State-of-the-art in implementing algorithms for the (ordinary) discrete logarithm problem

Antoine JOUX Reynald LERCIER

SCSSI CELAR

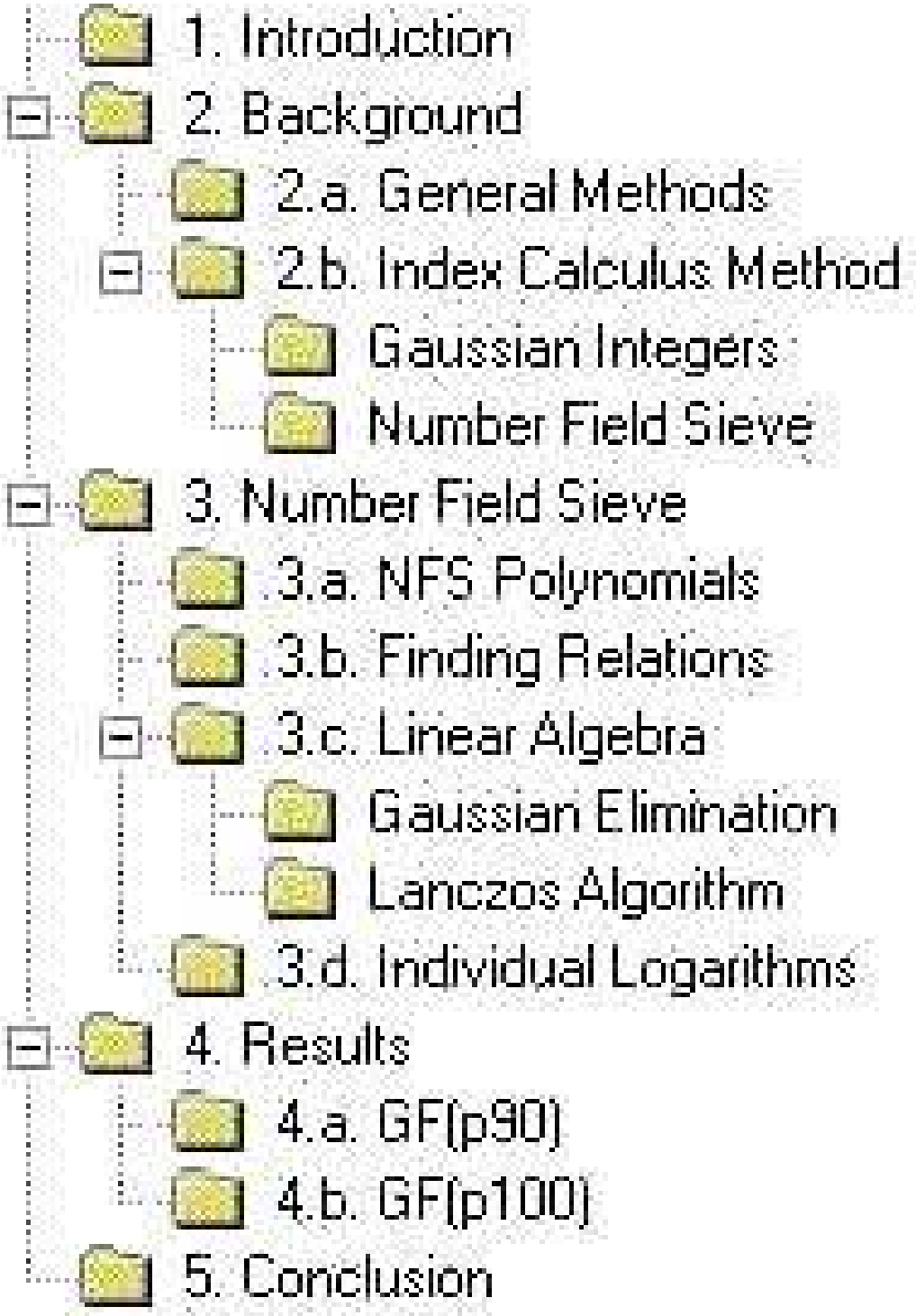
Issy-les-Moulineaux Bruz

Antoine.Joux@ens.fr lercier@celar.fr

FRANCE

November 1999

Plan


1. Introduction
2. Background
2.a. General Methods
2.b. Index Calculus Method
Gaussian Integers
Number Field Sieve
3. Number Field Sieve
3.a. NFS Polynomials
3.b. Finding Relations
3.c. Linear Algebra
Gaussian Elimination
Lanczos Algorithm
3.d. Individual Logarithms
4. Results
4.a. GF(p90)
4.b. GF(p100)
5. Conclusion

Discrete logarithm problem in $\mathbb{F}_p$

Th. : Let p be a prime, then $(\mathbb{F}_p^*, \times)$ is cyclic.

So, there exists an element $g \in \mathbb{F}_p^*$ such that,
for any element $y \in \mathbb{F}_p^*$,
there exists an integer x such that,
$$y = g^x.$$

Notation : $x = \log_g(y)$.

Algorithmical considerations :

Computing y from x is "easy" : algorithms whose complexity is polynomial in $\log(p)$.

Computing x from y is "hard" : algorithms whose complexity is sub-exponential in $\log(p)$.

Cryptographic implications

Computing discrete logarithms is of first importance in the field of cryptography.

Among others, it is :

- a tool for studying shift registers sequences (R. Lidl and H. Niederreiter. *Finite Fields*, volume 20 of *Encyclopedia of Mathematics and its Applications*. Addison–Wesley, 1983).
- the basis for many public-key cryptosystems, the first one being the Diffie Hellman key exchange algorithm
  (W. Diffie and M. Hellman. *New directions in cryptography*, volume 22 of *IEEE Trans, Inform, Theory*. 1976).

General methods

Following Shoup, the best algorithm that one can think of to compute discrete logarithms in a "general group" of prime cardinality $O(p)$ is of complexity at best $O(p^{\frac{1}{2}})$
(V. Shoup, *Lower Bounds for Discrete Logarithms and Related Problems. Advances in Cryptology, EUROCRYPT'97*, volume 1223 of *Lecture Notes in Computer Science*. 1997).

Precisely for $\mathbb{F}_p^*$:

1. Thanks to the Pohlig-Hellman algorithm, computing x is equivalent to compute discrete logarithms in cyclic subgroups whose orders q are divisors of $p-1$.
2. Computing discrete logarithms in cyclic subgroups of $\mathbb{F}_p^*$ can be done in $O(q^{\frac{1}{2}})$ thanks to Baby-Giant steps methods, or better, Pollard-ρ type methods.

Index calculus method

Taking advantage of the specificity of $\mathbb{F}_p^*$, there exists much better algorithms to compute discrete logarithms.

These algorithms are so called "index-calculus" algorithms.

Three main families for $\mathbb{F}_p$.

Original Method : Due to Kraitchik ? or Western and Miller (1968) ? Complexity equal to $L_p[\frac{1}{2}, c]$.

Gaussian Integer method : Due to Coppersmith, Odlyzko and Schroeppel (1986). Complexity equal to $L_p[\frac{1}{2}, 1]$.

General Number Field Sieve : Improved version by Schirokauer (1993) of a Gordon's proposal. Complexity equal to $L_p[\frac{1}{3}, \frac{64}{9}^{\frac{1}{3}}]$.

Here,

$$L_x[\alpha, c] = O\left(e^{(c+o(1))(\log x)^{\alpha}(\log\log x)^{1-\alpha}}\right).$$

Index calculus method
An overview

Basically, two main steps.

Step 1 : One fixes a subset $S = \{p_1, \ldots, p_{|S|}\}$ of $\mathbb{F}_p^*$ called the "factor base" and tries to write elements s of $\mathbb{F}_p$ whose logarithm is known as a product of elements of $|S|$. So, we have equations of the form

$$\log_g s = \sum_{p \in S} e_p \log_g p.$$

When enough such relations are collected, one obtains the quantities $\log_g p$ via a linear inversion modulo $p - 1$.

Step 2 : To find $\log_g y$, one tries random integers z until $g^z y$ is a product of elements of S. Then

$$\log_g y = -z + \sum_{p \in S} e_p \log_g p.$$

Gaussian Integer method

Idea : Mapping between an imaginary quadratic number field and $\mathbb{F}_p$.

Precisely : One chooses a small integer r such that $-r = m^2$ in $\mathbb{F}_p$. Then, we write $m = n/d$ in $\mathbb{F}_p$ with $n, d \simeq O(\sqrt{p})$ and we consider "the number fields" $\mathbb{Q}(\alpha)$, $\mathbb{Q}(\beta)$ respectively defined by

$$f_\alpha(X) = dX - n \text{ and } f_\beta(X) = X^2 + r.$$

Respective integer rings noted $\mathcal{O}_\alpha \simeq \mathbb{Z}$ and $\mathcal{O}_\beta$.

Mapping from the ideals of $\mathcal{O}_\alpha$ to $\mathbb{F}_p$:
Obvious, just $(a + b\alpha) \to a + bm$.

Mapping from the ideals of $\mathcal{O}_\beta$ to $\mathbb{F}_p$:
Easy when the ideal class group of $\mathcal{O}_\beta$ has cardinality 1 since

$$\forall \text{ ideal } \mathfrak{p} \text{ of } O_\beta,\ \exists (a_\mathfrak{p}, b_\mathfrak{p}) \in \mathbb{Z}^2,\ \mathfrak{p} = (a_\mathfrak{p} + b_\mathfrak{p}\beta).$$

Then, the mapping is $\mathfrak{p} \to a_\mathfrak{p} + b_\mathfrak{p} m$.

Gaussian Integer method
Algorithm

Thanks to the previous mappings, one sets

$$S = \{p \in \mathbb{Z}, p \text{ prime } < B_\alpha\} \cup$$
$$\{a_\mathfrak{p} + b_\mathfrak{p} m, \mathfrak{p} \text{ prime ideal of } O_\beta, \mathrm{Norm}(\mathfrak{p}) < B_\beta\}$$
$$\cup \{u \in \mathcal{O}_\beta, \mathrm{Norm}(u) = 1\}.$$

And step 1 is reformulated as follows :

Step 1 : Looks for couples (a, b) such that $(a + b\alpha)$ and $(a + b\beta)$ are both smooth. Then

$$v \prod_{p \in S} p^{e_p} = a + bm = u \prod_{\mathfrak{p} \in S} (a_\mathfrak{p} + b_\mathfrak{p} m)^{e_\mathfrak{p}}.$$

When enough such relations are collected, one obtains the quantities $\log_g p$, $\log_g v$, $\log_g u$ and $\log_g a_\mathfrak{p} + b_\mathfrak{p} m$ via a linear inversion done modulo $p - 1$.

Gaussian Integer method
Records

1991 : 192 bits (58 digits) by B.A. Lamacchia and A.M. Odlyzko, about 2 months on a DEC VAX 8850.

1996 : 85 digits by D. Weber, on about 120 workstations using their idle time of about 30 mips year.

1998 : 90 digits by A. Joux and R. Lercier, about 5 months on a PPro 180MHz based PC.

Number Field Sieve

Idea : Generalizing the Gaussian Integer method to arbitrary Number Fields $\mathbb{Q}(\alpha)$ or $\mathbb{Q}(\beta)$.

We still have

$$(a+b\alpha) = \prod_{\mathfrak{p}\in S} \mathfrak{p}^{e_\mathfrak{p}} \text{ and } (a+b\beta) = \prod_{\mathfrak{p}\in S} \mathfrak{p}^{e_\mathfrak{p}},$$

but ...

Problem : Mapping ideals of $\mathcal{O}_\alpha$ (resp. $\mathcal{O}_\beta$) to $\mathbb{F}_p$ is a difficult task. Mainly because the Unit and Class Groups are no more obvious.

In practice, we cannot even compute them ! This yields a so called Obstruction Group.

Moreover, we can't get rid of ideals and the inversion step done modulo $p-1$ is no more valid as it is.

Fortunately, O. Schirokauer found a way to raise these obstructions by adding to relations what he called "maps".

Number Field Sieve Records

For any $\mathbb{F}_p$

1994 : 25 digits by D. Weber, about 1 day on a SPARC ELC.

1995 : 65 digits by D. Weber, on 130 workstations using their idle time of about 5 mips year.

1998 : 85 digits by D. Weber, on 100 workstations using their idle time of about 44.5 mips year.

1999 : 100 digits by A. Joux and R. Lercier, about 1 year on a Pentiun II 450MHz based PC.

For primes p with a special form

1998 : 129 digits $((739 \cdot 7^{149} - 736)/3)$ by D. Weber, about 1 month on a SPARC 20 workstation for computing *one* discrete logarithm.

Number Field Sieve
Main steps

- Finding "good" GNFS polynomials. New
- Sieving to find relations.
- Inverting the linear system. Improved
- Computing specific logarithms. New

Remark : With such a scheme, there is no more need to restart the linear algebra from scratch whenever a new logarithm must be computed.

NFS polynomials

$f_\alpha(X)$ and $f_\beta(X)$ must satisfy :

- $f_\alpha(X)$ and $f_\beta(X)$ are irreducible over $\mathbb{Z}$,
- $\mathrm{Gcd}(\mathrm{Coefficients}(f_\alpha)) = 1$,
- $\mathrm{Gcd}(\mathrm{Coefficients}(f_\beta)) = 1$,
- $f_\alpha(X) \neq \pm f_\beta(X)$,
- $\exists m \in \mathbb{F}_p^*,\ f_\alpha(m) = f_\beta(m) = 0$.

For efficiency purposes, we try some ideals $(a + b\alpha)$ or $(a + b\beta)$ of norms as small as possible $\Rightarrow$ $f_\alpha(X)$ and $f_\beta(X)$ must have coefficients as small as possible.

But, in any case,

$$\mathrm{Resultant}(f_\alpha(X), f_\beta(X)) = k \cdot p,\ k \in \mathbb{Z}^*.$$

Additional improvements :

- polynomials with real roots ?
- polynomials with many roots modulo small primes ?

Known methods

Gaussian Integer method :

$$f_\alpha(X) = a_1 X + a_0, \;\; f_\beta(X) = X^2 + r,$$

with $a_i \simeq p^{\frac{1}{2}}$ and $r = O(1)$.

Montgomery's method :

$$f_\alpha(X) = a_2 X^2 + a_1 X + a_0,$$
$$f_\beta(X) = b_2 X^2 + b_1 X + b_0,$$

with $a_i, b_i \simeq p^{\frac{1}{4}}$.

Base m method :

$$f_\alpha(X) = X - m, \;\; f_\beta(X) = \sum_{i=0}^{d} b_i X^i,$$

with $m, b_i \simeq p^{\frac{1}{d+1}}$ and $\sum_{i=0}^{d} b_i m^i = p$.

Generalization of the Gaussian Integer method

- Choose a polynomial f_β of degree $d+1$ with very small coefficients and a root m in $\mathbb{F}_p$.
- Let m be a root of $f_\beta(X)$ in $\mathbb{F}_p$.
  Applying LLL algorithm to the lattice

$$\begin{pmatrix} K & Km & K(m^2 \bmod p) & \cdots & K(m^d \bmod p) \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

we obtain a vector $\begin{pmatrix} 0 \\ a_0 \\ \vdots \\ a_d \end{pmatrix}$ with $a_i \simeq p^{\frac{1}{d+1}}$.

Then,

$$f_\alpha(m) = \sum_{i=0}^{d} a_i m^i = 0.$$

NFS relations

Let S_α, S_β, be a set of "small" prime ideals in $\mathcal{O}_\alpha$, $\mathcal{O}_\beta$.

Then, sieving techniques yields $|S_\alpha| + |S_\beta| + O(1)$ equations

$$(a + b\alpha) = \prod_{\mathfrak{p} \in S_\alpha} \mathfrak{p}^{e_\mathfrak{p}}, \ (a + b\beta) = \prod_{\mathfrak{p} \in S_\beta} \mathfrak{p}^{e_\mathfrak{p}}.$$

Thanks to a matrix inversion done modulo $p - 1$, we have

$$\forall \mathfrak{p} \in S_\alpha, \ \mathfrak{p} = I_\mathfrak{p}^{p-1} \prod_{(a,b)} (a + b\alpha)^{e_{a,b}} \text{ and}$$

$$\forall \mathfrak{p} \in S_\beta, \ \mathfrak{p} = I_\mathfrak{p}^{p-1} \prod_{(a,b)} (a + b\beta)^{e_{a,b}},$$

and it remains $|S_\beta| + O(1)$, $|S_\alpha| + O(1)$ supplementary equations given by

$$I^{p-1} = \prod_{(a,b)} (a + b\alpha)^{e_{a,b}}, I^{p-1} = \prod_{(a,b)} (a + b\beta)^{e_{a,b}}.$$

Schirokauer's ideas

For each equation

$$I^{p-1} = \prod_{(a,b)} (a + b\alpha)^{e_{a,b}},$$

one computes $O(1)$ number of linear "maps" $(\lambda_1, \ldots, \lambda_{O(1)})$ defined from $\mathbb{Q}(\alpha)$ to $\mathbb{Z}/(p-1)\mathbb{Z}$.

With a small linear arithmetic done modulo $(p-1)$, it is easy to compute some suitable combinations of the last $O(1)$ equations with any of the first $|S_\beta|$ equations to obtain $|S_\beta|$ equations such that their "map vectors" are equal to

$$(0, \ldots, 0).$$

Then, Schirokauer's results state that,

if Leopoldt's conjecture is true,

such equations are true for algebraic integers,

$$\exists \delta \in \mathcal{O}_\alpha,\ \delta^{p-1} = \prod_{(a,b)} (a + b\alpha)^{e_{a,b}}.$$

Sieving by vectors (1/2)

With sieving methods, instead of testing each principal ideal $(a + b\alpha)$ or $(a + b\beta)$, we mark in a sieving array multiples of small prime ideals $\mathfrak{p}$.

Since the prime ideals $\mathfrak{p}$ we consider are degree one prime ideal, there exists a root ρ of $f_\alpha(X) \bmod \pi$ where $\pi = \mathrm{Norm}(\mathfrak{p})$.

Then, multiples of $\mathfrak{p}$ define a lattice given by

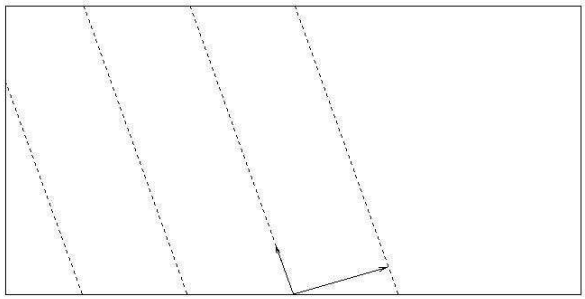
$$\begin{pmatrix} \rho & \pi \\ 1 & 0 \end{pmatrix}$$

Remark : In practice, it's important to use short vectors of this lattice to speed up the sieving.

Sieving by vectors (2/2)

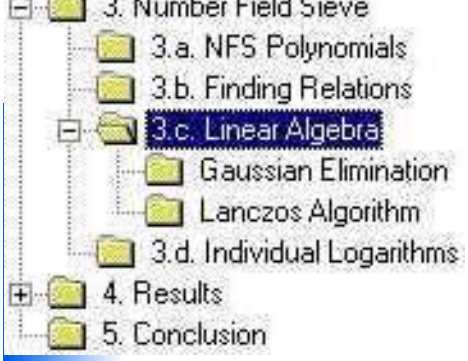
To sieve larger areas, one usually prefers to work with the sub-lattice defined by an ideal $\mathfrak{q}$. This technique is due to Pollard (1993).

Precisely, we only consider algebraic integers $a + b\alpha$ which are multiples of a prime ideal $\mathfrak{q}$. Such algebraic integers are on a lattice $L_{\mathfrak{q}}$.



Then, any multiples of prime ideals $\mathfrak{p}$ are on the sub-lattice $L_{\mathfrak{q}} \cap L_{\mathfrak{p}}$.

Remark : In practice, most of the time needed for such algorithms is spent in memory accesses.



Linear Algebra

At the end of the sieving stage, we have a (huge) matrix to invert. This matrix has 2 parts :

- a large but sparse part with very small entries (mainly ± 1),
- few dense columns whose elements are integers of size p due to Schirokauer's maps.

Two classical phases.

- A Structured Gaussian Elimination (introduced by Lamachia and Odlyzko in 1991) to significantly reduce the dimension of the linear system we have to solve.
- An iterative solver based on Lanczos' algorithm which was slightly modified in order to deal with Schirokauer's maps.

Structured gaussian elimination (1/2)

We propose a refinement which enables us to reduce the dimension of our systems by a factor of 1/5 to 1/10 (instead of 1/3 as it used to be).

We try to minimize the weight increase during a single elimination.

For a potential pivoting entry in a line of weight l and a column of weight c, the estimated variation is equal to

$$(c-1)((l-1)-1)-l = (l-2)(c-2)-2.$$

$\Rightarrow$ One chooses the entry with $(l-2)(c-2)$ minimal as a pivot.

Structured gaussian elimination (2/2)

After each step, $(l-2)(c-2)$ must be recomputed but thanks to an adapted representation of the matrix we can do it incrementally at high speed.

Precisely :

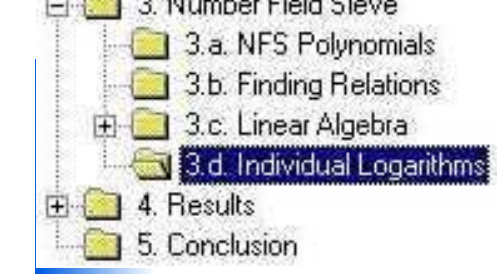
- We store the lines and the columns of the matrix to quickly get l and c.
- For each variable (except those appearing to often), we keep a pointer to the less heavy line which contains it and the corresponding value $(l-2)(c-2)$. We store these datas in a balanced binary tree.

Lanczos algorithm

It mainly consists of the multiplication of the matrix by a vector whose elements are integers of size p.

Therefore :

- Adding few dense columns due to Schirokauer's map implies only few modifications of the algorithm,
- To parallelize it, it's enough to parallelize the product matrix by vector. Which can be done very easily on a shared memory multi-processors computer.
  With n processors, we can expect to speed up the computation by a factor of n or, equivalently, to deal with matrixes $\sqrt{n}$ times larger.
- Most of the time is spent in integer additions.



Individual logarithms

To compute the discrete logarithm of any element $y \in \mathbb{F}_p$, we usually looks for two integers A and B of size $p^{\frac{1}{2}}$ such that

$$y = \frac{A}{B} \bmod p, \ A \text{ and } B \text{ smooth.}$$

The classical approach consists of testing many such couples (A, B) until one appears to be smooth.

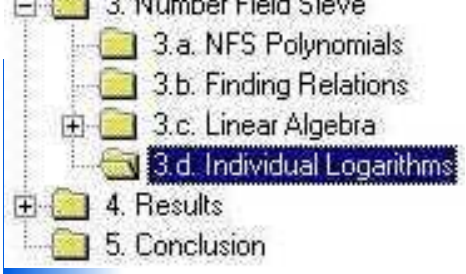
$\Rightarrow$ One can significantly improves the process using sieving techniques.

Individual logarithms using sieving techniques

Once reduced the lattice $\begin{pmatrix} y & p \\ 1 & 0 \end{pmatrix}$, we have integers A_1, B_1, A_2 and B_2 such that

$$\forall(\gamma,\delta)\in\mathbb{Z}^2,\ y=\frac{A_1}{B_1}=\frac{A_2}{B_2}=\frac{\gamma A_1+\delta A_2}{\gamma B_1+\delta B_2} \bmod p.$$

Then, the 'sieving by vectors" method applies to find integers γ and δ such that $\gamma A_1+\delta A_2$ and $\gamma B_1+\delta B_2$ are both smooth.



Individual logarithms using sieving techniques

Remarks :

- We use primes which split in $\mathbb{Q}_\alpha$ for the sieving but these primes are much larger than the factor base.
- Large primes are allowed on both sides.
- Whenever we need the discrete logarithm of a prime, we split it in prime ideals which can be considered as special-$\mathfrak{q}$ in a new sieving

Consequence : A discrete logarithm computation usually turns out to be a "tree" of sieving.

Results

Two discrete logarithm computations :

- Gaussian Integer method for $\mathbb{F}_{p_{90}}$ where $p_{90} = \lfloor 10^{89}\pi \rfloor + 156137$ ($(p_{90}-1)/2$ prime).
- General Number Field Sieve method for $\mathbb{F}_{p_{100}}$ where $p_{100} = \lfloor 10^{99}\pi \rfloor + 3932$ ($(p_{100}-1)/2$ prime).

Questions :

- What is the logarithm of y_{90}, $y_{90}+1$, $y_{90}+2$ and $y_{90}+3$ where $y_{90} = \lfloor 10^{89}e \rfloor$?
- What is the logarithm of y_{100}, $y_{100}+1$, $y_{100}+2$ and $y_{100}+3$ where $y_{100} = \lfloor 10^{99}e \rfloor$?

$\mathbb{F}_{p_{90}}$ (1/2)

Polynomials :

$$f_\beta(X) = X^2 + 2,$$
$$f_\alpha(X) = -2487489975172435043309669560589929434136978 27X + 436356663553236108573801834571320046705446681.$$

$$\Rightarrow \text{Units}(Q_\beta) = \{\pm 1\},\ h_\beta = 1.$$

Smoothness bounds :

$$B_\alpha = 1299709 \text{ (100000 primes)},$$
$$B_\beta = 350377 \text{ (30000 primes)},$$
$$B = 10^9.$$

Sieving : 6663097 equations, 4 months.

Full-Full	Full-Partial	Partial-Full	Partial-P
83519	235921	1377187	4966470

Linear algebra : Removing useless variables and equations $\Rightarrow$ 976062×674564 matrix.

- Structured Gaussian Elimination : 61136×61036, 5683954 entries, 1 day.
- Lanczos Algorithm : 3 weeks.

$\mathbb{F}_{p_{90}}$ (2/2)

Logarithms of small primes :

$$\log_2(3) = 19748300961167147793248409109777288916563744188340553316106369838771165445395903245301984 4,$$
$$\log_2(5) = 14911991564240187116623990082296104898497606799356947584954705853562521193399191674909789 2,$$
$$\vdots$$

Individual logarithms : 9 hours.

$$1299709^{92} y_{90} = A/B \bmod p,$$

with

$$A = 7 \cdot 167 \cdot 347 \cdot 421 \cdot 1049 \cdot 3067 \cdot 3079 \cdot 3319174077 \cdot 29326338174761 \cdot 82044163 \cdot,$$
$$B = 5^2 \cdot 17 \cdot 1187 \cdot 1877 \cdot 2039 \cdot 3019 \cdot 110863 \cdot 207589 \cdot 2903639 \cdot 5397737 \cdot 1064068253.$$

Then,

$$\log_2(y_{90}) = 17671380721142169627320482340716202723020579524499141574938447166779186585383741881010 93.$$

Total time : about 5 months.

$\mathbb{F}_{p_{100}}$ (1/2)

Polynomials :

$$f_\beta(X) = X^3 + 2,$$
$$f_\alpha(X) = 289054697525386277139299623663612X^2 + 347505963979820999503620050239250X + 132570685163002373007872440333058 3$$

$$\Rightarrow \begin{cases} 2 \text{ maps for } \mathbb{Q}(\alpha), \\ \text{Units}(Q_\beta) = \pm < \beta + 1 >, \ h_\beta = 1. \end{cases}$$

Smoothness bounds :

$B_\alpha = 8960467$ (600000 primes),
$B_\beta = 2750161$ (200000 primes),
no large prime.

Sieving : 2900000 equations, 8 months.

Linear algebra : Removing useless variables and equations $\Rightarrow 2592659 \times 816761$ matrix.

- Structured Gaussian Elimination : 172049×171061, 21747682 entries, 1 day.
- Lanczos Algorithm : 3 weeks (quadri-processors DEC alpha 500MHz).

$\mathbb{F}_{p_{100}}$ **(2/2)**

Logarithms of small primes :

$$\log_{67}(3) = 152316999516938936761928948080399004598295465153011753548948051201819212853392012263453297730973896 0,$$

$$\log_{67}(5) = 193775537281718231054135399213826792721338809102012017702986213046313481150182975740143065448645802,$$

$$\vdots$$

Individual logarithms : 1 day.

$$8960453^{48} y_{100} = A/B \bmod p,$$

with

$$\begin{aligned} A &= 2^4 \cdot 3361 \cdot 27107 \cdot 3182657 \cdot 4467119 \cdot \\ &\quad 11770463 \cdot 32918983 \cdot 6211385489629777, \\ B &= 613 \cdot 11317 \cdot 16427 \cdot 257473 \cdot 1188587 \cdot \\ &\quad 95157851 \cdot 124086068723 \cdot 293844018211. \end{aligned}$$

Then,

$$\log_{67}(y_{100}) = 120672227704822985297551351977014036214507903211557925279189901519568404075044457069062361119055307 1.$$

Total time : about 1 year.

Conclusion

- GNFS algorithm seems to be better than the Gaussian Integer Method for computing discrete logarithms modulo primes over 100 digits.
- Main differences with integer factorization :
  - Choosing polynomials.
  - Choice of parameters.
  - Linear algebra.
  - Individual Logarithms.
- The future ?