SEMICLASSICAL INVERSE SPECTRAL THEORY FOR SINGULARITIES OF FOCUS-FOCUS TYPE

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Abstract. We prove, assuming that the Bohr-Sommerfeld rules hold, that the joint spectrum near a focus-focus singular value of a quantum integrable system determines the classical Lagrangian foliation around the full focus-focus leaf. The result applies, for instance, to $\hbar$-pseudodifferential operators on cotangent bundles and Berezin-Toeplitz operators on prequantizable compact symplectic manifolds.

1. Introduction

The development of semiclassical and microlocal analysis since the 1960's now provides a strong theoretical background to discover new interactions between spectral theoretic and analytic methods, and geometric and dynamical ideas from symplectic geometry (see for instance Guillemin-Sternberg [16] and Zworski [35]). The theme of this paper is to study these interactions for an important class of singularities which appear in integrable systems: *focus-focus singularities* (of the associated Lagrangian foliation). The singular fibers corresponding to such singularities are pinched tori (Figure 2). Focus-focus singularities appear naturally also in algebraic geometry in the context of Lefschetz fibrations, where they are sometimes called *nodes*.

In this article we consider the joint spectrum of a pair of commuting semiclassical operators, in the case where the phase space is four-dimensional. We prove, assuming that the Bohr-Sommerfeld rules hold, that the joint spectrum in a neighborhood of a focus-focus singularity determines the classical dynamics of the associated system around the focus-focus fiber. This problem belongs to a class of inverse spectral questions which has attracted much attention in recent years, eg. [17, 31, 9], and which goes back to pioneer works of Bérard [1], Brüning-Heintze [2], Colin de Verdière [7, 8] and Guillemin-Sternberg [15], in the 1970s and 1980s. For recent surveys on inverse spectral problems see [32, 11]

The result applies as soon as one can show that the usual Bohr-Sommerfeld rules hold for the quantum system. This includes the cases

of $\hbar$-semiclassical pseudodifferential operators, as shown in [3] and [26]; it also includes the interesting case of Berezin-Toeplitz quantization, see [4].

Examples of quantum integrable systems, given by differential operators, with precisely one singularity of focus-focus type are the spherical pendulum, discussed by Cushman and Duistermaat in [10] (see also [28, Chapitre 2]), and the "Champagne bottle" [6]. Integrable systems given by Berezin-Toeplitz quantization are also common in the physics literature. An important example is the coupling of angular momenta (spins) [22] (see also [19, Section 8.3] for a proof that it is indeed a Berezin-Toeplitz system).

Many other integrable systems have focus-focus singularities, which are in fact the simplest integrable singularity with an isolated critical value. An infinite number of non-isomorphic systems with focus-focus singularities are provided by the so called *semitoric systems* constructed in [20].

The structure of the paper is as follows: in Section 2 we explain what we mean by a semiclassical operator, and recall the notion of integrable system in dimension four. In Section 3 we state our main theorem: Theorem 3.3. In Section 4 we review the construction of the so called Taylor series invariant, which classifies, up to isomorphisms, a semiglobal neighborhood of a focus-focus singularity. In Section 5 we prove Theorem 3.3.

2. Symplectic theory of integrable systems

Let (M, ω) be a smooth, connected 4-dimensional symplectic manifold.

2.1. Integrable systems.

An *integrable system* (J, H) on (M, ω) consists of two Poisson commuting functions $J, H \in \mathrm{C}^\infty(M; \mathbb{R})$ i.e. :

$$\{J, H\} := \omega(\mathcal{X}_J,\, \mathcal{X}_H) = 0,$$

whose differentials are almost everywhere linearly independent 1-forms. Here $\mathcal{X}_J, \mathcal{X}_H$ are the Hamiltonian vector fields induced by J, H, respectively, via the symplectic form ω: $\omega(\mathcal{X}_J, \cdot) = -\mathrm{d}J$, $\omega(\mathcal{X}_H, \cdot) = -\mathrm{d}H$.

For instance, let $M_0 = \mathrm{T}^*\mathbb{T}^2$ be the cotangent bundle of the torus $\mathbb{T}^2$, equipped with canonical coordinates (x_1, x_2, ξ_1, ξ_2), where $x \in \mathbb{T}^2$ and $\xi \in \mathrm{T}^*_x\mathbb{T}^2$. The linear system

$$(J_0, H_0) := (\xi_1, \xi_2)$$

is integrable.

An *isomorphism* of integrable systems (J, H) on (M, ω) and (J', H') on (M', ω') is a diffeomorphism $\varphi \colon M \to M'$ such that $\varphi^* \omega' = \omega$ and

$$\varphi^*(J', H') = (f_1(J, H), f_2(J, H))$$

for some local diffeomorphism (f_1, f_2) of $\mathbb{R}^2$. This same definition of isomorphism extends to any open subsets $U \subset M$, $U' \subset M'$ (and this is the form in which we will use it later). Such an isomorphism will be called *semiglobal* if U, U' are respectively saturated by level sets $\{J = \text{const}_1, H = \text{const}_2\}$ and $\{J' = \text{const}_1', H' = \text{const}_2'\}$.

If $F = (J, H)$ is an integrable system on (M, ω), consider a point $c \in \mathbb{R}^2$ that is a *regular value* of F, and such that the fiber $F^{-1}(c)$ is compact and connected. Then, by the action-angle theorem [12], a saturated neighborhood of the fiber is *isomorphic* in the previous sense to the above linear model on $M_0 = \mathrm{T}^* \mathbb{T}^2$. Therefore, all such regular fibers (called *Liouville tori*) are isomorphic in a neighborhood.

However, the situation changes drastically when the condition that c be regular is violated. For instance, it has been proved in [27] that, when c is a so-called *focus-focus* critical value (see Section 2.2 below), an infinite number of equations has to be satisfied in order for two systems to be semiglobally isomorphic near the critical fiber (see Section 4).

2.2. Focus-focus singularities.

Let $\mathcal{F}$ be the *associated singular foliation* to the integrable system $F = (J, H) \colon M \to \mathbb{R}^2$, the leaves of which are by definition the connected components of the fibers $F^{-1}(c)$. Let p be a critical point of F. We assume for simplicity that $F(p) = 0$, and that the (compact, connected) fiber $\Lambda_0 := F^{-1}(0)$ does not contain other critical points. A focus-focus singularity p is characterized by Eliasson's theorem [14, 30] as follows: there exist symplectic coordinates (x, y, ξ, η) in a neighborhood W around p in which (q_1, q_2), given by

$$(1) \qquad q_1 = x\eta - y\xi, \; q_2 = x\xi + y\eta$$

is a momentum map for the foliation $\mathcal{F}$: one has $F = g(q_1, q_2)$ for some local diffeomorphism g of $\mathbb{R}^2$ defined near the origin (the critical point p corresponds to coordinates $(0, 0, 0, 0)$). One of the major characteristics of focus-focus singularities is the existence of a *Hamiltonian action of S^1* that commutes with the flow of the system, in a neighborhood of the singular fiber that contains p [34, 33]. Such singularities are also very natural candidates for a topological study of singular Lagrangian fibrations [23].

3. Main Theorem: inverse spectral theory for focus-focus singularities

Let (M, ω) be a 4-dimensional connected symplectic manifold.

3.1. **Semiclassical operators.** Let $I \subset (0, 1]$ be any set which accumulates at 0. If $\mathcal{H}$ is a complex Hilbert space, we denote by $\mathcal{L}(\mathcal{H})$ the set of linear (possibly unbounded) selfadjoint operators on $\mathcal{H}$ with a dense domain.

A space Ψ of *semiclassical operators* is a subspace of $\prod_{\hbar \in I} \mathcal{L}(\mathcal{H}_\hbar)$ equipped with a $\mathbb{R}$-linear map

$$\sigma \colon \Psi \to \mathrm{C}^\infty(M; \mathbb{R}),$$

called the *principal symbol map.* If $P = (P_\hbar)_{\hbar \in I} \in \Psi$, the image $\sigma(P)$ is called the *principal symbol of P.*

We say that two semiclassical operators $(P_\hbar)_{\hbar \in I}$ and $(Q_\hbar)_{\hbar \in I}$ *commute* if for each $\hbar \in I$ the operators $P_\hbar$ and $Q_\hbar$ commute.

3.2. **Semiclassical spectrum.** Let $P = (P_\hbar)_{\hbar \in I}$ and $Q = (Q_\hbar)_{\hbar \in I}$ be semiclassical commuting operators on Hilbert spaces $(\mathcal{H}_\hbar)_{\hbar \in I}$, where at each $\hbar \in I$ the operators have a common dense domain $\mathcal{D}_\hbar \subset \mathcal{H}_\hbar$ such that $P_\hbar(\mathcal{D}_\hbar) \subset \mathcal{D}_\hbar$ and $Q_\hbar(\mathcal{D}_\hbar) \subset \mathcal{D}_\hbar$.

For fixed $\hbar$, the *joint spectrum* of $(P_\hbar, Q_\hbar)$ is the support of the joint spectral measure (see Figure 1 for an example). It is denoted by $\mathrm{JointSpec}(P_\hbar, Q_\hbar)$. If $\mathcal{H}_\hbar$ is finite dimensional, then

$$\mathrm{JointSpec}(P_\hbar,\, Q_\hbar) = \Big\{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \exists v \neq 0,\ P_\hbar v = \lambda_1 v,\ Q_\hbar v = \lambda_2 v \Big\}.$$

The *joint spectrum* of P, Q is the collection of all joint spectra of $(P_\hbar, Q_\hbar)$, $\hbar \in I$. It is denoted by $\mathrm{JointSpec}(P, Q)$. For convenience of the notation, we will also view the joint spectrum of P, Q as a set depending on $\hbar$.

3.3. **Bohr-Sommerfeld rules.** Recall that the *Hausdorff distance* between two subsets A and B of $\mathbb{R}^2$ is

$$\mathrm{d}_H(A,\, B) := \inf\{\epsilon > 0 \mid A \subseteq B_\epsilon \text{ and } B \subseteq A_\epsilon\},$$

where for any subset X of $\mathbb{R}^2$, the set X_ϵ is

$$X_\epsilon := \bigcup_{x \in X} \{m \in \mathbb{R}^2 \mid \|x - m\| \leqslant \epsilon\}.$$

If $(A_\hbar)_{\hbar \in I}$ and $(B_\hbar)_{\hbar \in I}$ are sequences of subsets of $\mathbb{R}^2$, we say that

$$A_\hbar = B_\hbar + \mathcal{O}(\hbar^N)$$

if there exists a constant $C > 0$ such that

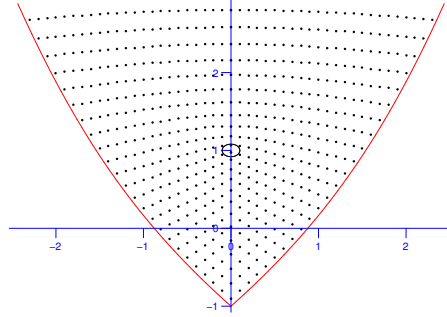$$\mathrm{d}_H(A_\hbar,\, B_\hbar) \leqslant C\hbar^N$$

for all $\hbar \in I$.

FIGURE 1. Joint spectrum of the quantum spherical pendulum when $\hbar = \frac{1}{10}$ (figure taken from [28]).

Definition 3.1. Let $F := (J, H)\colon M \to \mathbb{R}^2$ be an integrable system on a 4-dimensional connected symplectic 4-manifold. Let P and Q be commuting semiclassical operators with principal symbols $J, H\colon M \to \mathbb{R}$. Let $U \subset \mathbb{R}^2$ be an open set. We say that $\mathrm{JointSpec}(P,\, Q)$ *satisfies the Bohr-Sommerfeld rules on* U if for every regular value $c \in U$ of F there exists a small ball $\mathrm{B}(c, \epsilon_c) \subset U$ centered at c such that

$$\mathrm{JointSpec}(P,\, Q) \cap \mathrm{B}(c, \epsilon_c) = g_\hbar(2\pi\hbar\mathbb{Z}^2 \cap D) \cap \mathrm{B}(c, \epsilon_c) + \mathcal{O}(\hbar^2)$$

with

$$g_\hbar = g_0 + \hbar g_1,$$

where g_0, g_1 are smooth maps defined on a bounded open set $D \subset \mathbb{R}^2$, g_0 is a diffeomorphism into its image, and the components of $g_0^{-1} = (\mathcal{A}_1, \mathcal{A}_2)$ form a basis of action variables. We call $(g_\hbar)^{-1}$ an *affine chart* for $\mathrm{JointSpec}(P,\, Q)$.

For instance, Bohr-Sommerfeld rules are known to hold for pseudo-differential operators on cotangent bundles [3, 26], or for Toeplitz operators on prequantizable compact symplectic manifolds [4]. It would be interesting to formalize the minimal semiclassical category where Bohr-Sommerfeld rules are valid.

Remark 3.2. If $(g_\hbar)^{-1}$ is an affine chart for $\mathrm{JointSpec}(P,Q)$ and $B \in \mathrm{GL}(2, \mathbb{Z})$ then $B \circ (g_\hbar)^{-1}$ is again an affine chart.

3.4. **Main Theorem.** Let $\mathrm{CIS}(M, \omega)$ be the set of *classical integrable systems*

$$F = (J, H) \colon M \to \mathbb{R}^2$$

on the connected 4-dimensional symplectic manifold (M, ω), such that F is a proper map. Let $\mathrm{QIS}(M, \omega)$ be the set of *quantum integrable systems* given as pairs of commuting semiclassical operators (P, Q) whose principal symbols, say $(J, H) = \sigma(P, Q)$, form an integrable system $F \in \mathrm{CIS}(M, \omega)$. Let $\mathcal{P}(\mathbb{R}^2)$ the set of subsets of $\mathbb{R}^2$, and consider the following diagram:

$$\mathrm{QIS}(M, \omega) \xrightarrow{\mathrm{JointSpec}} \mathcal{P}(\mathbb{R}^2)$$
$$\downarrow \sigma$$
$$\mathrm{CIS}(M, \omega)$$

Theorem 3.3. *Let (M, ω) and (M', ω') be connected 4-dimensional symplectic manifolds. Let (P, Q) and (P, Q') be quantum integrable systems on (M, ω) and (M', ω'), respectively, which have a focus-focus singularity at points $p \in M$ and $p' \in M'$ respectively. Suppose that $c_0 := \sigma(P, Q)(p) = \sigma(P', Q')(p')$ and that there exists a neighborhood U of c_0 such that $\mathrm{JointSpec}(P, Q)$ and $\mathrm{JointSpec}(P', Q')$ satisfy the Bohr-Sommerfeld rules on U. If*

$$\mathrm{JointSpec}(P, Q) \cap U = (\mathrm{JointSpec}(P', Q') \cap U) + \mathcal{O}(\hbar^2),$$

then there are saturated neighborhoods $\mathcal{V}, \mathcal{V}'$ of the singular fibers of $\sigma(P, Q)$ and $\sigma(P', Q')$ respectively, such that the restrictions $\sigma(P, Q)|_{\mathcal{V}}$ and $\sigma(P', Q')|_{\mathcal{V}'}$ are isomorphic as integrable systems.

Remark 3.4. The method of using Bohr-Sommerfeld rules to recover topological invariants appeared in [24]. The idea of recovering the symplectic isomorphism class from the semiclassical spectrum was used for one-degree of freedom pseudodifferential operators in [29] and for Toeplitz operators on Kähler manifolds by Le Floch [18]. For higher dimensional systems the articles [5, 19] prove general results, which in particular concern elliptic and transversally-elliptic singularities. We are not aware of other works in this direction for more general integrable systems but we hope that the ideas of the present paper will prove hopeful in higher dimensions (in particular in the study of the semiclassical inverse spectral theory of semitoric integrable systems [21]).

4. Taylor series invariant at focus-focus singularity

We use here the notation of Section 2.2. In particular $p \in M$ is a focus-focus point, and W is a small neighborhood of p. Fix $A' \in \Lambda_0 \cap (W \setminus \{p\})$ and let Σ denote a small 2-dimensional surface transversal to $\mathcal{F}$ at the point A'. Since the Liouville foliation in a small neighborhood of Σ is regular for both F and $q = (q_1, q_2)$, there is a diffeomorphism φ from a neighborhood U of $F(A') \in \mathbb{R}^2$ into a neighborhood of the origin in $\mathbb{R}^2$ such that $q = \varphi \circ F$. Thus there exists a smooth momentum map $\Phi = \varphi \circ F$ for the foliation, defined on a neighborhood $\Omega = F^{-1}(U)$ of Λ_0, which agrees with q on W. Write $\Phi := (H_1, H_2)$ and $\Lambda_c := \Phi^{-1}(c)$. Note that $\Lambda_0 = \mathcal{F}_p$. It follows from (1) that near p the H_1-orbits must be periodic of primitive period 2π, whereas the vector field $\mathcal{X}_{H_2}$ is hyperbolic with a local stable manifold (the (ξ, η)-plane) transversal to its local unstable manifold (the (x, y)-plane). Moreover, $\mathcal{X}_{H_2}$ is *radial*, meaning that the flows tending towards the origin do not spiral on the local (un)stable manifolds.

Suppose that $A \in \Lambda_c$ for some regular value c.

Definition 4.1. Let $\tau_2(c) > 0$ be the smallest time it takes the Hamiltonian flow associated with H_2 leaving from A to meet the Hamiltonian flow associated with H_1 which passes through A. Let $\tau_1(c) \in [0, 2\pi)$ be the time that it takes to go from this intersection point back to A, closing the trajectory.

In Definition 4.1, the existence of τ_2 is ensured by the fact that the flow of H_2 is a quasiperiodic motion always transversal to the S^1-orbits generated by H_1.

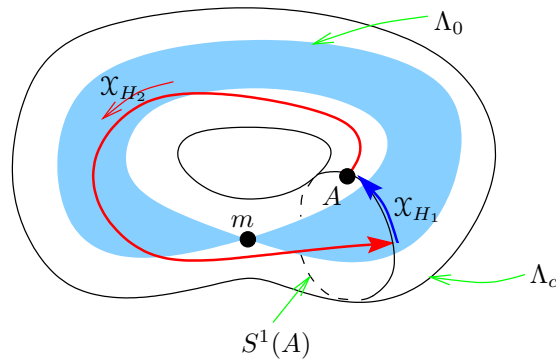The commutativity of the flows ensure that $\tau_1(c)$ and $\tau_2(c)$ do not depend on the initial point A.

FIGURE 2. Construction of the "periods" $\tau_j(c)$ around a focus-focus singularity

Write $c = (c_1, c_2) = c_1 + \mathrm{i}\, c_2$ $(c_1, c_2 \in \mathbb{R})$, and let $\log c$ be a fixed determination of the logarithmic function on the complex plane. Let

$$(2) \qquad \left\{ \begin{array}{rcl} \sigma_1(c) & = & \tau_1(c) - \mathrm{Im}(\log c) \\ \sigma_2(c) & = & \tau_2(c) + \mathrm{Re}(\log c), \end{array} \right.$$

where Re and Im respectively stand for the real and imaginary parts of a complex number. Vũ Ngọc proved in [27, Proposition 3.1] that σ_1 and σ_2 extend to smooth and single-valued functions in a neighborhood of 0 and that the differential 1-form $\sigma := \sigma_1 \, \mathrm{d}c_1 + \sigma_2 \, \mathrm{d}c_2$ is closed. Notice that if follows from the smoothness of σ_1 that one may choose the lift of τ_1 to $\mathbb{R}$ such that $\sigma_1(0) \in [0, 2\pi)$. This is the convention used throughout.

Following [27, Definition 3.1] , let S be the unique smooth function defined around $0 \in \mathbb{R}^2$ such that

$$(3) \qquad\qquad\qquad \mathrm{d}S = \sigma, \quad S(0) = 0.$$

The Taylor expansion of S at $(0, 0)$ is denoted by $(S)^\infty$.

Definition 4.2. The expansion $(S)^\infty$ is a formal power series in two variables with vanishing constant term, and we call it the *Taylor series invariant of (J, H) at the focus-focus point c_0.*

Theorem 4.3 ([27])**.** *The Taylor series invariant $(S)^\infty$ charaterizes, up to symplectic isomorphisms, a semiglobal saturated neighborhood of the singular fiber of the focus-focus singularity p.*

It is interesting to notice that, in the famous case of the spherical pendulum, the Taylor series invariant was recently explicitly computed [13].

5. Proof of Theorem 3.3

In view of Theorem 4.3, we wish to prove that the symplectic invariant $(S)^\infty$ is determined by the joint spectrum. The proof is organized in several statements. Throughout we use the notation of Section 4. Let $(P, Q) \in \mathrm{QIS}(M, \omega)$ and $(P, Q') \in \mathrm{QIS}(M', \omega')$ be quantum integrable systems which have a focus-focus singularity at points $p \in M$ and $p' \in M'$, respectively. Suppose that $c_0 := \sigma(P, Q)(p) = \sigma(P', Q')(p')$ and that there exists a neighborhood U of c_0 such that $\mathrm{JointSpec}(P, Q)$ and $\mathrm{JointSpec}(P', Q')$ satisfy the Bohr-Sommerfeld rules on U. Assume that

$$\mathrm{JointSpec}(P, Q) \cap U = (\mathrm{JointSpec}(P', Q') \cap U) + \mathcal{O}(\hbar^2)$$

Let $\sigma(P, Q) = (J, H) := F \colon M \to \mathbb{R}^2$ and let $\sigma(P', Q') = (J', H') := F' \colon M' \to \mathbb{R}^2$.

Lemma 5.1. ([25, Proposition 1]) *Let $(f_\hbar)^{-1}$ and $(g_\hbar)^{-1}$ be two affine charts for $\mathrm{JointSpec}(P,Q)$, both defined on a ball B around c_0. Let $(f_0)^{-1}, (g_0)^{-1}$ denote the principal symbols of $(f_\hbar)^{-1}$ and $(g_\hbar)^{-1}$ respectively. Then there is a constant matrix $C \in \mathrm{GL}(2,\mathbb{Z})$ such that for all $c \in B$, we have that*

$$\mathrm{d}(g_0)^{-1}(c) = C \cdot (\mathrm{d}(f_0)^{-1}(c)).$$

Proof. We recall the proof for the reader's convenience. Let $\Sigma_\hbar := \mathrm{JointSpec}(P_\hbar, Q_\hbar)$. Let $c \in U$, and $(f_\hbar)^{-1}$, $(g_\hbar)^{-1}$ be two affine charts of Σ defined on a ball B around c. Any open ball around c contains, for $\hbar$ small enough, at least one element of $\Sigma_\hbar$. Therefore, there exists a family $\lambda_\hbar \in \Sigma_\hbar \cap B$ such that $\lim_{\hbar \to 0} \lambda_\hbar = c$. Let $k \in \mathbb{Z}^2$ and let $\lambda'_\hbar$ be a family of elements of $\Sigma_\hbar \cap B$ such that

$$(f_\hbar)^{-1}(\lambda_\hbar) = (f_\hbar)^{-1}(\lambda'_\hbar) + \hbar k + \mathcal{O}(\hbar^2).$$

Then, as $\hbar$ tends to zero, $\frac{\lambda'_\hbar - \lambda_\hbar}{\hbar}$ tends towards a limit $v \in \mathbb{R}^2$ which satisfies $k = \mathrm{d}(f_0)^{-1}(c)v$. Since $\lambda_\hbar$ and $\lambda'_\hbar$ are in $\Sigma_\hbar$, there is a family $k'_\hbar \in \mathbb{Z}^2$ such that

(4)
$$\left(\frac{(g_\hbar)^{-1}(\lambda'_\hbar) - (g_\hbar)^{-1}(\lambda_\hbar)}{\hbar} \right) = k'_\hbar + \mathcal{O}(\hbar).$$

The left-hand side of (4) has limit $\mathrm{d}(g_0)^{-1}(c)v$ as $\hbar \to 0$. Therefore $k'_\hbar$ is equal to a constant integer k' for small $\hbar$, and we have $k' = \mathrm{d}(g_0)^{-1}(c)(\mathrm{d}(f_0)^{-1}(c))^{-1}k$, which implies that

$$\mathrm{d}(g_0)^{-1}(c)(\mathrm{d}(f_0)^{-1}(c))^{-1} \in \mathrm{GL}(2,\mathbb{Z}).$$

Since $\mathrm{GL}(2,\mathbb{Z})$ is discrete, the conclusion of the lemma follows. $\square$

Next we proceed in several steps.

Step 1. First we normalize the systems $F := (J, H) \colon M \to \mathbb{R}^2$ and $F' := (J', H') \colon M' \to \mathbb{R}^2$ at the focus-focus singular points, which doesn't change the Taylor series invariant. In order to do this, let $\varphi \colon \Omega \to (\mathbb{R}^4, \omega_0)$ be a symplectomorphism into its image, where Ω is a neighborhood of the singular point p and ω_0 is the standard symplectic form on $\mathbb{R}^4$, such that $F \circ \varphi^{-1} = g(q_1, q_2)$ near $(0,0,0,0)$ (it exists by Eliasson's Theorem, see Section 2.2). Here g is some local diffeomorphism. Similarly let $\varphi' \colon \Omega' \to (\mathbb{R}^4, \omega_0)$ be a local symplectomorphism, where Ω' is a small neighborhood of the singular point p' such that $F' \circ (\varphi')^{-1} = g'(q_1, q_2)$. Here g' is some local diffeomorphism.

By replacing F by the integrable system $g^{-1} \circ F$ and F' by $(g')^{-1} \circ F'$, defined respectively on semiglobal neighborhoods $\mathcal{V}, \mathcal{V}'$ we may assume

that $c_0 = 0$ and that g and g' are both the identity near the origin, that is, we may assume that

$$F \circ \varphi^{-1} = (q_1, q_2), \qquad F' \circ (\varphi')^{-1} = (q_1, q_2)$$

near $(0, 0, 0, 0)$.

Step 2. Denote by B_r the set of regular values of F and F' simultaneously. Since the focus-focus critical value $c_0 = 0$ is isolated, there exists a small ball U around 0 such that $\dot{U} := U \setminus \{0\} = \mathrm{B}_r \cap U$.

Let $c \in \dot{U}$; let $\Lambda_c := F^{-1}(c)$, which is a Liouville torus. Let (δ_1, δ_2) be loops in Λ_c that form a basis of cycles in $H_1(\Lambda_c, \mathbb{R})$. Let

$$\mathcal{A}_j(c) = \int_{\delta_j(c)} \nu, \quad j = 1, 2$$

be the action integrals, where ν is a 1-form such that $\mathrm{d}\nu = \omega$. Similarly, let $\Lambda'_c := (F')^{-1}(c)$, let (δ'_1, δ'_2) be a basis of cycles in Λ'_c, and let

$$\mathcal{A}_j(c) = \int_{\delta'_j(c)} \nu', \quad j = 1, 2.$$

Write $\mathcal{A} := (\mathcal{A}_1, \mathcal{A}_2)$ and $\mathcal{A}' := (\mathcal{A}'_1, \mathcal{A}'_2)$. It follows from the action-angle theorem that $\mathcal{A}$ and $\mathcal{A}'$ are local diffeomorphisms of $\mathbb{R}^2$ defined near c.

Lemma 5.2. *There exists a matrix $B \in \mathrm{GL}(2, \mathbb{Z})$ such that we have the following relation between the integrals above :*

$$\mathcal{A}(c) = B \circ \mathcal{A}'(c) + \mathrm{constant},$$

for all $c \in \dot{U}$.

Proof. Fix $c \in \dot{U}$. Let $(g_\hbar)^{-1}, (g'_\hbar)^{-1}$ be affine charts near c for the spectra $\mathrm{JointSpec}(P, Q)$ and $\mathrm{JointSpec}(P', Q')$ respectively. By Remark 3.2 we may assume that $(g_0)^{-1} = (\mathcal{A}_1, \mathcal{A}_2)$ and $(g'_0)^{-1} = (\mathcal{A}'_1, \mathcal{A}'_2)$. Since the joint spectra are equal modulo $\mathcal{O}(\hbar^2)$, $(g'_\hbar)^{-1}$ is also an affine chart for $\mathrm{JointSpec}(P, Q)$. Therefore by Lemma 5.1 there is a constant matrix $B \in \mathrm{GL}(2, \mathbb{Z})$ such that for all c' near c,

$$\mathrm{d}(g_0)^{-1}(c') = B \cdot (\mathrm{d}(g'_0)^{-1}(c')).$$

Since B is constant in a neighborhood of c, it does not depend on $c \in \dot{U}$, which proves the lemma.   □

By replacing (δ'_1, δ'_2) by $(B^{-1})(\delta'_1, \delta'_2)$ we may assume that the matrix B in (5) is the identity matrix.

Step 3. Consider the Hamiltonian vector field $\mathcal{X}_J$. Notice that J is a momentum map for an S^1-action on U (recall that $J \circ \varphi^{-1} = q_1$ on Ω). Recall the times τ_1, τ_2 in Definition 4.1. Let $\gamma_1(c)$ be the 2π-periodic orbit of $\mathcal{X}_J$ and let $\gamma_2(c)$ be the loop constructed as the flow of the vector field $\tau_1 \mathcal{X}_J + \tau_2 \mathcal{X}_H$. The pair (γ_1, γ_2) is a basis of the homology group $\mathrm{H}_1(\Lambda_c, \mathbb{R})$. Similarly define $\mathcal{X}_{J'}$, $\mathcal{X}_{H'}$, τ_1', τ_2', and (γ_1', γ_2'). We have that

$$(\gamma_1, \gamma_2) = C(\delta_1, \delta_2)$$

and

$$(\gamma_1', \gamma_2') = C'(\delta_1', \delta_2')$$

for some matrices C, $C' \in \mathrm{GL}(2, \mathbb{Z})$.

Step 4. Let $\mathcal{I}_1, \mathcal{I}_2$ be the actions corresponding to γ_1, γ_2. Then by Lemma 5.2 we have that

$$\mathrm{d}(\mathcal{I}_1, \mathcal{I}_2) = C\mathrm{d}(\mathcal{A}_1, \mathcal{A}_2) = C\mathrm{d}(\mathcal{A}_1', \mathcal{A}_2') = C \cdot (C')^{-1}\mathrm{d}(\mathcal{I}_1', \mathcal{I}_2').$$

We want to show that $C = C'$. We write

$$\mathrm{d}(\mathcal{I}_1, \mathcal{I}_2) = \begin{pmatrix} a & b \\ \alpha & \beta \end{pmatrix} \mathrm{d}(\mathcal{I}_1', \mathcal{I}_2'),$$

where $a, b, \alpha, \beta \in \mathbb{Z}$ are constant. The actions $\mathcal{I}_1, \mathcal{I}_1'$ are well-defined on $\dot{U}$ because they come from the S^1-action. But $\mathcal{I}_2, \mathcal{I}_2'$ are not single-valued on $\dot{U}$ because of monodromy (this follows from (2)). We have that:

$$\mathrm{d}\mathcal{I}_1 = a\mathrm{d}\mathcal{I}_1' + b\mathrm{d}\mathcal{I}_2'. \tag{5}$$

Hence $b = 0$ (since otherwise $\mathcal{I}_2'$ would be single-valued), and

$$\mathrm{d}\mathcal{I}_2 = \alpha\mathrm{d}\mathcal{I}_1' + \beta\mathrm{d}\mathcal{I}_2'. \tag{6}$$

Since $\begin{pmatrix} a & b \\ \alpha & \beta \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$ we must have that

$$\begin{pmatrix} a & b \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \\ \alpha & \pm 1 \end{pmatrix}.$$

Step 5. We will use the following well-known result, see for instance [12, 26].

Lemma 5.3. *Let $\mathcal{I} = \int_{\gamma_\alpha} \nu$ where $\mathrm{d}\nu = \omega$, $F = (J, H)\colon M \to \mathbb{R}^2$ be a proper integrable system on a connected symplectic 4-manifold (M, ω), and $\gamma_c \subset \Lambda_c = F^{-1}(c)$ is a smooth family of loops drawn on Λ_c, where c varies in a small ball of regular values of F. Let $c \mapsto x(c)$ and $y \mapsto y(c)$ be smooth functions such that the oriented loop γ_c is homologous*

to the $[0,1]$-*orbit of the flow of the vector field* $x\mathcal{X}_J + y\mathcal{X}_H$. *Then* $\mathrm{d}\mathcal{I} = x\mathrm{d}c_1 + y\mathrm{d}c_2$.

From Lemma 5.3 we obtain

$$\mathrm{d}\mathcal{I}_2 = \tau_1\mathrm{d}c_1 + \tau_2\mathrm{d}c_2, \quad \mathrm{d}\mathcal{I}'_2 = \tau'_1\mathrm{d}c_1 + \tau'_2\mathrm{d}c_2,$$

and $\mathrm{d}\mathcal{I}_1 = \mathrm{d}\mathcal{I}'_1 = 2\pi\,\mathrm{d}c_1$. From equation (5) we get

$$\mathrm{d}c_1 = a\mathrm{d}c_1$$

and hence $a = 1$. From equation (6) we get that

$$\tau_1\mathrm{d}c_1 + \tau_2\mathrm{d}c_2 = 2\pi\alpha\mathrm{d}c_1 + \beta(\tau'_1\mathrm{d}c_1 + \tau'_2\mathrm{d}c_2),$$

and therefore

$$\begin{cases} \tau_1 = 2\pi\alpha + \beta\tau'_1, & 0 \leqslant \tau_1 < 2\pi; \quad 0 \leqslant \tau'_1 < 2\pi; \\ \tau_2 = \beta\tau'_2 \end{cases}$$

Hence

$$2\pi\,|\alpha| = |\tau_1 - \tau'_1| < 2\pi,$$

which implies $\alpha = 0$. On the other hand, it follows from (2) that

$$\sigma_2 - \mathrm{Re}(\log c) = \beta(\sigma'_2 - \mathrm{Re}(\log c)).$$

Hence

$$(1 - \beta)\mathrm{Re}(\log c) = \sigma_2 - \beta\sigma'_2,$$

which implies, since σ_2 and σ'_2 extend smoothly to U, that $\beta = 1$. Therefore we have proven:

$$\begin{pmatrix} a & b \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that there exist some constants k_1, k_2 such that $\mathcal{I}_1 = \mathcal{I}'_1 + k_1$ and $\mathcal{I}_2 = \mathcal{I}'_2 + k_2$ on $\dot{U}$.

Step 6. Now from (2) (see also [27]) we deduce that there are constants K, K' such that

$$(7) \qquad \begin{cases} \mathcal{I}_2(c) = K - \mathrm{Re}(c\log c - c) + S(c) \\ \mathcal{I}'_2(c) = K' - \mathrm{Re}(c\log c - c) + S'(c) \end{cases}$$

Hence

$$\mathrm{d}S(c) = \mathrm{d}S'(c),$$

and therefore the symplectic invariants $(S)^\infty$ and $(S')^\infty$ are equal. The result now follows from Theorem 4.3.

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