RENNES 14/06/11

1. Hopf-formulas of HAMILTON-JACOBI EQUATIONS

We consider the initial-value problem

$$(CP) \quad \begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

MAIN HYPOTHESES

(H1) $H: \mathbb{R}^n \to \mathbb{R}$ convex, $\lim_{|p| \to +\infty} \frac{H(p)}{|p|} = +\infty$

(H2) $g: \mathbb{R}^n \to \mathbb{R}$ Lipschitz continuous.

For fixed $(x,t) \in \mathbb{R}^n \times (0,+\infty)$ we set

$$\mathcal{A}(x,t) = \{ y \in W^{1,\infty}([0,t], \mathbb{R}^n) : y(t) = x \}$$

and consider the problem of minimizing the cost functional

$$J(y;t) = \int_0^t H^*(y'(s))\,ds + g(y(0))$$

$\forall y \in \mathcal{A}(x,t)$.

DEF 1.1

The function $u: \mathbb{R}^n \times [0,t] \to \mathbb{R}$ defined as

$$u(x,t) = \inf_{y \in \mathcal{A}(t,x)} J(y;t)$$

is called the value function of our

minimization problem

THEOREM 1.1 (Hopf formula)

The value function u satisfies

$$u(t,x) = \min_{z \in \mathbb{R}^N} \left\{ t\, H^*\left(\frac{x-z}{t}\right) + g(z) \right\} \quad \text{(HF)}$$

$\forall\, (x,t) \in \mathbb{R}^N \times [0,+\infty)$.

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Remark 1

The inequality $\geq$ (HF) follows from the convexity and the fact that the geodesic are straight lines:

Indeed:

"$\geq$" Define $z = x - \int_0^t z(s)\,ds$

Then $u(x,t) \overset{*}{\geq} \inf_{z(\cdot)} \left\{ t\, H^*\left(\frac{1}{t}\int_0^t z(s)\,ds\right) + g(z) \right\}$

$$= \inf_z \left\{ t\, H^*\left(\frac{x-z}{t}\right) + g(z) \right\}.$$

$*$ Jensen's Inequality.

"$\leq$" Consider constant controls $z(\cdot)$ of the form $z(s) = \frac{x-z}{t}$ $\quad 0 \leq s \leq t$

We obtain

$$u(x,t) \leq \inf_z \left\{ t\, H^*\left(\frac{x-z}{t}\right) + g(z) \right\}$$

Remark 2

LEMMA 1.1 (FUNCTIONAL IDENTITY)

For each $x \in \mathbb{R}^n$ and $0 \leq s \leq t$ we have

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ (t-s)\, H^*\!\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}.$$

(it is nothing that the DYNAMIC PROGRAMMING PRINCIPLE).

LEMMA 1.2 (Lipschitz continuity)

The function u is Lipschitz continuous in $\mathbb{R}^n \times [0,+\infty)$ and

$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}.$$

THEOREM 1.2 (Hopf-Lax formula as solution).

The function u defined by the Hopf-Lax formula is Lipschitz continuous, is differentiable a.e in $\mathbb{R}^n \times (0,+\infty)$ and solves

$$\begin{cases} u_t + H(Du) = 0 & \text{a.e } (x,t) \in \mathbb{R}^n \times (0,+\infty) \\ u(x,0) = g(x) & x \in \mathbb{R}^n \end{cases}$$

the property of solving (CP) almost everywhere, is not enough to characterize the value function - Indeed such a problem can have more than one solution in the class of Lipschitz continuous functions as the the next example shows.

EXAMPLE 1.1

The problem

$$\begin{cases} u_t + u_x^2 = 0 & (x,t) \in \mathbb{R} \times [0,+\infty) \\ u(x,0) = 0 & x \in \mathbb{R}. \end{cases}$$

admits the solution $u \equiv 0$. However for any $a > 0$, the function u_a defined as

$$u_a(x,t) = \begin{cases} 0 & \text{if } |x| \geq at \\ a|x| - a^2 t & \text{if } |x| < at \end{cases}$$

is a Lipschitz function satisfying the equation a.e together its initial condition.

The above example show that the property of solving the equation a.e is too weak and does not suffice to provide a satisfactory notion of generalized solution. It is therefore desirable to find some additional conditions which ensures uniqueness and characterizes the Lipschitz continuous

solutions of the equation.

2.1 SEMICONCAVITY OF HOPF'S SOLUTION

In this section we consider a further regularity property of the solution to (CP) given by Hopf's formula, known as SEMICONCAVITY. We first give a definition.

DEFINITION 2.1

Let S be a subset of $\mathbb{R}^n$. We say that a function $u : S \to \mathbb{R}$ is semiconcave with linear modulus if there exists $c > 0$ such that

$$(*) \quad \lambda u(y_1) + (1-\lambda) u(y_2) - u(\lambda y_1 + (1-\lambda) y_2) \leq \lambda(1-\lambda) C_1 |y_1 - y_2|$$

for any pair $y_1, y_2 \in S$ such that $[y_1, y_2] \subseteq S$ and for any $\lambda \in [0,1]$.

PROPOSITION 2.1

Given $u : A \to \mathbb{R}$, with $A \subseteq \mathbb{R}^n$ open set, and given $c \geq 0$, the following properties are equivalent:

a) inequality $(*)$ is satisfied.

b) the function $y \mapsto u(y) - C_1 |y|^2$ is concave in every convex subset of A;

c) $u \in C(A)$ and satisfies $u(y+h) + u(y-h) - 2u(y) \leq 2C_1 |h|^2$ for any y, h such that $[y-h, y+h] \subseteq A$;

d) $\forall \nu \in \mathbb{R}^n$, $|\nu| = 1$ we have $\partial^2_{\nu\nu} u \leq 2C$ in the sense of distributions.

LEMMA 2.1

Suppose there exists a constant c such that

$$g(x+z) - 2g(x) + g(x-z) \leq c|z|^2$$

$\forall\ x, z \in \mathbb{R}^N$. Then u defined by the Hopf's formula satisfies:

$$u(x+z,t) - 2u(x,t) + u(x-z,t) \leq c|z|^2$$

for all $x, z \in \mathbb{R}^N$, $t > 0$.

REMARK 1: One can show that u is semiconcave also w.r.t. t. (see COR. 1.5.4 CANNARSA & SINESTRARI).

PROOF OF LEMMA 2.1

Choose $y \in \mathbb{R}^n$ so that $u(x,t) = tH^*\left(\frac{x-y}{t}\right) + g(y)$. Then putting $y+z$, $y-z$ in the Hopf-Lax formulas for $u(x+z,t)$ and $u(x-z,t)$ we find.

$$u(x+z,t) - 2u(x,t) + u(x-z,t) \leq$$

$$\leq \left[t\, H^*\left(\frac{x-y}{t}\right) + g(y+z)\right] - 2\left[t\, H^*\left(\frac{x-y}{t}\right) + g(y)\right]$$

$$+ \left[t\, H^*\left(\frac{x-y}{t}\right) + g(y+z)\right]$$

$$\leq g(y+z) - 2g(y) + g(y-z) \leq c|z|^2 \quad \blacksquare$$

DEFINITION 2.1

A C^2 convex function $H: \mathbb{R}^n \to \mathbb{R}$ is called UNIFORMLY convex (with constant $\theta>0$) if

$$\sum_{i,j=1}^{n} H_{p_i p_j}(p)\xi_i\xi_j \geq \theta|\xi|^2 \quad \forall p,\xi \in \mathbb{R}^n.$$

We now prove that even if g is not semiconcave, the uniform convexity of H forces u to become semiconcave for $t>0$: this is a kind of mild regularizing effect for the Hopf-Lax solution of (CP).

LEMMA 2.2

Suppose H is uniformly convex (with the constant $\theta>0$) and u defined by the Hopf-Lax formula. Then

$$u(x+z,t)-2u(x,t)+u(x-z,t) \leq \frac{1}{\theta t}|z|^2 \quad \forall x,z \in \mathbb{R}^n,\ t>0.$$

PROOF

By applying Taylor formula and the uniform convexity we get

$$H\left(\frac{p_1+p_2}{2}\right) \leq \frac{1}{2}H(p_1)+\frac{1}{2}H(p_2)-\frac{\theta}{8}|p_1-p_2|^2$$

CLAIM H^* satisfies the estimate

$$\frac{1}{2}H^*(q_1)+\frac{1}{2}H^*(q_2) \leq H^*\left(\frac{q_1+q_2}{2}\right)+\frac{1}{8\theta}|q_1-q_2|^2$$

Indeed

$$H^*(z) = \sup_{y \in \mathbb{R}^n} \{ z \cdot y - H(y) \}$$

Thus $H^*(z) \geq z \cdot y - H(y) \quad \forall y, \forall z$

In particular $H(y) \geq z \cdot y - H^*(z) \quad \forall y, \forall z$.

Let $\varepsilon > 0$ and q_1, q_2 be such that

$$H(p_1) \leq - H^*(q_1) + q_1 \cdot p_1 + \varepsilon$$

$$H(p_2) \leq - H^*(q_2) + q_2 \cdot p_2 + \varepsilon .$$

Let $h(p) = \frac{\theta}{8} |p|^2$. We have

$$h^*(q) = \sup_q \{ p \cdot q - \frac{\theta}{8} |q|^2 \} = \frac{2}{\theta} |q|^2$$

Thus

$$- H^*\left(\frac{q_1 + q_2}{2}\right) + \left(\frac{q_1 + q_2}{2}\right)\left(\frac{p_1 + p_2}{2}\right) - \frac{2}{\theta} \left|\frac{q_1 - q_2}{4}\right|^2$$

$$+ \left(\frac{q_1 - q_2}{4}\right)(p_1 - p_2) \leq - \frac{H^*(q_1)}{2} + \frac{q_1 p_1}{2} + \frac{\varepsilon}{2}$$

$$- \frac{H^*(q_2)}{2} + \frac{q_2 p_2}{2} + \frac{\varepsilon}{2}$$

$$\Rightarrow H^*\left(\frac{q_1 + q_2}{2}\right) + \frac{1}{8\theta} |q_1 - q_2|^2 \geq \frac{1}{2} H^*(q_1) + \frac{1}{2} H^*(q_2) + \varepsilon .$$

Let $\varepsilon \to 0$ and we prove the claim. $\square$

Now choose y so that $u(x,t) = t\, H^*\left(\frac{x-y}{t}\right) + g(y)$.

Then using the same value of y in the Hopf - Lax formula for $u(x+z,t)$ and $u(x-z,t)$ we get

$$u(x+z,t) - 2u(x,t) + u(x-z,t) \leq$$

$$\leq \left[t\, H^*\left(\frac{x+z-y}{t}\right) + g(y)\right] - 2\left[t\, H^*\left(\frac{x-y}{t}\right) + g(y)\right]$$

$$+ \left[t\, H^*\left(\frac{x-z-y}{t}\right) + g(y)\right] \leq 2t\,\frac{1}{8\theta}\left|\frac{2z}{t}\right|^2 \leq \frac{|z|^2}{\theta t}.$$

$\blacksquare$

We now show that the semiconcavity property singles out Hopf's solution among all Lipschitz continuous solutions of the HJ equation.

<u>THEOREM 2.1</u>

Let $H \in C^2$ be convex and let $u_1, u_2 \in Lip(\mathbb{R}^n \times [0,$ be solution of (CP) such that $\forall t > 0$ and $i = 1, 2$,

$$u_i(x+h,t) + u_i(x-h,t) - 2u_i(x,t) \leq C\left(1+\frac{1}{t}\right)|h|^2 \qquad \forall x, h \in \mathbb{R}^n$$

for some $C > 0$. Then $u_1 = u_2$ in $\mathbb{R}^n \times [0,+\infty)$.

We recall an elementary algebraic property.

LEMMA 2.3

Let A, B be two symmetric $n\times n$ matrices. Suppose that $0 \le A \le \Lambda I$ and $B \le k I$ for some $\Lambda, k > 0$. Then

$$\operatorname{trace}(AB) \le n k \Lambda .$$

PROOF.

Let $\{e_1, .., e_n\}$ be an orthonormal basis of eigenvectors of B and let $\{\mu_1, .., \mu_n\}$ be the corresponding eigenvalues. Then our assumptions imply

$$\mu_i \le k \ , \quad 0 \le \langle e_i, A e_i\rangle \le \Lambda \quad \forall i = 1, .., n$$

It follows

$$\operatorname{trace}(AB) = \sum_i \langle e_i, AB e_i\rangle = \sum_i \mu_i \langle e_i, A e_i\rangle$$

$$\le k \sum_{i=1}^{n} \langle e_i, A e_i\rangle \le n k \Lambda$$

PROOF OF THEOREM 2.1

Setting $\bar{u}(x,t) = u_1(x,t) - u_2(x,t)$ we have

$$\bar{u}_t(x,t) = H(Du_2(x,t)) - H(Du_1(x,t)) = - b(x,t)\nabla\bar{u}(x,$$

where

$$b(x,t) = \int_0^1 DH(r Du_2(x,t) + (1-r) Du_1(x,t))\,dr$$

Let $\Phi : \mathbb{R} \to [0, +\infty]$ be a C^1 function to be fixed later. Setting $v = \Phi(\bar{u})$ we obtain

for a.e (t,x)

$$v_t(x,t) + b(x,t) \cdot Dv(x,t) = 0.$$

Let $\tau \in C^\infty(\mathbb{R}^n)$ be a nonnegative function, $\operatorname{supp} \tau \subseteq B(0,1)$ and $\int_{\mathbb{R}^n} \tau \, dx = 1$.

We define

$$u_i^\varepsilon(x,t) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} u_i(y,t)\, \tau\left(\frac{x-y}{\varepsilon}\right) dy$$

$$= u_i * \tau^\varepsilon$$

$$\tau^\varepsilon = \frac{1}{\varepsilon^n} \tau\left(\frac{x}{\varepsilon}\right).$$

We have : $u_i^\varepsilon \in C^\infty$ with respect to x and satisfy

i) $|Du_i^\varepsilon| \leq \operatorname{Lip}(u_i)$

ii) $\forall t>0$, $Du_i^\varepsilon \to Du_i$ a.e. as $\varepsilon \to 0$.

In addition u_i^ε satisfies the semiconcavity estimate $\forall \varepsilon > 0$

Therefore

$$D^2 u_i^\varepsilon(x,t) \leq C\left(1+\frac{1}{t}\right) I \quad \forall (x,t) \in \mathbb{R}^n \times (0,+\infty)$$

Setting

$$b_\varepsilon(x,t) = \int_0^1 DH\left(r\, Du_2^\varepsilon(x,t) + (1-r)\, Du_1^\varepsilon(x,t)\right) dr$$

We can rewrite the equation $v_t + b\,Dv = 0$ in the form

$$v_t + \operatorname{div}(v b_\varepsilon) = (\operatorname{div} b_\varepsilon)\, v + (b_\varepsilon - b)\, Dv$$

Let us set

$$R = \max\{\, |DH(p)| : |p| \leq \max(\mathrm{Lip}(u_1), \mathrm{Lip}(u_2)) \}$$

$$\Lambda = \max\{\, \|D^2H(p)\| : |p| \leq \max(\mathrm{Lip}(u_1), \mathrm{Lip}(u_2)) \}$$

We have

$$\operatorname{div} b_\varepsilon = \int_0^1 \sum_{k,l=1}^{n} \frac{\partial^2 H}{\partial p_k \partial p_l}\left(r Du_2^\varepsilon + (1-r) Du_1^\varepsilon\right)\left(r \frac{\partial^2 u_2^\varepsilon}{\partial x_k \partial x_l} + (1-r)\frac{\partial^2 u_1^\varepsilon}{\partial x_k \partial x_l}\right) dr$$

$$\leq n \Lambda C \left(1 + \frac{1}{t}\right).$$

(we have applied Lemma 2.3 and the fact that u_i^ε are semiconcave).
Given $(x_0, t_0) \in \mathbb{R}^n \times [0, +\infty)$; let us introduce the function

$$E(t) = \int_{B(x_0, R(t_0 - t))} v(x,t)\,dx \qquad 0 \leq t \leq t_0$$

and the cone

$$\mathcal{C} = \{ (x,t) : 0 \leq t \leq t_0,\ x \in B(x_0, R(t_0 - t)) \}$$

Then E is Lipschitz continuous and satisfies for a.e $t>0$,

$$E'(t) = \frac{d}{dt}\int_0^{R(t_0-t)}\int_{\partial B_\rho(x_0)} v(x,t)\, d\sigma\, dt$$

$$= -R\int_{\partial B(x_0,R(t_0-t))} v(x,t)\, d\sigma + \int_{B(x_0,R(t_0-t))} v_t\, dx$$

$$= -\int_{\partial B(x_0,R(t_0-t))} R\, v(x,t)\, d\sigma + \int_{B(x_0,R(t_0-t))} \{-\operatorname{div}(v b_\varepsilon) + (\operatorname{div} b_\varepsilon) v + (b_\varepsilon - b) Dv\} dx$$

$$= -\int_{\partial B(x_0,R(t_0-t))} v\,(R + b_\varepsilon\cdot\nu)\, d\sigma + \int_{B(x_0,R(t_0-t))} (\operatorname{div} b_\varepsilon) v + (b_\varepsilon - b)\cdot Dv\, dx$$

$$\leq \int_{B(x_0,R(t_0-t))} \{(\operatorname{div} b_\varepsilon) v + (b_\varepsilon - b)\cdot v\}\, dx$$

$$\leq n\Lambda C\left(1+\frac{1}{t}\right)E(t) + \int_{B(x_0,R(t_0-t))} (b_\varepsilon - b)\cdot v\, dx.$$

Let $\varepsilon \to 0$ and we get

$$E'(t) \leq n\Lambda C\left(1+\frac{1}{t}\right)E(t) \quad \forall t\in\,]0,t_0[\ \text{a.e.}$$

We know choose Φ in the definition of v. We fix $\tau>0$ and define Φ such that $\Phi(z) = 0$ if $|z| \leq \tau\,(Lip(u_1)+Lip(u_2))$ and $\Phi(z)>0$ otherwise.

Then the assumption that $u_1 = u_2$ for $t=0$ implies that $\sigma(t,x)=0$ if $t \le \tau$.
Thus by applying Gronwall's inequality,

$$0 \le E(t) \le E(\tau) \exp \int_\tau^t n \lambda C \left(1+\frac{1}{s}\right) ds = 0$$

$\Rightarrow$ $E(t) = 0$ $\forall t \in [0, t_0]$. Hence

$$|u_2 - u_1| \le \tau \left(\mathrm{Lip}(u_1) + \mathrm{Lip}(u_2)\right) \text{ on } \mathcal{C}.$$

By the arbitrary of $\tau > 0$ we deduce that $u_1 \equiv u_2$ in $\mathcal{C}$ and in particular $u_1(x_0,t_0) = u_2(x_0,t_0)$

$\square$

COROLLARY 2.1.
Let H, g satisfy (H1), (H2). Suppose in addition that $H \in C^2(\mathbb{R}^n)$ and that either H is uniformly convex or g is semiconcave. Then there exists a unique $u \in \mathrm{Lip}([0,\infty) \times \mathbb{R}^n)$ which solves (CP) a.e. and which satisfies

$$u(x+h,t) - 2u(x,t) + u(x-h,t) \le C'\left(1+\frac{1}{t}\right)|h|^2$$
$$x, h \in \mathbb{R}^n,\ t>0.$$

for some $C'>0$. In addition u is given by Hopf's formula taking the Legendre transform of H.

EXAMPLES

1) Consider

$$(*)\begin{cases} u_t + \frac{1}{2}|Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u = |x| & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Here $H(p) = \frac{1}{2}|p|^2$ and $H^*(q) = \frac{1}{2}|q|^2$.
The Hopf-Lax formula for the unique weak solution of $(*)$ is

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ \frac{|x-y|^2}{2t} + |y| \right\}$$

Assume $|x| > t$. Then

$$D_y\left(\frac{|x-y|^2}{2t} + |y|\right) = \frac{y-x}{t} + \frac{y}{|y|} \qquad y \neq 0$$

and this expression equals if $x = y + \frac{y}{|y|}t$,
Thus $u(x,t) = |x| - \frac{t}{2}$ if $|x| > t$. If $|x| \le t$ the min. is attained at $y = 0$. Consequently

$$u(x,t) = \begin{cases} |x| - \frac{t}{2} & \text{if } |x| \geq t \\ \frac{|x|^2}{2t} & \text{if } |x| \le t \end{cases}$$

Observe that the solution becomes semiconcave at times $t > 0$ even though the initial function $g(x) = |x|$ is not semiconcave

3. Hopf-Lax formula revisited.

Consider

$$(CP) \quad \begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,T] \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

THEOREM 3.1 (Hopf-Lax formula as viscosity solution)

Assume (H1), (H2). Then u is a viscosity solution of (CP).

PROOF 1. Let $\varphi \in C^\infty(\mathbb{R}^n \times (0,+\infty))$ and assume that $u - \varphi$ has a local max at $(x_0,t_0) \in \mathbb{R}^n \times (0,+\infty)$. According to LEMMA 1.1 we have

$$u(x_0,t_0) = \min_{x \in \mathbb{R}^n} \left\{ (t_0-t) H^*\left(\frac{x_0-x}{t_0-t}\right) + u(x,t) \right\}$$

for each $0 \le t < t_0$. Thus $\forall\, 0 \le t < t_0,\ x \in \mathbb{R}^n$

$$u(x_0,t_0) \le (t_0-t) H^*\left(\frac{x_0-x}{t_0-t}\right) + u(x,t).$$

Since $u-\varphi$ has a local max at (x_0,t_0) we have

$$u(x_0,t_0) - \varphi(x_0,t_0) \ge u(x,t) - \varphi(x,t)$$

$$\forall\, (x,t) \in B((x_0,t_0),r)$$

Combining the above estimates we find

$$\varphi(x_0,t_0) - \varphi(x,t) \le (t_0-t) H^*\left(\frac{x_0-x}{t_0-t}\right)$$

Assume that (x,t) is a point of differentiability of u. Then there exists a sequence $(x_n,t_n) \to (x,t)$ such that u is differentiable at (x_n,t_n) and

$$u_t(x_n,t_n) + H(Du(x_n,t_n)) \geq 0 \quad \forall n$$

Since u is locally Lipsch. cont, up to subsequences

$$(D(x_n,t_n), u_t(x_n,t_n)) \to (Du\cdot(x,t), u_t(x,t))$$

so that, letting $n \to +\infty$ we get

$$u_t(x,t) + H(Du(x,t)) \geq 0 \quad \forall p \in D^{-}u(x)$$

$\blacksquare$

<u>Remark</u>: A generalized semiconcave solution of (HJE) satisfies $u_t + H(Du) = 0$ $\forall (x,t)$ & $\forall p \in D^{-}_{(x,t)}u(x,t)$.