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Counting the Number of Points on Elliptic Curves over Finite Fields: Strategies and Performances

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Abstract. Cryptographic schemes using elliptic curves over finite fields require the computation of the cardinality of the curves. Dramatic progress have been achieved recently in that field. The aim of this article is to highlight part of these improvements and to describe an efficient implementation of them in the particular case of the field $GF(2^n)$, for $n \leq 500$.

1 Introduction

Elliptic curves have been used successfully to factor integers [23, 33], and prove the primality of large integers [5, 12, 3]. Moreover they turned out to be an interesting alternative to the use of $\mathbb{Z}/N\mathbb{Z}$ in cryptographical schemes [30, 18]. Elliptic curve cryptosystems over finite fields have been built, see [4, 27]; some have been proposed in $\mathbb{Z}/N\mathbb{Z}$, N composite [20, 11, 38]. More applications were studied in [16, 19]. The interested reader should also consult [28].

In order to perform key exchange algorithms using an elliptic curve E over a finite field K, the cardinality of E must be known. The first suggestions in that direction were to use supersingular curves for which the cardinality is easy to compute [30, 18, 15, 4, 27]. But these curves turned out to be disastrous, since the discrete logarithm problem can be reduced to the discrete logarithm problem over an extension field of K of small degree [26]. For non supersingular curves, no reduction algorithm is known in general and the only known attack on such schemes is to use a variant of Pollard's algorithm [13] and this algorithm has exponential running time. Hence, it appears promising to use these curves since we can achieve the same level of confidence one has with $\mathbb{Z}/N\mathbb{Z}$ with much shorter keys.

Two types of finite fields $GF(q)$ have been suggested. The first one considers curves over $GF(p)$ where p is a large prime, the second one curves defined over $GF(2^n)$ where n is some integer. It is possible to use the properties of complex multiplication as stated in [3] to build an elliptic curve with cardinality satisfying some properties [35, 19, 31, 32, 21, 7]. On the other hand, one can use random curves and try to compute its cardinality. It was not until recently that Schoof's polynomial time algorithm for solving this problem could be efficiently implemented and give satisfactory results. The aim of this paper is to give some hints on how this was made possible and to give some precise timings on randomly selected curves.

Since there are industrial applications for elliptic curves over $GF(2^n)$ [13, 28], we will focus on this case. We will briefly compare the running time of our implementation with that of the case $q = p$, p large.

The stucture of this paper is as follows. Section 2 recalls basic facts on elliptic curves. Section 3 describes Schoof's algorithm in a synthetic way using the contributions of Atkin, Elkies, and Couveignes–Morain. We will present some strategies combining these ideas. Some details of the implementation are given in Section 4; precise timings on random curves for various fields are also given.

2 Elliptic curves over finite fields

We recall well known properties of elliptic curves. All these can be found in [41] (see also [28]).

Let $K = GF(q) = GF(p^n)$ be a finite field of characteristic p. The general equation of an elliptic curve is given as:

$$\mathcal{F}(X, Y, Z) := Y^2Z + a_1XYZ + a_3YZ^2 - (X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3) = 0$$

where the a_i's are in K and the discriminant Δ defined by

$$d_2 = a_1^2 + 4a_2, d_4 = 2a_4 + a_1a_3, d_6 = a_3^2 + 4a_6, d_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2,$$

$$c_4 = d_2^2 - 24d_4, \Delta = -d_2^2d_8 - 8d_4^3 - 27d_6^2 + 9d_2d_4d_6$$

is invertible in K. The j-invariant of the curve is $j(E) = c_4^3/\Delta$.

It is possible to define on the set of points $E(K)$ of E

$$E(K) = \{(x, y) \in K^2, \mathcal{F}(x, y, 1) = 0\} \cup \{O_E\}$$

an abelian law using the so-called *tangent-and-chord* method, O_E being the neutral element $(0, 1, 0)$. We refer to the references given above for the precise equations of the law.

Let m denote the cardinality of the set $E(K)$ of points on E. Then, it is well known that $m = q+1-t$ where t is an integer satisfying $|t| \leq 2\sqrt{q}$.

3 Counting the number of points

3.1 Torsion points

Let E be an elliptic curve and let N be an integer. Define $E[N]$ as the set of points of $E(\overline{K})$ of order N. When N is prime to p, then $E[N]$ is isomorphic to $(\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})$ and when $N = p^e$, it is either $\{O_E\}$ or $(\mathbb{Z}/p^e\mathbb{Z})$.

It can be shown that there exists a polynomial $f_N(X)$ in $\mathbb{Q}[a_6][X]$ of degree

$$d_N = \begin{cases} (N^2-1)/2 & \text{if } (N,p) = 1, N \text{ odd} \\ (N^2-4)/2 & \text{if } (N,p) = 1, N \text{ even} \\ (p^{2e}-p^e)/2 & \text{if } N = p^e \end{cases}$$

such that $P = (X, Y, 1)$ is in $E[N]$ if and only if $f_N(X) = 0$ in $\overline{K}$. The polynomial f_N is called *division polynomial*.

3.2 Schoof's algorithm

Schoof's algorithm [39] uses the properties of the Frobenius π_E which maps E onto itself and which sends a point $(X, Y, 1)$ in $E(\overline{K})$ to $(X^q, Y^q, 1)$. It is known that this endomorphism has characteristic equation

$$\pi^2 - t\pi + q = 0 \tag{1}$$

where t is related to the cardinality m of $E(K)$ via $m = q + 1 - t$.

Let ℓ be a prime number. Equation (1) is still valid when π_E is restricted to the group $E[\ell]$, and equivalently

$$\pi^2 - t\pi + q \equiv 0 \bmod \ell. \tag{2}$$

We can find $t_\ell \equiv t \bmod \ell$ by finding which value of τ, $0 \leq \tau < \ell$, satisfies

$$(X^{q^2}, Y^{q^2}) + q(X, Y) = \tau(X^q, Y^q)$$

in $GF(q)[X, Y]/(\mathcal{F}(X, Y, 1), f_\ell(X))$. If we know $t \bmod \ell$ for enough ℓ such that

$$\prod \ell > 4\sqrt{q}$$

then we can determine t using the Chinese remaindering theorem.

3.3 An overview of the improvements of Atkin and Elkies

Though Schoof's algorithm has polynomial running time, its implementation was rather inefficient, due to the size of the polynomials involved. However, Atkin first and then Elkies devised theoretical and practical improvements. We suppose from now on that we want to compute $t_\ell \equiv t \bmod \ell$, ℓ a prime number prime to p.

Firstly, Atkin [1] explained how to use the properties of the modular polynomial $\Phi_\ell(X,Y)$ modulo p to get a list of possible values of t_ℓ. The polynomial $\Phi_\ell(X,Y)$ is symmetric in X and Y and has degree $\ell+1$. The polynomial $\Phi(X) = \Phi_\ell(X, j(E))$ describes the cyclic subgroups of E. It can have basically two splitting in K: $(11r \dots r)$ with $\ell - 1 = rs$ or $(r \dots r)$ with $\ell + 1 = rs$ (there are two particular cases described in the paper which are rare and we omit the relevant details for the sake of simplicity). In the first case, ℓ is said to be an *Elkies prime* and an *Atkin prime* in the second. In each case r is the order of α/β where α and β are the roots of

$$\pi^2 - t\pi + q \equiv 0 \bmod \ell$$

and lie in $GF(\ell)$ if ℓ is an Elkies prime (and thus $t^2 - 4q$ must be a square modulo ℓ) and in $GF(\ell^2)$ otherwise (implying that $t^2 - 4q$ is not a square modulo ℓ). Once r is known, there are $\varphi(r)$ possible values of τ_ℓ and in many cases, this value is much less than ℓ; we denote by $c(\ell)$ the number of possible values of $t \bmod \ell$. It remains to combine these values in a clever way, using a *match & sort* technique described in [1]. This paper contains also many ideas concerning the alternative use of other modular equations, that turn out to be essential in practice, but that we do not want to describe here (for this see also [36]).

Elkies remarked that when $t^2 - 4q$ is a square modulo ℓ, then $f_\ell(X)$ has a factor $g_\ell(X)$ of degree $(\ell-1)/2$. Moreover, π_E has an eigenspace associated with g_ℓ, which means that we now look for some k, $1 \le k < \ell$ such that

$$(X^q, Y^q) = k(X, Y)$$

in $GF(q)[X,Y]/(\mathcal{F}(X,Y,1), g_\ell(X))$. This change was crucial, because it was then possible to use polynomials of degree $(\ell-1)/2$ rather than of degree $(\ell^2-1)/2$. Elkies gave an algorithm to compute g_ℓ using further properties of modular equations. Another approach was given in [8].

Atkin [2] gave his own solution to the problem of computing $g_\ell(X)$ using more modular equations and modular forms. Though rather tricky to implement, his approach is very fast in practice.

Recently, Couveignes and Morain showed how to use powers of small Elkies primes [10].

All these ideas are also decribed in [40] and were implemented [2, 22, 36, 37]. The results are striking, the record being that of the computation of the cardinality of a curve modulo a prime p of 425 digits (annoucement of V. Müller during the Dagstuhl seminar in october, 1994).

The only remaining problem was that these ideas could not work when $\ell > p$, which is the case when $p = 2$. As a matter of fact, the theory of Atkin and Elkies remains valid, but the algorithm used to compute $g_\ell(X)$ cannot work, since it requires using Newton's formulas to recover the coefficients of the polynomial and these formulas involve inverting small numbers in K. Couveignes solved this problem in his thesis [9], using formal groups as a powerful tool. The first successful implementation of these ideas is due to Lercier and Morain [24].

3.4 Couveignes's algorithm

Couveignes's algorithm [9] works in any characteristic $p > 0$. We simplify the exposition in the case $p = 2$.

When ℓ is an Elkies prime, we known that the initial curve

$$E : y^2 + xy = x^3 + a_6, \tag{3}$$

is isogenous to a curve

$$E^* : y^2 + xy = x^3 + a_6^* \tag{4}$$

that can be easily computed. The difficulty lies in the computation of the isogeny I from E to E^* defined by

$$I(x,y) = (U(x), V(x,y)) = \left(\frac{g(x)}{h^2(x)}, \frac{k(x)}{yh^3(x)} \right). \tag{5}$$

Then $h(x)$ is the factor of the division polynomial we are looking for.

Setting $t = -x/y$ and $s = -1/y$, the formal groups defined by (3) is the set of pairs (t,s) satisfying

$$t^3 + ts + a_6 s^3 = s$$

where t and s are formal series in $K((\tau))$. A morphism M from E to E^* satisfies the equality

$$M((t_1(\tau), s_1(\tau)) + (t_2(\tau), s_2(\tau))) = M(t_1(\tau), s_1(\tau)) + M(t_2(\tau), s_2(\tau)) \tag{6}$$

in $K((\tau))$. This equation is not sufficient to get I since there are much more morphisms than isogenies.

Since I can be written as (5), letting $z(\tau) = s(\tau)/t(\tau)$, U is a formal series such that

$$U(\tau) = U(z(\tau)) = z(\tau) \frac{\hat{h}^2(z(\tau))}{\hat{g}(z(\tau))} \tag{7}$$

with $\hat{h}$, a polynomial of degree $(\ell-1)/2$ and $\hat{g}$, a polynomial of degree ℓ. We write $U(\tau) = \tau + \sum_{i=2}^{\infty} u_i \tau^i$ and we find the u_i's coefficient by coefficient. If i is not a power of 2, we look for u_i such that the equality

$$U((\tau, s(\tau)) + (A\tau, s(A\tau))) = (U(\tau), s^*(U(\tau))) + (U(A\tau), s^*(U(A\tau))), \tag{8}$$

holds up to τ^{i+1}, A being a constant in the field chosen as described in [9]; when i is a power of 2, we do the same thing using

$$U((\tau, s(\tau)) + (\tau, s(\tau))) = (U(\tau), s^*(U(\tau))) + (U(\tau), s^*(U(\tau))), \tag{9}$$

We have to compute $4\ell+1$ terms of $U(\tau)$ in order to get $4\ell+2$ terms once substituted in $z^*(\tau) = s^*(t)/t$ and finally have $2\ell+1$ terms as a series in $z(\tau) = s(\tau)/\tau$, to be able to recognize

$$U(z) = \frac{z g(z)^2}{h(z)}$$

with the Massey–Berlekamp algorithm [25].

3.5 A synthetic description of the algorithm

The general algorithm runs as follows: we use two variables M_u and M_l which contain respectively the product of primes ℓ for which τ_ℓ is known and for which τ_ℓ is in some subset of possible values. Typically, M_u contains Elkies primes and M_l Atkin primes. The variable M will contain the current number of combinations to be tried. The general procedure is:

procedure `SEA`(K, E)

1. $\ell := 1$; $M_u := 1$; $M_l := 1$; $M := 1$;
2. **while** $(M_u \times M_l < 4\sqrt{q})$ **or** $(M > \mathcal{M})$ **do**
   (a) $\ell :=$ nextprime(ℓ);
   (b) compute $\Phi(X) = \Phi_\ell(X, j(E))$ and find the number ν of roots of Φ in K;
   (c) **if** $\nu = 2$ (ℓ is an Elkies prime) **then** `ElkiesCase`(ℓ);
   (d) **if** $\nu = 0$ (ℓ is an Atkin prime) **then** `AtkinCase`(ℓ);
3. use the match and sort technique to finish the computations.

The constant $\mathcal{M}$ is a bound on the number of combinations we will have to do in the last step. More details on its choice will be given later.

The core of the computations consists in the two procedures `ElkiesCase` and `AtkinCase`:

procedure `ElkiesCase`(ℓ)

1. compute a factor $g_\ell(X)$ of $f_\ell(X)$ of degree $(\ell-1)/2$ using Atkin's algorithm if $p > \ell$ and Couveignes's if $p < \ell$;
2. find an eigenvalue of π_E related to g_ℓ, i.e., $1 \leq k < \ell$ such that $(X^p, Y^p) = k(X, Y)$ in $GF(q)[X, Y](\mathcal{F}(X, Y, 1), g_\ell(X))$; deduce from this that $t \equiv (k^2 + p)/k \bmod \ell$.
3. $M_u := M_u \times \ell$;

procedure `AtkinCase`(ℓ)

1. find the least r such that $X^{q^r} \equiv X \bmod \Phi$ and set $c(\ell) = \varphi(r)$; $M := M \times c(\ell)$.
2. $M_l := M_l \times \ell$;

3.6 A more elaborate strategie

Let us give a variant of the algorithm we described above using four more constants $\mathcal{A}$, $\mathcal{E}$, $\mathcal{S}$ and $\mathcal{C}$ that will reflect the choice of possible strategies:

(c) **if** $\nu = 2$ **and** $\ell \leq \mathcal{E}$ **then**
  1. `ElkiesCase`(ℓ);
  2. **for** $n := 2$ **while** $\ell^{n-1}d \leq \mathcal{C}$ **do** compute $t \bmod \ell^n$; $M_u := M_u \times \ell$;

else if $(\ell^2 - 1)/2 \leq \mathcal{S}$ **then for** $n := 1$ **while** $(\ell^{2n} - \ell^{2n-2})/2 \leq \mathcal{S}$ **do**
  compute $t \bmod \ell^n$ using Schoof's original algorithm;

else if $\ell \leq \mathcal{A}$ **then** `AtkinCase`(ℓ);

In the above description, d is the order of the eigenvalue k modulo ℓ and the quantity $\ell^{n-1}d$ represents the degree of a factor of $f_{\ell^n}(X)$, see [10]; $(\ell^{2n} - \ell^{2n-2})/2$ is the degree of a factor of $f_{\ell^n}(X)$.

This presentation captures many possible strategies. First of all, setting $\mathcal{E} = \mathcal{A} = 0$ yields Schoof's original algorithm. Setting $\mathcal{E} = 0$ gives Atkin's first algorithm [1]. Introducing $\mathcal{C}$ makes it possible to use the ideas of [10]. We will detail the constants of our implementation in the next section.

4 Implementation and results

4.1 General remarks

We note that almost all the ideas (and tricks) of Atkin are still valid when the characteristic is 2. The first implementation of part of the above ideas is described in [29], which contains many interesting details.

4.2 Basic arithmetic

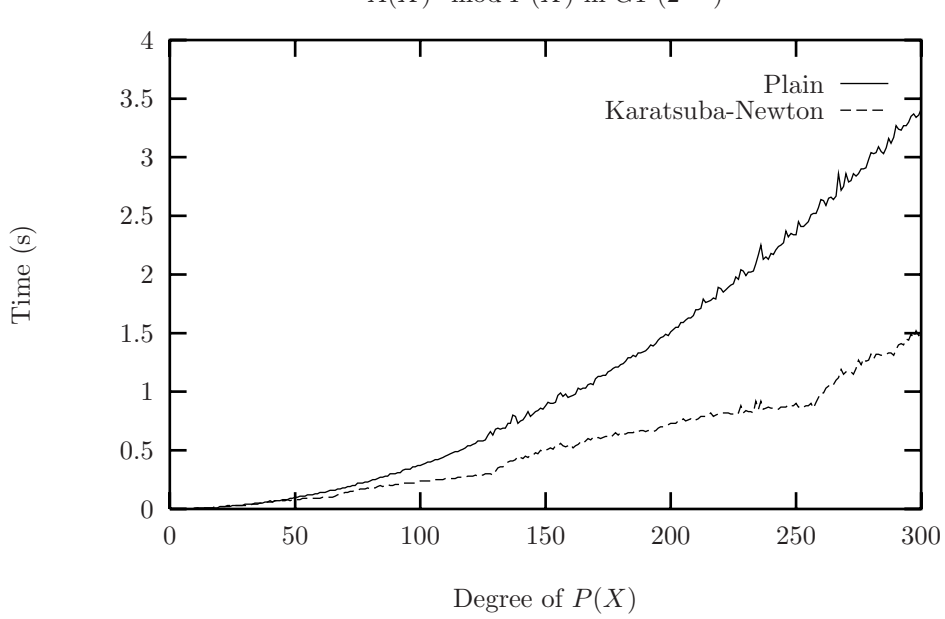
Our implementation is based on the library `GFM` written by F. Chabaud [6] (on top of `BigNum` – cf. [14]), and improved by the authors. It represents $GF(2^n)$ as the residue class ring $GF(2)[T]/(T^n + f(T))$ where $f(T)$ is a polynomial of degree smaller than n such that $T^n + f(T)$ is irreducible over $GF(2)$. In practice – in the range $1 \leq n \leq 500$ – we were always able to find a suitable f of degree less than 15.

The algorithm spends most of the time doing multiplications of elements in the field. To speed up this operation, we first perform the multiplication of two polynomials with coefficients in $GF(2)$ using a table storing all the products PQ of polynomials P and Q of degree at most 7 (at the expense of a storage of 128 kilo-bytes). Then we reduce this polynomial of degree at most $2n - 2$ modulo $T^n + f(T)$ using a second table storing the coefficients of $q(T)f(T)$ for all $q(T)$ of degree smaller than 15 (at the expense of a storage of 256 kilo-bytes too). We give below a table containing precise timings (in seconds) for performing 10^6 operations. All benchmarks have been done on a DecAlpha 3000/500.

n	Squaring	Multiplication	Inversion
65	5.2	27.4	567
89	5.3	30.8	834
105	5.8	33.6	994
155	7.3	63.8	1963
196	8.5	101.6	2835

4.3 Polynomial arithmetic

One of the main costs of the algorithm is the computation of $X^{2^n} \bmod f(X)$ where $f(X) \in GF(2^n)[X]$. As squaring of polynomials of degree d can be performed in $O(d)$ squarings in $GF(2^n)$, we have to improve the reduction of a polynomial $g(X)$ (of degree at most $2d$) modulo a polynomial $f(X)$ (of degree d). This usually costs $O(d^2)$ multiplications in K [17], but can be improved using Newton's method and Karatsuba's algorithm as described for instance in [34]. Precise benchmarks will be given in the final version.



4.4 Timings

In [13], the authors give running times for curves defined over $GF(2^{65})$, $GF(2^{89})$ and $GF(2^{105})$. We used these fields as benchmarks for our implementation. We took the 50 curves defined as $y^2 + xy = x^3 + a_6$ where $a_6 \in GF(2)[T]$ and $2 \leq a_6(2) \leq 51$ (none of such coefficient a_6 belongs to a smaller extension of $GF(2^{65})$, $GF(2^{89})$ and $GF(2^{105})$).

We give: ℓ_{max}, the maximal prime used; the number of U (resp. L) primes; $\#M$, the number of combinations; the cumulated time for X^q, X^{q^r}, Schoof's algorithm; computing g_ℓ and k when ℓ is Elkies; the time for the match and sort program; the total time. For each category, we give the minimal, maximal and average values.

Consider first $K = GF(2^{65})$. In this case, $\mathcal{M} = \infty$.

$\mathcal{E}=\infty, \mathcal{C}=64, \mathcal{S}=84, \mathcal{A}=0$				$\mathcal{E}=0, \mathcal{C}=0, \mathcal{S}=0, \mathcal{A}=\infty$			
	min	max	avg		min	max	avg
$\ell_{\max}$	19	53	35	$\ell_{\max}$	31	31	31
$\#U$	6	9	7	$\#U$	1	3	2
$\#L$	0	10	4	$\#L$	8	10	9
$\#M$	1	1	1	$\#M$	$1.02\,10^3$	$1.47\,10^6$	$2.79\,10^5$
X^q	1.5	18.9	7.2	X^q	4.7	4.9	4.8
X^{q^r}	0.0	0.0	0.0	X^{q^r}	1.1	3.9	3.0
Schoof	0.0	71.5	17.4	Schoof	0.0	0.0	0.0
g	17.4	337.0	106.0	g	0.0	0.0	0.0
k	5.1	27.3	14.7	k	0.0	0.0	0.0
M − S	0.0	0.1	0.1	M − S	0.1	2.1	1.3
Total	26.9	410.0	146.0	Total	7.2	10.5	9.1

These tables show immediately that throwing away Atkin primes is really a bad idea. Playing with the different parameters finally yields the following best results for our three fields. In each case, one has $\mathcal{A} = \infty$:

	$GF(2^{65})$ $\mathcal{E}=2,\mathcal{C}=2,\mathcal{S}=0$			$GF(2^{89})$ $\mathcal{E}=3,\mathcal{C}=4,\mathcal{S}=0$			$GF(2^{105})$ $\mathcal{E}=3,\mathcal{C}=4,\mathcal{S}=24$		
	min	max	avg	min	max	avg	min	max	avg
$\ell_{\max}$	29	29	29	37	41	39	43	43	43
$\#U$	1	4	2	1	6	3	4	6	5
$\#L$	6	9	7	6	12	9	8	10	9
$\#M$	10^3	$3.7\,10^5$	$5.8\,10^4$	$7.7\,10^2$	$2.8\,10^7$	$2.4\,10^6$	$1.5\,10^5$	$7.1\,10^8$	$6.6\,10^7$
X^q	3.8	4.0	3.9	10.2	14.9	12.5	22.4	24.8	23.3
X^{q^r}	1.3	3.2	2.2	3.8	12.0	8.1	11.5	18.3	14.7
Schoof	0.0	0.0	0.0	0.0	0.0	0.0	0.0	30.0	12.9
g	0.0	1.1	0.4	0.0	2.5	1.0	0.0	2.4	1.0
k	0.1	0.2	0.1	0.3	0.6	0.5	0.3	0.8	0.6
M − S	0.1	1.7	1.1	0.2	5.9	2.5	0.5	18.8	5.7
Total	6.1	8.8	7.7	17.9	32.2	24.6	43.0	73.9	58.1

A dynamic strategy When ℓ is an Elkies prime, the cost of procedure `ElkiesCase` turns out to be greater than that of `AtkinCase`. In order to have a program as fast as possible, it is sometimes better to treat an Elkies prime as an Atkin prime. This motivates our *dynamic* strategy. Let L denote the least prime such that $\prod_{\ell \le L} \ell > 4\sqrt{q}$ and denote by $\tilde{c}(\ell)$ an upper bound on $c(\ell)$. An upper bound for the number of combinations is then $\prod_{\ell \le L} \tilde{c}(\ell)$. The program runs as above and as soon as for the current prime ℓ, one has

$$\left(\prod_{l<\ell} c(l)\right)\left(\prod_{\ell \le l \le L} \tilde{c}(l)\right) < \mathcal{M}$$

one decides to treat the remaining primes as Atkin primes.

We can compute an upper bound $\tilde{c}(\ell)$ for $c(\ell)$ as $\tilde{c}(\ell) = \max\{\varphi(r)\}$ where $r \mid \ell - \varepsilon$ and $(q/\ell) = (-1)^{(\ell-\varepsilon)/r}$ for all choices of ε in $\{\pm 1\}$. (Note that $\varepsilon = +1$ if ℓ is an Elkies prime and -1 otherwise.)

The results of this strategy are as follows, with $\mathcal{A} = \infty$ and $\mathcal{E} = \infty$ in all cases:

	$GF(2^{65})$ $\mathcal{C}=16,\mathcal{S}=4,\mathcal{M}=10^6$			$GF(2^{89})$ $\mathcal{C}=16,\mathcal{S}=4,\mathcal{M}=10^8$			$GF(2^{105})$ $\mathcal{C}=32,\mathcal{S}=12,\mathcal{M}=10^{10}$		
	min	max	avg	min	max	avg	min	max	avg
$\ell_{\max}$	29	29	29	37	41	37	41	43	42
$\#U$	2	4	2	2	5	3	2	5	3
$\#L$	6	8	80	7	10	9	9	12	10
$\#M$	10^3	$3.7\,10^5$	$5.3\,10^4$	$3\,10^3$	$2.7\,10^7$	$2.9\,10^6$	$7.4\,10^4$	$9.4\,10^8$	$8.3\,10^7$
X^q	3.3	3.5	3.4	9.0	12.3	9.4	15.6	22.2	20.2
X^{q^r}	1.1	2.7	1.9	3.2	8.6	5.3	8.7	15.9	12.3
Schoof	0.0	3.3	1.8	0.0	3.4	1.7	0.0	11.0	2.5
g	0.0	0.2	0.1	0.0	0.9	0.2	0.0	2.1	0.7
k	0.1	0.2	0.2	0.3	0.9	0.6	0.4	2.2	0.9
M − S	0.1	1.5	1.0	0.2	5.7	2.5	0.6	27.3	6.7
Total	6.0	10.8	8.3	15.4	25.0	19.6	32.0	65.2	43.3

This shows that this strategy is useful in the last case only. The final version of this paper will describe a new strategy, called the *wait-and-see* strategy: first compute and store all $X^q \bmod \Phi$ until enough information can be gathered; then decide which procedure to use so as to minimize the total time needed.

4.5 Comparison with the case $GF(p)$

For the sake of comparisons, we give some timings when using the field $GF(p)$ where p is the least prime greater than 2^{65} (resp. 2^{89}, etc.). We considered the 50 random curves of equation $y^2 = x^3 + x + b$ for

$1 \leq b \leq 50$ for each of these primes. In all cases, $\mathcal{A} = \infty$.

	$p = 2^{65} + 131$ $\mathcal{E} = 7, \mathcal{C} = 0, \mathcal{S} = 0$			$p = 2^{89} + 29$ $\mathcal{E} = \infty, \mathcal{C} = 0, \mathcal{S} = 3$			$p = 2^{105} + 39$ $\mathcal{E} = \infty, \mathcal{C} = 0, \mathcal{S} = 3$		
	min	max	avg	min	max	avg	min	max	avg
ℓ_{max}	29	31	30	41	47	41	43	53	47
$\#U$	2	6	3	5	13	7	4	12	8
$\#L$	5	9	7	1	8	5	3	10	6
$\#M$	$1.7\,10^3$	$6.5\,10^5$	$1.0\,10^5$	8	$1.9\,10^6$	$7.2\,10^4$	$3.2\,10^1$	$2.6\,10^7$	$1.0\,10^6$
X^q	5.0	7.0	6.8	19.1	30.1	20.7	31.7	48.4	39.3
X^{q^r}	1.4	2.8	2.3	0.3	8.2	3.3	0.4	15.7	6.6
Schoof	0.0	0.0	0.0	0.0	0.2	0.1	0.0	0.2	0.1
g	0.0	0.0	0.0	0.1	0.9	0.4	0.1	1.6	0.6
k	0.0	0.1	0.0	0.7	12.9	4.4	1.2	19.8	9.1
M − S	0.3	1.3	0.6	0.4	2.5	0.6	0.5	10.1	1.3
Total	7.4	11.3	10.3	26.0	46.0	30.1	44.0	76.9	58.7

4.6 Records

In [29, 28], the authors gave timings for larger fields $GF(2^{155})$ and $GF(2^{195})$. For these fields and for larger fields (the last one being the current record, as of November 1994), our implementation gave the following timings, for the curve:

$$E_X : y^2 + xy = x^3 + T^{16} + T^{14} + T^{13} + T^9 + T^8 + T^7 + T^6 + T^5 + T^4 + T^3.$$

	$GF(2^{155})$	$GF(2^{196})$	$GF(2^{300})$	$GF(2^{400})$	$GF(2^{500})$	$GF(2^{601})$
ℓ_{max}	59	73	109	173	179	241
$\#U$	6	11	18	26	27	29
$\#L$	11	10	11	13	11	23
$\#M$	$2\,10^7$	10^8	$3\,10^8$	$2.5\,10^9$	$1.3\,10^7$	$2.1\,10^{10}$
X^q	121	440	3221	92643	29137	109708
X^{q^r}	42	127	356	94965	6106	52885
Schoof	0	69	0	186607	65799	240091
g	24	580	22974	1119077	518697	3139250
k	19	141	3613	774895	492213	1392113
M − S	10	23	56	27088	3609	1728
Total	217	1381	30221.1	2511000	1112093	4935776

5 Conclusion

It should be apparent from the preceding tables that the implementation of Schoof's algorithm in characteristic 2 is somewhat slower than in large characteristic. There is still room for many improvements in that direction. We think that the situation might evolve very rapidly soon.

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