s) + \eta q_1(s, \theta),$$

and define $\langle q_\eta \rangle_{\Lambda_a}$ to be the average of q_η over the invariant torus Λ_a, with respect to the natural smooth measure. When $\alpha > 0$, let us define the set $\mathcal{G}_{\alpha,\eta} \subset \langle q_\eta \rangle_{\Lambda_a}([-1, 1])$ according to (7.26)*. Then the measure of the complement of $\mathcal{G}_{\alpha,\eta}$ in $\langle q_\eta \rangle_{\Lambda_a}([-1, 1])$ is small when α and η are sufficiently small (and d is kept fixed). When $F_0 \in \mathcal{G}_{\alpha,\eta}$ and η is small enough, we have that for any neighborhood W of the finite set $\langle q_\eta \rangle_{\Lambda_a}^{-1}(F_0)$ there exists a constant $C(W) > 0$ such that*

$$\mathrm{dist}\,(Q_{\infty,\eta}(a), F_0) \geq \frac{1}{C(W)}, \quad a \in \complement W.$$

Here $Q_{\infty,\eta}(a)$ is the range of the limit of $\langle q_\eta \rangle_T$, as $T \to \infty$, along Λ_a.

It follows from Proposition 7.4 that for $F_0 \in \mathcal{G}_{\alpha,\eta}$, the assumptions of Theorems 1.1 and 1.2 (see also Theorems 5.2 and 6.1) are satisfied. An application of, say, Theorem 1.1 gives then the complete asymptotic expansions of all the eigenvalues of $-h^2\Delta + i\epsilon q_\eta$ in rectangles in the spectral complex plane, that are of the form

$$|\mathrm{Re}\, z - 1| \leq \frac{h^\delta}{\mathcal{O}(1)}, \quad |\mathrm{Im}\, z - \epsilon \mathrm{Re}\, F_0| \leq \frac{\epsilon h^\delta}{\mathcal{O}(1)}, \quad \delta > 0,$$

when $\epsilon = \mathcal{O}(h^\delta)$ is bounded from below by a fixed positive power of h. We refrain from formulating precisely the final result, as it is immediately obtained from the statement of Theorem 1.1.

7.3 Perturbations of completely integrable systems.

We now come to discuss the perturbed situation and before doing so, we shall pause to recall the statement of the isoenergetic KAM theorem.

We consider

$$p_\lambda = p_0 + \lambda p_1, \quad 0 < \lambda \ll 1, \tag{7.27}$$

where p_0 and p_1 are holomorphic bounded functions in a tubular neighborhood of T^*M, that are real on the real domain. We assume that the H_{p_0}-flow is completely integrable, and let $\Lambda_0 \subset p_0^{-1}(0)$ be an analytic H_{p_0}-invariant Diophantine Lagrangian torus. Take then the analytic action-angle transformation

$$\kappa : \mathrm{neigh}(\Lambda_0, T^*M) \to \mathrm{neigh}(\xi = 0, T^*\mathbf{T}^2), \tag{7.28}$$

such that $\kappa(\Lambda_0)$ is the zero section in $T^*\mathbf{T}^2$, and

$$p_0 \circ \kappa^{-1} = p_0(\xi) = a \cdot \xi + \mathcal{O}(\xi^2), \quad a = (a_1, a_2) \in \mathbf{R}^2,$$

with a satisfying the Diophantine condition (1.15), where we assume that $N_0 > 1$. Composing p_λ in (7.27) with the inverse of κ in (7.28), we get a function

$$p_\lambda(x, \xi) = p_0(\xi) + \lambda p_1(x, \xi),$$

analytic in a fixed complex neighborhood of $\xi = 0$. An application of the implicit function theorem shows that the energy surface $p_0^{-1}(0)$ takes the form $\xi_2 = f(\xi_1)$, where f is an analytic function near $0 \in \mathbf{R}$, with $f(0) = 0$, $f'(0) = -a_1/a_2$. We assume that the isoenergetic condition holds:

$$f''(0) \neq 0. \tag{7.29}$$

The condition (7.29) means that for $|\mu| \ll 1$, the H_{p_0}-invariant tori $\Lambda_\mu : \xi_1 = \mu$, $\xi_2 = f(\mu)$ can be parametrized by the corresponding rotation numbers $f'(\mu)$.

Theorem 7.5 *(The isoenergetic KAM theorem). Make the isoenergetic assumption* (7.29) *and let us define for $\alpha > 0$ sufficiently small and $d > 0$ fixed,*

$$K_\alpha = \left\{ \mu \in \mathrm{neigh}(0, \mathbf{R});\ \left| f'(\mu) - \frac{p}{q} \right| \geq \frac{\alpha}{q^{2+d}},\ p \in \mathbf{Z},\ q \in \mathbf{N} \right\}.$$

Assume that $0 < \lambda \ll \alpha^2$. Then there exists a C^∞-map

$$\Psi_\lambda : \mathbf{T}^2 \times \mathrm{neigh}(0, \mathbf{R}) \to \mathrm{neigh}(\xi = 0, T^*\mathbf{T}^2) \tag{7.30}$$

analytic in the angular variable (uniformly with respect to the other factor), depending analytically on $\lambda > 0$, and with $\mathrm{neigh}(0, \mathbf{R})$ *and* $\mathrm{neigh}(\xi = 0, T^*\mathbf{T}^2)$ *in* (7.30) *uniform in λ, such that for each $\mu \in K_\alpha$, the set*

$$\Lambda_{\mu,\lambda} = \{\Psi_\lambda(x, \mu); x \in \mathbf{T}^2\} \subset T^*\mathbf{T}^2 \tag{7.31}$$

is a uniformly analytic Lagrangian torus $\subset p_\lambda^{-1}(0)$, close to $\Lambda_\mu \subset p_0^{-1}(0)$. The restriction of $\exp(tH_{p_\lambda})$ *to $\Lambda_{\mu,\lambda}$ is conjugated to the flow of a constant vector field with the rotation number $f'(\mu)$ on $\mathbf{T}^2$, by means of an analytic diffeomorphism, depending analytically on λ and smoothly on $\mu \in$* $\mathrm{neigh}(0, \mathbf{R})$. *Moreover, relative to a sufficiently small neighborhood of $\xi = 0$ in $T^*\mathbf{T}^2$, the Liouville measure of the complement of the union of the invariant tori $\Lambda_{\mu,\lambda}$, $\mu \in K_\alpha$, in $p_\lambda^{-1}(0)$, is $\mathcal{O}(\alpha)$.*

Remark. The classical theorem of Kolmogorov on the existence of an invariant torus for the perturbed Hamiltonian, close to any given Diophantine invariant torus for the initial integrable system, is stated and proved in great detail in [3]. The isoenergetic version of the Kolmogorov theorem is stated explicitly in [7], but without a proof, and a direct argument showing how to derive the isoenergetic version of the theorem from the usual one is described in [17]. The paper [33] gives a complete proof of the full KAM theorem and establishes, in particular, the C^∞–dependence of the KAM tori on the frequencies, in the sense of Whitney. The statement of Theorem 7.5 above can be extracted from the references [5] (see, in particular, Lemma 2.3 of that paper), and especially, [8]. The latter paper derives Theorem 7.5 from the ordinary KAM theorem of [33]. Finally, we would also like to mention an interesting article [14], which provides a complete proof of the isoenergetic KAM theorem. It seems, however, that only a continuous, rather than a Whitney–smooth, dependence of the KAM tori on the rotation numbers, is obtained in that work.

Remark. A more refined version of the isoenergetic KAM theorem, which also follows from the arguments of [8], gives Cantor families of Diophantine invariant tori with prescribed rotation numbers, in energy surfaces $p_\lambda^{-1}(E)$, when $E \in \mathrm{neigh}(0, \mathbf{R})$, with an analytic dependence on E.

We shall apply Theorem 7.5 to the situation described in the beginning of this section, with $p_0 = p$. Let us recall that the energy surface $p^{-1}(0)$ is foliated by the Lagrangian tori Λ_a, $a \in J\backslash S$, and that the rotation number of Λ_a, $\omega(a)$, is assumed to be not identically constant on any connected component of $J\backslash S$. As before, let q be a real-valued bounded analytic function on T^*M, and define the function $\langle q\rangle_{\Lambda_a}$ as the average of q over the H_p–invariant tori Λ_a, $a \in J\backslash S$. Assume that the assumptions of Theorem 7.2 are satisfied and let $F_0 \in \mathcal{G}_\alpha$, with the latter set defined in (7.16), (7.10). Then

$$\langle q\rangle_{\Lambda_a}^{-1}(F_0) = \{a_1, \dots , a_L; a_j \in \widetilde{\Omega}_\alpha\},$$

and

$$\mathrm{dist}(Q_\infty(a), F_0) \geq 1/\mathcal{O}(1), \text{ when } a \text{ is away from the } a_j,\ j = 1, \dots , L. \tag{7.32}$$

It follows from the equivalence of the assumptions (1.24) and (2.53), noticed in section 2, together with Lemma 2.2, that for any sufficiently small H_p–invariant δ-neighborhood W_δ of $\cup_{j=1}^{L}\Lambda_{a_j}$ in T^*M we have, for $\rho \in p^{-1}((-\widetilde{\delta}(\delta), \widetilde{\delta}(\delta)))\backslash W_\delta$, provided that $\widetilde{\delta}(\delta) > 0$ is small enough,

$$\left|\langle q\rangle_{\delta^{-N_1},K,p}(\rho) - F_0\right| \geq \frac{\delta}{\mathcal{O}(1)}, \quad N_1 \in \mathbf{N}\backslash\{0\}. \tag{7.33}$$

Let now p_1 be a bounded analytic and real-valued function on T^*M, and let us consider

$$p_\lambda = p + \lambda p_1, \quad \lambda > 0.$$

An application of Theorem 7.5 shows that for λ/α^2 small enough, for each $1 \leq j \leq L$, there exists a smooth family of tori $\Lambda^{(j)}_{a,\lambda}$, $a \in \mathrm{neigh}(a_j, \mathbf{R})$, close to Λ_{a_j}, such that if $a \in \Omega_{\alpha,j}$, where

$$\Omega_{\alpha,j} = \left\{ a \in \mathrm{neigh}(a_j, \mathbf{R}),\ \left| \omega(a) - \frac{p}{q} \right| \geq \frac{\alpha}{2q^{2+d}},\ p \in \mathbf{Z},\ q \in \mathbf{N} \right\},$$

then $\Lambda^{(j)}_{a,\lambda} \subset p_\lambda^{-1}(0)$ is a Lagrangian torus, on which the H_{p_λ}–flow is quasi-periodic, with the rotation number $\omega(a)$. As before, we may remark that the measure of the set $\mathrm{neigh}(a_j, \mathbf{R}) \backslash \Omega_{\alpha,j}$ is small when $\alpha > 0$ is small.

When $1 \leq j \leq L$, we define a smooth function of $a \in \mathrm{neigh}(a_j, \mathbf{R})$, $\langle q \rangle_{\Lambda^{(j)}_{a,\lambda}}$, obtained by averaging q over the tori $\Lambda^{(j)}_{a,\lambda}$. Then we have, in the C^1-sense, as $\lambda \to 0$,

$$\langle q \rangle_{\Lambda^{(j)}_{a,\lambda}} \to \langle q \rangle_{\Lambda^{(j)}_{a,\lambda=0}},$$

and therefore, when $a \in \mathrm{neigh}(a_j, \mathbf{R})$, we get

$$\left| d_a \langle q \rangle_{\Lambda^{(j)}_{a,\lambda}} \right| \geq \frac{\alpha}{2}, \quad 1 \leq j \leq L,$$

provided that λ is small enough. Following (7.10), let us define

$$\mathcal{E}_{\alpha,F_0} = \complement \bigcup_{j=1}^{L} \langle q \rangle_{\Lambda^{(j)}_{a,\lambda}} \left(\mathrm{neigh}(a_j, \mathbf{R}) \backslash \Omega_{\alpha,j} \right). \tag{7.34}$$

Then the measure of the complement of $\mathcal{E}_{\alpha,F_0}$ is small with α, and if $F \in \mathcal{E}_\alpha$, then for each $1 \leq j \leq L$, there exists $b_j \in \mathrm{neigh}(a_j, \mathbf{R})$, such that $b_j \in \Omega_{\alpha,j}$ and $\langle q \rangle_{\Lambda^{(j)}_{b_j,\lambda}} = F$, $1 \leq j \leq L$. Hence the sets $\Lambda^{(j)}_{b_j,\lambda} \subset p_\lambda^{-1}(0)$, $1 \leq j \leq L$, are uniformly Diophantine Lagrangian tori, and the basic assumptions (1.13)–(1.15) are satisfied for the H_{p_λ}–flow.

To be able to apply Theorems 1.1 and 1.2 to an operator satisfying the general assumptions of the introduction and having the leading symbol $p_{\epsilon,\lambda} = p + \lambda p_1 + i\epsilon q + \mathcal{O}(\epsilon^2)$, we shall now discuss the construction of a weight function $\widetilde{G}$ for $p_{\epsilon,\lambda}$, satisfying the conclusion of Proposition 2.3. Outside a small neighborhood of $\cup_{j=1}^{L} \Lambda^{(j)}_{b_j,\lambda}$, we shall take $\widetilde{G} = G_{T,p}$, where $G_{T,p} = G_T$ is defined in (1.18), using the unperturbed symbol p. When working locally near a fixed H_{p_λ}–invariant Diophantine torus $\Lambda^{(j)}_{b_j,\lambda}$, which we identify with $\xi = 0$ in $T^*\mathbf{T}^2$, we follow (2.37) and set

$$\widetilde{G} = (1-\chi_\mu)G_{T,p} + \chi_\mu G. \tag{7.35}$$

Here G is defined as in (2.13) and $0 < \mu \ll 1$ is to be chosen. An application of (2.39) gives that uniformly in $\mu > 0$ we have

$$q - H_{p_\lambda}\widetilde{G} = \langle q\rangle + (1-\chi_\mu)\left(\mathcal{O}(T\xi^N) + \mathcal{O}(\xi^\infty + T^{-\infty})\right) + \mathcal{O}(T\xi^N) + \mathcal{O}_T(\lambda),$$

provided that $|T| \leq \mathcal{O}_N(1)\,|\xi|^{-N}$. Here we have also used that

$$H_{p_\lambda}G_{T,p} = q - \langle q\rangle_{T,K,p_\lambda} + \mathcal{O}_T(\lambda).$$

As in section 2, we take first N fixed but sufficiently large depending on N_1. Choose then $\mu \sim \delta$, with $\delta > 0$ sufficiently small but fixed as in (7.33), and put $T = \mu^{-N_1}$. Then in the region where $0 < \chi_\mu < 1$, we have

$$q - H_{p_\lambda}\widetilde{G} - F = \langle q\rangle(\xi) - \langle q\rangle(0) + \mathcal{O}(\xi^{N-N_1}) + \mathcal{O}_T(\lambda). \tag{7.36}$$

On the other hand, away from the union of the tori $\Lambda^{(j)}_{b_j,\lambda}$, we are in the region where $\widetilde{G} = G_{T,p}$, and there we have, as $\lambda \to 0$,

$$q - H_{p_\lambda}\widetilde{G} - F = \langle q\rangle_{T,K,p} - F_0 + \mathcal{O}_T(\lambda) - o(1). \tag{7.37}$$

Using (7.36) and (7.37), together with (7.33), we conclude that with this choice of $\widetilde{G}$, the conclusion of Proposition 2.3 is satisfied for p_λ, provided that $\lambda > 0$ is taken sufficiently small, depending on the parameters N_1 and δ. Hence the results of Theorem 1.1 and Theorem 1.2 apply in this perturbative situation as they stand and give complete spectral results for an operator satisfying the general assumptions of the introduction, and with the leading symbol

$$p_{\epsilon,\lambda} = p + \lambda p_1 + i\epsilon q + \mathcal{O}(\epsilon^2),$$

in rectangles of the form (1.25) and (1.26), for each $F \in \mathcal{E}_{\alpha,F_0}$.

The discussion of this section is summarized in the following theorem.

Theorem 7.6 *Let us keep all the general assumptions on the operator P_ϵ from the introduction, and write the leading symbol of P_ϵ as*

$$p_\epsilon = p + i\epsilon q + \mathcal{O}(\epsilon^2),$$

near $p^{-1}(0)\cap T^*M$. *Let us assume for simplicity that* q *is real-valued. We assume that the* H_p*–flow is completely integrable, so that we have a decomposition* (7.1). *When* $a\in J\backslash S$, *we let* $\omega(a)$ *stand for the rotation number of the invariant torus* Λ_a, *and assume that* $a\mapsto\omega(a)$ *is analytic and not identically constant on any connected component of* $J\backslash S$. *When* $a\in J$, *let us also introduce the compact sets* $Q_\infty(a)\subset\mathbf{R}$, *defined as in* (2.46), *and recall from* (1.19) *that*

$$\frac{1}{\epsilon}\mathrm{Im}\,\left(\mathrm{Spec}(P_\epsilon)\cap\{z;|\mathrm{Re}\,z|\leq\delta\}\right)\subset\left[\inf\bigcup_{a\in J}Q_\infty(a)-o(1),\sup\bigcup_{a\in J}Q_\infty(a)+o(1)\right],$$

as ϵ, h, $\delta\to 0$.

We then define a function $\langle q\rangle_{\Lambda_a}$, $a\in J\backslash S$, *obtained by averaging* q *over the invariant tori* Λ_a, *and assume that* $\langle q\rangle_{\Lambda_a}$ *depends analytically on* $a\in J\backslash S$, *and extends continuously to* J. *Assume next that* $a\mapsto\langle q\rangle_{\Lambda_a}$ *is not identically constant on any connected component of* $J\backslash S$. *When* $\alpha>0$ *and* $d>0$, *let* $\mathcal{F}_{\alpha,d}\subset\cup_{a\in J}Q_\infty(a)$ *be the set obtained by removing from* $\cup_{a\in J}Q_\infty(a)$ *the following set*

$$\left(\bigcup_{\mathrm{dist}(a,S)<\alpha}Q_\infty(a)\right)\bigcup\left(\bigcup_{a\in J\backslash S,\,|\omega'(a)|<\alpha}Q_\infty(a)\right)\bigcup\left(\bigcup_{a\in J\backslash S,\,|d\langle q\rangle_{\Lambda_a}|<\alpha}Q_\infty(a)\right)$$
$$\bigcup\left(\bigcup_{a\in J\backslash S,\,\omega(a)\text{ is not }(\alpha,d)-\text{Diophantine}}Q_\infty(a)\right).$$

Then Theorem 1.1 *applies to give complete asymptotic expansions for all the eigenvalues of* P_ϵ *in a rectangle of the form*

$$\left[-\frac{h^\delta}{\mathcal{O}(1)},\frac{h^\delta}{\mathcal{O}(1)}\right]+i\epsilon\left[F_0-\frac{h^\delta}{\mathcal{O}(1)},F_0+\frac{h^\delta}{\mathcal{O}(1)}\right],\quad F_0\in\mathcal{F}_{\alpha,d},\tag{7.38}$$

when $\delta>0$ *and* $h^K\leq\epsilon=\mathcal{O}(h^\delta)$, $K\gg 1$. *An analogous statement is obtained from Theorem* 1.2 *in the case when* $h^{1/3-\delta}\leq\epsilon\ll 1$, $\delta>0$. *The measure of the complement of* $\mathcal{F}_{\alpha,d}\subset\cup_{a\in J}Q_\infty(a)$ *is small, when* $\alpha>0$ *is small and* d *is fixed, provided that the measure of*

$$\left(\bigcup_{a\in J\backslash S,\omega(a)\in\mathbf{Q}}Q_\infty(a)\right)\bigcup\left(\bigcup_{a\in S}Q_\infty(a)\right)$$

is sufficiently small, depending on α, d.

When p_1 is an analytic function in a tubular neighborhood of T^*M, real on the real domain, with $p_1(x,\xi) = \mathcal{O}(m(\mathrm{Re}\,(x,\xi))$ in the case when $M = \mathbf{R}^2$, and $p_1(x,\xi) = \mathcal{O}(\langle\xi\rangle^m)$ in the manifold case, let $P_{\epsilon,\lambda}$, $\lambda > 0$, be an operator satisfying all the general assumptions of the introduction, and with the leading symbol

$$p_{\epsilon,\lambda} = p_\lambda + i\epsilon q + \mathcal{O}(\epsilon^2), \quad p_\lambda = p + \lambda p_1.$$

When $F_0 \in \mathcal{F}_{\alpha,d}$, let $\Lambda_{a_j} \subset p^{-1}(0)$, $a_j \in J\backslash S$, $1 \leq j \leq L$, be the (α, d)-Diophantine tori, with $|\omega'(a_j)| \geq \alpha$, and such that the average of q along Λ_{a_j} is equal to F_0, $1 \leq j \leq L$. When $\lambda/\alpha^2 \ll 1$, we introduce a Cantor family of Diophantine Lagrangian tori $\Lambda^{(j)}_{a,\lambda} \subset p_\lambda^{-1}(0)$, close to Λ_{a_j}, $1 \leq j \leq L$, whose existence is guaranteed by the isoenergetic KAM Theorem 7.5. *We then define a function $a \mapsto \langle q\rangle_{\Lambda^{(j)}_{a,\lambda}}$, obtained by averaging q over the tori $\Lambda^{(j)}_{a,\lambda}$, and recall from Theorem* 7.5 *that $\langle q\rangle_{\Lambda^{(j)}_{a,\lambda}}$ extends to a smooth function on a full neighborhood of a_j, $1 \leq j \leq L$. Let $\mathcal{E}_{\alpha,F_0}$ be the set obtained by removing from $\cup_{j=1}^L \langle q\rangle_{\Lambda^{(j)}_{a,\lambda}}(\mathrm{neigh}(a_j,\mathbf{R}))$ the set given by*

$$\bigcup_{j=1}^{L} \langle q\rangle_{\Lambda^{(j)}_{a,\lambda}}\Big(a \in \mathrm{neigh}(a_j,\mathbf{R});\ \omega(a) \textit{ is not } (\alpha/2,d) - \textit{Diophantine}\Big). \tag{7.39}$$

Then the measure of the complement of $\mathcal{E}_{\alpha,F_0}$ is small when α is small and d is kept fixed, and when $F \in \mathcal{E}_{\alpha,F_0}$ and $h^K \leq \epsilon = \mathcal{O}(h^\delta)$, $K \gg 1$, $\delta > 0$, Theorem 1.1 *applies and gives complete asymptotic expansions for all the eigenvalues of $P_{\epsilon,\lambda}$ in a rectangle of the form* (7.38)*, with F_0 replaced by F, provided that λ is sufficiently small. A similar result holds by applying Theorem* 1.2 *to $P_{\epsilon,\lambda}$, in the case when $h^{1/3-\delta} \leq \epsilon \leq \epsilon_0$, $0 < \epsilon_0 \ll 1$.*

In the last theorem and elsewhere in the paper, we have tried to find windows in the spectral band where all the eigenvalues of P_ϵ can be described asymptotically. We shall finally discuss a reformulation of Theorem 7.6 which permits us to describe a larger subset of the spectral band where the conclusion of the theorem will hold uniformly. At the same time, we shall let J be a graph rather than an interval (although this is not essential).

The completely integrable case. We make the same assumptions of complete integrability as in Theorem 7.2. We also assume that $p^{-1}(0)$ decomposes into a disjoint union

$$p^{-1}(0) = \bigcup_{\Lambda\in J} \Lambda,$$

where Λ are closed connected H_p-invariant sets. We assume that J is a finite connected graph with S denoting the set of vertices. We assume that the union of edges $J \setminus S$ has a natural real-analytic structure and that every $\Lambda \in J \setminus S$ is an analytic torus depending analytically on Λ with respect to that structure.

We identify each edge of J analytically with a real bounded interval and this determines a distance on J in the natural way. As in (2.43), we assume the continuity property:

$$\begin{aligned}&\text{For every } \Lambda_0 \in J \text{ and every } \epsilon > 0,\ \exists\, \delta > 0,\ \text{such that if} \\ &\Lambda \in J,\ \operatorname{dist}(\Lambda, \Lambda_0) < \delta,\ \text{then } \Lambda \subset \{\rho \in p^{-1}(0);\ \operatorname{dist}(\rho, \Lambda_0) < \epsilon\}.\end{aligned} \tag{7.40}$$

For $\Lambda \in J\setminus S$, let $\omega(\Lambda)$ be the corresponding rotation number, depending analytically on Λ. Assume

$$\omega(\Lambda) \text{ is not identically constant on any open edge.} \tag{7.41}$$

Assume for simplicity that q is real-valued. (In the general case, simply replace q below by $\operatorname{Re} q$.) We also assume

$$\langle q\rangle(\Lambda) \text{ is not identically constant on any open edge.} \tag{7.42}$$

Here $\langle q\rangle(\Lambda)$ denotes the average of $q_{|\Lambda}$. As before, we also assume that $\langle q\rangle$ extends continuously to J.

For every $\Lambda \in J$, define the closed interval $Q_\infty(\Lambda)$ as in Proposition 7.1, so that $Q_\infty(\Lambda) = \{\langle q\rangle(\Lambda)\}$, when $\Lambda \in J \setminus S$ and $\omega(\Lambda) \notin \mathbf{Q}$. Again, by Lemma 2.4 we know that $\{(\Lambda, z);\ \Lambda \in K,\ z \in Q_\infty(\Lambda)\}$ is closed whenever K is a closed subset of J. In particular, $\bigcup_{\Lambda\in K} Q_\infty(\Lambda)$ is compact for every closed subset $K \subset J$. As we have seen in the case of surfaces of revolution in subsection 7.2, it may happen that

$$\bigcup_{\Lambda\in\omega^{-1}(\mathbf{Q})\cup S} Q_\infty(\Lambda)$$

has a small measure compared to that of $\langle q\rangle(J \setminus S)$.

We shall first define the uniformly good values in $\mathbf{R}$. Fix $d > 0$. Given $\alpha, \beta, \gamma > 0$, we say that $r \in \mathbf{R}$ is (α, β, γ)-good if the following hold:

– r is not in the union of all $Q_\infty(\Lambda)$, with $\operatorname{dist}(\Lambda, S) \le \alpha$ or with $\omega(\Lambda)$ not (α, d)-Diophantine (in the sense of (7.6)).

– If $r = \langle q\rangle(\Lambda)$, then $|d_\Lambda\langle q\rangle(\Lambda)|$, $|d_\Lambda\omega(\Lambda)| \ge \alpha$.

– Let $\langle q\rangle^{-1}(r) = \{\Lambda_1, ..., \Lambda_L\}$, with $\Lambda_j \in J \setminus S$, $\operatorname{dist}(\Lambda_j, S) > \alpha$. Then

$$\operatorname{dist}\Big(r, \bigcup_{\substack{\Lambda\in J,\\ \operatorname{dist}(\Lambda,\cup_1^L\Lambda_j)>\beta}} Q_\infty(\Lambda)\Big) > \gamma.$$

If r is not (α, β, γ)-good we say that it is (α, β, γ)-bad. Choosing successively α, β, γ sufficiently small, we see that the measure of the set of (α, β, γ)-bad values in $\mathbf{R} \setminus \bigcup_{\Lambda \in \omega^{-1}(\mathbf{Q}) \cup S} Q_\infty(\Lambda)$ can be made arbitrarily small.

The close to completely integrable case. We now replace p by p_λ as in (7.27) and Theorems 7.5, 7.6. Near each Λ_j with $\langle q \rangle(\Lambda_j)$ (α, β, γ)-good, we apply the isoenergetic KAM theorem 7.5, with λ small enough depending only on α, β, γ and uniformly with respect to Λ, and get a family of KAM-tori $\Lambda_{j,\lambda,\mu}$.

We say that $F_0 \in \mathbf{R}$ is $(\alpha, \beta, \gamma, \lambda)$-good if:

– It is a λ-perturbation of an (α, β, γ)-good value r with $\langle q \rangle^{-1}(r) = \{\Lambda_1, ..., \Lambda_L\}$.

– There exist KAM-tori $\Lambda_{j,\lambda}$ close to Λ_j, that are $(\alpha/2, d)$-Diophantine, such that $F_0 = \langle q \rangle_{\Lambda_{j,\lambda}}$ for $j = 1, ..., L$.

Then the set of $(\alpha, \beta, \gamma, \lambda)$-good values in any bounded interval has a measure that tends to that of the set of (α, β, γ)-good values in the same interval, when λ tends to zero. Moreover the conclusions of Theorems 1.1 and 1.2 hold uniformly when λ is small enough depending on α, β, γ, and F_0 is $(\alpha, \beta, \gamma, \lambda)$-good.

8 Barrier top resonances: the non-resonant case

The purpose of this section is to illustrate Theorem 1.2 by applying it to the problem of distribution of semiclassical resonances for the Schrödinger operator. The discussion here will be analogous to the corresponding treatments in [28], [23], and [24], and for this reason, the following presentation will be somewhat less detailed.

Consider

$$P = -h^2\Delta + V(x), \quad p(x, \xi) = \xi^2 + V(x), \quad (x, \xi) \in T^*\mathbf{R}^2, \tag{8.1}$$

where V is an analytic potential satisfying satisfying the general assumptions of section 7 of [23], that allow us to define the resonances of P in a fixed sector in the fourth quadrant. Let $V(0) = E_0$, $V'(0) = 0$ and assume that we are in the barrier top case, so that $V''(0) < 0$. In this section, we shall also assume for simplicity that V is an even function. The Taylor expansion of p in a suitable system of linear symplectic coordinates then takes the form

$$p(x, \xi) - E_0 = \sum_{j=1}^{2} \frac{\lambda_j}{2}(\xi_j^2 - x_j^2) + p_4(x) + \dots, \tag{8.2}$$

where $\lambda_j > 0$, $j = 1, 2$, and p_4 is a homogeneous polynomial of degree 4.

We shall assume that we are in the non-resonant case so that

$$\lambda \cdot k \neq 0, \quad 0 \neq k \in \mathbf{Z}^2. \tag{8.3}$$

In this case, the result of [26] gives all resonances of P in the open disc $D(E_0, h^\delta) = \{z \in \mathbf{C}; |z - E_0| < h^\delta\}$, for any $\delta > 0$. They are given by

$$z_k = E_0 + f(h(k - \theta_0)h; h), \quad k \in \mathbf{N}^2,$$

where $\theta_0 \in \left(\frac{1}{2}\mathbf{Z}\right)^2$ is fixed, and $f(\theta; h)$ is a smooth function of $\theta \in \mathrm{neigh}(0, \mathbf{R}^2)$, with $f(\theta; h) \sim f_0(\theta) + hf_1(\theta) + \ldots$. We have

$$f_0(\theta) = -i\lambda \cdot \theta + \mathcal{O}(\theta^2).$$

The purpose of this section is to show how Theorem 1.2 allows us to go further away from the real axis and obtain a description of resonances that are at a distance $\sim \epsilon_0$, $0 < \epsilon_0 \ll 1$, away from the real axis.

As in [23], [24], in order to study the resonances of P near E_0, we perform the complex scaling, which near $(0,0)$ is given by $x = e^{i\pi/4}\widetilde{x}$, $\xi = e^{-i\pi/4}\widetilde{\xi}$, $\widetilde{x}$, $\widetilde{\xi} \in \mathbf{R}^2$. Using (8.2) and dropping the tildes from the notation, we get a new operator with the leading symbol

$$\frac{1}{i}\left(p_2(x,\xi) - ip_4(x) + \ldots\right) =: \frac{1}{i}q(x,\xi), \quad (x,\xi) \to (0,0). \tag{8.4}$$

Here

$$p_2(x,\xi) = \sum_{j=1}^{2} \frac{\lambda_j}{2}(x_j^2 + \xi_j^2) \tag{8.5}$$

is the harmonic oscillator. We shall be interested in eigenvalues E of the operator

$$Q(x, hD_x; h) = q(x, hD_x) + \mathcal{O}(h),$$

with $|E| \sim \epsilon^2$, $0 < \epsilon \ll 1$. An additional rescaling $x = \epsilon y$, $h^\delta \leq \epsilon \leq 1$, $0 < \delta < 1/2$, gives

$$\frac{1}{\epsilon^2}Q(x, hD_x; h) = \frac{1}{\epsilon^2}Q(\epsilon(y, \widetilde{h}D_y); h), \quad \widetilde{h} = \frac{h}{\epsilon^2} \ll 1.$$

Viewed as an $\widetilde{h}$-pseudodifferential operator, $\epsilon^{-2}Q(x, hD_x; h)$ has the leading symbol

$$p_2(x,\xi) - i\epsilon^2 p_4(x) + \mathcal{O}(\epsilon^4),$$

to be considered for (x, ξ) in a bounded region.

The H_{p_2}–flow is completely integrable, and as in [23], we introduce the action-angle coordinates $I_j \geq 0$ and $\tau_j \in \mathbf{R}/2\pi\mathbf{Z}$, $j = 1, 2$, for p_2, given by

$$x_j = \sqrt{2I_j}\cos\tau_j, \quad \xi_j = -\sqrt{2I_j}\sin\tau_j.$$

We also remark that the non-resonant condition (8.3) implies that the H_{p_2}–flow is ergodic on each invariant torus $\Lambda_\mu \subset p_2^{-1}(1)$, given by

$$I_1 = \mu, \quad I_2 = \frac{1}{\lambda_2} - \frac{\lambda_1}{\lambda_2}\mu, \quad \mu \notin \left\{0, \frac{1}{\lambda_1}\right\},$$

Therefore, the flow average

$$\langle p_4 \rangle_T = \frac{1}{T}\int_0^T p_4 \circ \exp(tH_{p_2})\, dt$$

along the torus Λ_μ converges to the torus average of p_4, $\langle p_4 \rangle = \langle p_4 \rangle_\infty$, as $T \to \infty$.

When computing the flow and torus averages, it will be convenient to work in the symplectic coordinates (y, η) given by

$$y = \frac{1}{\sqrt{2}}(x - i\xi), \quad \eta = \frac{1}{i\sqrt{2}}(x + i\xi).$$

In these coordinates $p_2 = \sum_{j=1}^{2} i\lambda_j y_j \eta_j$, and the flow is given by

$$\exp(tH_{p_2})(y, \eta) = (e^{it\lambda_1}y_1, e^{it\lambda_2}y_2, e^{-it\lambda_1}\eta_1, e^{-it\lambda_2}\eta_2).$$

Writing

$$x^\alpha = \sum_{0 \leq k \leq \alpha} a_{k\alpha} y^k \eta^{\alpha - k}, \quad |\alpha| = 4, \quad a_{k\alpha} = \frac{i^{|\alpha - k|}}{4}\begin{pmatrix}\alpha \\ k\end{pmatrix},$$

we immediately see that as $T \to \infty$, the flow average $\langle x^\alpha \rangle_T$ converges to

$$\langle x^\alpha \rangle = \sum_{\substack{0 \leq k \leq \alpha \\ 2k = \alpha}} a_{k\alpha} y^k \eta^{\alpha - k}.$$

It follows that if

$$p_4(x) = \sum_{|\alpha| = 4} v_\alpha x^\alpha,$$

then

$$\langle p_4\rangle(y,\eta) = \frac{-1}{4}\left(6v_{(4,0)}y_1^2\eta_1^2 + 4v_{(2,2)}y_1y_2\eta_1\eta_2 + 6v_{(0,4)}y_2^2\eta_2^2\right).$$

This function is in involution with p_2, and can be expressed in terms of the action variables as follows,

$$\langle p_4\rangle = \frac{1}{4}\left(6v_{(4,0)}I_1^2 + 4v_{(2,2)}I_1I_2 + 6v_{(0,4)}I_2^2\right).$$

Now $p_2 = \lambda_1 I_1 + \lambda_2 I_2$, and therefore we see that for $I_1I_2 \neq 0$, the differentials $dp_2 = (\lambda_1, \lambda_2)$ and $d\langle p_4\rangle = -(3v_{(4,0)}I_1 + v_{(2,2)}I_2, 3v_{(0,4)}I_2 + v_{(2,2)}I_1)$ are linearly dependent precisely when

$$(\lambda_1 v_{(2,2)} - 3\lambda_2 v_{(4,0)})I_1 = (\lambda_2 v_{(2,2)} - 3\lambda_1 v_{(0,4)})I_2. \tag{8.6}$$

When $I_1 = 0$ or $I_2 = 0$, the question of linear independence of the differentials should be examined directly in the (y,η)–coordinates, and we then easily see that we have the linear dependence when $I_1I_2 = 0$. We also compute that $\langle p_4\rangle$ restricted to $p_2^{-1}(1)$ has the critical values $A_1 = \frac{3}{2}v_{(0,4)}\lambda_2^{-2}$, corresponding to $I_1 = 0$, $A_2 = \frac{3}{2}v_{(4,0)}\lambda_1^{-2}$, corresponding to $I_2 = 0$, and a third value A_3 which occurs when the line (8.6) intersects the quadrant $I_1 > 0$, $I_2 > 0$, i.e., when

$$(\lambda_1 v_{(2,2)} - 3\lambda_2 v_{(2,2)})(\lambda_2 v_{(2,2)} - 3\lambda_1 v_{(0,4)}) > 0.$$

We are now in a position to apply Theorem 1.2 to the operator

$$\frac{1}{\epsilon^2}Q(x, hD_x) - 1,$$

with ϵ^2 considered as a small perturbation parameter and using $\widetilde{h} = \frac{h}{\epsilon^2}$ as the new semiclassical parameter. Here we should also remark that the Birkhoff normal form construction of section 3 goes through in the present case assuming only (8.3) and that the Diophantine condition is not required—see also [37] for the classical Birkhoff construction in the non-resonant case, near a stable equilibrium point.

Theorem 8.1 *Consider the operator P in* (8.1) *with the leading symbol*

$$p(x,\xi) - E_0 = \sum_{j=1}^{2} \frac{\lambda_j}{2}\left(\xi_j^2 - x_j^2\right) + x_1^4 + \mathcal{O}(x^6), \quad (x,\xi) \to (0,0).$$

Assume that the non-resonance condition $\langle \lambda, k\rangle \neq 0$, $0 \neq k \in \mathbf{Z}^2$, *holds. Let us put* $A_1 = 0$, $A_2 = \frac{3}{2}\lambda_1^{-2}$. *Assume that* $0 < \delta < \frac{1}{4}$. *Then the resonances of* P *in the domain*

$$\{z \in \mathbf{C}; h^{\delta} \ll |z - E_0| \leq \epsilon_0\}\backslash \bigcup_{j=1}^{2}\{z \in \mathbf{C}; \left|\operatorname{Re} z - E_0 - A_j \left|\operatorname{Im} z\right|^2\right| < \eta \left|\operatorname{Im} z\right|^2\}, \tag{8.7}$$

where $0 < \epsilon_0 = \epsilon_0(\eta) \ll 1$ *and* $\eta > 0$ *is arbitrary small but fixed, are given by*

$$\sim E_0 - i \sum_{n=0}^{\infty} h^n \epsilon^{2-2n} \widetilde{p}_n^{(\infty)} \left(\frac{h}{\epsilon^2}\left(k - \frac{\alpha}{4}\right) - \frac{S}{2\pi}, \epsilon\right), \quad k \in \mathbf{Z}^2,$$

with $\widetilde{p}_n^{(\infty)}(\xi, \epsilon)$ *analytic in* $\xi \in \operatorname{neigh}(0, \mathbf{C}^2)$ *and smooth in* $\epsilon \in \operatorname{neigh}(0, \mathbf{R})$. *We have* $S \in \mathbf{R}^2$ *and* $\alpha \in \mathbf{Z}^2$ *are fixed, and we choose* $\epsilon > 0$ *with* $|E - E_0| \sim \epsilon^2$.

Remark. Combining Theorem 8.1 with the arguments of section 7 of [28], we obtain that the result of [26] extends to a set of the form

$$D(E_0, \epsilon_0)\backslash \bigcup_{j=1}^{2} \left\{z; \operatorname{Re} z - E_0 \in -[A_j - \eta, A_j + \eta](\operatorname{Im} z)^2\right\}.$$

Here $D(E_0, \epsilon_0) = \{z, |z - E_0| < \epsilon_0\}$, with $0 < \epsilon_0 = \epsilon_0(\eta) \ll 1$, and $\eta > 0$ is arbitrary but fixed.

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