Fast p-adic arithmetic for (hyper)elliptic AGM point counting algorithms

R. Lercier

DGA & University of Rennes — France

email : reynald.lercier(at)m4x.org
www : http://perso.univ-rennes1.fr/reynald.lercier/

Counting Points: Theory, Algorithms and Practice

Montréal, April 2010

Motivation

- In 2000, Satoh and Mestre independently proposed very efficient p-adic methods for counting points on elliptic and hyperelliptic curves in $\mathbb{F}_{p^n}$.
- Numerous improvements finally made decrease the complexity in time from $O(n^{3+o(1)})$ to $O(n^{2+o(1)})$.
- We focus on the choice of good basis for p-adic unramified extensions, especially we consider p-adic analogues of the normal elliptic basis introduced by Couveignes and L. in 2009 for $\mathbb{F}_{p^n}$.

Outline

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3. p-adic Elliptic Periods
   - Normal basis
   - Multiplication Tensor

Elliptic curves

$O(n^{3+o(1)})$ in time, $O(n^3)$ in space:

- T. Satoh. *The canonical lift of an ordinary elliptic curve over a finite field and its point counting*. J. Ramanujan Math. Soc., 15, 2000.
- M. Fouquet, P. Gaudry R.J. Harley. *An extension of Satoh's algorithm and its implementation*. J. Ramanujan Math. Soc., 2000.
- B. Skjernaa, *Satoh's algorithm in characteristic 2*. Math. Comp., 2000.

$O(n^{3+o(1)})$ in time, $O(n^2)$ in space:

- F. Vercauteren, B. Preneel, J. Vandewalle. *A Memory Efficient Version of Satoh's Algorithm*. EUROCRYPT 2001, LNCS.
- J.F. Mestre. *AGM pour le genre 1 et 2*. Lettre à Gaudry et Harley. December 2000.

Elliptic curves

$O(n^{2.5+o(1)})$ time, $O(n^2)$ in space:

- T. Satoh, B. Skjernaa, Y. Taguchi. *Fast computation of canonical lifts of elliptic curves and its application to point counting*. August 2001.
- T. Satoh. *On p-adic point counting algorithms for elliptic curves over finite fields*. ANTS-V. July 2002.
- H.Y. Kim, J.Y. Park, J.H. Cheon, J.H. Park, J.H. Kim, S.G. Hahn. *Fast elliptic curve point counting using gaussian normal basis*. ANTS-V. July 2002.
- P. Gaudry. *A comparison and a combination of SST and AGM algorithms for counting points of elliptic curves in characteristic 2*. ASIACRYPT 2002.

Elliptic curves

$O(n^{2+o(1)})$ in time, $O(n^2)$ in space:

- D. Bernstein. *Re: Elliptic Curve Point Counting: 32003 bits*. Number Theory Mailing List, November 2002.
- R. Lercier, D. Lubicz. *Counting points on elliptic curves over finite fields in quadratic time*. EUROCRYPT 2003.
- R.J. Harley, *Algorithmes avancés pour l'arithmétique des courbes*. PHD thesis, 2003.
- F. Vercauteren, *Computing Zeta functions of curves over finite fields*. PHD thesis, 2003.

Hyperelliptic curves of small genus

Genus 2, $O(n^{3+o(1)})$ in time, $O(n^2)$ in space:

- J.F. Mestre. *AGM pour le genre 1 et 2*. Lettre à Gaudry et Harley, 2000.
- P. Gaudry. *Algorithms for counting points on curves*. ECC, Waterloo, 2001.
- J.F. Mestre. *Algorithmes pour compter des points en petite caractéristique en genre 1 et 2*. Talk at the cryptographic seminar of Rennes, 2002.

$O(n^{2+o(1)})$ in time, $O(n^2)$ in space:

- R. Lercier, D. Lubicz. *A quasi-quadratic time algorithm for hyperelliptic curve point counting*. The Ramanujan Journal, 2006.
- R. Carls, D. Lubicz. *A p-adic quasi-quadratic point counting algorithm*. Int. Math. Res. Journal, 2008.

Outline

p-adic numbers

p-adic norm $|\cdot|_p$ of $r \in \mathbb{Q}^*$ is $|r|_p = p^{-\rho}$ $(r = p^\rho u/v,\ p \nmid u,\ p \nmid v)$.

Field of p-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ w.r.t. $|\cdot|_p$,

$$\sum_{i=\rho}^{\infty} a_i p^i, \quad a_i \in \{0, 1, \ldots, p-1\}, \quad \rho \in \mathbb{Z}.$$

p-adic integers $\mathbb{Z}_p$ is the ring with $|\cdot|_p \leqslant 1$ or $\rho \geqslant 0$.

$\mathbb{F}_p \cong \mathbb{Z}_p/M$ where M is the unique maximal ideal

$$M = \{x \in \mathbb{Q}_p \mid |x|_p < 1\} = p\mathbb{Z}_p\,.$$

Def. Let π_m be the projection from $\mathbb{Z}/p^{m+1}\mathbb{Z}$ onto $\mathbb{Z}/p^m\mathbb{Z}$, then a p-adic integer is a sequence $x = (x_1, x_2, \ldots, x_m, \ldots)$ with $x_m \in \mathbb{Z}/p^m\mathbb{Z}$ and such that $\pi_m(x_{m+1}) = x_m$.

p-adic field extensions

K extension of $\mathbb{Q}_p$ of degree n with
valuation ring $\mathbb{Z}_q$ and maximal ideal $M_{\mathbb{Z}_q} = \{x \in K \mid |x|_K < 1\}$.

Def. The Teichmuller Lift is the map $\omega : \mathbb{F}_q \to \mathbb{Z}_q$ defined by $\omega(0) = 0$ and for $x \neq 0$, $\omega(x)$ is the unique $q-1$-th root of one in $\mathbb{Z}_q$ such that $\pi(\omega(x)) = x$ with π the canonical projection of $\mathbb{Z}_q$ to $\mathbb{F}_q$.

Def. The semi-Witt decomposition of $x \in \mathbb{Z}_q$ is the unique sequence $(x_i)_{i\geq 0}$ of $\mathbb{F}_q$ such that $x = \sum_{i\geq 0} \omega(x_i) p^i$.

The Galois group of (unramified) $K/\mathbb{Q}_p$ is cyclic with generator Frobenius substitution σ and σ modulo $M_{\mathbb{Z}_q}$ equals to the small Frobenius on $\mathbb{F}_q$.

Prop. Let $(x_i)_{i\geq 0}$ be the semi-Witt decomposition of a p-adic x, then $x^\sigma = \sum_{i\geq 0} \omega(x_i)^p p^i$.

Basis

Polynomial Basis. Let $\mathbb{F}_q \cong \mathbb{F}_p[t]/(\overline{F}(t))$, let $F(t)$ be any lift of $\overline{F}(t)$ to $\mathbb{Z}_p[t]$, then K can be constructed as

$$K \cong \mathbb{Q}_p[t]/(F(t)) .$$

Such a choice yields a basis $\{1, t, \ldots, t^{n-1}\}$.

Multiplication, at precision m, costs $T_{m,n} = O((nm)^{1+o(1)})$.

Gaussian Normal Basis (GNB). For cyclic Galois extension $K/\mathbb{Q}_p$, there exists elements α which yields basis of the form $\{\alpha, \alpha^{\sigma}, \ldots, \alpha^{\sigma^{n-1}}\}$.

Def. For some r such that $\exists$ a primitive r-th root of unity γ in $\mathbb{Z}/(nr+1)\mathbb{Z}$ and such that $\alpha = \sum_{i=0}^{r-1} \zeta^{\gamma^i}$ (where $\zeta^{nr+1} = 1$) generates a *gaussian normal basis* over $\mathbb{Q}_p$ of type r.

In this case, $T_{m,n} = O((r\ nm)^{1+o(1)})$.

$O(n^{3+o(1)})$ time complexity

A first algorithm by Satoh, improved by Vercauteren to obtain a $O(n^2)$ in space. Another algorithm by Mestre for $\mathbb{F}_{2^n}$, based on AGM.

Algorithm 1: AGM

input : An (ordinary) elliptic curve $E/\mathbb{F}_{2^n} : y^2 + xy = x^3 + \alpha$
output: The trace c of E

`// Lift phase`
$a := 1 + 8\alpha \in \mathbb{Z}_q$; $b := 1 \in \mathbb{Z}_q$;
for $i := 1$ **to** $\lceil \frac{n}{2} \rceil + 2$ **do**
  $a, b := \frac{a+b}{2}, \sqrt{ab}$

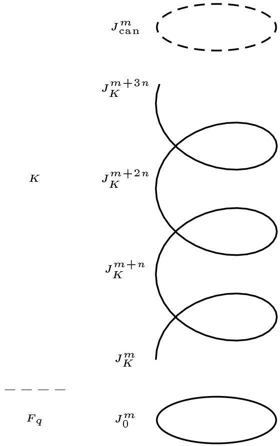
`// Norm phase`
A := a; B := b;
for $i := 1$ **to** n **do**
  $a, b := \frac{a+b}{2}, \sqrt{ab}$
return $\frac{A}{a}$ mod 2^n *as a signed integer in* $[-2\sqrt{2^n}, 2\sqrt{2^n}]$.

AGM iterations

- An AGM step is an isogeny of degree 2 between elliptic curves.
- Repeatedly, we get the following sequence

$$J^1_{K_q} \xrightarrow{\sigma^1} \cdots \xrightarrow{\sigma^{m-1}} J^m_{K_q} \xrightarrow{\sigma^m} \cdots \xrightarrow{\sigma^{m+n-1}} J^{m+n}_{K_q} \quad .$$

- Then, $(J^{m+in}_{K_q})_i$ converges to J^m_{can}, the canonical lift of J^m_0.



$O(n^{2+o(1)})$ time complexity

Lift phase. First,

$$\begin{cases} a_{i+1} & = & \frac{a_i+b_i}{2}, \\ b_{i+1} & = & \sqrt{a_i b_i}, \end{cases}$$

can be replaced via $c_i = a_i/b_i$ by $c_{i+1} = \frac{2+c_i}{2\sqrt{c_i}}$.

Second,

$$c_{i+1} = c_i^{\sigma}.$$

Consequently, one must solve at precision $n/2 + O(1)$,

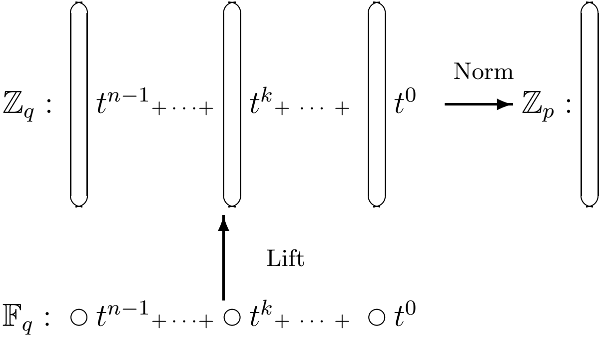
$$\boxed{4x(x^{\sigma})^2 = (1+x)^2}\ .$$

This equation is an equation of the form $\phi(x, x^{\sigma})$ where $\phi(x, y)$ is a polynomial.

Norm phase. We simply have,

$$\boxed{c = N_{\mathbb{Z}_{2^n}/\mathbb{Z}_2}\left(\frac{2c_{\lceil n/2\rceil+3}}{1+c_{\lceil n/2\rceil+3}}\right)}\ .$$

Fast "lift" and "norm" algorithms



Newton iteration

To compute the root of a polynomial $f(x)$ from

$$f(x + p^w \delta) = f(x) + p^w \delta \frac{\partial f}{\partial x}(x) + O(p^{2w}).$$

Algorithm 2: Newton

input : x_0 s.t. $f(x_0) \equiv 0 \bmod p^{2k+1}$ where $k = v(\partial f / \partial x(x_0))$ and $m \in \mathbb{N}$.

output: x a solution of $f(x) \mod p^m$.

if $m \leqslant 2k+1$ **then**
  return x_0

$x := \mathtt{Newton}(x_0, \lceil \frac{m}{2} \rceil + k)$;

$V := f(x) \mod p^m$; $\Delta_x := \partial f / \partial x(x) \mod p^{w-k}$;

return $x - V/\Delta_x$

Remark. Very fast in practice. For polynomials with $O(1)$ terms of degree $O(1)$, time complexity is $O(T_{m,n})$.

Generalized Newton iterations

One generalizes Newton alg. to eq. of the form $\phi(x, x^\sigma) = 0$. Based on

$$\phi(x + p^w\delta, (x + p^w\delta)^\sigma) = \phi(x, x^\sigma) + p^w\delta\frac{\partial\phi}{\partial x}(x, x^\sigma) + p^w\delta^\sigma\frac{\partial\phi}{\partial y}(x, x^\sigma) + O(p^{2w}).$$

Algorithm 3: NewtonLift

input : x_0 s.t. $\phi(x_0, x_0^\sigma) \equiv 0 \mod p^{2k+1}$ where $k = v(\partial\phi/\partial y(x_0))$ and $m \in \mathbb{N}$.
output: x a solution of $\phi(x, x^\sigma) \mod p^m$.

if $m \leqslant 2k+1$ **then**
  return x_0
$w := \lceil \frac{m}{2} \rceil + k$; $x :=$ `NewtonLift`(x_0, w);
Lift x to $\mathbb{Z}_q/p^m\mathbb{Z}_q$; $y := x^\sigma \mod p^m$;
$\Delta_x := \partial_x\phi(x, y) \mod p^{w-k}$; $\Delta_y := \partial_y\phi(x, y) \mod p^{w-k}$;
$V := \phi(x, y) \mod p^m$;
$a, b :=$ `ArtinSchreierRoot`$(-V/(p^{w-k}\Delta_y), -\Delta_x/\Delta_y, w-k, n)$;
return $x + p^{w-k}(1-a)^{-1}b$

Remark. `ArtinSchreierRoot` is a "black box" which solves equations of the form $x^\sigma = ax + b$, a and b in $\mathbb{Z}_q$.

Artin-Schreier equations with Normal Basis

- For all $k \in \mathbb{N}$, $x^{\sigma^k} \equiv a_k x + b_k \mod p^w$.
- $x^{\sigma^n} = x$, which means that $(1 - a_n)x = b_n$.
- A classical "square and multiply" composition formula, $\forall k, k' \in \mathbb{Z}^2$,

$$x^{\sigma^{k+k'}} = a_k^{\sigma^{k'}} a_{k'} x + a_k^{\sigma^{k'}} b_{k'} + b_k^{\sigma^{k'}}.$$

Algorithm 4: ArtinSchreierRoot

input : Eq. $x^\sigma = ax + b$ in $\mathbb{Z}_q/p^m\mathbb{Z}_q$, m and ν in $\mathbb{N}$.
output: A and B s.t. $x^{\sigma^\nu} = Ax + B \mod p^m$.

if $\nu = 1$ **then**
  return a, $b \mod p^m$
$w := \lfloor \nu/2 \rfloor$; A, $B :=$ `ArtinSchreierRoot`(a, b, w);
$A, B := AA^{\sigma^w}, BA^{\sigma^w} + B^{\sigma^w} \mod p^m$;
if $\nu \equiv 1 \mod 2$ **then**
  $A, B := Aa^\sigma, bA^\sigma + B^\sigma \mod p^m$
return A, B;

Complexity is $O(T_{m,n} \log n)$.

Norm computation with Normal Basis

- A square and multiply approach suggested by Kedlaya.
- Combine, from $a_0 = a$, quantities of the form

$$a_{i+1} := a_i^{\sigma^{2^i}} a_i \text{ for } i = 0, \ldots, \lfloor \log_2 n \rfloor \,.$$

Algorithm 5: Norm

input : a in $\mathbb{Z}_q$ and a precision m in $\mathbb{N}$.
output: $\mathrm{N}_{K/\mathbb{Q}_p}(a) \bmod p^m$.

```
i := n; j := 0, r := 1, s := a;
while i > 0 do
    if i ≡ 1 mod 2 then r := s r^(σ^(2^j));
    if i > 1 then s := s s^(σ^(2^j));
    j := j + 1; i := ⌊i/2⌋;
return r;
```

Complexity is $O(T_{m,n} \log n)$.

Timings for counting points on elliptic curves defined over $\mathbb{F}_{2^n}$ (GNB)

On a 731 MHz Alpha EV6 CPU (2002 timings).

n	GNB type 1		
	Lift	Norm	Total
1018	2.5s	1.5s	4s
2052	10s	7s	17s
4098	1mn	45s	1mn 45
8218	6mn 30	4mn 30	11mn
16420	34mn	23mn	57mn
32770	3h 17	2h 18	5h 35
65538	15h 45	13h 20	1d 5
100002	1d 18	1d 16	3d 10

Lifting the Frobenius at precision m [Satoh-Harley]

Computing x^σ in a polynomial basis is a costly task.

One lifts $\overline{F}(t)$ at precision m to the minimal polynomial F of $\omega(t)$ with

$$F(t^p) = \prod_{i=0}^{p-1} F(t\zeta^i) \text{ with } \zeta^p = 1.$$

This can be done by Newton iterations in $O(pT_{m,n} \log n)$.

It follows that $t^\sigma = t^p$ and

$$x^\sigma = \sum_{i=0}^{n-1} x_i t^{ip} = \sum_{j=0}^{p-1} \left(\sum_{0 \leqslant pk+j<n} x_{pk+j} t^k \right) C_j(t) \bmod F(t)\,.$$

With $C_j(t) = t^{jp} \bmod F(t)$ precomputed, a $O(p\, T_{m,n})$ complexity.

Artin-Schreier equations without Normal Basis [Harley-Gaudry]

A two-fold recursive algorithms to doubling the precision.

Algorithm 6: ArtinSchreierRoot

input : Eq. $x^\sigma = ax + b$ in $\mathbb{Z}_q/p^m\mathbb{Z}_q$ with $|b|_K < 1$, m in $\mathbb{N}$.
output: A $x \in \mathbb{Z}_q$ s.t. $x^\sigma = ax + b \mod p^m$.

if $m = 1$ **then**
  return $b^{\bar{\sigma}}$
$N := \lfloor m/2 \rfloor$; $M := m - N$;
$x_0 :=$ `ArtinSchreierRoot`(a, b, N);
$\beta := (x_0^\sigma - ax_0 - b)/p^N \mod p^M$;
$x_1 :=$ `ArtinSchreierRoot`(a, β, M);
return $x_0 + p^N x_1 \mod p^m$

Let $T(n)$ be the running time for precision m, then

$$T(m) \leqslant 2T(m/2) + (pnm)^{1+o(1)} \Rightarrow T(m) = O(p\, T_{m,n} \log m)\,.$$

Norm computation without Normal Basis

For $\alpha \in \mathbb{Q}_p$,

$$\mathrm{N}_{K/\mathbb{Q}_p}(\alpha) = p^{n\,\mathrm{ord}_p(\alpha)}\,\mathrm{N}_{K/\mathbb{Q}_p}(\alpha/p^{\mathrm{ord}_p(\alpha)})\,.$$

For α a unit, let $\alpha = \sum_{i=0}^{n-1} a_i t^i$, then

$$\mathrm{N}_{K/\mathbb{Q}_p}(\alpha) = \mathrm{Res}(F(t), \sum_{i=0}^{n-1} a_i t^i)\,.$$

The resultant $\mathrm{Res}(F(t), \sum_{i=0}^{n-1} a_i t^i)$ can be computed in softly linear time using a variant of Moenck's fast extended GCD algorithm.

Complexity is $O(T_{m,n} \log n)$, mostly due to multiplications of 2×2 matrices with (polynomial) coefficients in $\mathbb{Z}_p[t]$, at precision m.

Harley's timings

Measured on a 750 MHz Alpha EV6 (Nov. 2002, NMBRTHRY mailing list) .

Bits	Point counting		Precomputation	
	Lift	Norm	Lift	Norm
197	0.04	0.04	0.01	-
409	0.26	0.25	0.04	0.01
571	0.76	0.61	0.11	0.02
1000	2.46	1.43	0.35	0.08
2003	15.2	7.71	2.02	0.86
4001	1m 33	52	12	11
8009	9m 30	6m 20	1m 21	2m 09
16001	59m	48m 56	9m 06	31m 42
32003	6h 9m	6h 41m	1h 4m	5h 58m
130020	?	67h 17m	?	?

Remark. Asymptotically fast lifts, but still a $O(n^{2+1/3} \log n \log \log n)$ norm computation (after Satoh).

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Some remarks

It is expected that normal basis (with fast multiplication tensors), even if it does not change the asymptotic complexity, yield faster point counting algorithms :

- it supresses the computation of the lift F in $\mathbb{Q}_p[t]$ of the definition polynomial $\overline{F}(t)$ for $\mathbb{F}_q$,
- it supresses the p factor in the complexity of some parts of the algorithm, especially the `ArtinSchreierRoot` routine,
- it is expected that $\mathbb{Z}_q/\mathbb{Z}_p$ norms can be computed faster.

Maybe more important, we may hope that memory requirements are slightly lowered too.

But, it is hopeless to expect that a Gaussian normal basis of small type r exists for many degree n : in general $r \simeq n^3 \log^2(np)$ [Adleman-Lenstra 1986].

Elliptic Normal Basis (Finite Fields)

For $\mathbb{F}_q$, we made use of torsion points on elliptic curves instead of roots of unity to obtain analogues of Gaussian normal basis.

Theorem (Couveignes-L.)

To every couple (q, n) with q a prime power and $n \geqslant 2$ an integer s.t. $n_q \leqslant \sqrt{q}$, one can associate a normal basis $\Theta(q, n)$ of the degree n extension of $\mathbb{F}_q$ such that the following holds:

- *There exists an algorithm that multiplies two elements given in $\Theta(q, n)$ at the expense of $\tilde{O}(n \log q)$ elementary operations.*

This can be easily extend to a result without any restriction on q and n.

Remark: Here n_q is such that

- $v_\ell(n_q) = v_\ell(n)$ if ℓ is prime to $q - 1$, $v_\ell(n_q) = 0$ if $v_\ell(n) = 0$,
- $v_\ell(n_q) = \max(2v_\ell(q - 1) + 1, 2v_\ell(n))$ if ℓ divides both $q - 1$ and n.

A p-adic generalisation

- Let $E/\mathbb{Q}_p$ be an elliptic curve given by

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3\,.$$

- If A, B and C are three pairwise distinct points in $E(\mathbb{Q}_p)$, we define

$$\Gamma(A,B,C) = \frac{y(C-A) - y(A-B)}{x(C-A) - x(A-B)}\,.$$

- We define a function $u_{A,B} \in \mathbb{Q}_p(E)$ by $u_{A,B}(C) = \Gamma(A,B,C)$.

  It has degree two with two simple poles, at A and B.

Ingredient 1: Residue fields of divisors on elliptic curves

Let E be an elliptic curve defined over $\mathbb{Q}_p$.

- Assume $E(\mathbb{Q}_p)$ contains a cyclic subgroup $\mathcal{T}$ of order n
  (find such a curve mod p and lift it, with $\mathcal{T}$, to $\mathbb{Q}_p$) .
- Let $I : E \to E'$ be the degree n cyclic isogeny with kernel $\mathcal{T}$
- Take a in $E'(\mathbb{Q}_p)$ s.t. $\hat{I}(a) \neq O_E$.
- Let $\mathcal{P}$ be the fibre $I^{-1}(a) = \sum_{t \in \mathcal{T}} [b + t]$, a simple divisor over $\mathbb{Q}_p$.
- Then, $\phi(b) - b \in \mathcal{T}$ (where ϕ is the Frobenius map).

Under some mild condition, $\phi(b) - b$ is a generator of $\mathcal{T}$ and the n geometric points above a are defined on a degree n extension K of $\mathbb{Q}_p$ (and permuted by Galois action).

K is the residue extension of $\mathbb{Q}_p(E)$ at $\mathcal{P}$.

p-adic Elliptic Normal Basis

Coming back to the functions u_{AB}, we choose for A and B consecutive points in $\mathcal{T}$.

For $k \in \mathbb{Z}/d\mathbb{Z}$, we more precisely set

$$u_k = \mathfrak{a} u_{kt,(k+1)t} + \mathfrak{b}$$

($\mathfrak{a}$ and $\mathfrak{b}$, constants chosen such that $\sum u_k = 1$),

and we evalute the u_k's at b.

Lemma (A normal basis)

The system $\Theta = (u_k(b))_{k\in\mathbb{Z}/d\mathbb{Z}}$ is a $\mathbb{Q}_p$ normal basis of K.

Ingredient 2: Relations among elliptic functions

We can prove the following identities (with Taylor expansions at poles)

$$
\begin{aligned}
\Gamma(A,B,C) &= \Gamma(B,C,A) = -\Gamma(B,A,C) - a_1 \\
&= -\Gamma(-A,-B,-C) - a_1\,, \\
u_{A,B} + u_{B,C} + u_{C,A} &= \Gamma(A,B,C) - a_1\,,
\end{aligned}
$$

and

$$
\begin{aligned}
u_{A,B}u_{A,C} &= x_A + \Gamma(A,B,C)u_{A,C} + \Gamma(A,C,B)u_{A,B} \\
&\quad + a_2 + x_A(B) + x_A(C)\,, \\
u_{A,B}^2 &= x_A + x_B - a_1 u_{A,B} + x_A(B) + a_2\,,
\end{aligned}
$$

where

- $\tau_A : E \longrightarrow E$ denotes the translation by A,
- and in $\mathbb{Q}_p(E)$, $x_A = x \circ \tau_{-A}$ and $y_A = y \circ \tau_{-A}$.

A fast multiplication algorithm

$$
\begin{aligned}
u_{A,B}u_{A,C} &= x_A + \Gamma(A,B,C)u_{A,C} + \Gamma(A,C,B)u_{A,B} \\
&\quad + a_2 + x_A(B) + x_A(C)\,, \\
u_{A,B}^2 &= x_A + x_B - a_1 u_{A,B} + x_A(B) + a_2\,.
\end{aligned}
$$

This yields a multiplication tensor for Θ with quasi-linear complexity,

$$
\vec{\alpha} \times \vec{\beta} = (\mathfrak{a}^2 \overrightarrow{\iota}) \star \Big((\vec{\alpha} - \sigma(\vec{\alpha})) \diamond (\vec{\beta} - \sigma(\vec{\beta})) \Big) +
$$

$$
\overrightarrow{u_R}^{(-1)} \star \Big((\vec{u}_R \star \vec{\alpha}) \diamond (\vec{u}_R \star \vec{\beta}) - (\mathfrak{a}^2 \vec{x}_R) \star \Big((\vec{\alpha} - \sigma(\vec{\alpha})) \diamond (\vec{\beta} - \sigma(\vec{\beta})) \Big) \Big)\ .
$$

Notations :

- $\vec{\alpha} \star \vec{\beta}$, the convolution product $(\vec{\alpha} \star_j \vec{\beta})_j$, with $\vec{\alpha} \star_j \vec{\beta} = \sum_i \alpha_i \beta_{j-i}$.
- $\sigma(\vec{\alpha}) = (\alpha_{i-1})_i$, the cyclic shift of $\vec{\alpha}$.
- $\vec{\alpha} \diamond \vec{\beta} = (\alpha_i \beta_i)_i$, the component-wise product.

Evaluations/interpolations

It consists in evaluations and interpolations at n points $r + kt$, where

$$r \in E(\mathbb{Q}_p) - E[d]\,.$$

Constants are

$$\begin{aligned}
\overrightarrow{\iota} &= (\iota_i)_{0\leqslant i\leqslant d-1} \text{ s.t. } x(b) = \textstyle\sum_{0\leqslant k\leqslant d-1} \iota_k \theta_k\,,\\
\vec{x}_R &= (x(r+kt))_{0\leqslant k\leqslant d-1}\,,\\
\vec{u}_R &= (u_0(r+kt))_{0\leqslant k\leqslant d-1}.
\end{aligned}$$

Fast convolutions

- Convolution and polynomial multiplication :

$$F(X) = \sum_{i=0}^{n-1} f_i X^i, \quad G(X) = \sum_{i=0}^{n-1} g_i X^i$$

Then :

$$\vec{h} = \vec{f} \star \vec{g} \Longleftrightarrow H(X) \equiv F(X)G(X) \mod (X^n - 1)$$

- FFT's speedup :

$$\vec{f} \star \vec{g} = \widehat{\hat{\vec{f}} \diamond \hat{\vec{g}}}^{(-1)}$$

Application to normal elliptic basis

$$(\mathfrak{a}^2\overrightarrow{\iota}) \star \Big((\vec{\alpha} - \sigma(\vec{\alpha})) \diamond (\vec{\beta} - \sigma(\vec{\beta}))\Big) +$$
$$\overrightarrow{u_R}^{(-1)} \star \Big((\vec{u}_R \star \vec{\alpha}) \diamond (\vec{u}_R \star \vec{\beta}) - (\mathfrak{a}^2\vec{x}_R) \star \Big((\vec{\alpha} - \sigma(\vec{\alpha})) \diamond (\vec{\beta} - \sigma(\vec{\beta}))\Big)\Big)$$

	"Dense" Polynomial Basis	Normal Elliptic Basis	"Sparse" Polynomial Basis
Product	4+3= 7 FFTs of lg. $2n$ $\simeq$ 14 FFTs of lg. n	3+5= 8 FFTs of lg. n	2+1= 3 FFTs of lg. $2n$ $\simeq$ 6 FFTs of lg. n
Squaring	3+3= 6 FFTs of lg. $2n$ $\simeq$ 12 FFTs of lg. n	2+4= 6 FFTs of lg. n	1+1= 2 FFTs of lg. $2n$ $\simeq$ 4 FFTs of lg. n

- Precompute FFTs for $\overrightarrow{\iota}$, $\overrightarrow{u_R}^{(-1)}$, $\vec{u}_R$ et $\vec{x}_R$,
- 3 direct FFTs, for $\vec{\alpha}$, $\vec{\beta}$ et $(\vec{\alpha} - \sigma(\vec{\alpha})) \diamond (\vec{\beta} - \sigma(\vec{\beta}))$,
- 5 inverse FFTs.

To conclude

It is expected that elliptic normal basis yields faster practical implementations of Satoh/Mestre's algorithms.

Especially, for p large enough such that the Hasse's bound $n \leqslant p + 1 + 2\sqrt{p}$ is satisfied.

For p very small, typ. $p = 2$, it is not clear that the extra $\log n$ penalty to pay for the existence of an elliptic normal basis will be too large.