Coupling of two incompressible fluids with a fixed interface

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Geometry

$\mathbf{u}_i = \mathbf{u}_i(\mathbf{x}_h, z) = (\mathbf{u}_{i,h}, w_i)$, $\quad \mathbf{u}_{i,h} = (u_{i,x}, u_{i,y})$: velocity of fluid

$i = 1, 2$, $\mathbf{x}_h = (x, y)$

$p_i = p_i(\mathbf{x}_h, z)$: pressure of fluid i,

Γ_1 top of fluid 1, Γ_2 bottom of fluid 2, Γ_{Int} : interface between both fluids

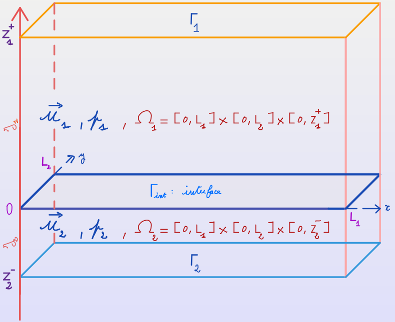


FIGURE – Computational box

Equations

The equations are the following.

$$
\begin{cases}
(\mathbf{u}_i \cdot \nabla)\mathbf{u}_i - \nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i & \text{in } \Omega_i, \\
\nabla \cdot \mathbf{u}_i = 0, & \text{in } \Omega_i, \\
\nu_i \dfrac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -C_D(\mathbf{u}_{i,h} - \mathbf{u}_{j,h})|\mathbf{u}_{i,h} - \mathbf{u}_{j,h}|, & \text{on } \Gamma_{Int}, \\
\nu_i \dfrac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -c_i(\mathbf{u}_{i,h} - \mathbb{V}_i) & \text{on } \Gamma_i, \\
\mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \Gamma_{Int} \cup \Gamma_i,
\end{cases}
\tag{1}
$$

where $\mathbf{x}_h = (x, y) \in \mathbb{T}_2$,

$$\mathbb{T}_2 = \frac{[0, L_1] \times [0, L_2]}{\mathbb{Z}^2},$$

is a two dimensional torus, which means that for the sake of the simplicity, we consider horizontal periodic boundary conditions :

$$\forall\, (n, k, q) \in \mathbb{N} \times \mathbb{Z} \times \mathbb{Z}, \quad D^n \mathbf{u}(x + kL_1, y + qL_2, z) = D^n \mathbf{u}(x, y, z),$$

in the sense of the distributions.

Plan of the following

- Define the functional spaces,
- Set the variational formulation,
- Show à priori estimates,
- Define an approximated problem in order to regularize the system, that we linearize,
- Get an existence result by fixed point,
- Set numerical algorithms,
- Perform numerical simulations : in particular we will study the impact of the roughness coefficient C_D on the convergence of the algorithms.

Functional spaces

Recall the the interface Γ_{Int} is given by

$$\Gamma_{Int} = \{(\mathbf{x}_h, 0), \mathbf{x}_h \in \mathbb{T}_2\}.$$

The boundaries Γ_i are given by

$$\Gamma_1 = \{(\mathbf{x}_h, z_1^+), \mathbf{x}_h \in \mathbb{T}_2\},$$

the top of fluid 1,

$$\Gamma_2 = \{(\mathbf{x}_h, z_2^-), \mathbf{x}_h \in \mathbb{T}_2\},$$

The bottom of fluid 2. For the simplicity we set

$$J_1 = [0, z_1^+], \quad J_2 = [z_2^-, 0],$$

where $z_1^+ > 0$ and $z_2^- < 0$. In other word, the domains Ω_i can be defined

$$\Omega_i = \mathbb{T}_2 \times J_i,$$

although for pratical calculations

$$\Omega_i = [0, L_1] \times [0, L_2] \times J_i.$$

Functional spaces

Let

$$\mathcal{W}_i = \{\mathbf{u} \in C^\infty(\mathbb{T}_2 \times J_i),\ \mathbf{u} \cdot \mathbf{n}_i|_{\Gamma_{Int} \cap \Gamma_i} = 0\},$$

equipped with the norm

$$||\mathbf{u}||_{i,1} = ||\nabla \mathbf{u}||_{L^2(\mathbb{T}_2 \times J_i)} + ||\mathrm{tr}\mathbf{u}||_{L^2(\Gamma_i)},$$

where $\mathbf{u} \to \mathrm{tr}\mathbf{u}$ denotes the trace operator, which will not systematically mentionned.
Let W_i denotes the completion of $\mathcal{W}_i$,

$$\mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2, \quad W = W_1 \times W_2.$$

Remark

In view of a mixed formulation velocity-pressure, we dot consider spaces for velocities with zero divergence.

Pressures will be seek in :

$$X = L^2(\mathbb{T}_2 \times J_1) \times L^2(\mathbb{T}_2 \times J_2).$$

Diffusion :

$$A(\mathbf{U},\mathbf{V}) = \nu_1 \int_{\mathbb{T}_1\times J_1} \nabla \mathbf{u}_1 \cdot \nabla \mathbf{v}_1 + \nu_2 \int_{\mathbb{T}_2\times J_2} \nabla \mathbf{u}_2 \cdot \nabla \mathbf{v}_2.$$

We denote by a the continous operator $W \to W'$ given by :

$$\langle a(\mathbf{U}),\mathbf{V}\rangle = A(\mathbf{U},\mathbf{V}).$$

Transport :

$$B(\mathbf{U},\mathbf{V},\mathbf{W}) = B_1(\mathbf{u}_1,\mathbf{v}_1,\mathbf{w}_1) + B_2(\mathbf{u}_2,\mathbf{v}_2,\mathbf{w}_2),$$

where

$$B_i(\mathbf{u}_i,\mathbf{v}_i,\mathbf{w}_i) = \frac{1}{2}\left(\int_{\mathbb{T}_2\times J_i} (\mathbf{u}_i\cdot\nabla)\mathbf{v}_i\cdot\mathbf{w}_i - \int_{\mathbb{T}_2\times J_i} (\mathbf{u}_i\cdot\nabla)\mathbf{w}_i\cdot\mathbf{v}_i.\right).$$

We also will consider $b : W \times W \to W'$ which satisfies

$$\langle b(\mathbf{U},\mathbf{V}),\mathbf{W}\rangle = B(\mathbf{U},\mathbf{V},\mathbf{W}).$$

Variational formulation

Pressure :

$$N(P,\mathbf{V}) = \langle n(p),\mathbf{V}\rangle = -\int_{\mathbb{T}_2\times J_1} p_1\nabla\cdot\mathbf{v}_1 - \int_{\mathbb{T}_2\times J_2} p_2\nabla\cdot\mathbf{v}_2.$$

Friction terms :

$$\langle g(\mathbf{U},\mathbf{V}),\mathbf{W}\rangle = G(\mathbf{U},\mathbf{V},\mathbf{W}) = C_D\int_{\Gamma_{Int}} |\mathbf{u}_{i,h}-\mathbf{u}_{j,h}|(\mathbf{v}_{i,h}-\mathbf{v}_{j,h})\cdot(\mathbf{w}_{i,h}-\mathbf{w}_{j,h})$$

$$h(\mathbf{U}),\mathbf{V})\rangle = H(\mathbf{U},\mathbf{V}) = c_1\int_{\Gamma_1}(\mathbf{u}_{1,h}-\mathbb{V}_1)\cdot\mathbf{v}_{1,h}+c_2\int_{\Gamma_2}(\mathbf{u}_{2,h}-\mathbb{V}_2)\cdot\mathbf{v}_{2,h}$$

Source term :

$$\langle\mathbf{F},\mathbf{V}\rangle = \int_{\mathbb{T}_2\times J_1}\mathbf{f}_1\cdot\mathbf{v}_1 + \int_{\mathbb{T}_2\times J_2}\mathbf{f}_2\cdot\mathbf{v}_2$$

Remark

Notice that for all $\mathbf{V}, \mathbf{U} \in W$,

$$B(\mathbf{V}, \mathbf{U}, \mathbf{U}) = 0,$$

when $\nabla \cdot \mathbf{u}_i = 0$,

$$B_i(\mathbf{u}_i, \mathbf{u}_i, \mathbf{v}_i) = \int_{\mathbb{T}_2 \times J_i} (\mathbf{u}_i \cdot \nabla)\mathbf{u}_i \cdot \mathbf{v}_i,$$

and for all $P \in X$,

$$N(P, \mathbf{U}) = 0.$$

Moreover, for all $\mathbf{U} \in W$,

$$G(\mathbf{U}, \mathbf{U}, \mathbf{U}) = C_D \int_{\Gamma_{Int}} |\mathbf{u}_1 - \mathbf{u}_2|^3 \geq 0,$$

Variational formulation

Definition

We say that $(\mathbf{U}, P) = [(\mathbf{u}_1, \mathbf{u}_2), (p_1, p_2)] \in W \times X$ is a weak solution to Problem (1) if :

$\forall (\mathbf{V}, Q) \in W \times X,$

$$\begin{cases} \underbrace{B(\mathbf{U}, \mathbf{U}, \mathbf{V})}_{transport} + \underbrace{A(\mathbf{U}, \mathbf{V})}_{diffusion} + \underbrace{H(\mathbf{U}, \mathbf{V})}_{top,bottom} + \underbrace{G(\mathbf{U}, \mathbf{U}, \mathbf{V})}_{interface} + \underbrace{N(P, \mathbf{V})}_{pressure} = \underbrace{\langle \mathbf{F}, \mathbf{V} \rangle}_{sources}, \\ \underbrace{(Q, \nabla \cdot \mathbf{U}) = 0}_{incompressibility}. \end{cases}$$

In other words, Problem (1) can be written as : $\mathbf{U} \in W$, and

$$\begin{cases} b(\mathbf{U}, \mathbf{U}) + a(\mathbf{U}) + h(\mathbf{U}) + g(\mathbf{U}, \mathbf{U}) + n(P) = \mathbf{F} \in W', \\ \nabla \cdot \mathbf{U} = 0 \quad \text{in } X, \end{cases}$$

A priori estimates

Proposition

Let $(\mathbf{U}, P)$ *be any weak solution of the problem* (1). *Then* $\mathbf{U}$ *satisfies the energy equality :*

$$\underbrace{A(\mathbf{U},\mathbf{U}) + H(\mathbf{U},\mathbf{U})}_{\textit{yields the norm in } W} + \underbrace{G(\mathbf{U},\mathbf{U})}_{\geq 0} = \langle \mathbf{F}, \mathbf{U} \rangle,$$

leading to the estimate

$$\|\mathbf{U}\|_W^2 \leq A(\underbrace{\|\mathbf{F}\|_{W'}^2}_{\textit{source}} + \underbrace{\|\mathbb{V}_1\|_{L^2(\Gamma_1)}^2}_{\textit{top}} + \underbrace{\|\mathbb{V}_2\|_{L^2(\Gamma_2)}^2}_{\textit{bottom}}) = \mathcal{R}_U \qquad (2)$$

where the constant A *is given by*

$$A = A\left(\nu_1, \nu_2, c_1, c_2\right),$$

which does not depend on C_D.

Proposition

Let $(\mathbf{U}, P)$ *be any weak solution of the problem* (1). *Then* $(\mathbf{U}, P)$ *satisfies the estimate :*

$$||P||_X^2 \leq \quad B[(1 + C_D)||\mathbf{U}||_W^4 + ||\mathbf{U}||_W^2 + ||\mathbf{F}||_{W'}^2 + \|\mathbb{V}_1\|_{L^2(\Gamma_1)}^2 + \|\mathbb{V}_2\|_{L^2(\Gamma_2)}^2]$$

where

$$B = B(\nu_1, \nu_2, c_1, c_2)$$

In particular

$$||P||_X^2 \leq \quad B'[(1 + C_D)(||\mathbf{F}||_{W'}^4 + \|\mathbb{V}_1\|_{L^2(\Gamma_1)}^4 + \|\mathbb{V}_2\|_{L^2(\Gamma_2)}^4) + ||\mathbf{F}||_{W'}^2 + \|\mathbb{V}_1\|_{L^2(\Gamma_1)}^2 + \|\mathbb{V}_2\|_{L^2(\Gamma_2)}^2] = \mathcal{R}_P \quad (3)$$

Regular linear system

We aim to regularize athe system by replacing the incompressiblity condition $\nabla \cdot \mathbf{u}_i = 0$ by the equation

$$-\varepsilon \Delta p_i + \nabla \cdot \mathbf{u}_i = 0,$$

for a given $\varepsilon > 0$ with suitable boundary conditions, then we linearize it for a given $\mathbf{V} \in W$, which yields the system

$$\begin{cases} (\mathbf{v}_i \cdot \nabla)\mathbf{u}_i - \nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i & \text{in } \Omega_i, \\ -\varepsilon \Delta p_i + \nabla \cdot \mathbf{u}_i = 0 & \text{in } \Omega_i, \\ \nu_i \dfrac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -C_D(\mathbf{u}_{i,h} - \mathbf{u}_{j,h})|\mathbf{v}_{i,h} - \mathbf{v}_{j,h}| & \text{on } \Gamma_{Int}, \\ \dfrac{\partial p_i}{\partial \mathbf{n}_i} = 0 & \text{on } \Gamma_{Int}, \\ \nu_i \dfrac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -c_i(\mathbf{u}_{i,h} - \mathbf{V}_i) & \text{on } \Gamma_i, \\ p_i = 0 & \text{on } \Gamma_i, \\ \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \Gamma_{Int} \cup \Gamma_i, \end{cases} \tag{4}$$

Regular linear system

in terms of operators :

$$\begin{cases} \underbrace{b(\mathbf{V},\mathbf{U})}_{\textit{linearized}} + a(\mathbf{U}) + \underbrace{g(\mathbf{V},\mathbf{U})}_{\textit{linearized}} + h(\mathbf{U}) + n(P) = \mathbf{F} & \text{in } W', \\ \underbrace{\varepsilon \tilde{a}(P) + d(\mathbf{U}) = 0}_{\textit{regularization of incompressibility}} & \text{in } H_0', \end{cases}$$

where

$$H_0 = \left\{ Q = (q_1, q_2), \quad q_i \in H^1(\Omega_i), \quad q_i = 0 \text{ on } \Gamma_i \right\},$$

$$\langle \tilde{a}(P), Q \rangle = \int_{\Omega_1} \nabla p_1 \cdot \nabla q_1 + \int_{\Omega_2} \nabla p_1 \cdot \nabla q_2,$$

$$\langle d(\mathbf{U}), Q \rangle = -\int_{\Omega_1} \mathbf{u}_1 \cdot \nabla q_1 - \int_{\Omega_2} \mathbf{u}_2 \cdot \nabla q_2.$$

Remark

In the Freefem code, we take $p_1|_{\Gamma_1} = 1$ *bar and* $p_2|_{\Gamma_2} = 3$ *bars.*

Fixed point process

Being given $\mathbf{U} \in W$, there exists a unique $P = P_\varepsilon(\mathbf{U}) \in H_0$ s.t.

$$\varepsilon \tilde{a}(P) + d(\mathbf{U}) = 0 \quad \text{in } H_0',$$

so that Problem (4) gets the linear problem in $\mathbf{U}$:

$$b(\mathbf{V},\mathbf{U}) + a(\mathbf{U}) + g(\mathbf{V},\mathbf{U}) + h(\mathbf{U}) + n(P_\varepsilon(\mathbf{U})) = \mathbf{F} \quad \text{in } W', \quad (5)$$

Lemma

Given any $\mathbf{V} \in W$*, problem* (5) *has a unique solution* $\mathbf{U} = \mathbf{U}_\varepsilon(\mathbf{V}) \in B(0,\mathcal{R}_U)$ *s.t.* $P_\varepsilon(\mathbf{U}_\varepsilon(\mathbf{V})) \in B(0,\mathcal{R}_P)$*. Moreover,*

$$\left\{ \begin{array}{c} B(0,\mathcal{R}_U) \to B(0,\mathcal{R}_U) \\ \mathbf{V} \to \mathbf{U}_\varepsilon(\mathbf{V}), \end{array} \right.$$

has a fixed point $\mathbf{U}_\varepsilon$*, which solves in* W' *:*

$$b(\mathbf{U}_\varepsilon,\mathbf{U}_\varepsilon) + a(\mathbf{U}_\varepsilon) + g(\mathbf{U}_\varepsilon,\mathbf{U}_\varepsilon) + h(\mathbf{U}_\varepsilon) + n(P_\varepsilon(\mathbf{U}_\varepsilon)) = \mathbf{F}.$$

Limit as $\varepsilon \to 0$

Let $p_\varepsilon = P_\varepsilon(\mathbf{U}_\varepsilon)$. Note that the familly $(\mathbf{U}_\varepsilon, p_\varepsilon)_{\varepsilon>0}$ is bounded in $W \times X$.

Theorem

There exists $(\mathbf{U}, p) \in W \times X$, a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ s.t. $\varepsilon_n \to 0$ and s.t $(\mathbf{U}_{\varepsilon_n}, p_{\varepsilon_n})_{n\in\mathbb{N}}$ weakly converges to in $W \times X$ to $(\mathbf{U}, p)$ and s.t. $(\mathbf{U}, p)$ is a weak solution to Problem (1).

We conjecture that the solution is unique when $\mathbf{F}$ is small enough as well as the roughness coefficient C_D.

Simple recurrence algorithm

$$b(\mathbf{U}_n, \mathbf{U}_{n+1}) + a(\mathbf{U}_{n+1}) + g(\mathbf{U}_n, \mathbf{U}_{n+1}) + h(\mathbf{U}) + n(P_{n+1}) = \mathbf{F}.$$

Double recurrence algorithm

Let

$$\langle \tilde{g}(\mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \mathbf{V}), \mathbf{W} \rangle =$$

$$C_D \int_{\Gamma_{Int}} |\mathbf{u}_{i,h}^{(1)} - \mathbf{u}_{j,h}^{(1)}|^{1/2} |\mathbf{u}_{i,h}^{(2)} - \mathbf{u}_{j,h}^{(2)}|^{1/2} (\mathbf{v}_{i,h} - \mathbf{v}_{j,h}) \cdot (\mathbf{w}_{i,h} - \mathbf{w}_{j,h})$$

$$b(\mathbf{U}_n, \mathbf{U}_{n+1}) + a(\mathbf{U}_{n+1}) + \tilde{g}(\mathbf{U}_{n-1}, \mathbf{U}_n, \mathbf{U}_{n+1}) + h(\mathbf{U}) + n(P_{n+1}) = \mathbf{F}.$$

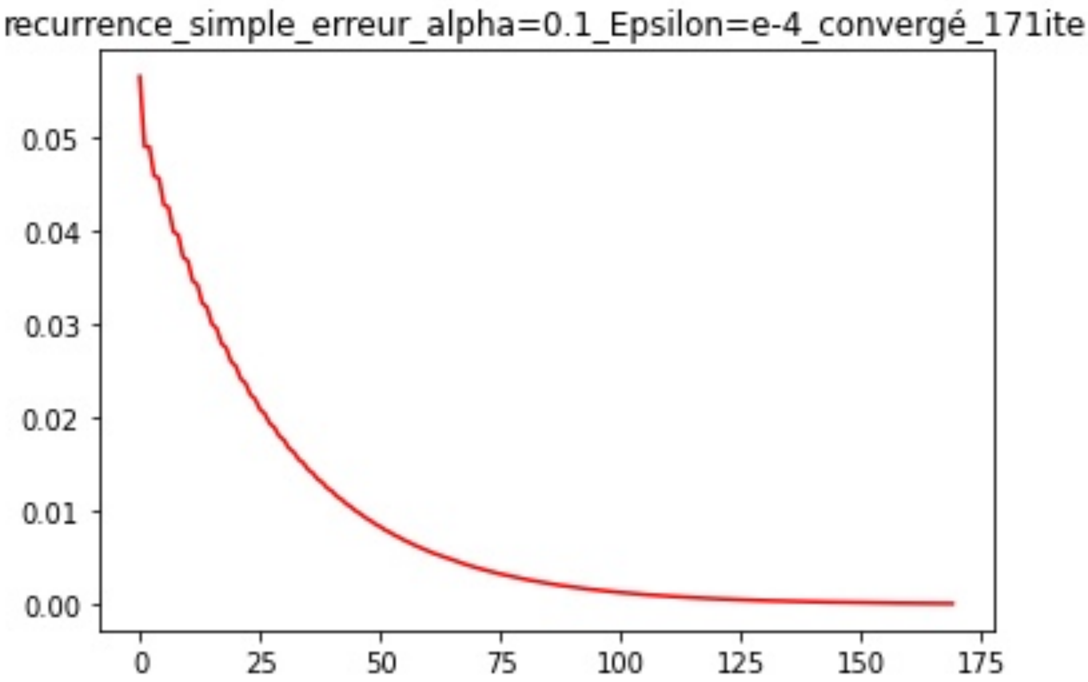
2D simulations with FreeFem

Parameters :

- $L = 100m$, $z_1 = 50m$, $z_2 = 30m$,
- Dirichlet BC for the velocities at Γ_1 and Γ_2, which amounts to Navier's conditions for large c_i's : $\mathbf{u}_1|_{\Gamma_1} = (10, 0)$ (in ms^{-1}), $\mathbf{u}_2|_{\Gamma_2} = (0, 0)$,
- $p_1|_{\Gamma_1} = 101325\, Pa$ (standard atmospheric pressure), $p_2|_{\Gamma_2} = 300000\, Pa \approx 3 bars$,
- $\mathbf{F}_1 = (0, 5)$, light convection as Boussinesq force, with $\delta T \approx 20^\circ C$, $\mathbf{F}_2 = (0, -\rho g) = (0, -10000)$ gravity,
- Viscosities : $nu1h = 0.01, nu1v = 1\, m^2 s^{-1}$, $nu2h = 100, nu2v = 100\, m^2 s^{-1}$ (vertical and horizontal viscosities) $nupa = 1e-4, nupo = 1e-6$ (elliptic regularized pressure).

Relative error :

$$\delta U_n = \frac{||\mathbf{U}_{n+1} - \mathbf{U}_n||_{L^2}}{||\mathbf{U}_n||_{L^2}}$$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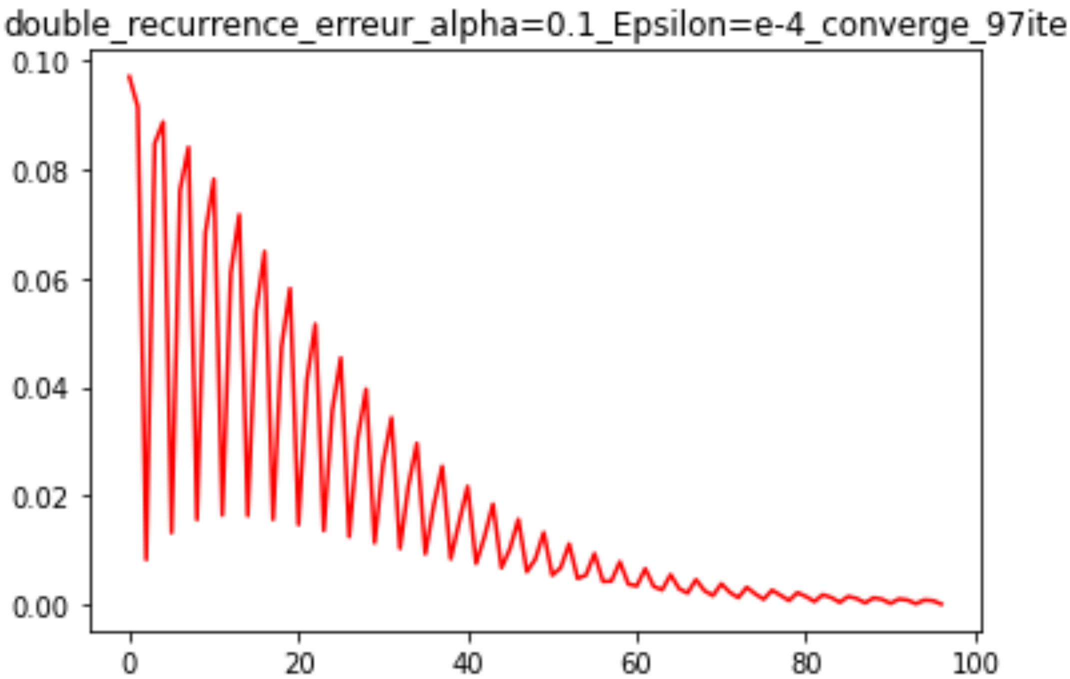
75
100
125
150
175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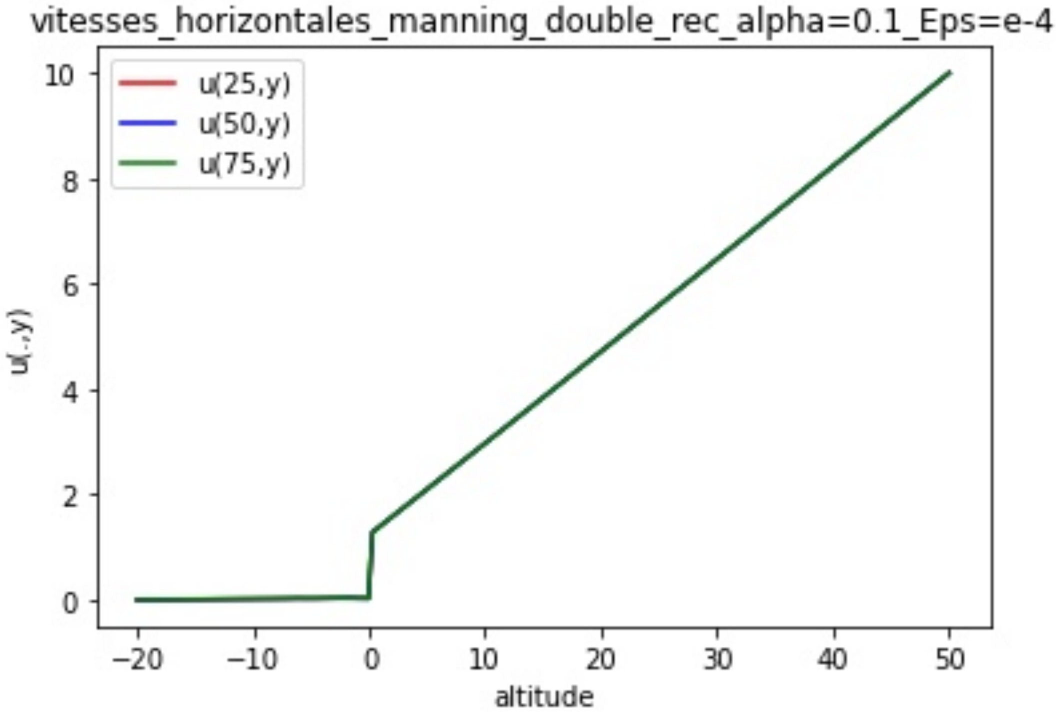
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u(75,y)
u(.,y)
altitude
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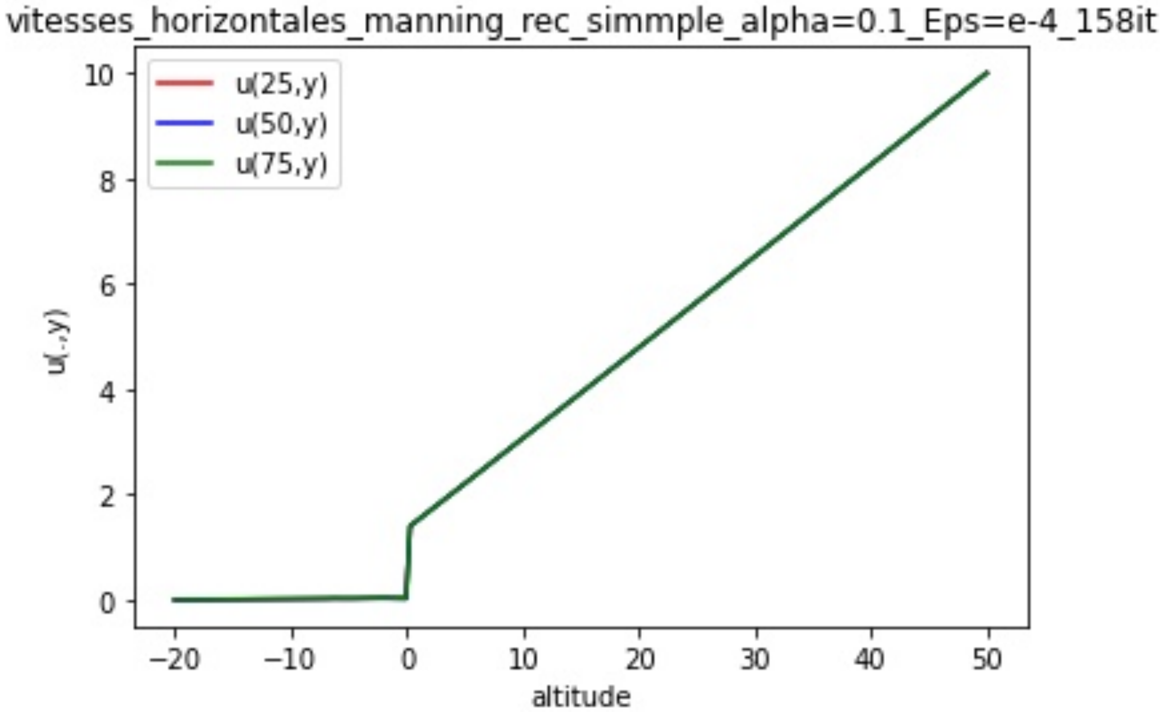
2
4
6
8
10
−20
−10
0
10
20
30
40
50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u(50,y)
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altitude
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4
6
8
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20
30
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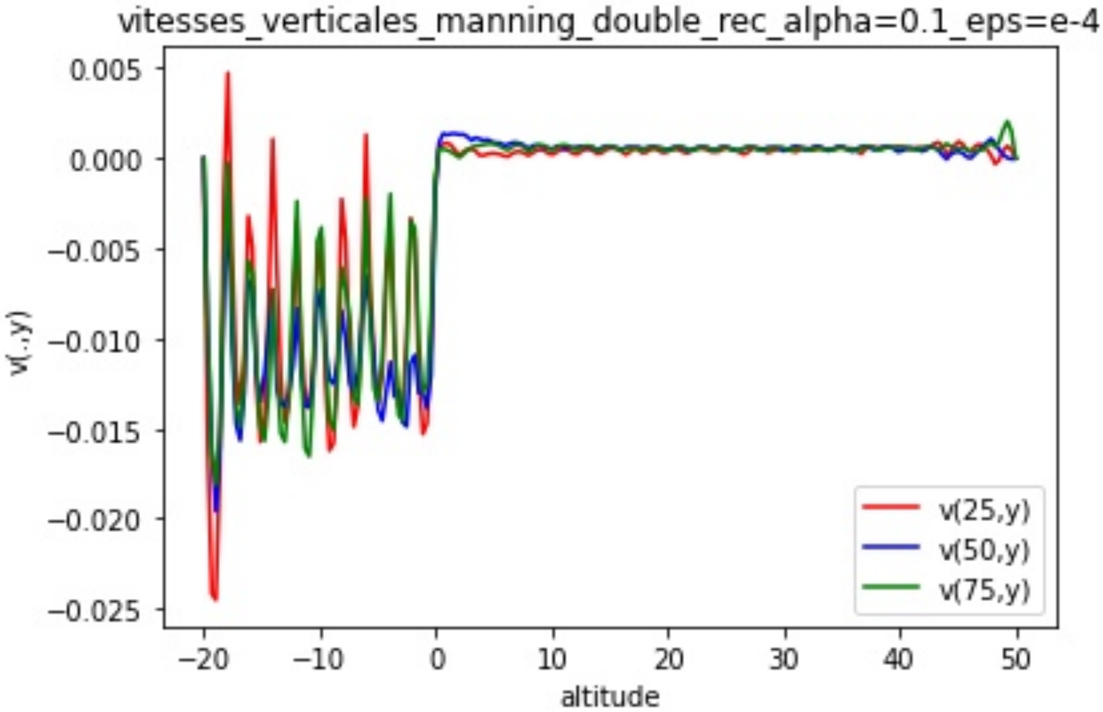
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0.000
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−0.025
v(.,y)
−20
−10
0
10
20
30
40
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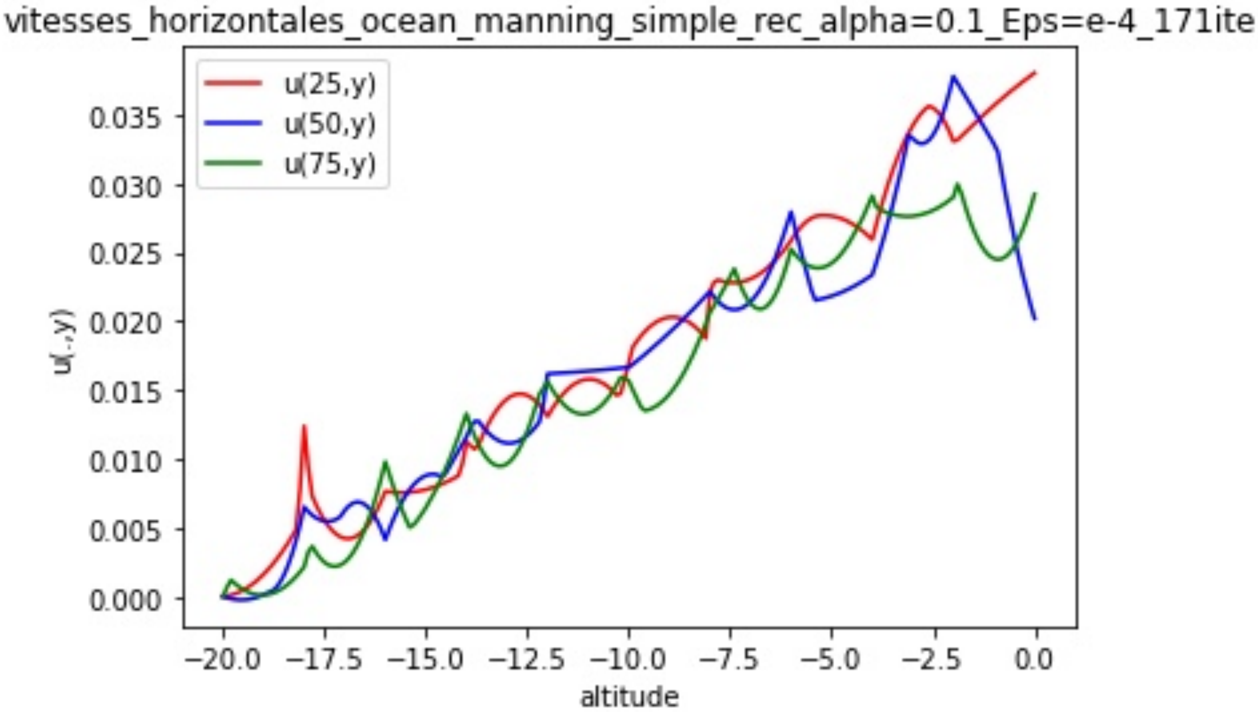
altitude
v(25,y)
v(50,y)
v(75,y)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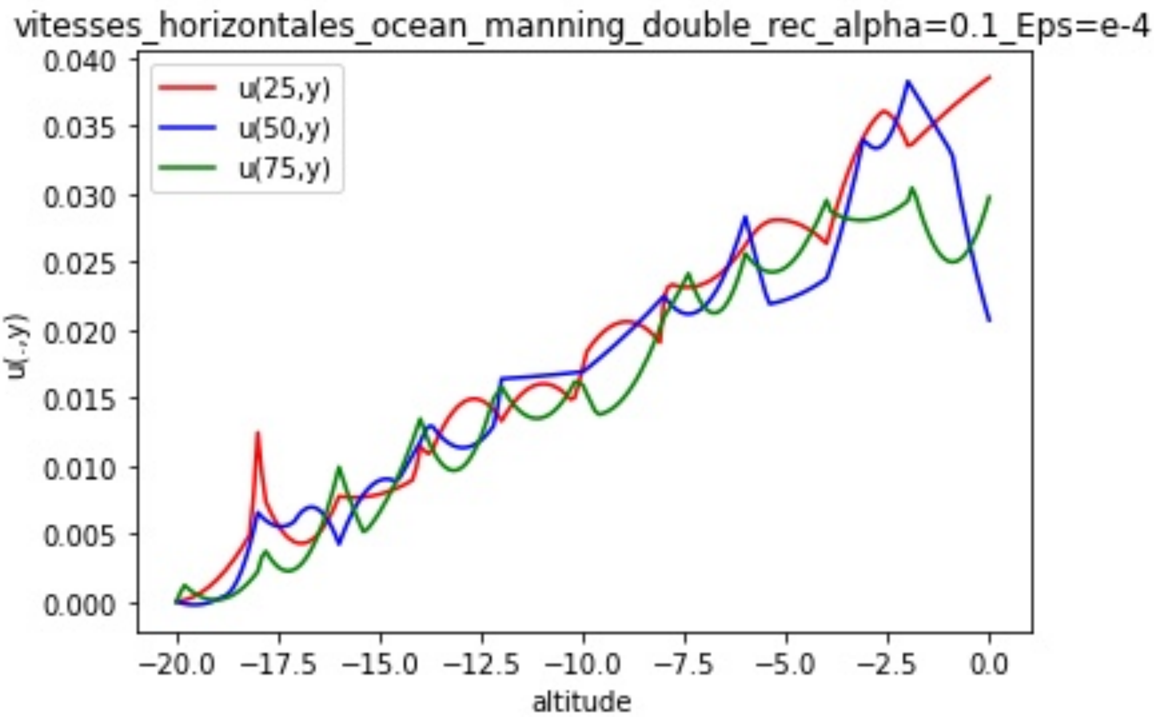
u(.,y)
altitude



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u(75,y)
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0.030
0.025
0.020
0.015
0.010
0.005
0.000
u(.,y)
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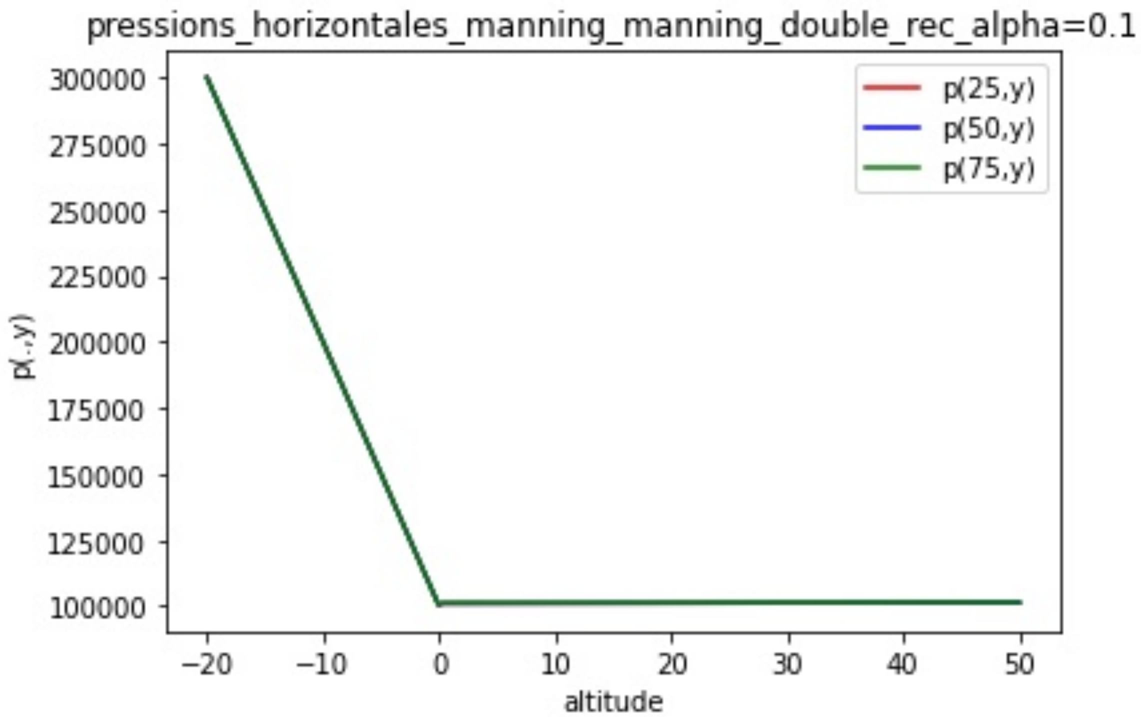
altitude



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100000
p(.,y)
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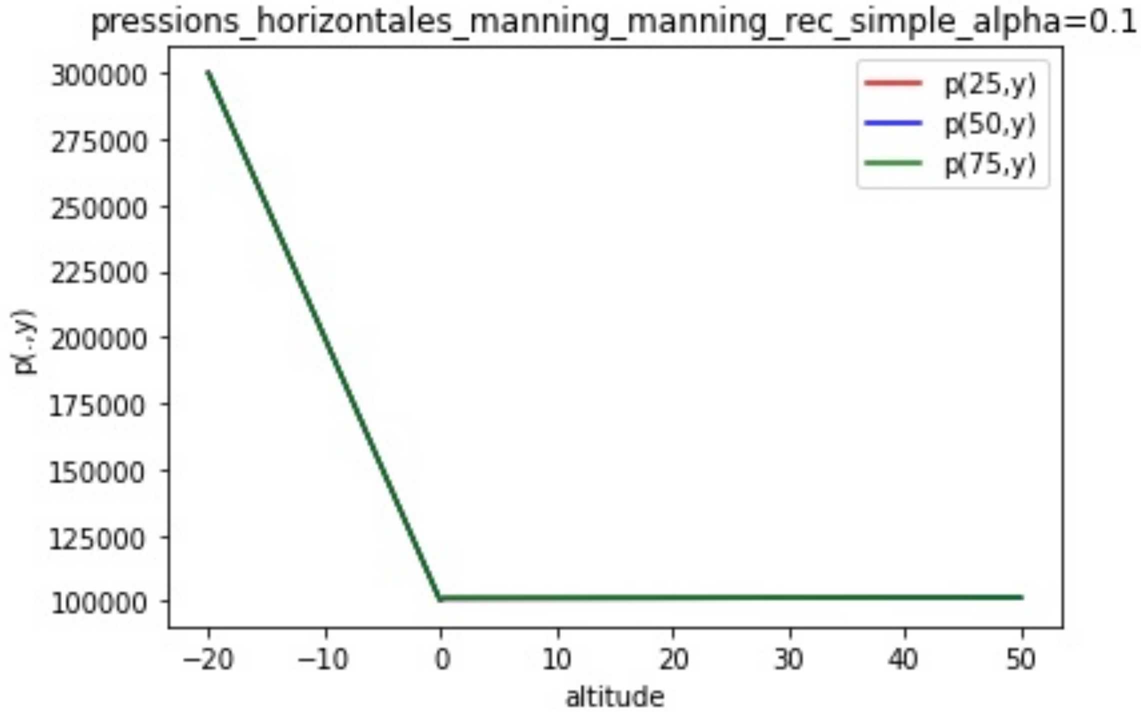
50
altitude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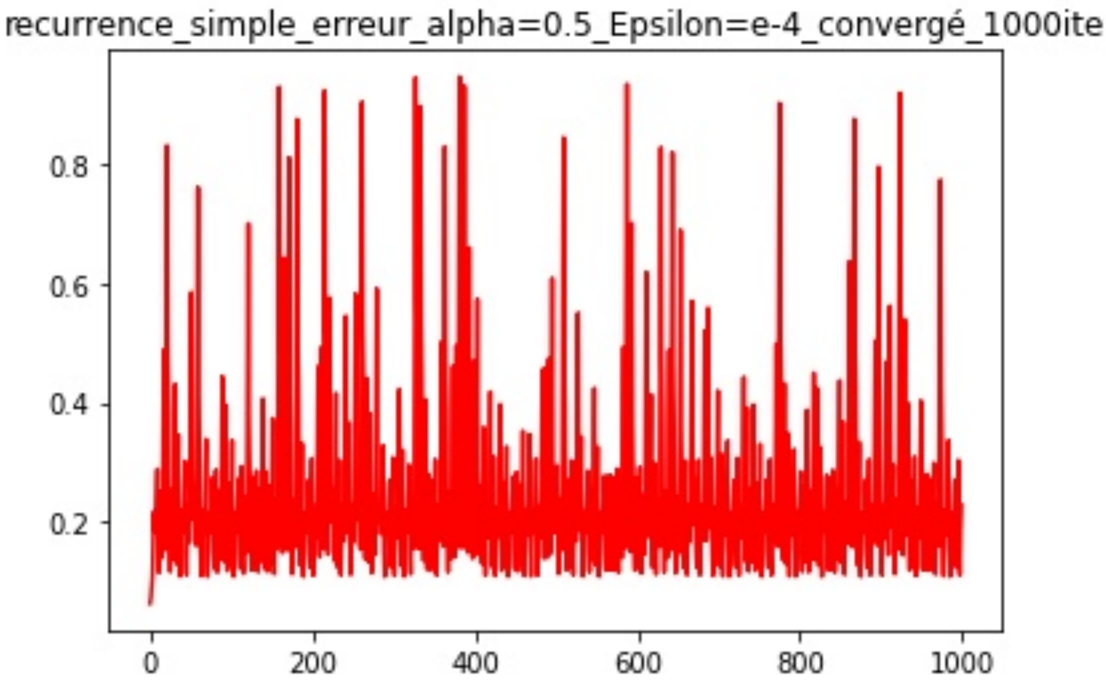
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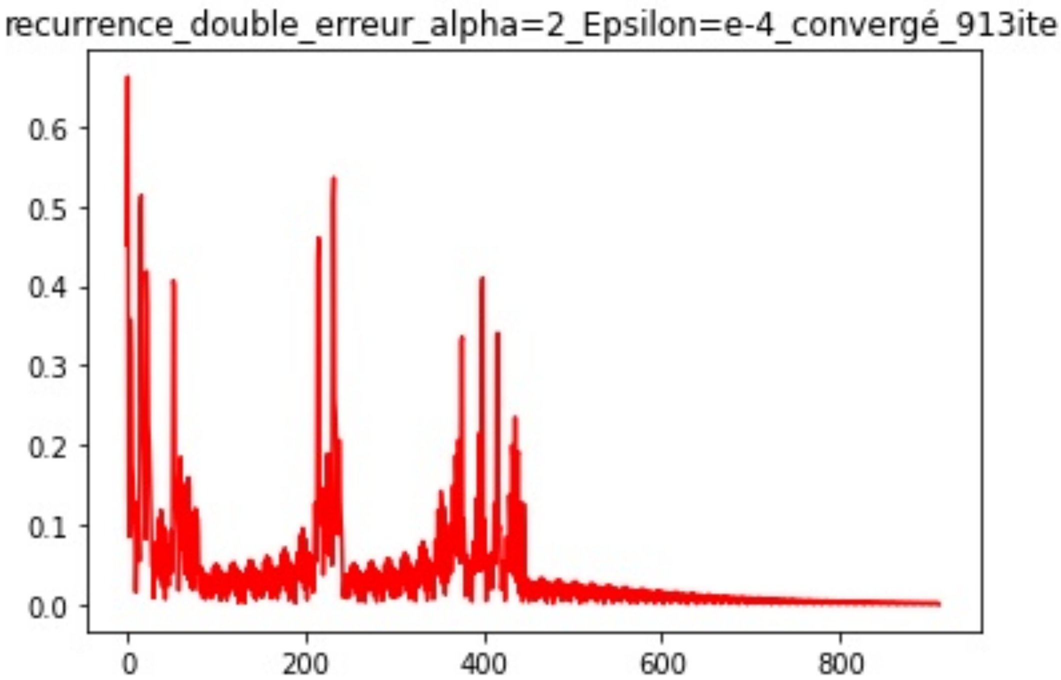
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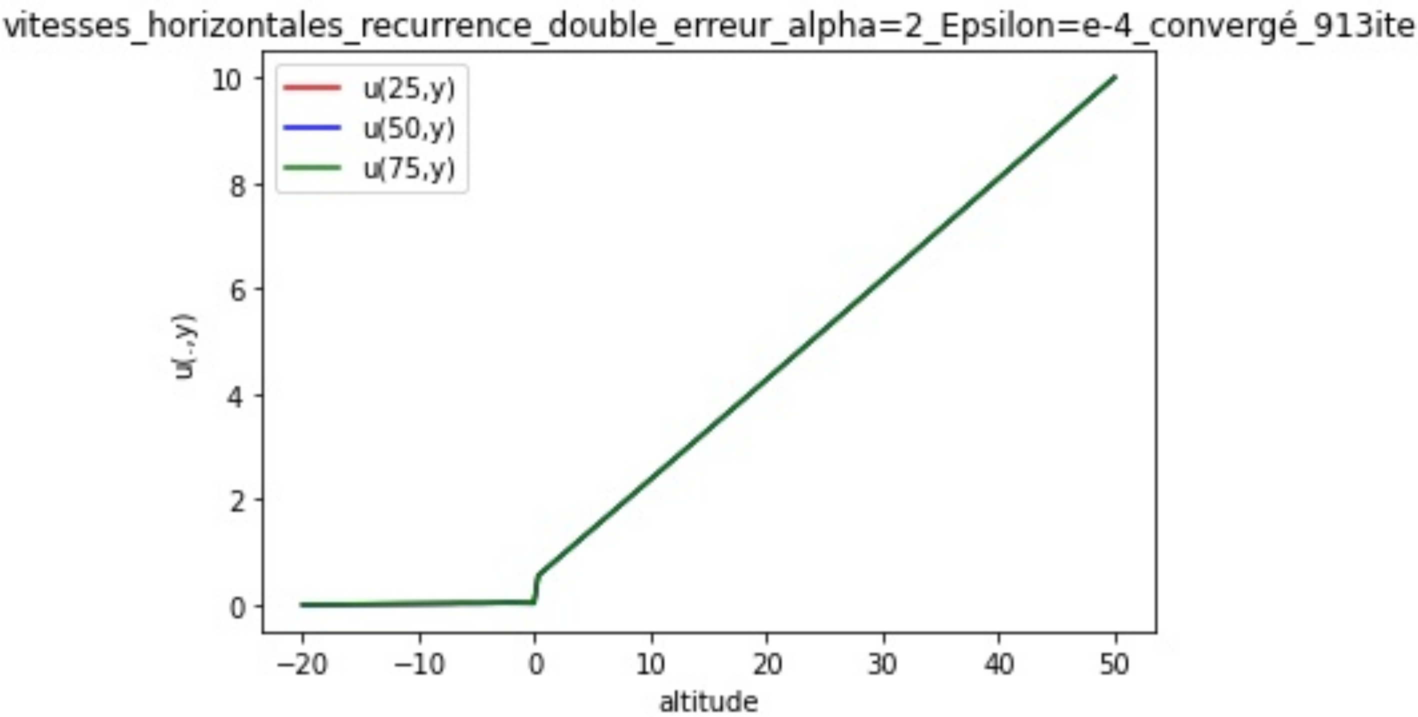
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u(.,y)
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40
50

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