

Stokes and Navier-Stokes equations with friction laws at the boundary of the domain and coupling of two fluids

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Problem 1. $\mathbf{u} = (u_1, \dots)$: fluid velocity, p : pressure. Ω : flow domain, Γ : boundary of Ω .

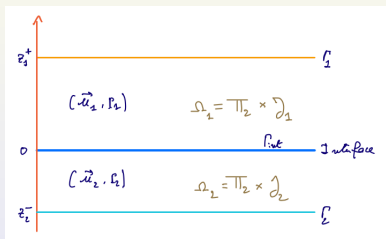
Navier-Stokes equations with a friction law

$$\left\{ \begin{array}{ll} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in }]0, +\infty[\times \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in }]0, +\infty[\times \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0 & \text{on }]0, +\infty[\times \Gamma, \\ -\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} = \mathbf{u} H(|\mathbf{u}|) & \text{on }]0, +\infty[\times \Gamma, \\ \mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) & \text{in } \Omega, \end{array} \right.$$

- $\nu > 0$ is the kinematic viscosity,
- $\nabla \cdot = \text{div}$ is the divergence operator, Δ the Laplacian,
- $\Omega \subset \mathbb{R}^3$ is an open bounded Lipschitz domain, $\Gamma = \partial\Omega$,
- \mathbf{f} is an external force, \mathbf{u}_0 the initial velocity.

Problem 2.

Coupling of two fluids thru a rigid interface (rigid lid hypothesis)



- $(\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{u}_1(\mathbf{x}_h, z_1), \mathbf{u}_2(\mathbf{x}_h, z_2))$, $\mathbf{u}_i = (\mathbf{u}_{i,h}, w)$,
 $\mathbf{u}_{i,h} = (u_{i,x}, u_{i,y})$, velocities
- $\mathbf{x}_h \in \mathbb{T}_2$, two dimensional torus, $z_1 \in J_1 = [0, z_1^+]$,
 $z_2 \in J_2 = [z_2^-, 0]$, $z_1^+ > 0$ and $z_2^- < 0$,
- Interface : $\Gamma_{int} = \{(\mathbf{x}_h, 0), \mathbf{x}_h \in \mathbb{T}_2\}$,
- Top : $\Gamma_1 = \{(\mathbf{x}_h, z_1^+), \mathbf{x}_h \in \mathbb{T}_2\}$. Bottom :
 $\Gamma_2 = \{(\mathbf{x}_h, z_2^-), \mathbf{x}_h \in \mathbb{T}_2\}$,

$$\left\{ \begin{array}{ll} -\nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i & \text{in } \mathbb{T}_2 \times J_i, \\ \nabla \cdot \mathbf{u}_i = 0 & \text{in } \mathbb{T}_2 \times J_i, \\ w_i = \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \mathbb{T}_2 \times \Gamma_{int}, \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -\alpha (\mathbf{u}_{i,h} - \mathbf{u}_{j,h}), & \text{on } \mathbb{T}_2 \times \Gamma_{int} \ (i \neq j), \\ \mathbf{u}_i = 0 & \text{on } \mathbb{T}_2 \times \Gamma_i. \end{array} \right.$$

Questions :

- Behavior of the solutions when $\alpha \rightarrow \infty$?
- Stable numerical scheme and simulations

Problem 1. Friction law : assumptions

$$-\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}|_{\Gamma} = \mathbf{u}H(|\mathbf{u}|)|_{\Gamma} = \mathbf{g}(\mathbf{u}). \text{ Assume}$$

- **(H₁)** : $\forall \mathbf{v} \in \mathbb{R}^3$, $0 < C_1 \leq H(|\mathbf{v}|) \leq C_2(1 + |\mathbf{v}|)$ or $C_1|\mathbf{v}|^\alpha \leq H(|\mathbf{v}|) \leq C_2(1 + |\mathbf{v}|)$, ($0 < \alpha < 1$),
- **(H₂)** : $H \in C^1(\mathbb{R})$, $H' \in L^\infty(\mathbb{R})$, and $H' \geq 0$ a.e.

Typical examples

$$H(|\mathbf{v}|) = C\tau = \alpha : \text{Navier's Law}$$

$$H(|\mathbf{v}|) = C_D|\mathbf{v}| : \text{Glauker-Manning's law}$$

C_D : drag coefficient.

Problem 1. Energy balance

Assume $\mathbf{u}_0 \in L^2(\Omega)$, $\mathbf{u}_0 \cdot \mathbf{n}|_\Gamma = 0$, $\mathbf{f} \in L^2_{loc}(\mathbb{R}_+, H^{-1}(\Omega)^d)$.

Let (\mathbf{u}, p) such that $\nabla \cdot \mathbf{u} = 0$, $\mathbf{u} \cdot \mathbf{n}|_\Gamma = 0$ be a "strong" solution to the Navier-Stokes equations. Then, $\forall t > 0$,

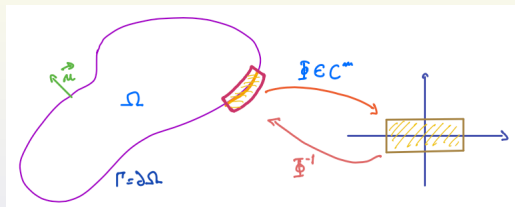
$$\frac{1}{2} \int_{\Omega} |\mathbf{u}(t, \mathbf{x})|^2 d\mathbf{x} + \nu \int_0^t \int_{\Omega} |\nabla \mathbf{u}(s, \mathbf{x})|^2 d\mathbf{x} ds + \int_0^t \int_{\Gamma} |\mathbf{u}(s, \mathbf{x})|^2 H(|\mathbf{u}(s, \mathbf{x})|) d\mathbf{x} ds = \frac{1}{2} \int_{\Omega} |\mathbf{u}_0(\mathbf{x})|^2 d\mathbf{x} + \int_0^t \langle \mathbf{f}(s), \mathbf{u}(s, \cdot) \rangle ds$$

Hence for all $T > 0$, as $\mathbf{v} \rightarrow \|\nabla \mathbf{v}\|_{\Omega; 0,1}^2 + \|\mathbf{v}\|_{\Gamma; 0,p}$ is a norm equivalent to $\|\cdot\|_{\Omega; 1,2}$ on $H^1(\Omega)^3$, $1 < p \leq 4$,

$$\|\mathbf{u}\|_{L^2([0,T], H^1(\Omega)^d)} + \|\mathbf{u}\|_{L^\infty([0,T], L^2(\Omega)^d)} \leq C(\|\mathbf{u}_0\|_{\Omega; 0,2}, H, \|\mathbf{f}\|_{L^2([0,T], H^{-1}(\Omega)^d)}, \nu),$$

Problem 1. Function space

$\mathbf{u} \cdot \mathbf{n}|_{\Gamma} = 0$. Let $\Omega \subset \mathbb{R}^d$, bounded, of class C^m ($m \geq 1$, $d = 2, 3$)



$$\mathcal{W}_m = \{\varphi \in C^m(\overline{\Omega})^d, \varphi \cdot \mathbf{n}|_{\Gamma} = 0\},$$

$$W = \{\mathbf{u} \in H^1(\Omega)^d, \mathbf{u} \cdot \mathbf{n}|_{\Gamma} = 0\}$$

$$\|\mathbf{u}\|_W = \left(\int_{\Omega} |\nabla \mathbf{u}|^2 + \int_{\Gamma} |\mathbf{u}|^2 \right)^{\frac{1}{2}} \approx \|\mathbf{u}\|_{\Omega;1,2}$$

Theorem

(Chacón-Lewandowski. 2014) $\mathcal{W}(\Omega)$ is dense in $W(\Omega)$.

Problem 1. Properties of the Friction law

Let

$$G : \begin{cases} W \rightarrow W' \\ \mathbf{v} \rightarrow G(\mathbf{v}) \end{cases}$$

$$\forall \mathbf{w} \in W, \quad \langle G(\mathbf{v}), \mathbf{w} \rangle = \int_{\Gamma} \mathbf{v} \cdot \mathbf{w} H(|\mathbf{v}|) d\sigma.$$

Lemma

Assume that **(H₁)** and **(H₂)** hold. Then G satisfies, for all $\mathbf{v}, \mathbf{w} \in W$:

- $\|G(\mathbf{v})\|_{W'} \leq C(1 + \|\mathbf{v}\|_W^2)$,
- $C\|\mathbf{v}\|_{\Gamma;0,2}^2 \leq \langle G(\mathbf{v}), \mathbf{v} \rangle$ (coercivity),
- $0 \leq \langle G(\mathbf{v}) - G(\mathbf{w}), \mathbf{v} - \mathbf{w} \rangle$ (monotony),
- $\|G(\mathbf{v}) - G(\mathbf{w})\|_{W'} \leq C(\|\mathbf{v}\|_W + \|\mathbf{w}\|_W)\|\mathbf{v} - \mathbf{w}\|_W$,
- G is continuous and compact.

Problem 1. Transport-diffusion operators

Let

$$B : \begin{cases} W \times W \rightarrow W' \\ (\mathbf{v}, \mathbf{w}) \rightarrow B(\mathbf{v}, \mathbf{w}) \end{cases}, \quad \forall \mathbf{z} \in W, \langle B(\mathbf{v}, \mathbf{w}), \mathbf{z} \rangle = \int_{\Omega} (\mathbf{v} \cdot \nabla) \mathbf{w} \cdot \mathbf{z}.$$

Lemma

$\forall \mathbf{v}, \mathbf{w} \in W,$

- $\|B(\mathbf{v}, \mathbf{w})\|_{W'} \leq \|\mathbf{v}\|_W \|\mathbf{w}\|_W,$
- *When $\nabla \cdot \mathbf{v} = 0,$ then $\langle B(\mathbf{v}, \mathbf{w}), \mathbf{w} \rangle = 0.$*

$$A : \begin{cases} W \rightarrow W' \\ \mathbf{v} \rightarrow A\mathbf{v} \end{cases}, \quad \forall \mathbf{w} \in W, \langle A\mathbf{v}, \mathbf{w} \rangle = \nu \int_{\Omega} \nabla \mathbf{v} : \nabla \mathbf{w}.$$

Momentum Equation : find \mathbf{u}, p such that

$$\frac{\partial \mathbf{u}}{\partial t} + B(\mathbf{u}, \mathbf{u}) + A\mathbf{u} + G(\mathbf{u}) + \nabla p = f, \quad \nabla \cdot \mathbf{u} = 0$$

Problem 1. Estimate for the pressure

$Q = [0, T] \times \Omega$. The pressure is in $L^{4/3}(Q)$. More specifically : Let (u, p) be a strong solution to the NSE. Then

$$\left(\int_Q |p(t, \mathbf{x})|^{4/3} d\mathbf{x} dt \right)^{3/4} = \|p\|_{Q;0,4/3} \leq C(T, H, \mathbf{u}_0, \mathbf{f}, \nu).$$

Essential ingredient of the proof. Solve

$$\begin{cases} -\Delta \Psi = p|p|^{-2/3} & \text{in } \Omega, \\ \frac{\partial \Psi}{\partial \mathbf{n}} = 0 & \text{on } \Gamma, \\ \int_{\Omega} \Psi(\mathbf{x}) d\mathbf{x} = 0 \end{cases}$$

then take $\mathbf{w} = \nabla \Psi$ as test function in the equation for the velocity.

Consequence :

$$\partial_t \mathbf{v} \in L^{4/3}([0, T], W_4'), \quad W_4 = W \cap H^4(\Omega)^4$$

Problem 1. Variational formulation and existence result

Weak solution : Find

$$(\mathbf{u}, p) \in L^2(W) \cap L^\infty(L^2) \times L^{4/3}(Q), \quad \text{s.t. } \partial_t \mathbf{u} \in L^{4/3}([0, T], W_4'),$$

such that $\forall (\mathbf{v}, q) \in L^4(W_4) \times L^2(L^2)$,

$$\int_0^T \langle \partial_t \mathbf{u}(t) + B(\mathbf{u}, \mathbf{u}) + A(\mathbf{u}) + G(\mathbf{u}), \mathbf{v} \rangle dt - \int_0^T \int p(t, \mathbf{x}) \nabla \cdot \mathbf{v}(t, \mathbf{x}) d\mathbf{x} dt +$$

$$\int_0^T \int q(t, \mathbf{x}) \nabla \cdot \mathbf{u}(t, \mathbf{x}) d\mathbf{x} dt = \int_0^T \langle \mathbf{f}, \mathbf{v} \rangle dt$$

Theorem

Chacón-Lewandowski, 2014. The NSE have a weak solution

$$(\mathbf{u}, p) \in L^2(W) \cap L^\infty(L^2) \times L^{4/3}(Q).$$

Inspired by the work of Bulíček, M. and Málek, J. and Rajagopal, K. R. 2007.

Problem 1. Proof by Regularization

Let

$$P_\varepsilon : \begin{cases} W \rightarrow H^1(\Omega) \\ \mathbf{v} \rightarrow p_\varepsilon = P_\varepsilon(\mathbf{v}) \end{cases}$$

where

$$\begin{cases} -\varepsilon \Delta p_\varepsilon + \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega, \\ \frac{\partial p_\varepsilon}{\partial \mathbf{n}} = 0 & \text{on } \Gamma, \quad \int_\Omega p_\varepsilon(\mathbf{x}) d\mathbf{x} = 0 \end{cases}$$

and for a given mollifier $\rho_\alpha = \rho_\alpha(\mathbf{x})$, for $\mathbf{v} \in W$, $\bar{\mathbf{v}}$ its extension to $H^1(\mathbb{R}^3)^3$,

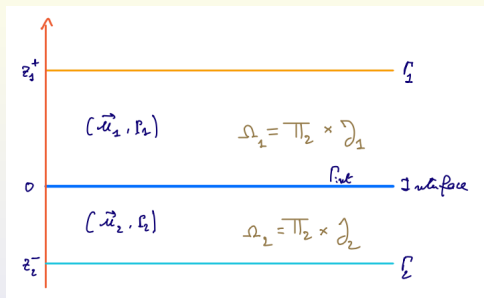
$$V_\alpha(\mathbf{v})(\mathbf{x}) = (\rho_\alpha \star \bar{\mathbf{v}})(\mathbf{x}) = \int_{\mathbb{R}^3} \rho_\alpha(\mathbf{y} - \mathbf{x}) \bar{\mathbf{v}}(\mathbf{y}) d\mathbf{y}$$

Then solve

$$\frac{\partial \mathbf{u}}{\partial t} + B(V_\alpha(\mathbf{u}), \mathbf{u}) + A\mathbf{u} + G(\mathbf{u}) + \nabla P_\varepsilon(\mathbf{u}) = f.$$

and take the limit as $(\alpha, \varepsilon) \rightarrow (0, 0)$.

Problem 2. Recall : coupled problem with friction



$$(\text{Co./Fr.}) : \left\{ \begin{array}{ll}
 -\nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i & \text{in } \mathbb{T}_2 \times J_i, \\
 \nabla \cdot \mathbf{u}_i = 0 & \text{in } \mathbb{T}_2 \times J_i, \\
 w_i = \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \mathbb{T}_2 \times \Gamma_{int}, \\
 \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -\alpha (\mathbf{u}_{i,h} - \mathbf{u}_{j,h}), & \text{on } \mathbb{T}_2 \times \Gamma_{int} \ (i \neq j), \\
 \mathbf{u}_i = 0 & \text{on } \mathbb{T}_2 \times \Gamma_i.
 \end{array} \right.$$

Problem 2. Functional framework

$$\mathcal{W}_i = \{\mathbf{u} \in C^\infty(\mathbb{T}_2 \times J_i), \mathbf{u}|_{\Gamma_i} = 0, \mathbf{u} \cdot \mathbf{n}_i|_{\Gamma_{int}} = 0, \nabla \cdot \mathbf{u}_i = 0\},$$

equipped with $\|\mathbf{u}\|_{i,1} = \|\nabla \mathbf{u}\|_{L^2(\mathbb{T}_2 \times J_i)}$. Let

$$W_i = \overline{\mathcal{W}_i}^{\|\cdot\|_{i,1}}, \quad W = W_1 \times W_2,$$

$$W_0 = \{(\mathbf{u}_1, \mathbf{u}_2) \in W, \mathbf{u}_{1,h}|_{\Gamma_{int}} = \mathbf{u}_{2,h}|_{\Gamma_{int}} \text{ a.e. in } \Gamma_{int}\},$$

$$\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2), \mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2) \in W,$$

$$\Lambda(\mathbf{U}, \mathbf{V}) = \nu_1 \int_{\mathbb{T}_2 \times J_1} \nabla \mathbf{u}_1 \cdot \nabla \mathbf{v}_1 + \nu_2 \int_{\mathbb{T}_2 \times J_2} \nabla \mathbf{u}_2 \cdot \nabla \mathbf{v}_2.$$

$$W_0 = \ker \operatorname{tr}_{\Gamma_{int}}(u_1 - u_2), \quad \operatorname{codim}(W_0) = 1,$$

$$W = W_0 \oplus^\perp \operatorname{vect} \Phi, \quad \Phi = (\phi_1, \phi_2)$$

Problem 2. Weak Solutions

Let

$$\int_{\mathbb{T}_2 \times J_1} \mathbf{f}_1 \cdot \mathbf{v}_1 + \int_{\mathbb{T}_2 \times J_2} \mathbf{f}_2 \cdot \mathbf{v}_2 = (\mathbf{F}, \mathbf{V})$$

Definition

(weak solution) A couple $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2) \in W$ is a weak solution to Problem (Co./Fr.) when $\forall \mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2) \in W$,

$$\Lambda(\mathbf{U}, \mathbf{V}) + \alpha \int_{\Gamma_{int}} (\mathbf{u}_{1,h} - \mathbf{u}_{2,h}) \cdot (\mathbf{v}_{1,h} - \mathbf{v}_{2,h}) = (\mathbf{F}, \mathbf{V}).$$

Theorem

Problem (Co./Fr.) has a unique weak solution, denoted \mathbf{U}_α .

Straightforward by Lax-Milgram Theorem.

Question : behavior of $(\mathbf{U}_\alpha)_{\alpha>0}$ when $\alpha \rightarrow \infty$?

Problem 2. Energy balance and consequence

The solution \mathbf{U}_α of Problem (Co./Fr.) verifies the energy balance :

$$\Lambda(\mathbf{U}_\alpha, \mathbf{U}_\alpha) + \alpha \int_{\Gamma_{int}} |\mathbf{u}_{1,h,\alpha} - \mathbf{u}_{2,h,\alpha}|^2 = (\mathbf{F}, \mathbf{U}_\alpha)$$

In particular :

- The family $(\mathbf{U}_\alpha)_{\alpha>0}$ is bounded in W uniformly in α ,
- $\int_{\Gamma_{int}} |\mathbf{u}_{1,h,\alpha} - \mathbf{u}_{2,h,\alpha}|^2 \rightarrow 0$ as $\alpha \rightarrow \infty$.

Consequence : There exists $(\alpha_n)_{n \in \mathbb{N}}$, $\alpha_n \xrightarrow[n \rightarrow \infty]{} +\infty$, $\mathbf{U} \in W_0$, such that $(\mathbf{U}_{\alpha_n})_{n \in \mathbb{N}}$ weakly converges to \mathbf{U} in W .

Problem 2. Limit problem when $\alpha \rightarrow \infty$: continuous BC's

$$(Co./Con.) : \left\{ \begin{array}{ll} -\nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i & \text{in } \mathbb{T}_2 \times J_i, \\ \nabla \cdot \mathbf{u}_i = 0 & \text{in } \mathbb{T}_2 \times J_i, \\ w_i = \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \mathbb{T}_2 \times \Gamma_{int}, \\ \mathbf{u}_1 = \mathbf{u}_2, & \text{on } \mathbb{T}_2 \times \Gamma_{int} \\ \mathbf{u}_i = 0 & \text{on } \mathbb{T}_2 \times \Gamma_i. \end{array} \right.$$

Theorem

(Legeais-Lewandowski, *Applied Maths Letters*, 2022.) The family $(\mathbf{U}_\alpha)_{\alpha > 0}$ strongly converges in W to the unique weak solution $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2) \in W_0$ of Problem (Co./Con.), in the sense :

$$\forall \mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2) \in W_0, \quad \Lambda(\mathbf{U}, \mathbf{V}) = (\mathbf{F}, \mathbf{V}).$$

Problem 2. Algorithm -Initialization

Initialization : We solve the following initial problem on the upper part to get a first field $(\mathbf{u}_1^{\alpha,0}, P_1^{(0)})$:

$$-\nu_1 \Delta \mathbf{u}_1^{\alpha,0} + \nabla P_1^{(0)} = \mathbf{f}_1, \quad (1)$$

$$\nabla \cdot \mathbf{u}_1^{\alpha,0} = 0, \quad (2)$$

$$\nu_1 \frac{\partial \mathbf{u}_{1,h}^{\alpha,0}}{\partial \mathbf{n}_1} \Big|_{\Gamma_{int}} = -\alpha \mathbf{u}_{1,h}^{\alpha,0}, \quad \mathbf{u}_{1,h}^{\alpha,0} \cdot \mathbf{n}_1 \Big|_{\Gamma_{int}} = 0. \quad (3)$$

$$\mathbf{u}_{1,h}^{\alpha,0} \Big|_{\Gamma_1} = 0, \quad (4)$$

Problem 2. Schwarz like Algorithm-iterations

Once the upper field $(\mathbf{u}_1^{\alpha,n}, P_1^{(n)})$ is calculated for a given $n \geq 0$, determine the bottom field $(\mathbf{u}_2^{\alpha,n}, P_2^{(n)})$ by solving :

$$-\nu_2 \Delta \mathbf{u}_2^{\alpha,n} + \nabla P_2^{(n)} = \mathbf{f}_2, \quad \nabla \cdot \mathbf{u}_2^{\alpha,n} = 0, \quad (5)$$

$$\nu_2 \frac{\partial \mathbf{u}_{2,h}^{\alpha,n}}{\partial \mathbf{n}_2} \Big|_{\Gamma_I} = -\alpha(\mathbf{u}_{2,h}^{\alpha,n} - \mathbf{u}_{1,h}^{\alpha,n}), \quad \mathbf{u}_{2,h}^{\alpha,n} \cdot \mathbf{n}_2 = 0. \quad (6)$$

$$\mathbf{u}_{2,h}^{\alpha,n} \Big|_{\Gamma_2} = 0, \quad (7)$$

and go from n to $n+1$ in calculating $(\mathbf{u}_1^{\alpha,n+1}, P_1^{(n+1)})$ by solving

$$-\nu_1 \Delta \mathbf{u}_1^{\alpha,n+1} + \nabla P_1^{(n+1)} = \mathbf{f}_1, \quad \nabla \cdot \mathbf{u}_1^{\alpha,n+1} = 0, \quad (8)$$

$$\nu_1 \frac{\partial \mathbf{u}_{1,h}^{\alpha,n+1}}{\partial \mathbf{n}_1} \Big|_{\Gamma_{int}} = -\alpha(\mathbf{u}_{1,h}^{\alpha,n+1} - \mathbf{u}_{2,h}^{\alpha,n}), \quad \mathbf{u}_{1,h}^{\alpha,n+1} \cdot \mathbf{n}_1 = 0 \quad (9)$$

$$\mathbf{u}_{1,h}^{\alpha,n+1} \Big|_{\Gamma_1} = 0. \quad (10)$$

Problem 2. Numerical convergence

The simulations were carried out with the software FreeFem ++

α	n
10	4026
10^2	984
10^3	408
10^4	221
10^5	137
10^6	54
10^9	9

To check the numerical convergence of the method, we study the error term $\|\mathbf{U}^{\alpha,n+1} - \mathbf{U}^{\alpha,n}\|_{L^2}$. On the left, for a given α , we have the first value of n for which

$$\|\mathbf{U}^{\alpha,n+1} - \mathbf{U}^{\alpha,n}\|_{L^2} < 10^{-3}.$$

The method is always converging, and the convergence is almost instantaneous for large α ($> 10^6$).

The complete theoretical proof of the convergence is in progress.

Problem 2. Simulations

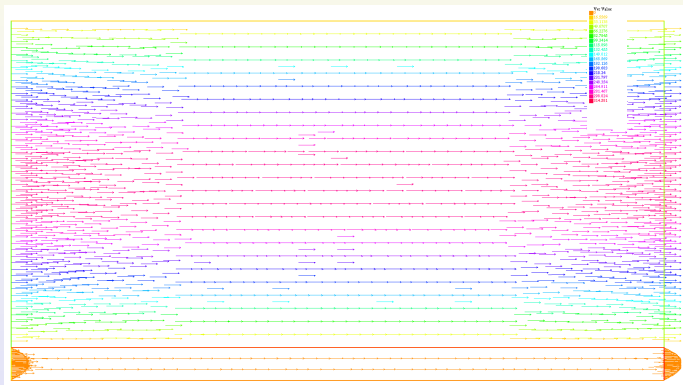


FIGURE – Velocities, $f_1 = (1, -1) = f_2$, $n = 9$, $\alpha = 10^9$