UNIVERSITE PIERRE & MARIE CURIE
EXCELLENCE A PARIS

PMA

Institut Formation Doctorale

THESE DE DOCTORAT DE L'UNIVERSITE PIERRE ET MARIE CURIE

Spécialité :
Mathématiques

Présentée par

Nathalie Krell

Pour obtenir le grade de

DOCTEUR EN SCIENCES DE L'UNIVERSITÉ PIERRE ET MARIE CURIE

Sujet de thèse

Quelques développements récents en théorie des fragmentations.

Soutenue publiquement le **30 juin 2008** devant le jury composé de

M. Jean	**BERTOIN**	Université Paris–VI	Directeur
M. Yuri	**GOLUBEV**	Université de Provence	Rapporteur
M. Marc	**HOFFMANN**	Université Paris Est	Directeur
M. Andreas	**KYPRIANOU**	University of Bath	Rapporteur
M. Alain	**ROUAULT**	Université de Versailles	Président du jury
M. Lorenzo	**ZAMBOTTI**	Université Paris–VI	Examinateur

Remerciements

Je tiens tout d'abord à exprimer ma profonde reconnaissance à Jean Bertoin. Ce fut un grand privilège d'apprendre sous sa direction et je le remercie pour ses précieux conseils scientifiques, sa rigueur mathématique et sa grande clarté.

J'ai eu la chance d'avoir un deuxième directeur en cours de thèse. Je ne remercierai jamais assez Marc Hoffmann pour sa disponibilité, sa confiance en moi et en notre projet ainsi que ses judicieux conseils scientifiques et humains.

Je tiens à remercier Yuri Golubev et Andreas Kyprianou d'avoir accepté de rapporter ma thèse, et de faire le déplacement afin d'assister à ma soutenance. Je les remercie aussi de l'intérêt qu'ils ont manifesté pour mon travail. Merci encore à Andreas Kyprianou pour ses précieuses remarques.

Je suis très honorée par la participation d'Alain Rouault et de Lorenzo Zambotti à mon jury de thèse.

Je remercie aussi toute l'équipe administrative et technique du laboratoire de probabilités pour leur aide.

Parmi les enseignants qui m'ont particulièrement marquée, je tiens aussi à exprimer ma profonde gratitude envers Mme Tomasine, ainsi qu'à Mme et M. Helmer qui m'ont fait découvrir la beauté des mathématiques. Bien sûr il y a aussi les professeurs de Cachan et de Paris 7, en particulier Laurent Desvillettes et Dominique Picard, qui ont toujours cru en moi et m'ont soutenue tout au long de ces années.

J'ai eu l'opportunité au cours de ma thèse de passer quelques mois à l'université du Chili à Santiago. J'ai eu la chance de pouvoir y travailler avec Joaquim Fontbona et Servet Martinez, ainsi que de faire la connaissance de Adriana, José, Léo et d'autres expatriés en particulier Hélène et Sylvie.

Une mention spéciale à toute la petite équipe formée d'anciens thésards et postdoc de Jean Bertoin, qui m'a parfaitement intégrée et fait profiter de son expérience: Amaury, Anne-Laure, Bénédicte, Christina, Grégory et Julien.

Dans le même ordre d'idée, un grand merci aussi à Francesco, Ivan et Jean-Stephane.

Mes remerciements vont aussi à tous ceux qui ont partagé mon quotidien à Chevaleret: ceux du 3D1: Abass, Amel, Bruno, François, Paul, Sophie, Vincent pour ne citer que les plus récents et les plus présents. Une dédicace spéciale à Paul et François pour l'aide à la réalisation des figures et à la résolution des problèmes LaTeX. Ainsi qu'à ceux du quatrième Fabien, Fernando, Guillaume, Joseph, Juan-Carlos, Olivier, et enfin, et pas des moindres, ceux du cinquième étage en particulier François, Karim, Marc et Maxime. Je tiens également à remercier François et Maxime pour leur aide à la relecture de ma thèse et à la préparation des exercices de l'ENSAE.

Même s'il ne font pas partie du labo, ils ont partagé ces années de thèse: Arno, Claire, Cédric, Guillaume, Julien, Marie, Nicolas et Philippe.

Je conclurai mes remerciements par mes amis, bien sûr Mimi, mais aussi Camille,

Céline, Jojo, Luc, Micheline, et enfin mes parents, grand-parents et ma soeur, qui m'ont tous soutenue tout au long de cette aventure.

Je dédie tout particulièrement ce travail à Julien à qui je dois beaucoup.

Contents

Introduction et présentation des résultats

Le sujet principal de cette thèse de doctorat est l'étude de diverses quantités reliées à une famille particulière de processus, les processus de fragmentation. Ces processus sont destinés à modéliser un objet de masse unité se fragmentant au cours du temps. Un exemple concret de ce phénomène nous est fourni par l'industrie minière, où de gros blocs de pierre sont concassés dans une succession de broyeurs afin de pouvoir extraire le minerai de cuivre de chacun des fragments. Cet exemple est développé plus en détails dans [21] et les références s'y trouvant. Les fragmentations se retrouvent aussi dans de nombreux autres domaines que ce soit en science de l'informatique (par exemple fragmentation d'un disque dur), en physique des particules ou encore en physique nucléaire (avec la fission nucléaire). Bien d'autres champs d'application sont explicités dans [24].

Mathématiquement parlant, les processus de fragmentation ont été introduits par Kolmogorov [46] en 1941. Par la suite, un de ses élèves, Filippov, [35] a donné les propriétés de base du modèle. Ce modèle a été repris dans les années 80 par Brennan et Durrett ([26] et [27]) qui ont donné des théorèmes de convergence de la mesure empirique dans le cas de fragmentations binaires. Depuis 2001, Bertoin a contribué de manière importante à la compréhension de ce modèle et des phénomènes s'y rattachant, par exemple dans [14], [15], [16], et surtout à travers sa récente monographie [17]. Dans les dernières années, une abondante littérature s'est développée sur ce sujet, citons par exemple les articles de Abraham et Delmas [1], Basdevant [7], Berestycki [8], Haas [38], Ghorbel et Huillet [36], Gnedin et Pitman [37], Miermont et Schweinsberg [59], Bertoin et Martinez [21], Bertoin et Rouault [22].

Ce travail se compose de quatre chapitres. Le premier est consacré à l'étude de fragments dont la taille décroît de manière exponentielle. Il est constitué de l'article [52] auquel j'ai ajouté la démonstration de la proposition 1.2 ainsi qu'un appendice donnant une démontration du fait que la martingale M_t est bornée dans $\mathrm{L}^2(\mathbb{P})$ et utilisant les fragmentations en partitions. Dans le deuxième chapitre, on construit un processus de Markov auto-similaire qui généralise les fragmentations classiques

autorisant en particulier la taille des descendants à être plus grande que celle de leurs parents. Ce travail a fait l'objet d'un article accepté dans "Séminaire de probabilités" [51]. Dans le troisième chapitre, on s'intéresse à une estimation statistique de la mesure de Lévy du subordinateur associé à la fragmentation lorsque l'on observe la fragmentation uniquement à un temps d'arrêt donné. Plus précisément on observe les fragments seulement à l'instant où ils atteingnent une taille inférieure à un seuil fixé. Le corps de ce chapitre est l'article coécrit avec Marc Hoffmann [42]. Enfin, dans un quatrième chapitre, on étudie le coût énergétique d'une succession de fragmentations. Ce dernier travail a été réalisé en collaboration avec Joaquim Fontbona et Servet Martinez. Ces chapitres sont autonomes et rédigés en anglais.

Dans ce qui suit, nous allons dans un premier temps énoncer des définitions et propriétés utiles à la compréhension de la suite ; les principaux résultats de cette thèse seront ensuite donnés dans les parties 0.0.2 à 0.0.5.

0.0.1 Processus de fragmentation

Fragmentation en intervalles.

On désigne par "fragmentation en intervalles" un processus $(F(t))_{t\geq 0}$ à valeurs dans l'espace $\mathcal{U}$ des ouverts de $(0,1)$. Tout ouvert de $\mathbb{R}$ pouvant s'écrire de manière unique comme une union au plus dénombrable d'intervalles ouverts disjoints, on peut écrire

$$F(t) := \bigcup_{i\in I} J_i(t),$$

où I est un ensemble d'indices fini ou dénombrable et les $(J_i(t))_{i\in I}$ des intervalles ouverts disjoints (et donc définis de manière unique). Les $(J_i(t))_{i\in I}$ sont les fragments de notre processus de fragmentation. Dans cette partie, on va s'intéresser exclusivement aux fragmentations en intervalles homogènes, mais il existe aussi des fragmentations en intervalles auto-similaires (c'est-à-dire telles que le temps de fragmentation d'un intervalle dépend de sa taille).

Une fragmentation en intervalles homogène est un processus de Markov à valeurs dans $\mathcal{U}$ qui vérifie deux propriétés clés. La première est la propriété de branchement : différents fragments ont une évolution indépendante. La seconde est la propriété d'homogénéité, qui stipule qu'à un facteur de changement d'échelle près, la loi de la fragmentation est indépendante de la taille du fragment initial.

Plus précisément, si $\mathbb{P}$ est la loi de la fragmentation F initialisée par $F(0) = (0,1)$, alors pour tout $s,t \geq 0$, conditionnellement à l'ouvert $F(t) = \bigcup_{i\in I} J_i(t)$, la fragmentation $F(t+s)$ a la même loi que $F^1(s) \cup F^2(s) \cup ...$ où pour chaque i, $F^i(s)$ est un sous ensemble de $J_i(t)$ ayant la même loi que l'image de $F(s)$ par l'application affine canonique $(0,1) \to J_i(t)$.

Basdevant a montré dans [7] que la loi de la fragmentation en intervalles F est entièrement caractérisée par deux constantes d'érosion c_r et c_l et par une mesure de dislocation ν sur $\mathcal{U}$ telle que

$$\nu((0,1)) = 0,$$

et

$$\int_{\mathcal{U}} (1 - u_1)\nu(dU) < \infty,$$

où u_1 désigne la taille du plus grand intervalle composant l'ouvert générique U.

La fonction suivante, étroitement reliée à ν, sera souvent utilisée par la suite :

$$\kappa(q) := \int_{\mathcal{U}} \left(1 - \sum_{j=1}^{\infty} u_j^{q+1}\right) \nu(dU) \qquad \forall q > \underline{p} \tag{0.0.1}$$

où $(u_1, u_2, ...)$ désigne la suite des tailles réordonnée en ordre décroissant des intervalles ouverts composant l'ouvert U et où $\underline{p}$ désigne le plus petit réel tel que κ soit fini ; c'est-à-dire

$$\underline{p} := \inf \left\{ p \in \mathbb{R} : \int_{\mathcal{U}} \sum_{j=2}^{\infty} u_j^{p+1} \nu(dU) < \infty \right\}.$$

Dans le premier chapitre, on va s'intéresser aux fragmentations propres ; c'est-à-dire aux fragmentations sans perte de masse. Cette dernière condition est équivalente à ce que les coefficients c_r et c_l soient nuls et

$$\sum_{i=1}^{\infty} u_i = 1 \quad \text{pour} \quad \nu - \text{presque tout} \quad U \in \mathcal{U}.$$

Fragment marqué.

Soit $x \in (0,1)$ et $I_x(t)$ l'intervalle ouvert de l'ouvert aléatoire $F(t)$ contenant x, et soit $|I_x(t)|$ sa taille. Soit V une variable aléatoire uniforme sur $[0,1]$, indépendante de la fragmentation.

Bertoin a montré dans [14] que

$$\xi(t) := -\log |I_V(t)|, \quad t \geq 0, \tag{0.0.2}$$

est un subordinateur dont l'exposant de Laplace $\kappa(q)$ est défini en (0.0.1) (autrement dit $\forall \lambda > \underline{p}\ \mathbb{E}(e^{-\lambda \xi(t)}) = e^{-t\kappa(\lambda)}$) et la mesure de Lévy est donnée par

$$\pi(dx) := e^{-x} \sum_{i=1}^{\infty} \nu(-\log s_i \in dx).$$

Cette propriété remarquable permet de jeter des ponts entre les processus de fragmentation et des objets très classiques en théorie des probabilités, les subordinateurs.

Elle se révèlera fondamentale dans notre premier chapitre. Une des principales difficultés techniques que l'on va rencontrer dans le deuxième chapitre, où l'on construit un processus plus général que les fragmentations, va être de trouver un analogue à $(\xi(t))_{t\geq 0}$, qui ne sera malheureusement plus un subordinateur.

En revanche dans le troisième chapitre, on va fortement utiliser les propriétés de $(\xi(t))_{t\geq 0}$. L'objectif y est d'estimer l'exposant κ. Dans le cas de la fragmentation binaire cela est équivalent à l'estimation de ν, mais pas dans le cas général. [1]

Fragmentation de masse.

On s'intéresse ici aux fragmentations à valeurs dans l'ensemble des partitions de masse

$$\mathcal{S}^{\downarrow} := \left\{ s = (s_1, s_2, ...) \mid s_1 \geq s_2 \geq ... \geq 0 \ , \sum_{i=1}^{\infty} s_i \leq 1 \right\}.$$

Si on considère

$$X(t) = (|J_1(t)|, ..., |J_n(t)|, ...)^{\downarrow}$$

le réarrangement en ordre décroissant de la taille des fragments d'une fragmentation en intervalles de mesure de dislocation ν, on obtient une fragmentation de masse de mesure de dislocation $|\nu|^{\downarrow}$ l'image de ν par l'application $U \to |U|^{\downarrow}$. Inversement à partir d'une fragmentation de masse $X = ((X_1(t), ..., X_n(t), ...)_{t\geq 0})$, on peut toujours construire une fragmentation en intervalles F tel que

$$\begin{aligned} F(t) = \ & (0, X_1(t)) \cup (X_1(t), X_1(t) + X_2(t)) \cup (X_1(t) + X_2(t), X_1(t) + X_2(t) + X_3(t)) \\ & \cup .... \cup (X_1(t) + ... + X_n(t), X_1(t) + ... + X_n(t) + X_{n+1}(t)) \cup .... \end{aligned}$$

On peut aussi se référer à l'article de Basdevant dans [7] pour les bijections existantes entre les différents types de fragmentation, celles entre les fragmentations de masse et celle de partition, ainsi que la bijection entre les fragmentations en intervalles et celles en composition.

Dans la suite, nous nous intéresserons aux fragmentations de masse autosimilaires (et non pas seulement homogènes). Un formalisme qui se révèlera particulièrement fructueux est celui des arbres marqués. Pour ceci on va reprendre l'approche de Bertoin dans [17] afin de construire une famille de variables aléatoires qui va permettre de définir la fragmentation.

[1] Comme l'atteste l'exemple suivant : la mesure de dislocation de la loi $(VU, (U - VU), 1 - U, 0, 0, ...)$ avec U et V deux lois uniformes indépendantes va avoir le même κ que la mesure de dislocation qui avec une probabilité $1/2$ va donner $(U, 1 - U, 0, 0, ...)$ et avec une probabilité $1/2$ va donner $(VU, (U - VU), (WU - U), 1 - WU, 0, 0, ...)$ avec U, V et W trois lois uniformes indépendantes. En effet, il est facile de voir que dans les deux cas, on a

$$\kappa(p) = \frac{1}{p+1} + \frac{2}{(p+1)^2} \quad \forall p > -1.$$

Définition 0.1. *Soit ν une mesure de probabilité sur $\mathcal{S}^{\downarrow}$ et soient deux familles indépendantes de variables i.i.d. indexées par les noeuds de l'arbre de Ulam-Harris $\mathbb{T} := \cup_{n\in\mathbb{N}}\mathbb{N}^n$: $(\overline{\xi}_u, u \in \mathcal{U})$ et $(\mathbf{e}_u, u \in \mathcal{U})$, où pour $u \in \mathcal{U}$ $\overline{\xi}_u = (\widetilde{\xi}_{ui})_{i\in\mathbb{N}}$ a pour loi ν, et $(\mathbf{e}_{ui})_{i\in\mathbb{N}}$ est une suite de variables exponentielles i.i.d. de paramètre 1. On définit par récurrence*

$$\xi_\emptyset := 1, \qquad a_\emptyset := 0, \qquad \zeta_\emptyset := x^{-\alpha}\mathbf{e}_\emptyset,$$

et pour $u \in \mathcal{U}$ et $i \in \mathbb{N}$:

$$\xi_{ui} := \widetilde{\xi}_{ui}\xi_u, \qquad a_{ui} := a_u + \zeta_u, \qquad \zeta_{ui} := \xi_{ui}^{-\alpha}\mathbf{e}_{ui}.$$

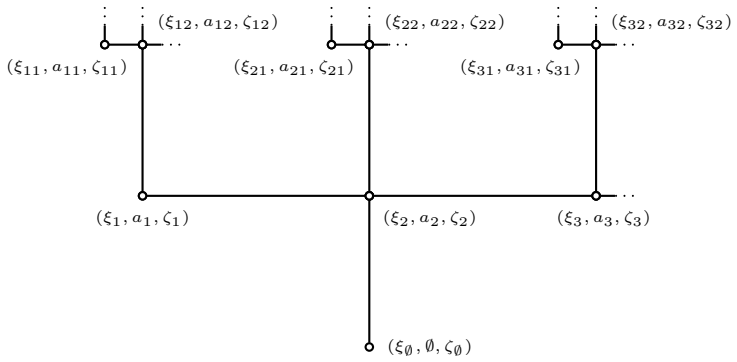

FIG. 1 – Arbre généalogique.

Ici ξ_u représente la taille du fragment, a_u représente son temps de naissance et ζ_u sa durée de vie. On peut observer que le paramètre d'auto-similarité n'a d'incidence que sur la durée de vie du fragment. Ainsi, pour $\alpha > 0$, ce sont les gros fragments qui vont avoir tendance à se fragmenter le plus vite. Il est aussi important de noter que ce sont les lois exponentielles qui sont utilisées pour la durée de vie ce qui garantit le caractère Markovien du processus. On définit le processus $\mathbf{X} = (\mathbf{X}(t), t \geq 0)$ comme étant le processus qui vérifie l'égalité suivante entre mesures ponctuelles sur $(0, \infty)$:

$$\sum_{i=1}^{\infty} \delta_{\mathbf{X}_i(t)} = \sum_{u\in\mathcal{U}} \mathbb{1}_{\{t\in I_u\}}\delta_{\xi_u}, t \geq 0,$$

avec $I_u = [a_u, a_u + \zeta_u)$ l'intervalle de vie du fragment indexé par u. $X(t)$ est un processus de fragmentation de masse, de mesure de dislocation ν.

On va donner un exemple explicitant le lien entre ces deux points de vue avec l'exemple d'une fragmentation ayant pour mesure de dislocation

$$\nu(ds) = \frac{1}{3}\left(\delta_{(1/2,1/2,0,0,\ldots)}(ds) + \delta_{(1/3,1/3,1/3,0,0,\ldots)}(ds) + \delta_{(2/3,1/3,0,0,\ldots)}(ds)\right).$$

Soit $(T_i)_{i\geq 1}$ une suite de v.a. i.i.d. de loi commune la loi exponentielle de paramètre 1. La figure 2 ci-dessous représente un exemple d'arbre marqué défini précédemment pour un tel ν, pour lequel on a simplifié les notations : $L_1 = T_1 + T_2$, $L_2 = T_1 + T_3$,

$L_3 = T_1 + T_2 + T_4$ et $L_4 = T_1 + T_2 + T_5$. La figure 3 représente une version continue de la fragmentation.

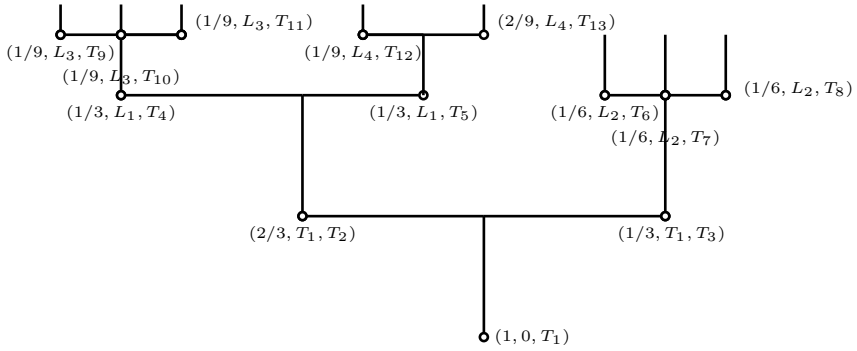

FIG. 2 – *L'arbre marqué associé à la fragmentation.*

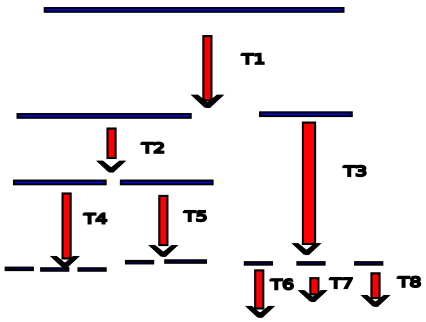

FIG. 3 – *Fragmentation en temps continu.*

Ces deux points de vue vont nous être utile afin de généraliser le processus de fragmentation en un processus auto-similaire autorisant les descendants à être plus grands que leurs parents dans le chapitre 2.

0.0.2 Spectre multifractal et taux précis de décroissance d'une fragmentation homogène

Dans ce premier travail, on considère des fragmentations en intervalles homogènes et on va généraliser un résultat de Berestycki [9] qui avait étudié l'ensemble :

$$G_v := \left\{ x \in (0,1) \ : \ \lim_{t\to\infty} \frac{1}{t} \log |I_x(t)| = -v \right\}$$

pour un $v > 0$, afin d'en déterminer la dimension de Hausdorff. On rappelle que, pour $s \geq 0$, la s-mesure de Hausdorff d'un ensemble $F \subset \mathbb{R}$ est

$$\mathcal{H}^s(F) := \lim_{\delta\to 0} \inf \left\{ \sum_{i\in J} |I_i|^s, \ |I_i| < \delta, \ F \subset \cup_{i\in J} I_i, \ J \text{ dénombrable} \right\}.$$

La dimension de Hausdorff de F est alors

$$dim(F) := \inf\{s \geq 0 : \ \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(F) \ = \infty\}.$$

Ici, nous déterminons celle de $G_{(v,a,b)}$:

$$G_{(v,a,b)} := \left\{ x \in (0,1) \ : \ a \leq \liminf_{t\to\infty} e^{vt}|I_x(t)| \leq \limsup_{t\to\infty} e^{vt}|I_x(t)| \leq b \right\},$$

pour un $v > 0$, $0 < a < b$ fixés, avec $I_x(t)$ l'intervalle ouvert de $F(t)$ contenant x. Pour cela, nous allons d'abord étudier celle de l'ensemble

$$\Lambda_{(v,a,b)} = \left\{ x \in (0,1) \ : \ ae^{-vt} < |I_x(t)| < be^{-vt} \ \forall t \geq 0 \right\},$$

pour $0 < a < 1 < b$. On utilise aussi le subordinateur $\xi(t)$ (défini en (0.0.2)) qui est naturellement associé à la fragmentation et plus précisément on s'intéresse au processus $Y_t = vt - \xi(t)$ qui est un processus de Lévy sans saut positif ayant pour exposant de Laplace $\psi(\lambda) = v\lambda - \kappa(\lambda)$, avec κ défini en (0.0.1). Pour ce processus, on introduit les notations suivantes : pour $\beta > 0$, on note le premier temps de sortie de Y_t de l'intervalle $(0, \beta)$:

$$T_\beta = \inf\{t : \ Y_t \notin (0, \beta)\}.$$

On définit $W : \mathbb{R}_+ \to \mathbb{R}_+$ la fonction d'échelle comme étant l'unique fonction continue ayant pour transformée de Laplace :

$$\int_0^\infty e^{-\lambda x} W(x)dx = \frac{1}{\psi(\lambda)} \quad , \ \lambda > \varphi(0),$$

où $\varphi(\bullet)$ est l'inverse à droite de $\psi(\bullet)$. Pour $q \in \mathbb{R}$, soit $W^{(q)} : \mathbb{R}_+ \to \mathbb{R}_+$ la fonction continue telle que pour tout $x \in \mathbb{R}_+$

$$W^{(q)}(x) := \sum_{k=0}^{\infty} q^k W^{*k+1}(x),$$

où $W^{*n} = W * ... * W$ est le n-ième produit de convolution de la fonction W. On définit

$$\rho_\beta := \inf\{q \geq 0 \ ; \ W^{(-q)}(\beta) = 0\}.$$

Soit $(\mathcal{F}_t)_{t\geq 0}$ la filtration naturelle associée à la fragmentation et soit la filtration $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(I_V(t))$ où V est une variable uniforme sur $(0,1)$ indépendante de la fragmentation. D'après l'article de Lambert [56], on obtient que, sous une hypothèse

technique, le processus

$$D_t := e^{\rho_\beta t}\ \mathbf{1}_{\{t<T_\beta\}}\ \frac{W^{(-\rho_\beta)}(Y_t)}{W^{(-\rho_\beta)}(x)}$$

est une martingale pour la filtration $(\mathcal{G}_t)_{t\geq 0}$. Pour simplifier les notations, on introduit

$$\rho := \rho_{\log(b/a)}$$

et

$$h(t) := W^{(-\rho)}(t-\log a)\mathbf{1}_{\{t\in(\log a,\log b)\}}$$

pour tout $t\in\mathbb{R}$, et $h(-\infty)=0$.

En projetant la martingale D_t (pour $x=\log(1/a)$ et $\beta=\log(b/a)$) sur la sous-filtration $(\mathcal{F}_t)_{t\geq 0}$ on obtient la martingale

$$M_t := \frac{e^{\rho t}}{h(0)}\sum_{i\in\mathbb{N}} h\left(vt+\log|J_i^\dagger(t)|\right)|J_i^\dagger(t)|$$

associée aux fragments $\{J_i(t)^\dagger\}$. Ici $\{J_i(t)^\dagger\}$ désigne les bons fragments, c'est-à-dire les intervalles appartenant à $G(t)$, avec

$$G(t) := \{I_x(t):\ x\in(0,1)\ \text{ et }\ ae^{-vs}<|I_x(s)|<be^{-vs}\ \ \forall\ s\leq t\}.$$

Il est clair que $\sharp G(t)$ est fini pour tout temps t. Soit $\zeta=\inf\{t:\ G(t)=\emptyset\}$. On a alors le :

Théorème 0.1. *Si $v>\rho$ et sous l'hypothèse (1.2.6) , on a :*

1. *La martingale M_t est bornée dans* $\mathrm{L}^2(\mathbb{P})$.
2. *De plus, conditionnellement à $\zeta=\infty$, on a :* $\lim_{t\to\infty} M_t>0$.

De manière heuristique, on voit que M_t se comporte comme $\sharp\{G(t)\}\ e^{(\rho-v)t}$ quand t est grand, si bien que pour montrer le deuxième point du théorème ci-dessus on doit avoir au moins $e^{(v-\rho)t}$ bons intervalles au temps t. En fait, il s'agit bien du bon ordre de grandeur comme le montre la proposition suivante :

Proposition 0.1. *Sous l'hypothèse (1.2.6) et si $v>\rho$ alors conditionnellement à $\{\zeta=\infty\}$, on a :*

$$\lim_{t\to\infty}\frac{1}{t}\log\sharp G(t) = \ v-\rho \qquad p.s.$$

Ce résultat généralise celui de Bertoin et Rouault (Corollaire 2 dans [22]). Il donne aussi une idée de la dimension de Hausdorff de l'ensemble $\Lambda_{(v,a,b)}$. Pour cela, on introduit la dimension de boîte ("box-counting") de cet ensemble qui va être la limite quand δ tend vers 0 de $\log N(\delta)/\log(1/\delta)$ où $N(\delta)$ désigne le nombre minimal d'intervalles de taille au plus δ qu'il faut pour recouvrir $\Lambda_{(v,a,b)}$. Intuitivement, ce

nombre est assez proche de $\sharp G(t)$ si bien que la dimension de boite de $\Lambda_{(v,a,b)}$ est $1 - v/\rho$. On verra que c'est aussi la dimension de Hausdorff de cet ensemble.

Il est aussi à noter qu'un autre corollaire de la proposition précédente concerne la mesure aléatoire

$$\sigma_t := \frac{e^{\rho t}}{h(0)} \sum_{i \in \mathbb{N}} h\left(vt + \log|J_i^\dagger(t)|\right) |J_i^\dagger(t)| \delta_{\log(1/a)+vt+\log|J_i^\dagger(t)|}$$

associée à la configuration $J^\dagger(t) = \{|J_i^\dagger(t)|\}$ de la taille des bons intervalles, ainsi que la mesure moyenne associée σ_t^*. Cette dernière est définie par la formule

$$\int_0^\infty f(x)\sigma_t^*(dx) = \mathbb{E}\int_0^\infty f(x)\sigma_t(dx)$$

pour toute fonction continue et à support compact f. Comme M_t est une martingale, σ_t^* est une mesure de probabilité. On obtient alors le résultat suivant :

Corollaire 0.1. *Sous l'Hypothèse (1.2.6) et avec $v > \rho$ on obtient :*

1. *La mesure σ_t^* converge faiblement quand $t \to \infty$, vers la mesure de probabilité*

$$\varrho(dy) := ch(y + \log a)h(\log(b) - y)dy$$

2. *Pour toute fonction mesurable bornée f :*

$$L^2 - \lim_{t\to\infty} \int_0^\infty f(x)\sigma_t(dx) = M_\infty \int_0^\infty f(x)\varrho(dx).$$

La proposition 0.1 nous permet de déterminer la dimension de Hausdorff de l'ensemble $\Lambda_{(v,a,b)}$. Plus précisement, on a :

Proposition 0.2. *Supposons (1.2.6) et $0 < a < b < 1$:*

- *Si $\rho > v$, alors :*
$$\Lambda_{(v,a,b)} = \emptyset \quad p.s.$$
- *Si $\rho < v$, alors : $\mathbb{P}(\Lambda_{(v,a,b)} \neq \emptyset) > 0$, et conditionnellement à $\Lambda_{(v,a,b)} \neq \emptyset$,*
$$dim(\Lambda_{(v,a,b)}) = 1 - \rho/v.$$

Ce qui peut paraître surprenant au premier abord c'est que c'est ce résultat permet de calculer facilement la dimension de Hausdorff de l'ensemble $G_{(v,a,b)}$. A priori ce dernier ensemble pourrait sembler beaucoup plus grand que le précédent mais il n'en est rien. Plus précisement on obtient :

Théorème 0.2. ***: Spectre Multifractal.*** *Supposons (1.2.6) :*

- *Si $\rho > v$, alors :*
$$G_{(v,a,b)} = \emptyset \quad p.s.$$

- *Si $\rho < v$, alors :*

$$dim(G_{(v,a,b)}) = 1 - \rho/v \quad p.s.$$

0.0.3 Chaîne de branchement Markovienne auto-similaire

Dans ce deuxième travail, on va construire un processus de Markov auto-similaire assez général. On veut lever l'hypothèse que les enfants doivent avoir une taille inférieure ou égale à celle de leurs parents (hypothèse toujours présente dans le cas des fragmentations classiques). Cette hypothèse peut sembler a priori assez peu importante, mais en fait elle implique de profonds changements dans les démonstrations. Par exemple, on perd le fait que la taille des fragments décroît avec le temps et donc ainsi le fait que le processus restreint aux fragments de taille supérieure ou égale à ϵ reste un processus Markovien.

Introduisons les notations et les définitions dont nous aurons besoin. Une mesure ponctuelle finie sur $\mathbb{R}_+^*$ est une somme finie de masses de Dirac $\mathbf{s} = \sum_{i=1}^n \delta_{s_i}$ où les s_i sont appelés les atomes de $\mathbf{s}$ et où n est un entier naturel. On note $\sharp\mathbf{s} = n = \mathbf{s}(\mathbb{R}_+^*)$ le nombre d'atomes de $\mathbf{s}$ et $\mathcal{M}_p(\mathbb{R}_+^*)$ pour l'espace des mesures ponctuelles finies sur $\mathbb{R}_+^*$. Soit $\alpha \geq 0$ et ν une **mesure de probabilité** sur $\mathcal{M}_p(\mathbb{R}_+^*)$. Le but de cette partie est de construire une chaîne de branchement Markovienne $\mathbf{X} = ((\sum_{i=1}^{\sharp\mathbf{X}(t)} \delta_{X_i(t)})_{t\geq 0})$ à valeurs dans $\mathcal{M}_p(\mathbb{R}_+^*)$, dont l'indice d'auto-similarité est α et ayant pour loi de reproduction ν. Comme pour les fragmentations, l'indice d'auto-similarité va jouer un rôle dans la vitesse avec laquelle les particules vont se reproduire. **En fait on va supposer que l'on vérifie les hypothèses de Malthus c'est-à-dire qu'il existe** $p_0 := \inf\{p : \kappa(p) = 0\} > 0$**, tel que**

$$\kappa(p) > 0 \quad \textbf{pour un} \quad p > p_0.$$

De plus on suppose que

$$\int_{\mathcal{M}_p(\mathbb{R}_+^*)} (\langle x^{p_0}, \mathbf{s}\rangle)^p \, \nu(d\mathbf{s})$$

est fini pour un $p > 1$**.**

Voici un exemple d'une mesure vérifiant les hypothèses de notre travail sans être la mesure de dislocation d'une fragmentation. Il est basé sur la loi de Dirichlet (confère le livre de Kingman [45]). Soit $n \geq 2$, $(\upsilon_1, ..., \upsilon_n)$ n nombre réel positifs et soit $\upsilon = \sum_{i=1}^n \upsilon_i$. On définit le simplexe Δ_n par

$$\Delta_n := \left\{ (p_1, p_2, ..., p_n) \in \mathbb{R}_+^n, \ \sum_{j=1}^n p_i = 1 \right\}.$$

Le loi de Dirichlet de paramètre $(\upsilon_1, ..., \upsilon_n)$ sur le simplexe Δ_n a pour densité :

$$f(p_1, ..., p_n) = \frac{\Gamma(\upsilon)}{\Gamma(\upsilon_1)...\Gamma(\upsilon_n)} p_1^{\upsilon_1 - 1} ... p_n^{\upsilon_n - 1} \mathbb{1}_{\{(p_1, p_2, ..., p_n) \in \Delta_n\}}.$$

On introduit la constante $a := \upsilon(\upsilon+1)/(\sum_{i=1}^n \upsilon_i(\upsilon_i+1)) > 1$. On définit la mesure ν comme ayant pour loi $(aX_1, ..., aX_n)$, où $(X_1, ..., X_n)$ est un vecteur aléatoire de loi la loi de Dirichlet de paramètre $(\upsilon_1, ..., \upsilon_n)$. On obtient aisément

$$\kappa(p) = a^p \frac{\Gamma(\upsilon)}{\Gamma(\upsilon + p)} \sum_{i=1}^n \frac{\Gamma(p + \upsilon_i)}{\Gamma(\upsilon_i)},$$

et il est clair que cette mesure rentre dans le cadre de ce travail (avec $p_0 = 1$).

On va introduire un arbre marqué dont les noeuds contiennent la taille ξ_u, le temps de naissance a_u et le temps de vie ζ_u d'un fragment d'étiquette $u \in \mathbb{T}$ (avec $\mathbb{T}$ le système d'étiquettes de Ulam-Harris $\mathbb{T} := \cup_{n=0}^{\infty} \mathbb{N}^n$).

Sous les hypothèses de Malthus, il existe un ou deux réels p_0 (défini précédemment) et éventuellement $p_1 > p_0$ tels que

$$\left(M_n^{(p_i)} := \sum_{|u|=n} \xi_u^{p_i}, n \geq 0 \right)$$

soit une martingale. On considère alors la martingale $M_n := M_n^{(p_0)}$. En utilisant des résultats de Biggins [25] concernant les processus de branchement on va montrer que cette martingale est bornée dans $\mathrm{L}^p(\mathbb{P})$. Cela va nous permettre de définir un processus continu à valeurs dans $\mathcal{M}_p(\mathbb{R}_+^*)$:

Définition 0.2. *On définit le processus* $\mathbf{X} = (\mathbf{X}(t), t \geq 0)$ *par*

$$\mathbf{X}(t) = \sum_{u \in \mathcal{U}} \mathbb{1}_{\{t \in I_u\}} \delta_{\xi_u}, t \geq 0, \tag{0.0.3}$$

avec $I_u := [a_u, a_u + \zeta_u)$ *si* $\xi_u > 0$ *et sinon* $I_u := \emptyset$.

Les premières propriétés de ce processus sont :

Théorème 0.3. *Sous l'hypothèse (2.4.2), le processus* $\mathbf{X}$ *prend ses valeurs dans l'ensemble* $\mathcal{M}_p(\mathbb{R}_+^*)$*, est une chaîne de Markov, c.a.d. la loi de* $\mathbf{X}(t+r)$ *conditionnellement à* $\mathbf{X}(r) = \mathbf{s}$ *est la même que la somme* $\sum \mathbf{X}^{(i)}(t)$ *où pour chaque* i*,* $\mathbf{X}^{(i)}(t)$ *est une copie indépendante de* $\mathbf{X}(t)$ *sous* $\mathbb{P}_{s_i}$.

On utilise une preuve totalement différente de celle de Bertoin dans [17] (dans le cas des fragmentations) car l'on perd ici la propriété que la taille des fragments décroît avec le temps et donc aussi le fait que le processus des fragments de taille supérieure ou égale à ϵ est Markovien.

Dans la suite, de même que dans [17], on va prouver un résultat de $\mathrm{L}^p(\mathbb{P})$ convergence de $M(t) = \sum_{u\in\mathcal{U}} \mathbb{1}_{\{t\in I_u\}}\xi_u^{p_0}$ l'analogue continu de M_n.

Comme dans le cas de la fragmentation, on peut construire une "branche" marquée et introduire $\chi(t)$ la taille de ce fragment marqué au temps t. Il est à noter que ce processus n'est pas nécessairement décroissant. Mais on peut tout de même en donner une représentation de Lamperti (confère (2.5.1)) ce qui permet d'établir la convergence en loi de $t^{1/\alpha}\chi(t)$ en utilisant un résultat de Bertoin et Yor [23] sur les processus de Markov auto-similaires (confère Proposition 2.2). Comme dans la littérature ([26], [27], [20], [17] et [52]) on s'intéresse à la configuration de masse $X(t) = \{(X_i(t))_{1\leq i\leq \sharp\mathbf{X}(t)}\}$ que l'on va encoder par la mesure de poids empirique

$$\sigma_t := \sum_{i=1}^{\sharp\mathbf{X}(t)} X_i^{p_0}(t)\delta_{t^{1/\alpha}X_i(t)}.$$

Afin de démontrer la convergence de cette mesure on va d'abord démontrer celle de la mesure moyenne associée σ_t^* définie par la formule

$$\int_0^\infty k(x)\sigma_t^*(dx) = \mathbb{E}\left(\int_0^\infty k(x)\sigma_t(dx)\right).$$

On obtient :

Théorème 0.4. *Sous les hypothèses de la proposition 2.2, on a que pour toute fonction continue bornée k :*

$$\mathrm{L}^p - \lim_{t\to\infty}\int_0^\infty k(x)\sigma_t(dx) = M_\infty\mathbb{E}(k(Y)) = \frac{M_\infty}{\alpha m}E(k(I)I^{-1}),$$

pour un $p > 1$ et avec M_∞ la limite de la martingale M_n et I défini en (2.6.1).

De plus en utilisant les résultats de Jagers [44] on donne des résultats plus précis concernant le processus intrinsèque $\{M_Q,\ Q\subset\mathcal{U}\}$, $M_Q = \sum_{u\in M}\xi_u^{p_0}$ (confère l'Appendice).

0.0.4 Analyse statistique d'une chaîne de fragmentation auto-similaire conservative.

Après la rédaction de mon premier article [52] ne sachant pas grand chose de la constante ρ (hormis dans des cas très spécifique qui ne vérifiaient pas les hypothèses dans lesquelles on se plaçait), je me suis demandée s'il n'y avait pas un moyen de l'estimer via un échantillon de variable aléatoire. Je me suis donc intéressée à l'estimation statistique pour les processus de fragmentation. J'ai commencé à travailler avec Marc Hoffmann. Un premier pas a été de comprendre la bonne manière de poser le problème statistique. Etant au Chili pour un échange de cinq mois au cours

de ma thèse, j'ai pu rencontrer des experts de l'industrie minière qui m'ont expliqué la manière dont on concassait les pierres afin d'avoir des fragments assez petits pour pouvoir leur appliquer certains procédés physico-chimiques. La manière dont on concasse le minerai semble pouvoir être modélisée par une fragmentation. Plus précisément on met de grosses pierres dans une machine qui va les concasser selon un processus de fragmentation (sans que l'on puisse voir ce qui se passe à l'intérieur de la machine) et qui libère les morceaux de pierre dès qu'ils ont une taille plus petite qu'un certain seuil ε (dans la pratique, dès qu'elles sont assez petites pour passer à travers un tamis dont la taille des trous mesure ε). Ainsi notre observable est $X_\varepsilon := (\xi_u, u \in \mathcal{U}_\varepsilon)$ avec

$$\mathcal{U}_\varepsilon := \big\{u \in \mathcal{U},\ \ \xi_{u-} \geq \varepsilon, \xi_u < \varepsilon\big\},$$

et $u-$ le parent du fragment d'étiquette u. Pour des raisons de simplification on va supposer que la fragmentation est conservative, c'est-à-dire sans perte de masse (mais des résultats sont donnés dans des cas plus généraux (confère la sous section 3.4.3)). Ainsi on va considérer dans un premier temps la mesure empirique

$$\mathcal{E}_\varepsilon(g) := \sum_{u \in \mathcal{U}_\varepsilon} \xi_u \, g(\xi_u/\varepsilon),$$

avec $g(\bullet)$ une fonction test.

Bertoin et Martinez ont montré dans [21] que sous des hypothèses assez faibles sur $\nu(\bullet)$, la mesure $\mathcal{E}_\varepsilon(g)$ converge vers

$$\mathcal{E}(g) := \frac{1}{c(\nu)} \int_0^1 \frac{g(a)}{a} \int_{\mathcal{S}^\downarrow} \sum_{i=1}^{\infty} s_i \, 1_{\{s_i < a\}} \nu(d\mathbf{s}) da$$

en probabilité quand $\varepsilon \to 0$, avec $c(\nu) = -\int_{\mathcal{S}^\downarrow} \sum_{i=1}^{\infty} s_i \log s_i \, \nu(d\mathbf{s})$.

Notre but va être d'estimer

$$\pi(dx) := e^{-x} \sum_{i=1}^{\infty} \nu(-\log s_i \in dx).$$

la mesure de Lévy du subordinateur $\xi(t)$ (associé à la fragmentation et défini en (0.0.2)). La mesure π intervient explicitement dans l'énergie $\mathcal{E}(g)$ comme le montrent les égalités suivantes :

$$\mathcal{E}(g) = \frac{1}{c(\pi)} \int_0^1 \frac{g(a)}{a} \pi(-\log a, +\infty) \, da = \frac{1}{c(\pi)} \int_0^{+\infty} g(e^{-x}) \pi(x, +\infty) \, dx.$$

D'ailleurs sauf dans le cas binaire, l'estimation de π ne permet toutefois pas de caractériser ν.

Notre premier résultat précise une vitesse pour la convergence de $\mathcal{E}_\varepsilon(g)$ vers $\mathcal{E}(g)$, qui allie technique de martingale et vitesse dans le théorème du renouvellement (nous

avons utilisé les résultats de Sgibnev [65]). On obtient :

Théorème 0.5. *Sous certaines hypothèses techniques, on a que pour tout $m > 0$ et $0 < \mu < \kappa$*

$$\sup_{g \in \mathcal{C}(m)} \mathbb{E}\big[\big(\mathcal{E}_\varepsilon(g) - \mathcal{E}(g)\big)^2\big] = o\big(\varepsilon^{\mu/(\mu+1)}\big).$$

avec $\mathcal{C}(m)$ la classe des fonctions continues

$$\mathcal{C}(m) := \big\{g(\bullet) : [0,1] \to \mathbb{R}, \ \ \|g\|_\infty := \sup_x |g(x)| \leq m\big\},$$

et avec $\|g\|_\infty := \sup_x |g(x)|$, $\Pi(\kappa)$ la classe des mesures de probabilité $\pi(dx)$ définies sur $[0, +\infty)$ telles que

$$\int_{[0,+\infty)} e^{\kappa x}\, \pi(dx) < +\infty.$$

Dans la limite $\kappa \to \infty$, on trouve sans surprise la vitesse $\sqrt{\varepsilon}$: heuristiquement, les fragments utilisés dans la construction de la mesure empirique $\mathcal{E}_\varepsilon(g)$ sont (essentiellement) de l'ordre de grandeur de ε, leur nombre étant alors de l'ordre de ε^{-1}, on peut s'attendre à une précision en $\sqrt{\varepsilon}$. Un point essentiel pour les applications statistiques est de pouvoir faire dépendre la fonction test $g(\bullet)$ de l'asymptotique ε. On montre en fait un résultat un peu plus précis lorsque π est absolument continue, où l'on quantifie la vitesse de convergence en fonction de la taille du support de $g(\bullet) = g_\varepsilon(\bullet)$ lorsque cette taille est petite avec ε.

On s'est aussi intéressé à une observation un peu plus fine qui correspond au fait que l'on observe X_ε avec une perte d'information dûe à un bruit systématique d'intensité σ petit devant ε, c'est à dire que l'observable est $X_{\varepsilon,\sigma} := \big(\xi^\sigma_u, \ u \in \mathcal{U}_{\varepsilon,\sigma}\big)$ avec

$$\mathcal{U}_{\varepsilon,\sigma} := \big\{u \in \mathcal{U}, \ \xi^\sigma_{u-} \geq \varepsilon, \ \xi^\sigma_u < \varepsilon\big\},$$

et

$$\xi^\sigma_u := \xi_u + \sigma U_u.$$

avec $(U_u, u \in \mathcal{U})$ identiquement distribuées et indépendantes de X_ε et telles que pour tout $u \in \mathcal{U}$,

$$|U_u| \leq 1 \ \text{ et } \ \mathbb{E}[U_u] = 0.$$

L'énergie devient alors :

$$\mathcal{E}_{\varepsilon,\sigma}(g) := \sum_{u \in \mathcal{U}_{\varepsilon,\sigma}} 1_{\{\xi^\sigma_u \geq t_\varepsilon\}}\, \xi^\sigma_u\, g\big(\xi^\sigma_u/\varepsilon\big),$$

où t_ε est un seuil petit (mais pas trop petit devant ε et plus grand que σ)

On obtient alors

Théorème 0.6. *Sous certaines hypothèses, on a alors que pour tout $m > 0$ et*

$0 < \mu < \kappa$

$$\sup_{g \in \mathcal{C}'(m)} \mathbb{E}\big[\big(\mathcal{E}_{\varepsilon,\sigma}(g) - \mathcal{E}_{\varepsilon}(g)\big)^2\big] = \mathcal{O}(\sigma^2 \varepsilon^{-2}).$$

avec $\mathcal{C}'(m)$ la classe des fonctions continûment différentiables $g(\bullet)$ telles que $g' \in \mathcal{C}(m)$.

On s'intéresse alors à deux cas. Tout d'abord au cas paramétrique. Pour $k \geq 1$, on va estimer le k-ième moment de π :

$$m_k(\pi) := \int_0^{+\infty} x^k \pi(x) dx.$$

On va montrer la convergence de l'estimateur $\widehat{m}_{k,\varepsilon}$ (défini en 3.3.5). On obtient alors le théorème suivant qui va donner une majoration de la vitesse :

Théorème 0.7. *Soit $0 < \kappa_2 \leq \infty$ et $\kappa_1 > \max\{1, \kappa_2\}$.*

Sous certaines hypothèses et pour $1 \leq \mu < \kappa_1$, la famille

$$\varepsilon^{-\mu\,\kappa_2/(\mu+1)(2\kappa_2+1)}\big(\widehat{m}_{k,\varepsilon} - m_k(\pi)\big)$$

est $\mathbb{P}_1$ tendue dès que

$$\pi \in \Pi(\kappa_1) \cap \mathcal{R}(\kappa_2)$$

et que $\sigma\varepsilon^{-3}$ reste borné, avec $\mathcal{R}(\kappa)$ la classe des densités qui restent bornées par $x^{\kappa-1}$ en l'origine.

Dans la limite $\kappa_1, \kappa_2 \to \infty$, on trouve la vitesse (attendue) $\sqrt{\varepsilon}$. En fait dans le cas binaire (hypothèse utilisée pour par souci de clarté) on va montrer (sous une hypothèse supplémentaire) que la vitesse $\sqrt{\varepsilon}$ est la vitesse optimale au sens minimax de la convergence de l'estimateur $m_k(\pi)$ (pour la définition précise de cela on se réfère à la définition 3.1).

Et enfin on s'est intéressé au cas non paramétrique et pour cela on cherche à estimer la fonction

$$a \rightsquigarrow \beta(a) := a^{-1}\pi(-\log a), \quad a \in (0,1).$$

En utilisant le théorème précédent concernant l'estimation des moments, on obtient :

Théorème 0.8. *Soit $1 < \kappa_1, \kappa_2 \leq \infty$.*

Sous certaines hypothèses et pour $1 \leq \mu < \min\{1, \kappa_1/2\}$, et tout $a \in (0,1)$, la famille

$$\varepsilon^{-\mu s/(\mu+1)(2s+3)}\big(\widehat{\beta}_{\varepsilon}(a) - \beta(a)\big)$$

est tendue sous $\mathbb{P}_1$ (avec $\widehat{\beta}_{\varepsilon}(a)$ définie en (3.3.9)), dès que

$$\pi \in \Sigma(s) \cap \Pi(\kappa_1) \cap \mathcal{R}(\kappa_2)$$

pour $s < \max\{N, \kappa_2 - 1\}$ *et* $\sigma\varepsilon^{-3}$ *borné, avec* $\Sigma(s)$ *la classe des densités höldériennes d'indice de régularité* s*.*

On notera que dans ce cas, la vitesse obtenue dans la limite $\kappa_1, \kappa_2 \to \infty$ est $\varepsilon^{2s/(2s+3)}$, qui s'interprète comme la vitesse d'un problème inverse d'ordre 1. On conjecture que ce résultat est optimal (au moins dans la limite κ_1 et κ_2 grands), mais son étude précise dépasse le cadre de ce troisième article.

0.0.5 La concaténation de deux processus de fragmentation.

Je me suis ensuite intéressée à un autre problème provenant de l'industrie. En effet comme je l'ai expliqué dans la sous-section précédente, dans l'industrie minière on va réussir à obtenir des fragments assez petits à partir d'une pierre en lui faisant subir une série de transformations. On met la pierre dans une machine qui va la concasser de manière à obtenir des fragments plus petits qu'un certain seuil puis on va les mettre dans une nouvelle machine afin d'obtenir des fragments encore plus petits et ainsi de suite. Les machines existent déjà et l'on ne peut donc pas modifier les paramètres de la fragmentation. Le seul paramètre sur lequel on peut influer est la taille du tamis que l'on met à la sortie de la machine, c'est-à-dire le seuil à partir duquel on fait sortir les fragments.

Ainsi une question naturelle qui se pose est pour deux fragmentations de mesure de dislocation ν et ν' (c'est-à-dire pour deux machines données) et une pierre de masse 1, à quelle taille faut-il fixer le tamis de la première machine afin de minimiser le coût énergétique des deux fragmentations afin d'avoir au final des fragments de tailles tous inférieures à un ϵ fixé.

Dans ce travail, on pose les bases permettant de répondre à cette question. On réécrit l'énergie de deux fragmentations successives comme la somme de deux énergies (celle de la première fagmentation et celle de la deuxième). On va alors être capable d'expliciter complétement cette énergie (confère Théorème 4.1) en fonction de la mesure de Lévy et de la mesure de renouvellement de $\xi(t)$ (subordinateur associé à la première fragmentation) pris sous une certaine probabilité $\widetilde{\mathbb{P}}$ ainsi que celle de $\xi'(t)$ (subordinateur associé à la deuxième fragmentation) pris sous une autre probabilité $\widetilde{\mathbb{P}}'$. On donnera ensuite différentes pistes afin de pouvoir poursuivre l'étude du problème initial.

0.0.6 Conclusion

Il est intéressant de noter que si l'on avait des résultats plus précis concernant le comportement des processus markoviens auto-similaires, il serait possible d'approfondir de nombreux résultats. En effet si l'on avait une vitesse dans la convergence de $T_b = \inf\{t \geq 0 : \ X(t) > b\}$ (pour un processus X de Markov fort semi-stable à valeur dans $(0, \infty)$) de la proposition 3 de [19], on pourrait obtenir une vitesse dans la convergence d'un estimateur de α.

De même si l'on voulait étendre le premier article à des fragmentations plus générales, auto-similaires ou non conservatives par exemple, il faudrait généraliser les résultats de Lambert [56] et trouver une martingale associée à un processus de Lévy ayant des sauts positifs et négatifs.

Afin de répondre à la question du quatrième article il faut arriver à pouvoir caractériser le comportement de l'énergie en fonction du paramètre η'. Cela semble assez difficile, car même dans le cas particulier traité dans la sous-section 4.4.3 la réponse ne semble pas aisée.

Chapter 1

Multifractal spectra and precise rates of decay in homogeneous fragmentations.

We consider a mass-conservative fragmentation of the unit interval. Motivated by a result of Berestycki [9], the main purpose of this work is to specify the Hausdorff dimension of the set of locations having exactly an exponential decay. The study relies on an additive martingale which arises naturally in this setting, and a class of Lévy processes constrained to stay in a finite interval.

Contents

The material of this chapter is appeared in *SPA* (cf. [52]).

1.1 Introduction.

Fragmentation appears in a wide range of phenomena in science and technology, such as degradation of polymers, colloids, droplets, rocks,... See the proceedings [24] for some applications in physics, for example [50] for computer science, [21] for mineral crushing, and works quoted in [9] for some further references. This work is a contribution to the study of the rates of decay of fragments. More precisely, our aim is to investigate the set of locations which have an exact exponential decay (see (1.1.1) below for a precise definition).

Roughly a homogeneous fragmentation of intervals $F(t)$ can be seen as a family of nested open sets in $(0,1)$ such that each interval component is spill independently of the others, independently of the way that spill before, and with the same law as that of the initial fragmentation (up to spatial rescaling). We will suppose that no loss of mass occurs during the process.

Let $x \in (0,1)$ and $I_x(t)$ be the interval component of the fragmentation $F(t)$ which contains x, and $|I_x(t)|$ its length. Bertoin showed in [14] that if V is a uniform random variable on $[0,1]$ which is independent of the fragmentation, then $\xi(t) := -\log|I_V(t)|$ is a subordinator entirely determined by the fragmentation characteristics. By the SLLN for a subordinator, there exists v_{typ} such that $\frac{\xi(t)}{t} \to v_{typ}$ $a.s.$, which means that $|I_V(t)| \approx e^{-v_{typ}t}$. Berestycki [9] computed the Hausdorff dimension of the set

$$G_v := \left\{ x \in (0,1) \ : \ \lim_{t\to\infty} \frac{1}{t} \log|I_x(t)| = -v \right\}$$

for all $v > 0$. In this article we shall rather consider for some $0 < a < b$ the set

$$G_{(v,a,b)} := \left\{ x \in (0,1) \ : \ a \leq \liminf_{t\to\infty} e^{vt}|I_x(t)| \leq \limsup_{t\to\infty} e^{vt}|I_x(t)| \leq b \right\}. \qquad (1.1.1)$$

Our goal is to compute the Hausdorff dimension of the set $G_{(v,a,b)}$. Our approach relies on some results on Lévy processes constrained to stay in a given interval.

Firstly we will recall background on fragmentations and Lévy processes. Secondly we will consider an additive martingale M which is naturally associated to the problem and obtain a criterion for uniform integrability. This is used in Section 1.4 to derive some limit theorems which may be of independent interest (see Engländer and Kyprianou [33] for a related approach in the setting of spatial branching processes). Finally we will compute the Hausdorff dimension of $G_{(v,a,b)}$ in Section 1.5.

1.2 Preliminaries.

1.2.1 Definition of fragmentation.

We will recall some facts about homogeneous interval fragmentations, which are mostly lifted from [9] , [14] and [16]. More precisely, we will consider fragmentations defined on the space $\mathcal{U}$ of open subsets of $(0,1)$. We shall use the fact that every element U of $\mathcal{U}$ has an interval decomposition, i.e. there exists a collection of disjoint open intervals $(J_i)_{i\in I}$, where the set of indices I can be finite or countable, such that $U = \cup_{i\in I} J_i$. Each interval component is viewed as a fragment.

A homogeneous interval fragmentation is a Markov process with values in the space $\mathcal{U}$ which enjoys two keys properties. First the branching property: different fragments have independent evolutions. Second, the homogeneity property: up to an obvious spacial rescaling, the law of the fragment process does not depend on the initial length of the interval.

Specifically, if $\mathbb{P}$ stands for the law of the interval fragmentation F started from $F(0) = (0,1)$, then for $s,t \geq 0$ conditionally on the open set $F(t) = \underset{i\in I}{\cup} J_i(t)$, the interval fragmentation $F(t+s)$ has the same law as $F^1(s) \cup F^2(s) \cup ...$ where for each i, $F^i(s)$ is a subset of $J_i(t)$ and has the same distribution as the image of $F(s)$ by the homothetic map $(0,1) \to J_i(t)$.

1.2.2 Poissonian construction of the fragmentation.

Recall that $\mathcal{U}$ denotes the space of open subsets of $(0,1)$, and set $\mathbf{1} = (0,1)$. For $U \in \mathcal{U}$,

$$|U|^{\downarrow} := (u_1, u_2, ...)$$

will be the decreasing sequence of the interval component lengths of U. For $U = (a_1, b_1) \in \mathcal{U}$, we define the affine transformation $g_U : (0,1) \to U$ given by $g_U(x) = a_1 + x(b_1 - a_1)$.

In this article we will only consider proper fragmentations (which means that the Lebesgue measure of $F(t)$ is equal to 1). In this case, Basdevant [7] has shown that the law of the interval fragmentation F is completely characterized by the so-called dislocation measure ν (corresponding to the jump-component of the process) which is a measure on $\mathcal{U}$ which fulfills the conditions

$$\nu(\mathbf{1}) = 0,$$

$$\int_{\mathcal{U}} (1 - u_1)\nu(dU) < \infty, \tag{1.2.1}$$

and

$$\sum_{i=1}^{\infty} u_i = 1 \qquad for\ \nu - almost\ every\ U\ \in \mathcal{U}.$$

This last assumption is imposed by the hypothesis of length-conservation and means that when a sudden dislocation occurs, the total length of the intervals is unchanged. Specialists will notice that the erosion rates of the fragmentation c_r and c_l are here equal to 0 for the same reason.

We now recall the interpretation of sudden dislocations of the fragmentation process in terms of atoms of a Poisson point process (see [7], [8]). Let ν be a dislocation measure fulfilling the preceding conditions. Let $K = ((\Delta(t), k(t)), t \geq 0)$ be a Poisson point process with values in $\mathcal{U} \times \mathbb{N}$, and with intensity measure $\nu \otimes \sharp$, where $\sharp$ is the counting measure on $\mathbb{N}$. As in [8], we can construct a unique $\mathcal{U}$-valued process $F = (F(t), t \geq 0)$ started from $(0, 1)$, with paths that jump only for times $t \geq 0$ at which a point $(\Delta(t), k(t))$ occurs, and then $F(t)$ is obtained by replacing the $k(t)$-interval $J_{k(t)}(t-)$ by $g_{J_{k(t)}(t-)}(\Delta(t))$. This point of view will be used in Section 1.3.

Some information about the dislocation measure ν and therefore about the distribution of the homogeneous fragmentation F is contained in the function:

$$\kappa(q) := \int_{\mathcal{U}} \left(1 - \sum_{j=1}^{\infty} u_j^{q+1}\right) \nu(dU) \qquad \forall q > \underline{p} \tag{1.2.2}$$

with $\underline{p}$ the smallest real number for which κ remains finite :

$$\underline{p} := \inf \left\{ p \in \mathbb{R} : \int_{\mathcal{U}} \sum_{j=2}^{\infty} u_j^{p+1} \nu(dU) < \infty \right\}.$$

We have that $-1 \leq \underline{p} \leq 0$ (because $\int_{\mathcal{U}} (1 - u_1)\nu(dU) < \infty$ and $\sum_{i=1}^{\infty} u_i = 1$ for ν-almost every $U\ \in \mathcal{U}$).

This point of view is the same as in [9] and [14], which deal with ranked fragmentation instead of interval fragmentation. In the latter the space $\mathcal{U}$ is replaced by the space of mass partitions

$$\mathcal{S}^{\downarrow} := \left\{ x = (x_1, x_2, ...) \mid x_1 \geq x_2 \geq ... \geq 0\ , \sum_{i=1}^{\infty} x_i \leq 1 \right\}.$$

For the precise link between these two fragmentations see [7].

1.2.3 An important subordinator.

Let $x \in (0, 1)$ and $I_x(t)$ be the interval component of the random open set $F(t)$ which contains x, and $|I_x(t)|$ its length. Let V be a uniform random variable on $[0, 1]$ which is independent of the fragmentation.

Bertoin showed in [14] that

$$\xi(t) := -\log |I_V(t)|, \quad t \geq 0, \tag{1.2.3}$$

is a subordinator, with Laplace exponent $\kappa(q)$ defined in (1.2.2) (i.e. $\mathbb{E}(e^{-\lambda\xi(t)}) = e^{-t\kappa(\lambda)}$ for all $\lambda > \underline{p}$). In order to interpret this as a Lévy-Khintchine formula, we introduce the measure

$$L(dx) := e^{-x} \sum_{j=1}^{\infty} \nu(-\log u_j \in dx), \qquad x \in (0, \infty).$$

It is easy to check that $\int \min(1, x) L(dx) < \infty$, thus L is the Lévy measure of a subordinator, and we can check that $\kappa(q) = \int_{(0,\infty)} (1 - e^{qx})\, L(dx)$.

In this article we shall consider the Lévy process $Y_t = vt - \xi(t)$. In order to apply certain results to this process, we will need to assume that its one-dimensional distributions are absolutely continuous. Let L^{ac} be the absolutely continuous part of the measure L. Tucker has shown in [67] that

$$\int_{\mathbb{R}_+} \frac{1}{1+x^2} L^{ac}(dx) = \infty, \tag{1.2.4}$$

ensures the absolute continuity of one-dimensional distribution of the Lévy process evaluated at any $t > 0$. As $\int \min(1, x) L(dx) < \infty$, the condition (1.2.4) is equivalent to :

$$L^{ac}([0, \epsilon)) = \infty \quad \text{for any} \quad \epsilon > 0. \tag{1.2.5}$$

Let ν_1 be the image of the measure ν by the map $U \to u_1$ (recall that u_1 is the length of the longest interval component of the open set U) and ν_1^{ac} be the absolutely continuous part of the measure ν_1. Throughout this work we will make the following assumption, which is easily seen to imply (1.2.5) (in fact we can even show that the two are equivalent):

$$\nu_1^{ac}([0, \epsilon)) = \infty \quad \text{for any} \quad \epsilon > 0. \tag{1.2.6}$$

In the next subsection, we will give some results about Lévy processes that will be needed in the sequel, and apply for $Y_t = vt - \xi(t)$.

1.2.4 An estimate for completely asymmetric Lévy processes.

For the next sections, we will need some technical notions about completely asymmetric Lévy processes. Therefore we recall some facts mostly lifted from [10] and [12]. Let $Y = (Y_t)_{t\geq 0}$ be a Lévy process with no positive jumps and $(\mathcal{E}_t)_{t\geq 0}$ the natural filtration associated to $(Y_t)_{t\geq 0}$. The case where Y is the negative of a subordinator is degenerate for our purpose and therefore will be implicitly excluded in the rest of the article. The law of the Lévy process started at $x \in \mathbb{R}$ will be denoted by $\mathbf{P}_x$ (so bold symbols $\mathbf{P}$ and $\mathbf{E}$ refer to the Lévy process while $\mathbb{P}$ and $\mathbb{E}$ refer to

the fragmentation), its Laplace transform is given by

$$\mathbf{E}_0(e^{\lambda Y_t}) = e^{t\psi(\lambda)}, \quad \lambda,\ t \geq 0,$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}$ is called the Laplace exponent.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be the right inverse of ψ (which exists because $\psi : \mathbb{R}_+ \to \mathbb{R}$ is convex with $\lim_{t\to\infty} \psi(\lambda) = \infty$), i.e. $\psi(\varphi(\lambda)) = \lambda \quad \forall \lambda \geq 0$.

Let us recall some important features on the two-sided exit problem (which is completely solved in [12]). For $\beta > 0$ we denote the first exit time from $(0, \beta)$ by

$$T_\beta = \inf\{t :\ Y_t \notin (0, \beta)\}. \tag{1.2.7}$$

Let $W : \mathbb{R}_+ \to \mathbb{R}_+$ be the scale function, that is the unique continuous function with Laplace transform:

$$\int_0^\infty e^{-\lambda x} W(x)dx = \frac{1}{\psi(\lambda)} \quad ,\ \lambda > \varphi(0).$$

For $q \in \mathbb{R}$, let $W^{(q)} : \mathbb{R}_+ \to \mathbb{R}_+$ be the continuous function such that for every $x \in \mathbb{R}_+$

$$W^{(q)}(x) := \sum_{k=0}^{\infty} q^k W^{*k+1}(x),$$

where $W^{*n} = W * ... * W$ denotes the nth convolution power of the function W (for more details about this see [10] or [12]). So that

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x)dx = \frac{1}{\psi(\lambda) - q} \quad ,\ \lambda > \varphi(q).$$

The next statement is about the asymptotic behavior of the Lévy process killed when it exits $(0, \beta)$ (point 1 and 2), which is taken from [12], and about the Lévy process conditioned to remain in $(0, \beta)$ (point 3, 4 and 5), which is taken from Theorem 3.1 (ii) and Proposition 5.1 (i) and (ii) in [56] :

Theorem 1.1. *Let us define the transition probabilities*

$$P_t(x, A) := \mathbf{P}_x(Y_t \in A, t < T_\beta) \text{ for } x \in (0, \beta) \text{ and } A \in \mathcal{B}((0, \beta)),$$

and the critical value

$$\rho_\beta := \inf\{q \geq 0\ ;\ W^{(-q)}(\beta) = 0\}, \tag{1.2.8}$$

Suppose that the one-dimensional distributions of the Lévy process are absolutely continuous. Then the following holds:

1. $\rho_\beta \in (0, \infty)$ *and the function* $W^{(-\rho_\beta)}$ *is strictly positive on* $(0, \beta)$

2. *Let* $\Pi(dx) := W^{(-\rho_\beta)}(\beta - x)dx$. *For every* $x \in (0, \beta)$:

$$\lim_{t\to\infty} e^{\rho_\beta t} P_t(x, .) = cW^{(-\rho_\beta)}(x)\Pi(.)$$

in the sense of weak convergence, where

$$c := \left(\int_0^\beta W^{(-\rho_\beta)}(y) W^{(-\rho_\beta)}(\beta - y)dy \right)^{-1}.$$

3. *The process*

$$D_t := e^{\rho_\beta t}\ \mathbf{1}_{\{t<T_\beta\}}\ \frac{W^{(-\rho_\beta)}(Y_t)}{W^{(-\rho_\beta)}(x)} \tag{1.2.9}$$

is a $(\mathbf{P}_x, (\mathcal{E}_t))$*-martingale.*

4. *The mapping* $(x, q) \mapsto W^{(q)}(x)$ *is of class* $\mathcal{C}^1$ *on* $(0, \infty) \times (-\infty, \infty)$.

5. *The mapping* $\beta \mapsto \rho_\beta = \inf\{q > 0 : W^{(-q)}(\beta) = 0\}$ *is strictly decreasing and of class* $\mathcal{C}^1$ *on* $(0, \infty)$.

Remark 1.1. *The definition of* ρ_β *is of course complicated, however in the simple case when* Y *is a standard Brownian motion, we have:*

$$\rho_\beta = \pi^2/\beta^2 \quad \textit{and} \quad W^{(-\rho_\beta)}(x) = \frac{\beta}{\pi} \sin\left(\frac{\pi}{\beta} x\right).$$

In the case where Y *is a standard stable process, the mapping of* $\beta \to \rho_\beta$ *is depicted in [11]. We also point at the more explicit lower bound (see Lemma 5 in [12]):*

$$\rho_a \geq 1/W(a),$$

Another lower bound will be given in Remark 1.4 below.

Remark 1.2. *The formula for the constant* c *in part 2. of Theorem 1.1 stems from the relation*

$$e^{\rho_\beta t} \frac{W^{(-\rho_\beta)}(y)}{W^{(-\rho_\beta)}(x)} P_t(x, dy) \underset{t\to\infty}{\sim} cW^{(-\rho_\beta)}(\beta - y)W^{(-\rho_\beta)}(y)dy.$$

Integrating over $(0, \beta)$ *and using the fact that* D_t *is a martingale yields the given expression.*

We also refer to the recent article of T. Chan and A. Kyprianou [29] for further properties of $W^{(-\rho_\beta)}$.

Now we have recalled the background that is needed to solve our problem.

1.3 An additive martingale.

Now we turn our attention to the main purpose of this article and consider a homogeneous interval fragmentation $(F(t), t \geq 0)$ and some real numbers $v > 0$ and $0 < a < b$. We are interested in the asymptotic set:

$$G_{(v,a,b)} = \left\{ x \in (0,1) \ : \ a \leq \liminf_{t\to\infty} e^{vt}|I_x(t)| \leq \limsup_{t\to\infty} e^{vt}|I_x(t)| \leq b \right\},$$

with $|I_x(t)|$ the length of the interval component of $F(t)$ which contains x.

In order to do that, we will have to consider first the non asymptotic set:

$$\Lambda_{(v,a,b)} = \left\{ x \in (0,1) \ : \ ae^{-vt} < |I_x(t)| < be^{-vt} \ \forall t \geq 0 \right\},$$

for $0 < a < 1 < b$.

In this section and in the next we will assume that $0 < a < 1 < b$.

We introduce some notation, that we will need in the rest of the article: define the set of the "good" intervals at time t as

$$G(t) := \{ I_x(t) : \ x \in (0,1) \ \text{ and } \ ae^{-vs} < |I_x(s)| < be^{-vs} \ \ \forall \ s \leq t \}. \tag{1.3.1}$$

Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of the interval fragmentation $(F(t), t \geq 0)$. Let $(\mathcal{G}_t)_{t\geq 0}$ be the enlarged filtration defined by $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(I_V(t))$ where V is a uniform variable independent of the fragmentation). We can remark that for all t we have $\mathcal{G}_t \subsetneq \mathcal{F}_t \vee \sigma\{V\}$, and $\mathcal{G}_\infty = \mathcal{F}_\infty \vee \sigma\{V\}$.

We recall that $\xi(t) = -\log|I_V(t)|$ is a subordinator. More precisely we are interested in the Lévy process with no positive jump $Y_t := vt - \xi(t) + \log(1/a)$, and use the results of preceding subsection for this Lévy process. We remark that its Laplace exponent $\psi(\lambda)$ is equal to $v\lambda - \kappa(\lambda)$, with κ defined in Subsection 1.2.3. Since we have supposed (1.2.6), the one-dimensional distributions of the Lévy process Y_t are absolutely continuous and we can apply Theorem 1.1.

For this Lévy process Y let

$$T := T_{\log(b/a)}$$

and

$$\rho := \rho_{\log(b/a)},$$

where T_β is defined in (1.2.7) and ρ_β is defined in (1.2.8). We stress that ρ depends on v, a, b and κ.

To simplify the notation, let also

$$h(t) := W^{(-\rho)}(t - \log a)\mathbf{1}_{\{t\in(\log a, \log b)\}}$$

for all $t \in \mathbb{R}$, and $h(-\infty) = 0$.

By rewriting (1.2.9) with the new notation we get a $(\mathcal{G}_t)$-martingale

$$D_t = e^{\rho t}\ \mathbf{1}_{\{t<T\}}\ \frac{h(vt + \log|I_V(t)|)}{h(0)}, \qquad t \geq 0.$$

If I is an interval component of $F(t)$, we define the "killed" interval $I^\dagger$ by $I^\dagger = I$ if I is good (i.e. $I \in G(t)$ with $G(t)$ defined in (1.3.1)), else by $I^\dagger = \emptyset$. Projecting the martingale D_t on the sub-filtration $(\mathcal{F}_t)_{t\geq 0}$, we obtain an additive martingale

$$M_t := \frac{e^{\rho t}}{h(0)} \int_0^1 h(vt + \log|I_x^\dagger(t)|)\ dx\ , \qquad t \geq 0.$$

We notice that if $y \in I_x(t)$, then $I_y(t) = I_x(t)$. Now we will consider the interval decomposition $(J_i(t), J_2(t), ...)$ of the open $F(t)$ (see subsection 1.2.1). We can rewrite M_t as:

$$M_t = \frac{e^{\rho t}}{h(0)} \sum_{i\in\mathbb{N}} h\left(vt + \log|J_i^\dagger(t)|\right)|J_i^\dagger(t)|. \tag{1.3.2}$$

We will use this expression in the rest of the article.

Finally, let the absorption time of M_t at 0 be

$$\begin{aligned} \zeta &:= \inf\{t : M_t = 0\} \\ &= \inf\{t : G(t) = \emptyset\}, \end{aligned}$$

with the convention $\inf\emptyset = \infty$.

Our first result is:

Theorem 1.2. *In the previous notation, with the assumptions (1.2.6) and if $v > \rho$ holds, then:*

1. *The martingale M_t is bounded in $\mathrm{L}^2(\mathbb{P})$.*
2. *Conditionally on $\zeta = \infty$, we have: $\lim_{t\to\infty} M_t > 0$.*

Remark 1.3. *We stress that as ρ depends on v, a, b and κ, the condition $v > \rho$ involves implicitly the parameters a and b. In particular it forces $b > 2a$, otherwise there would never be more than one "good" interval (as a fragment of size x will split into at least two different fragments and the smallest one will have a size at most equal to $x/2$), and as a consequence we would have $M_\infty = 0$ a.s., in contradiction with the uniform integrability of $M_\cdot$.*

The proof of Theorem 1.2.1. is given in the appendix.

In order to prove Theorem 1.2.2 we will first introduce some notation, then prove two lemmas, and after we will conclude.

Let I be an interval of $(0,1)$. The law of the homogeneous interval fragmentation started at I will be denoted by $\mathbb{P}_I$. We remark that $\mathbb{P}_I(M_\infty = 0|\zeta = \infty)$ only depends on the length of I. Therefore we define

$$g(x) := \mathbb{P}_I(M_\infty = 0|\zeta = \infty),$$

where I is an interval such that $|I| = x$. Let N be the integer part of $(2b-a)/a$. As we assume $v > \rho$, we have necessarily $b > 2a$ (see Remark 1.3), thus $N \geq 2$. Let $\eta := (b-a)N^{-1}$. We remark that $\eta < a$ and $b - a = N\eta$. Denote the first time when there are at least two good intervals by

$$T^F := \inf\{t : \sharp G(t) \geq 2\},$$

with the convention $\inf \emptyset = \infty$. We notice that T^F is an $(\mathcal{F}_t)$ stopping time as $\sharp G(t)$ is $\mathcal{F}_t$-adapted.

Lemma 1.1. *In the previous notation, supposing that (1.2.6) and $v > \rho$ hold, we get: for every open interval I*

$$\mathbb{P}_I(T^F = \infty|\zeta = \infty) = 0.$$

Proof. We notice that, as the martingale M_t is not identically 0 and is uniformly integrable, we have $\mathbb{P}_I(T^F = \infty|\zeta = \infty) < 1$ (because $M_\infty = 0$ when $T^F = \infty$).

Let I be an open interval such that $|I| \in (a,b)$, $t_0 := \log(2b/a)/v$ and $\epsilon := a^2/(2b^2)$. Thus

$$|I|(1-\epsilon) > a/2 \geq be^{-vt_0} \quad \text{and} \quad |I|\epsilon < b\epsilon \leq ae^{-vt_0}$$

therefore, if the dislocation of I produces at time t_0 an interval of length at least $|I|(1-\epsilon)$ then this interval is too large to be good and the remaining ones are too small to be good either. As a consequence we have

$$\mathbb{P}_I(M_{t_0} = 0) \geq \mathbf{P}_{\log|I|}(e^{-\xi(t_0)} > e^{-\log|I|}(1-\epsilon)) = \mathbf{P}(\xi(t_0) < -\log(1-\epsilon)),$$

by the homogeneous property of the fragmentation. Moreover since $\xi(t)$ is a subordinator, we get $p := \mathbf{P}(\xi(t_0) < -\log(1-\epsilon)) > 0$, therefore

$$\mathbb{P}_I(M_{t_0} = 0) \quad \geq \quad p > 0. \tag{1.3.3}$$

Additionally for every open interval I such that $|I| \in (a,b)$:

$$\mathbb{P}_I(\sharp G(t) = 1 \ \forall t \leq t_0) \leq 1 - \mathbb{P}_I(M_{t_0} = 0) \leq 1 - p.$$

Using the strong Markov property of the fragmentation and (1.3.3) we find by

induction that for all $k \in \mathbb{N}$:

$$\mathbb{P}_I(\sharp G(t) = 1 \ \forall t \leq kt_0) \leq (1-p)^k.$$

Therefore

$$\lim_{t\to\infty} \mathbb{P}_I(\sharp G(s) = 1 \ \forall s \leq t) = 0$$

and as a consequence

$$\mathbb{P}_I(T^F = \infty | \zeta = \infty) = 0.$$

□

Lemma 1.2. *In the previous notation, supposing that (1.2.6) and $v > \rho$ hold, we get:*

$$\sup_{a<x<b} g(x) = \max_{1\leq k\leq N} g(a + k\eta),$$

where $N = \lfloor (2b-a)/a \rfloor$ and $\eta = (b-a)/N$.

Proof. We will prove this lemma by induction.

The hypothesis of induction is for $n \leq N$:

$$(H)_n \ : \qquad \sup_{x\in(a,a+n\eta)} g(x) = \max_{1\leq k\leq n} g(a+\eta k).$$

$*$ The case $n = 1$: let I be an open interval such that $|I| \in (a, a+\eta)$. We work under $\mathbb{P}_I$ conditionally on "non-extinction" (which means conditionally on the event $\zeta = \infty$). Let

$$T^1 := \inf\{t \geq 0 | \ \exists J(t) \in G(t) \ : \ e^{vt}|J(t)| \notin (a, a+\eta)\},$$

with $G(t)$ defined in (1.3.1). The random time T^1 is an $(\mathcal{F}_t)$ stopping times. As the quantity $vt - (-\log|J(t)|)$ creeps upwards with probability equals to 1 and as $J(t) \in G(t)$ implies that $e^{vt}|J(t)| > a$, we get

$$T^1 = \inf\{t \geq 0 | \ \exists J(t) \in G(t) \ : \ e^{vt}|J(t)| = a+\eta\}.$$

Moreover by the choice of η we have $a + \eta < 2a$, which implies that there is at most one good interval whose length is always in $(a, a+\eta)$. Recall from Lemma 1.1 that $\mathbb{P}_I(T^F < \infty | \zeta = \infty) = 1$, thus

$$\mathbb{P}_I(T^1 < \infty | \zeta = \infty) = 1.$$

Using the strong Markov property at the stopping times T^1, we get

$$g(x) \leq g(a+\eta) \quad , \quad x \in (a, a+\eta),$$

thus $(H)_1$ holds.

$*$ The case $n+1$ (with $n+1 \leq N$): we suppose that the hypothesis of induction holds for all $k \leq n$.

Let I be an open interval such that $|I| \in (a+n\eta, a+(n+1)\eta)$. We work under $\mathbb{P}_I$ conditionally on "non-extinction". Let

$$T^n := \inf\{t \geq 0 |\ \exists J(t) \in G(t)\ :\ e^{vt}|J(t)| \notin (a+n\eta, a+(n+1)\eta)\},$$

with $G(t)$ defined in (1.3.1). The random time T^n is an $(\mathcal{F}_t)$ stopping times. As the quantity $e^{vt}|J(t)|$ grows only continuously, we get

$$T^n = \inf\{t \geq 0 |\ \exists J(t) \in G(t)\ :\ e^{vt}|J(t)| = a+(n+1)\eta \text{ or } e^{vt}|J(t)| \in (a, a+n\eta]\}.$$

Moreover by the choice of η we have $a+\eta < 2a$, which implies that there is at most one good interval which length is always in $(a+n\eta, a+(n+1)\eta)$. Additionally by Lemma 1.1, we get $\mathbb{P}_I(T^F < \infty | \zeta = \infty) = 1$, thus

$$\mathbb{P}_I(T^n < \infty | \zeta = \infty) = 1.$$

Using the strong Markov property at the stopping times T^n, we get

$$g(|I|) \ \leq\ \max\left(g(a+(n+1)\eta), \sup_{y \in (a, a+n\eta]} g(y)\right).$$

As this holds for every open interval I such that $|I| \in (a+n\eta, a+(n+1)\eta)$, by the hypothesis of induction, we have established $(H)_{n+1}$. □

Proof of Theorem 1.2.2. With Lemma 1.2, we get that there exists a integer k_0 in $[1, N]$ such that $g(a+\eta k_0) = \sup_{x \in (a,b)} g(x)$ (if two or more values of k, are possible, we choose the smallest one). Let x_0 be $a+\eta k_0$.

Additionally, with Lemma 1.1, we get $\mathbb{P}_{(0,x_0)}(T^F < \infty | \zeta = \infty) = 1$. Using the strong property of Markov for the stopping times T^F, and with $n \geq 2$ the random number of good intervals of the fragmentation at time T^F and with $\alpha_1, ..., \alpha_n$ the length of those intervals, we get:

$$g(x_0) \leq \mathbb{E}(g(\alpha_1)...g(\alpha_n)) \leq \mathbb{E}(g(x_0)^n) \leq g(x_0)^2.$$

As $g(x_0) < 1$ by the uniformly integrability of M_t, we get that $g(x_0) = 0$ and finally that $g \equiv 0$. □

1.4 Limit theorems.

In this section, we establish two corollaries of Theorem 1.2, which will be useful in the sequel.

Bertoin and Rouault (Corollary 2 in [22]) proved that

$$\lim_{t\to\infty}\frac{1}{t}\log\sharp\{I_x(t):\ ae^{-vt}<|I_x(t)|<be^{-tv}\}=C(v), \tag{1.4.1}$$

where $C(v):=(\Upsilon_v+1)v-\kappa(\Upsilon_v)$ and Υ_v is the reciprocal of v by κ' i.e, $\kappa'(\Upsilon_v)=v$ for $v\in(v_{min},v_{max})$. [1]

Here we deal with the more stringent requirement: $\forall s\leq t$, $|I_x(s)|\in(ae^{-sv},be^{-sv})$, and the next proposition gives the rates that we find in that case.

Proposition 1.1. *In the notation of the previous sections, with the assumptions (1.2.6) and if $v>\rho$ we get that conditionally on $\zeta=\infty$ (i.e. M is not absorbed at 0, or in a equivalent way $\Lambda_{(v,a,b)}\neq\emptyset$):*

$$\lim_{t\to\infty}\frac{1}{t}\log\sharp G(t)=\ v-\rho\qquad a.s. \tag{1.4.2}$$

Before proving this corollary we make the following remark

Remark 1.4. *It is interesting to compare the estimate found by Bertoin and Rouault and the present one (of course we have not considered the same set, nevertheless the two estimates are related). For this we show that for all $v\in(v_{min},v_{max})$ and a and b such that $\rho\geq v_{min}$ we have $C(v)\geq v-\rho$. In this direction we use results from [9] Section 1. Let $\Psi(p):=p\kappa'(p)-\kappa(p)$ for all $p>0$ with κ' the derivative of κ (this function is well defined because of the definition of $\underline{p}$ in Section 2 and because $\underline{p}\leq 0$). For every $p>0$, $\Psi'(p)=p\kappa''(p)\leq 0$ since κ is concave. As a consequence Ψ is decreasing. With the definition of Υ_v, we get that the function $v\in(v_{min},v_{max})\mapsto\Upsilon_v\in\mathbb{R}$ is decreasing, additionally $\Upsilon_{v_{min}}>0$, therefore the function $v\in(v_{min},v_{max})\mapsto g(\Upsilon_v)\in\mathbb{R}$ is increasing. Moreover $\Psi(\Upsilon_v)=C(v)-v$, hence for all $v\in(v_{min},v_{max})$:*

$$C(v)-v\geq C(v_{min})-v_{min}=-v_{min}.$$

Additionally as $\rho\geq v_{min}$, we finally obtain:

$$\forall v\in(v_{min},v_{max})\qquad C(v)\geq v-\rho.$$

As a consequence, we have checked that the rate of growth of $\sharp G(t)$ (defined in (1.3.1)) is lower that of $\sharp\{I_x(t):\ |I_x(t)|\in(ae^{-tv},be^{-vt})\}$, which was of course expected.

Proof. In this proof we work conditionally on $\zeta=\infty$ (i.e M is not absorbed at 0). Applying Theorem 1.2, we get $M_\infty>0$. In order to show that (1.4.2) holds, we will first look at the lower bound of the inequality, and then at the upper bound.

[1] Where v_{min} is the maximum of the function $p\mapsto\kappa(p-1)/p$ on $(\underline{p}+1,\infty)$ and $v_{max}:=\kappa'(\underline{p}^+)$ (see [9]).

• With the definition of M_t in (1.3.2), of $G(t)$ and of $J_i^\dagger(t)$ at the beginning of Section 1.3 and by the conditioning, there exists $t^{'} > 0$ such that for all $t \geq t^{'}$:

$$\frac{M_\infty}{2} \leq \frac{e^{\rho t}}{h(0)} \sum_{i\in\mathbb{N}} h(vt + \log(|J_i^\dagger(t)|))\ |J_i^\dagger(t)| \leq \frac{e^{\rho t}}{h(0)} \sum_{i\in\mathbb{N}} C_4\ be^{-vt}\ \mathbf{1}_{\{J_i(t)\in G(t)\}},$$

with C_4 as maximum of $h(.)$ on $[\log a, \log b]$. Hence for all $t \geq t^{'}$:

$$\sharp G(t) \geq\ e^{(v-\rho)t} \frac{h(0)}{2C_4 b} M_\infty,$$

and as a consequence, conditionally on $\zeta = \infty$,

$$\liminf_{t\to\infty} \frac{1}{t} \log \sharp G(t) \geq\ v - \rho. \tag{1.4.3}$$

• Secondly we will show the converse inequality.

Let $0 < a^{'} < a < 1 < b < b^{'}$, and $\rho^{'} := \rho_{\log(b^{'}/a^{'})}$. Denote the set of "good" intervals associated to $a^{'}$ and $b^{'}$ by:

$$G^{'}(t) := \{I_x(t) :\quad x \in (0,1)\ \text{ and }\ \ |I_x(s)|\ \in\ (a^{'}e^{-vs}, b^{'}e^{-vs})\quad \forall\ s \leq t\}.$$

Let $M_t^{'}$ be the martingale defined at the beginning of Section 1.3 (and denoted there by M) associated to $a^{'}, b^{'}$ instead of a, b. Plainly, if M_t is not absorbed at 0, then a fortiori $M_t^{'}$ is not absorbed at 0 either. Additionally, since $\log(b^{'}/a^{'}) > \log(b/a)$, and $\rho_{.}$ is strictly decreasing (see Theorem 1.1.5), we get $v > \rho > \rho^{'}$ and we may apply Theorem 1.2 for $a^{'}, b^{'}$ instead of a, b. We get $\lim_{t\to\infty} M_t^{'} = M_\infty^{'} > 0$.

With the definition (1.3.2) of M_t and with an analogue of the function $h(t)$, namely $t \in \mathbb{R}$

$$\varphi(t) := W^{(-\rho^{'})}(t + \log(1/a^{'}))\mathbf{1}_{\{t\in(\log a^{'}, \log b^{'})\}},$$

we get:

$$M_\infty^{'} \ =\ \lim_{t\to\infty} \frac{e^{\rho^{'}t}}{\varphi(0)} \sum_{i\in\mathbb{N}} \varphi(vt + \log|J_i(t)|)|J_i(t)|\ \mathbf{1}_{\{J_i(t)\in G^{'}(t)\}}.$$

Therefore there exists $t^{'} > 0$ such that for every $t \geq t^{'}$

$$2M_\infty^{'} \ \geq\ \frac{e^{\rho^{'}t}}{\varphi(0)} \sum_{i\in\mathbb{N}} \varphi(vt + \log|J_i(t)|)|J_i(t)|\ \mathbf{1}_{\{J_i(t)\in G^{'}(t)\}}$$

$$\geq\ \frac{e^{\rho^{'}t}}{\varphi(0)} \sum_{i\in\mathbb{N}} \varphi(vt + \log|J_i(t)|)\ a^{'}e^{-vt}\ \mathbf{1}_{\{J_i(t)\in G(t)\}}.$$

Since $(ae^{-vt}, be^{-vt}) \subsetneq (a^{'}e^{-vt}, b^{'}e^{-vt})$, we get by Theorem 1.1.1, that for all $x\ \in$

$[\log a, \log b]$: $\varphi(x) > 0$. Because $[\log a, \log b]$ is compact and $\varphi(.)$ is a continuous function,

$$\inf_{x\in[\log a,\log b]} \varphi(x) > 0.$$

Combining this with

$$C_5 := 2M_\infty^{'}\varphi(0)/\left(a^{'} \inf_{x\in[\log a,\log b]} \varphi(x)\right) < \infty,$$

we get for all $t \geq t^{'}$:

$$C_5 \geq e^{(\rho^{'}-v)t} \sum_{i\in\mathbb{N}} \mathbf{1}_{\{J_i(t)\in G(t)\}}$$

and thus

$$C_5 e^{(v-\rho^{'})t} \geq \sharp G(t).$$

Hence for all $a^{'}, b^{'}$ such that $0 < a^{'} < a < 1 < b < b^{'}$:

$$\limsup_{t\to\infty} \frac{1}{t} \log \sharp G(t) \leq v - \rho^{'}.$$

For $a^{'} \to a$ and $b^{'} \to b$ we get by the continuity of $\rho_{.}$:

$$\limsup_{t\to\infty} \frac{1}{t} \log \sharp G(t) \leq v - \rho.$$

□

Now we will give an other corollary, using the same method as that of Bertoin and Gnedin in [20]. We encode the configuration $J^\dagger(t) = \{|J_i^\dagger(t)|\}$ of the lengths of good intervals into the random measure

$$\sigma_t := \frac{e^{\rho t}}{h(0)} \sum_{i\in\mathbb{N}} h\left(vt + \log|J_i^\dagger(t)|\right) |J_i^\dagger(t)| \delta_{\log(1/a)+vt+\log|J_i^\dagger(t)|}$$

which has total mass M_t.

The associated mean measure σ_t^* is defined by the formula

$$\int_0^\infty f(x)\sigma_t^*(dx) = \mathbb{E}\left(\int_0^\infty f(x)\sigma_t(dx)\right)$$

which is required to hold for all compactly supported continuous functions f. Since M_t is a martingale, σ_t^* is a probability measure. More precisely the next proposition establishes the convergence of the mean measure σ_t^*, and then of σ_t itself.

Proposition 1.2. *In the notation of the previous sections, with the assumptions (1.2.6), and $v > \rho$ we get:*

1. *The measures σ_t^* converge weakly, as $t \to \infty$, to the probability measure*

$$\varrho(dy) := ch(y + \log a)h(\log(b) - y)dy$$

where $c > 0$ is the constant that appears in Theorem 1.1.5.

2. *For any bounded continuous f*

$$L^2 - \lim_{t\to\infty} \int_0^\infty f(x)\sigma_t(dx) = M_\infty \int_0^\infty f(x)\varrho(dx). \tag{1.4.4}$$

Proof. 1. Firstly we prove the convergence of the mean measures $\sigma_t^* \to \varrho$. Let f be a bounded continuous function. By definition we get:

$\int_0^\infty f(y)\sigma_t^*(dy)$

$$= \mathbb{E}\left(\int_0^1 f(\log(1/a) + vt + \log|I_x^\dagger(t)|)\frac{e^{\rho t}}{h(0)}h\left(vt + \log|I_x^\dagger(t)|\right)\mathbf{1}_{\{I_x^\dagger(t)\in G(t)\}}dx\right)$$

$$= \mathbf{E}_{\log(1/a)}\left(f(Y_t)e^{\rho t}\frac{h(Y_t + \log a)}{h(0)}\mathbf{1}_{\{t<T\}}\right),$$

with the definition of Y_t. Thus by the definition of P_t in Theorem 1.1, we get

$$\int_0^\infty f(y)\sigma_t^*(dy) = \int_0^{\log(b/a)} f(y)\frac{h(y + \log a)}{h(0)}e^{\rho t}P_t(\log(1/a), dy).$$

By Theorem 1.1.2, we get

$$\int_0^\infty f(y)\sigma_t^*(dy) \underset{t\to\infty}{\sim} c\int_0^{\log(b/a)} f(y)h(y + \log a)h(\log(b) - y)dy.$$

Therefore the measure σ_t^* converge weakly to the probability measure ϱ.

2. Now we show that the scaled empirical measures induced by $J(t)$ converge in the L^2-sense to the random measure $M_\infty\varrho$.

Let f_1 and f_2 be two continuous functions bounded from above by 1, and

$$S_t = \sum_{i,j} f_1(\log(1/a) + vt + \log|J_i^\dagger(t)|)\frac{e^{\rho t}}{h(0)}h\left(vt + \log|J_i^\dagger(t)|\right)|J_i^\dagger(t)|$$

$$\times f_2(\log(1/a) + vt + \log|J_j^\dagger(t)|)\frac{e^{\rho t}}{h(0)}h\left(vt + \log|J_j^\dagger(t)|\right)|J_j^\dagger(t)|.$$

We need to show that

$$\mathbb{E}\left(S_t\right) \to \left(\int_0^\infty f_1(x)\varrho(dx)\right)\left(\int_0^\infty f_2(x)\varrho(dx)\right)\mathbb{E}\left(M_\infty^2\right) \quad (1.4.5)$$

for f_1 and f_2 positive and bounded from above by 1. Indeed, suppose (1.4.5) is shown. Denote

$$A_t = \sum_j f_1(\log(1/a) + vt + \log|J_i^\dagger(t)|)\frac{e^{\rho t}}{h(0)}h\left(vt + \log|J_j^\dagger(t)|\right)|J_j^\dagger(t)|.$$

Take $f_2 = 1$ to conclude from (1.4.5) that

$$\lim_{t\to\infty}\mathbb{E}(A_t M_t) = \int_0^\infty f_1(x)\varrho(dx)\mathbb{E}\left(M_\infty^2\right).$$

Similarly, by setting $f_1 = f_2$ we get

$$\lim_{t\to\infty}\mathbb{E}\left(A_t^2\right) = \left(\int_0^\infty f_1(x)\varrho(dx)\right)^2\mathbb{E}\left(M_\infty^2\right).$$

Recalling that $\mathbb{E}(M_t^2) \to \mathbb{E}(M_\infty^2)$ and combining the above we get the desired

$$\lim_{t\to\infty}\mathbb{E}\left[\left(A_t - M_t\int_0^\infty f_1(x)\varrho(dx)\right)^2\right] = 0.$$

To prove (1.4.5) let us replace t by $t+s$ and condition on $J^\dagger = (|J_i^\dagger(s)|)_{i\in\mathbb{N}}$. We have two cases: write $i \sim_s j$ for the case where at time $t+s$ two coexisting intervals $J_i^\dagger(t+s)$ and $J_j^\dagger(t+s)$ stem from the same interval at time s, and $i \nsim_s j$ for the case these intervals are not included into the same interval component at time s. Therefore, with the notation

$$S_{t+s}^{(1)} := \mathbb{E}\left(\sum_{i\sim_s j} S_{t+s} \mid J^\dagger(s)\right) \quad \text{and} \quad S_{t+s}^{(2)} := \mathbb{E}\left(\sum_{i\nsim_s j} S_{t+s} \mid J^\dagger(s)\right)$$

we get:

$$S_{t+s}^{(1)} + S_{t+s}^{(2)} = \mathbb{E}(S_{t+s}|J^\dagger(s)).$$

For the studies of $S_{t+s}^{(1)}$ we use the homogeneous property of the fragmentation and the notation $I_0 = (0, \log(b/a))$, and get

$$
\begin{aligned}
&|S^{(1)}_{t+s}| \\
&\leq \sum_i |J_i^\dagger(s)|^2 e^{2\rho s}\mathbb{E}\left(\sum_j |J_j^\dagger(t)|e^{\rho t}\right)^2 \sup_{x\in I_0}\left(\frac{h(x+\log a)}{h(0)}\right)^2 \sup_{x\in I_0}|f_1(x)| \sup_{x\in I_0}|f_2(x)| \\
&\leq be^{(\rho-v)s}C_6,
\end{aligned}
$$

with

$$
C_6 := \sum_i |J_i^\dagger(s)|e^{\rho t}\mathbb{E}\left(\sum_j |J_j^\dagger(t)|e^{\rho t}\right)^2 \sup_{x\in I_0} h(x+\log(a))^2 \sup_{x\in I_0}|f_1(x)| \sup_{x\in I_0}|f_2(x)|/h(0)^2
$$

which is finite because

$$
\mathbb{E}\left(\sum_j |J_j^\dagger(t)|e^{\rho t}\right) = \mathbf{E}\left(\mathbf{1}_{\{t<T\}}e^{\rho t}\right) < \infty.
$$

Thus $S^{(1)}_{t+s} \to 0$ as $s \to \infty$ uniformly in t.

Now we look at $S^{(2)}_{t+s}$. We introduce the notation $y_k = |J_k^\dagger(s)|$. Write $i \searrow k$ if the length $|J_i^\dagger(t+s)|$ stems from y_k. By independence, the intervals which are included in the interval with length y_k and those which are included in the interval with length y_l evolve independently, thus gathering the lengths $|J_i^\dagger(t+s)|$ by the ancestors at time s yields

$$
S^{(2)}_{t+s} = \sum_{k\neq l}\left(\mathbb{E}\sum_{i\searrow k} \ldots\right)\left(\mathbb{E}\sum_{j\searrow l} \ldots\right).
$$

On the other hand, by self-similarity and convergence of the mean measures

$$
\begin{aligned}
&\mathbb{E}\left(\sum_{i\searrow k} e^{\rho s}\frac{h\left(vs+\log\left(y_k/a\right)\right)}{h(0)} y_k \; f_1(vt+\log(|J_i^\dagger(t)|)+vs+\log(y_k/a))\right. \\
&\left. e^{\rho t}\frac{h\left(vt+\log\left(|J_i^\dagger(t)|\right)+vs+\log(y_k/a)\right)}{h\left(vs+\log\left(y_k/a\right)\right)}|J_i^\dagger(t)| \;\middle|\; J^\dagger(s)\right) \\
&\underset{t\to\infty}{\to} \quad e^{\rho s}\frac{h\left(vs+\log\left(y_k/a\right)\right)}{h(0)} y_k\left(\int_0^\infty f_1(x)\varrho(dx)\right),
\end{aligned}
$$

and

$$\mathbb{E}\left(\sum_{j\searrow l} e^{\rho s}\frac{h\left(vs+\log\left(y_l/a\right)\right)}{h(0)}y_l\ f_2(vt+\log(|J_j^\dagger(t)|)+vs+\log(y_l/a))\right.$$

$$\left. e^{\rho t}\frac{h\left(vt+\log\left(|J_j^\dagger(t)|\right)+vs+\log(y_l/a)\right)}{h\left(vs+\log\left(y_l/a\right)\right)}|J_j^\dagger(t)|\ \middle|\ J^\dagger(s)\right)$$

$$\underset{t\to\infty}{\to}\quad e^{\rho s}\frac{h\left(vs+\log\left(y_l/a\right)\right)}{h(0)}y_l\left(\int_0^\infty f_2(x)\varrho(dx)\right).$$

Therefore by dominated convergence

$$\mathbb{E}\left(S^{(2)}_{t+s}\right)\ \underset{s\to\infty}{\sim}\ \left(\int_0^\infty f_1(x)\varrho(dx)\right)\left(\int_0^\infty f_2(x)\varrho(dx)\right)\mathbb{E}\Big(\sum_{k\neq l}\frac{e^{\rho s}}{h(0)}|J_k^\dagger(s)|$$

$$h\left(vs+\log(|J_k^\dagger(s)|/a)\right)\frac{e^{\rho s}}{h(0)}h\left(vs+\log(|J_l^\dagger(s)|/a)\right)|J_l^\dagger(s)|\Big).$$

Moreover with $C_7:=b\sup_{x\in I_0}|h(x+\log a)|^2/h(0)^2$, we get

$$\mathbb{E}\left(\sum_k e^{2\rho s}\frac{h\left(vs+\log\left(|J_k^\dagger(s)|/a\right)\right)^2}{h(0)^2}|J_k^\dagger(s)|^2\right)\le C_7\mathbb{E}\left(\sum_k e^{\rho s}|J_k^\dagger(s)|\right)e^{(\rho-v)s}$$

which goes to 0 when $s\to\infty$, as a consequence

$$\mathbb{E}\left(\sum_{k\neq l} e^{\rho s}\frac{h\left(vs+\log\left(|J_k^\dagger(s)|/a\right)\right)}{h(0)}|J_k^\dagger(s)|e^{\rho s}\frac{h\left(vs+\log\left(|J_l^\dagger(s)|/a\right)\right)}{h(0)}|J_l^\dagger(s)|\right)$$
$$\underset{s\to\infty}{\sim}\mathbb{E}\left(M_s^2\right).$$

□

1.5 The Hausdorff dimension.

In this section we use the notation and definitions of the previous sections. We recall that $\rho=\rho_{\log(b/a)}$, where $\rho_\cdot$ is define in (1.2.8). Let *dim* be the Hausdorff dimension. The aim of this section would be to proof the main theorem:

Theorem 1.3. ***: Multifractal spectrum.*** *Assume (1.2.6):*

- *if $\rho>v$ holds, then:*

$$G_{(v,a,b)}=\emptyset\quad a.s.$$

- *if $\rho < v$ holds, then:*

$$dim(G_{(v,a,b)}) = 1 - \rho/v \quad a.s. \tag{1.5.1}$$

Remark 1.5. *1. Berestycki in [9] has computed the Hausdorff dimension of the set*

$$G_v = \left\{ x \in (0,1) \mid \lim_{t\to\infty} \frac{1}{t} \log |I_x(t)| = -v \right\}.$$

He found that for $v \in (v_{min}, v_{max})$, $dim(G_v) = C(v)/v$ (with $C(v)$ defined at the beginning of section 1.4). In Remark 1.4 we have shown that for all $v \in (\max(v_{min}, \rho), v_{max})$ we have $C(v) \geq v - \rho$ and we can notice that the inequality is strict for $\rho > v_{min}$. As a consequence the set $G_{(v,a,b)}$ has a Hausdorff dimension smaller than that of G_v, and also smaller than that one could have infer from equality (1.4.1).

2. In the case $v > v_{typ}$, we have $Y_t/t \underset{t\to\infty}{\to} v - v_{typ} > 0 \quad a.s.$ and

$$\mathbf{P}_{\log(1/a)}(\inf\{t : \ Y_t \leq 0\} = \infty) > 0.$$

Thus $W^{(-q)}(\infty) = 0$ for all $q \geq 0$ and then $\lim_{\beta\to\infty} \rho_\beta = 0$. Moreover using the fact that, $\lim_{\beta\to 0} \rho_\beta = \infty$ and $\rho_\cdot$ is decreasing, we get that for all $v > v_{typ}$, there exist a and b such that $\rho_{\log(b/a)} < v$ and thus the fact that the set of good intervals is not empty.

The proof of this theorem use the non-asymptotic set $\Lambda_{(v,a,b)}$. In particular the key of the proof is the next proposition:

Proposition 1.3. *Assume (1.2.6) and $0 < a < b < 1$:*

- *if $\rho > v$ holds, then:*

$$\Lambda_{(v,a,b)} = \emptyset \quad a.s.$$

- *if $\rho < v$ holds, then: $\mathbb{P}(\Lambda_{(v,a,b)} \neq \emptyset) > 0$, and conditionally on $\Lambda_{(v,a,b)} \neq \emptyset$,*

$$dim(\Lambda_{(v,a,b)}) = 1 - \rho/v. \tag{1.5.2}$$

Proof. 1. Let $v > 0$ and a and b such that $v < \rho$. We define

$$N(t) := \sharp G(t),$$

with $G(t)$ defined in (1.3.1). We remark that

$$N(t) = \int_0^1 \frac{1}{|I_x(t)|} \mathbf{1}_{\{I_x(t) \in G(t)\}}(x) dx.$$

and in particular

$$\mathbb{E}(N(t)) \;=\; \mathbb{E}\left(\int_0^1 \frac{1}{|I_x(t)|}\mathbf{1}_{\{I_x(t)\in G(t)\}}(x)dx\right).$$

Additionally by (1.2.3), we get

$$\mathbb{E}(N(t)) \;=\; e^{vt}\ \mathbf{E}\left(e^{\xi(t)-vt}\ \mathbf{1}_{\{vs-\xi(s)-\log a\in(0,\log(b/a))\ \forall\ s\le t\}}\right).$$

With the notation $Y_t = vt-\xi(t)$ and P_t defined in Theorem 1.1 we rewrite the previous equality as:

$$\begin{aligned}\mathbb{E}(N(t)) &= e^{vt}\ \mathbf{E}_{\log(1/a)}\left(e^{-Y_t-\log a}\ \mathbf{1}_{\{t<T\}}\right)\\ &= \frac{1}{a}e^{(v-\rho)t}\ \int_0^{\log(b/a)} e^{-y+\rho t}P_t(\log(1/a),dy).\end{aligned}$$

By Theorem 1.1.2 we get

$$\mathbb{E}(N(t)) \underset{t\to\infty}{\sim} \frac{1}{a}e^{(v-\rho)t}\ c\ h(0)\ \int_0^{\log(b/a)} e^{-y}\quad \Pi(dy),$$

with Π defined in Theorem 1.1.

Finally as the function $y\mapsto e^{-y}\ h(\log(b)-y)$ is continuous, the integral above is a finite constant. Thus if $\rho > v$ then $\lim_{t\to\infty}\mathbb{E}(N(t)) = 0$, from which one concludes that $\lim_{t\to\infty} N(t) = 0$, i.e. $\Lambda_{(v,a,b)} = \emptyset$ *a.s.*

2. Now we deal with the case where a and b are such that $v > \rho$. We work conditionally on $\Lambda_{(v,a,b)} \neq \emptyset$ (or, equivalently, on the event $\zeta = \infty$, which has a positive probability by Theorem 1.2).

• Firstly, in order to prove the lower bound of the Hausdorff dimension of $\Lambda_{(v,a,b)}$, we will use the same method as Berestycki in [9] . We will divide this proof into three steps. Each step will begin with a star ($\star$). In the first step we will construct a subset $\underset{n\in\mathbb{N}}{\cap}\mathbb{G}_\delta(n)$ of $\Lambda_{(v,a,b)}$, which will be defined latter on (see (1.5.4)). In the second we shall obtain a lower bound of the Hausdorff dimension of this subset. In order to do that we will construct an increasing process indexed by $t\in(0,1)$, which only increases on $\underset{n\in\mathbb{N}}{\cap}\mathbb{G}_\delta(n)$, and which is Hölder continuous. In the last step we will conclude.

$\star$ As in [9] for $\delta > 0$ we define for all $n\in\mathbb{N}$, $H_\delta(n)$ as a multi-type branching process with each particle corresponding to a segment of $G(\delta n)$ and

$$G_\delta(n) := \underset{I\in H_\delta(n)}{\cup} I,$$

with $G(t)$ defined in (1.3.1) (i.e. $G_\delta(n) = G(\delta n)$).

We notice that the family $(G_\delta(n))_{n\in\mathbb{N}}$ is nested and that $\underset{n\in\mathbb{N}}{\cap} G_\delta(n) = \Lambda_{(v,a,b)}$.

Let $\epsilon > 0$, and fix $\epsilon^{'} > 0$ and $\eta > 0$ such that $\eta < \min(\epsilon, v-\rho)$. By Proposition 1.1, for this $\epsilon^{'} > 0$ and $\eta > 0$, we may find $t_0 > \max((1 + |\log(1-\epsilon^{'})|)/(\epsilon-\eta), \log(2)/(v-\rho-\eta))$ such that for all $t > t_0$:

$$\mathbb{P}(|t^{-1}\log(\sharp G(t)) - (v-\rho)| > \eta | \zeta = \infty) < \epsilon^{'}.$$

For each $t > 0$, we consider a variable $\widetilde{\chi}(t)$ whose law is given by

$$\mathbb{P}(\widetilde{\chi}(t) = 0) = \epsilon^{''},$$

and

$$\mathbb{P}(\widetilde{\chi}(t) = \lfloor e^{[(v-\rho)-\eta]t} \rfloor) = 1 - \epsilon^{''},$$

where $\lfloor . \rfloor$ is the integer part and $\epsilon^{''} := \mathbb{P}(|t^{-1}\log(\sharp G(t)) - (v-\rho)| > \eta | \zeta = \infty) < \epsilon^{'}$. Moreover by using that for all $x \geq 2$: $\log(x) - 1 \leq \log(\lfloor x \rfloor)$, we notice that

$$|t^{-1}\log(\mathbb{E}(\widetilde{\chi}(t))) - (v-\rho)| \leq \eta + t^{-1}(|\log(1-\epsilon^{'})| + 1).$$

Plainly $\widetilde{\chi}(t)$ is stochastically dominated by $\sharp G(t)$. Exactly as in [9] we can construct a true Galton-Watson tree $\mathbb{H}$ by thinning H_δ where $\delta > t_0$. More precisely the offspring distribution of $\mathbb{H}$ is given by the law of $\widetilde{\chi}(\delta)$. Let $m := \mathbb{E}(\widetilde{\chi}(\delta))$ be the expectation of the number of children of a particle. Therefore, we get

(a)
$$|\delta^{-1}\log m - (v-\rho)| < \epsilon. \tag{1.5.3}$$

(b) The family
$$(\mathbb{G}(n) := \underset{I\in\mathbb{H}(n)}{\cup} I)_{n\in\mathbb{N}} \tag{1.5.4}$$
is nested. The $\mathbb{G}(n)$ is the union of the interval of the n generation of $\mathbb{H}$.

(c) $\underset{n\in\mathbb{N}}{\cap} \mathbb{G}(n) \subseteq \Lambda_{(v,a,b)}$.
This last point makes sense because we work conditionally on $\zeta = \infty$.

$\star$ We fix $\epsilon > 0$. We choose $\delta > t_0$ as shown above and consider the tree $\mathbb{H}$. We define $Z(n)$ as the number of nodes of $\mathbb{H}$ at height n. By the theory of Galton-Watson processes, as we are working conditionally on the event $\Lambda_{(v,a,b)} \neq \emptyset$, we have that almost surely

$$m^{-n}Z(n) \to \mathcal{W} > 0.$$

Let σ be a node of our tree (thus it is also a subinterval of $(0,1)$). Fix an

interval $I \subset (0,1)$ and introduce

$$\mathbb{H}_I(n) := \{\sigma \in \mathbb{H}(n), \sigma \cap I \neq \emptyset\},$$

$$Z_I(n) := \sharp\, \mathbb{H}_I(n).$$

Define

$$x \to L_x := \lim_n m^{-n} Z_{(0,x)}(n), \quad x \in (0,1).$$

We will now state a lemma that we will use to conclude:

Lemma 1.3. *For each $\epsilon > 0$,*

(a) There exists a version $\widetilde{L}$ of $(L_x)_{x\in[0,1]}$ which is Hölder continuous of order α for any $\alpha < 1 - \rho/v - \epsilon$ for every $\epsilon > 0$.

(b) The process $\widetilde{L}$ only grows on the set $\underset{n\in\mathbb{N}}{\cap} \mathbb{G}(n)$.

Proof of Lemma 1.3. (a) Exactly as in [9] , we show the first point by verifying Kolmogorov's criterium (see [64] Theorem 2.1 p.26). Let $W(\sigma)$ be the "renormalized weight" of the tree rooted at σ, i.e.,

$$W(\sigma) := \lim_{n\to\infty} m^{-n}\, \sharp\{\sigma' \in \mathbb{H}(|\sigma| + n), \sigma' \subset \sigma\},$$

where $|\sigma|$ is the generation of σ.

By the definition of L we have for all $x > y \in (0,1)$:

$$|L_x - L_y| = \lim_{n\to\infty} m^{-n} Z_{(x,y)}(n), \quad x \in (0,1).$$

For any J open subinterval of $(0,1)$, let

$$\eta(J) := \sup\{n \in \mathbb{N} :\ e^{-v\delta n} \geq |J|\} = \lfloor -\log(|J|)/v\delta \rfloor.$$

For all x, y such that $x < y$ by the definition of $L_\cdot$, we get:

$$\begin{aligned}
&|L_x - L_y| \\
&= \lim_n m^{-\eta((x,y))} m^{-n+\eta((x,y))} \sum_{\sigma\in\mathbb{H}_{(x,y)}(\eta((x,y)))} \sharp\{\sigma' \in \mathbb{H}(|\sigma| + n - \eta((x,y))), \sigma' \subset \sigma\} \\
&\leq m^{-\eta((x,y))} \sum_{\sigma\in\mathbb{H}_{(x,y)}(\eta((x,y)))} W(\sigma),
\end{aligned}$$

and by the definition of $\eta(.)$:

$$\begin{aligned} |L_x - L_y| &\leq e^{\log m(\frac{1}{v\delta}\log(y-x)+1)} \sum_{\sigma\in\mathbb{H}_{(x,y)}(\eta((x,y)))} W(\sigma) \\ &\leq m|x-y|^{1-\epsilon-\rho/v} \sum_{\sigma\in\mathbb{H}_{(x,y)}(\eta((x,y)))} W(\sigma), \end{aligned}$$

by using (1.5.3). Moreover by the definition of good intervals, we have that for each n the sizes of intervals in $\mathbb{H}(n)$ have a lower bound given by $ae^{-v\delta n}$, so $a|J|e^{-v\delta}$ is a lower bound for the sizes of the intervals of $\mathbb{H}(\eta(J))$, and thus $Z_J(\eta(J)) \leq e^{v\delta}/a$. Therefore for all $\gamma > 1$ and all $J \subset (0,1)$ we have:

$$\begin{aligned} \mathbb{E}\left((\sum_{\sigma\in\mathbb{H}_J(\eta(J))} W(\sigma))^\gamma \right) &\leq \mathbb{E}((W_1 + ... + W_{\lfloor e^{v\delta}/a\rfloor+1})^\gamma) \\ &\leq \mathbb{E}((W_1 + ... + W_{\eta(J)+2})^\gamma) < \infty, \end{aligned}$$

where the W_i are i.i.d. with the same law as W. The finiteness comes from the existence of finite moments of all orders for W (see for example Theorem 3.4 p. 479 of Harris [39]).

(b) The second point is clear by the choice of L.

□

⋆ To prove that $dim\left(\underset{n\in\mathbb{N}}{\cap}\mathbb{G}(n)\right) \geq 1 - \rho/v - \epsilon$, it is enough to show that

$$\sum_i diam(U_i)^{1-\rho/v-\epsilon} > 0 \tag{1.5.5}$$

for any cover $\{U_i\}$ of $\underset{n\in\mathbb{N}}{\cap}\mathbb{G}(n)$, where $diam(U_i)$ is the diameter of U_i. Clearly, it is enough to assume that the $\{U_i\}$ are intervals, and by expanding them slightly and using the compactness of the closure of $\underset{n\in\mathbb{N}}{\cap}\mathbb{G}(n)$, we only need to check (1.5.5) if $\{U_i\}$ is a finite collection of open subintervals of $[0,1]$.

Let $\cup_{i=0}^N (l_i, r_i)$ be a cover of $\underset{n\in\mathbb{N}}{\cap}\mathbb{G}(n)$ (where the (l_i, r_i) are disjoints open intervals). Therefore

$$\sum_{i=1}^N |\widetilde{L}_{r_i} - \widetilde{L}_{l_i}| = \mathcal{W}.$$

Thus for all such covers with $\max_i(r_i - l_i)$ small enough

$$\mathcal{W} \leq k\sum_{i=0}^N (r_i - l_i)^{1-\rho/v-\epsilon}$$

and hence

$$dim(\Lambda_{(v,a,b)}) \geq dim(\underset{n\in\mathbb{N}}{\cap}\mathbb{G}(n)) \geq 1-\rho/v-\epsilon.$$

To get the lower bound of the Hausdorff dimension of $\Lambda_{(v,a,b)}$, we let ϵ tend to 0.

• Secondly, the upper bound for (1.5.2) is an easy corollary of the fact that the Hausdorff dimension is smaller than the box-counting dimension (see [34] p.36-43), using the cover $\underset{n\geq N}{\cup}\underset{i\in\theta_{v,a,b}(n)}{\cup}J_i(n)$, with $\theta_{v,a,b}(t)=\{i\in\mathbb{N}\mid J_i(t)\in G(t)\}$ (with $G(t)$ defined in Section 1.3).

□

Then we have the next corollary, which deals with the general case for a and b:

Corollary 1.1. *For $t^{'}\geq 0$ set*

$$\Lambda_{(v,a,b)}(t^{'}):=\left\{x\in(0,1)\ :\ ae^{-vt}<|I_x(t)|<be^{-vt}\ \forall t\geq t^{'}\right\}.$$

Assume (1.2.6), $0<a<b$ and $\rho<v$, then

$$\mathbb{P}(\Lambda_{(v,a,b)}(t^{'})\neq\emptyset)\underset{t^{'}\to\infty}{\to}1,$$

and

$$\mathbb{P}\left(dim(\Lambda_{(v,a,b)}(t^{'}))=1-\rho/v\ \ \middle|\ \Lambda_{(v,a,b)}(t^{'})\neq\emptyset\right)=1.$$

Proof. 1. The first part of the proof is a consequence of the homogeneity of the fragmentation and of Proposition 1.3.

2. Fix $\rho^{'}>\rho$. As $\lim_{\beta\to 0}\rho_\beta=\infty$, and, by Theorem 1.1.5, the application $\beta\to\rho_\beta$ is continuous and strictly decreasing, therefore there exists $\beta_0\in(1,b/a)$ such that $\rho^{'}=\rho_{\log(\beta_0)}$. Let $\epsilon:=(\beta_0-1)/(1+\beta_0)$, $a^{'}:=1-\epsilon$, $b^{'}:=1+\epsilon$, $x_0:=(\beta_0+1)(a+b/\beta_0)/4$ (notice that $x_0\in(a,b)$) and

$$p_0:=\mathbb{P}(dim(\Lambda_{(v,a^{'},b^{'})})\geq 1-\rho_{\log(b^{'}/a^{'})}/v).$$

By Proposition 1.3, we get that $p_0>0$. We notice that by the choice of $a^{'}$ and of $b^{'}$, we have $\rho_{\log(b^{'}/a^{'})}=\rho_{\log(\beta_0)}=\rho^{'}$.

Let I be an interval of $(0,1)$. The law of the homogeneous interval fragmentation started at I will be denoted by $\mathbb{P}_I$. We remark that $\mathbb{P}_I(dim(\Lambda_{(v,a,b)})\geq 1-\rho^{'}/v)$ only depends on the length of I. Thus we define

$$g_{a,b}(x):=\mathbb{P}_I(dim(\Lambda_{(v,a,b)})\geq 1-\rho^{'}/v),$$

where I is an interval such that $|I|=x$.

Let $x \in (x_0 a', x_0 b')$. We remark that by the choice of x_0 and as $1 < \beta_0 < b/a$ we have that $(x_0 a', x_0 b') \subset (a, b)$ and thus

$$g_{a,b}(x) \geq g_{x_0 a', x_0 b'}(x).$$

Moreover by the scaling property of the fragmentation we get that

$$g_{x_0 a', x_0 b'}(x) = \mathbb{P}(dim(\Lambda_{(v, a'/x, b'/x)}) \geq 1 - \rho_{\log((b'/x)/(a'/x))}/v) = p_0$$

Therefore

$$\inf_{x \in (x_0 a', x_0 b')} g_{a,b}(x) \geq p_0. \tag{1.5.6}$$

Let

$$B(t) = \{i : \quad x_0 a' < e^{vt} |J_i(t)| < x_0 b'\} \quad , \; n_t = \sharp B(t),$$

where $(J_1, J_2, ...)$ is the interval decomposition of $F(t)$.

Fix $t' \geq 0$. By applying the Markov property at time t' we get that

$$\begin{aligned}
&\mathbb{P}(dim(\Lambda_{(v,a,b)}(t')) < 1 - \rho'/v)) \\
&\leq \mathbb{E}\left(\prod_{i \in B(t')} \mathbb{P}_{J_i(t')}(dim(\Lambda_{(v, x_0 a', x_0 b')}) < 1 - \rho'/v) \right) \\
&\leq \mathbb{E}((1 - p_0)^{n_{t'}}),
\end{aligned}$$

by using (1.5.6). Therefore as $p_0 > 0$, $n_{t'} \underset{t' \to \infty}{\to} \infty$ (see (1.4.1)) and with the first part of the proof we can conclude.

□

Now we are able to proof our main result:

Proof of Theorem 1.3. Observe that for all $n \in \mathbb{N}$, we have

$$\Lambda_{(v,a,b)}(n) \subset G_{(v,a,b)} \subset \bigcap_{\epsilon > 0} \bigcup_{m \in \mathbb{N}} \Lambda_{(v, a-\epsilon, b+\epsilon)}(m). \tag{1.5.7}$$

We can notice that the second inclusion is actually an equality.

• First we consider the case where $\rho > v$. As the application $\beta \to \rho_\beta$ is continuous and strictly decreasing (see Theorem 1.1.5), there exists $\epsilon_0 > 0$ such that $v < \rho_{\log((b+\epsilon_0)/(a-\epsilon_0))} < \rho$. Moreover by (1.5.7)

$$G_{(v,a,b)} \subset \bigcup_{m \in \mathbb{N}} \Lambda_{(v, a-\epsilon_0, b+\epsilon_0)}(m),$$

therefore thanks to Proposition 1.3 and the homogeneous property of the fragmentation, we get the first part of the proof.

• Second we consider the case where $\rho < v$. Thanks the second inclusion and the corollary 1.1, we get that: for all $\epsilon \in (0, a)$,

$$dim(G_{(v,a,b)}) \leq dim(\underset{n\in\mathbb{N}}{\cup}\Lambda_{(v,a-\epsilon,b+\epsilon)}(n)) = \max_n dim(\Lambda_{(v,a-\epsilon,b+\epsilon)}(n)) \leq 1 - \rho_{\log(\frac{b+\epsilon}{a-\epsilon})}/v.$$

Then by the continuity of $\rho_{\cdot}$ (see Theorem 1.1.5), we get the uper bound of the Hausdorff dimension of $G_{(v,a,b)}$.

The lower bound of the Hausdorff dimension is a consequence of the first inclusion of (1.5.7), as $dim(\Lambda_{(v,a,b)}(n)) = 1 - \rho/v$ with a probability which goes to 1 when n goes to infinity.

□

1.6 Appendix

1.6.1 A partition fragmentation.

In this appendix we give a proof of Theorem 1.2.1. (Section 1.3).

For this, we use the method of Bertoin and Rouault in [22] for fragmentation, which goes back to Lyons and al. [58] for Galton-Watson processes, and tools taken from the article of Engländer, Harris and Kyprianou [32].

We first introduce the notations that we need and we define what a partition fragmentation Π is. Let $\mathcal{P}$ the space of partition of $\mathbb{N}$, and for every integer k, the block $\{1, ..., k\}$ is denoted by $[k]$. As in [22], we call discrete point measure on the space $\Omega := \mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$, any measure :

$$w = \sum_{(t,\pi,k)\in\mathcal{D}}^{\infty} \delta_{(t,\pi,k)},$$

where $\mathcal{D}$ is a subset of $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$ such that

$$\forall t' \geq 0 \ \ \forall n \in \mathbb{N} \ \ \sharp\Big\{(t,\pi,k) \in \mathcal{D} \mid t \leq t', \pi_{|[n]} \neq ([n], \emptyset, \emptyset, ...), k \leq n\Big\} < \infty$$

and for all $t \in \mathbb{R}$

$$w(\{t\} \times \mathcal{P} \times \mathbb{N}) \in \{0, 1\}.$$

Starting from an arbitrary discrete point measure ω on $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$, we will construct a nested partition $\Pi = (\Pi(t), t \geq 0)$ (which means that for all $t \geq t'$ $\Pi(t)$ is a finer partition of $\mathbb{N}$ than $\Pi(t')$). We fix $n \in \mathbb{N}$, the assumption that the point measure ω is discrete enables us to construct a step path $(\Pi(t, n), t \geq 0)$ with values in the space of partitions of $[n]$, which only jumps at times t at which the fiber $\{t\}\times\mathcal{P}\times\mathbb{N}$ carries an atom of ω, say (t, π, k), such that $\pi_{|[n]} \neq ([n], \emptyset, \emptyset, ...)$ and $k \leq n$. In that case, $\Pi(t, n)$ is the partition obtained by replacing the $k-th$ block of $\Pi(t-, n)$, denoted $\Pi_k(t-, n)$, by the restriction $\pi_{|\Pi_k(t-,n)}$ of π to this block, and leaving the

other blocks unchanged. Of course for all $t \geq 0$, $(\Pi(t,n), n \geq 0)$ is compatible (i.e. for every n, $\Pi(n,t)$ is a partition of $[n]$ such that the restriction of $\Pi(n+1,t)$ to $[n]$ coincide with $\Pi(n,t)$), as a consequence, there exists a unique partition $\Pi(t)$, such that for all $n \geq 0$ we have $\Pi(t)_{|[n]} = \Pi(t,n)$. With the terminology of [14], it is shown in [22] that this process Π is a (partition valued) homogeneous fragmentation.

One says that a block $B \subset \mathbb{N}$ has an asymptotic frequency, if the limit

$$|B| := \lim_{n\to\infty} n^{-1} card(B \cap [n])$$

exists. When every block of some partition $\pi \in \mathcal{P}$ has an asymptotic frequency, we write $|\pi| = (|\pi_1|, ...)$ and then $|\pi|^{\downarrow} = (|\pi_1|^{\downarrow}, ...) \in \mathcal{S}^{\downarrow}$ for the decreasing rearrangement of the sequence $|\pi|$. In the case where some block of the partition π does not have an asymptotic frequency, we decide that $|\pi| = |\pi|^{\downarrow} = \partial$, where ∂ stands for some extra point added to $\mathcal{S}^{\downarrow}$. **We stress that the process of ranked asymptotic frequencies $|\Pi|^{\downarrow}$ is a ranked fragmentation.**

Moreover, let ν be the dislocation measure associated to this ranked fragmentation (see Subsection 1.2.2). According to Theorem 2 in [14], there exists a unique measure μ on $\mathcal{P}$ which is exchangeable (i.e. invariant by the action of finite permutations on $\mathcal{P}$), and such that ν is the image of μ by the map that associate the decreasing rearrangement $|\pi|^{\downarrow}$ of the sequence of the asymptotic frequencies of the blocks of π, to π. Thanks to exchangeability, we get that for all measurable function $f : [0,1] \to \mathbb{R}_+$ such that $f(0) = 0$.

$$\int_{\mathcal{P}} f(|\pi_1|)\mu(d\pi) = \int_{\mathcal{S}^*} \sum_{i=1}^{\infty} s_i f(s_i)\nu(ds).$$

We denote the sigma-field generated by the restriction to $[0,t] \times \mathcal{P} \times \mathbb{N}$ by $\mathcal{G}_0(t)$. So $(\mathcal{G}_0(t))_{t\geq 0}$ is a filtration, and the nested partitions $(\Pi(t), t \geq 0)$ are $(\mathcal{G}_0(t))_{t\geq 0}$-adapted. We define also the sigma-field $(\mathcal{F}_0(t))_{t\geq 0}$ generated by the decreasing rearrangement $|\Pi(r)|^{\downarrow}$ of the sequence of the asymptotic frequencies of the blocks of $\Pi(r)$ for $r \leq t$. Of course $(\mathcal{F}_0(t))_{t\geq 0}$ is a sub-filtration of $(\mathcal{G}_0(t))_{t\geq 0}$.

Let $\mathcal{G}_1(t)$ the sigma-field generated by the restriction of the discrete point measure w to the fiber $[0,t] \times \mathcal{P} \times \{1\}$. So $(\mathcal{G}_1(t), t \geq 0)$ is a sub-filtration of $(\mathcal{G}_0(t), t \geq 0)$, and the first block of Π is $(\mathcal{G}_1(t), t \geq 0)$-measurable. Let $\mathcal{D}_1 \subseteq \mathbb{R}_+$ be the random set of times $r \geq 0$ for which the discrete point measure has an atom on the fiber $\{r\} \times \mathcal{P} \times \{1\}$, and for every $r \in \mathcal{D}_1$, denote the second component of this atom by $\pi(r)$.

We define the probability measure $\mathbf{P}^{\updownarrow}$ as the h-transform of $\mathbf{P}$ based on the martingale D_t (defined in Theorem 1.1 (3)):

$$d\mathbf{P}^{\updownarrow}_x|_{\mathcal{E}_t} = D_t d\mathbf{P}_x|_{\mathcal{E}_t}. \tag{1.6.1}$$

To simplify the notation, as in the section 1.3 we define for all $t \in \mathbb{R}$ $h(t) =$

$W^{(-\rho)}(t+\log(1/a))\mathbf{1}_{\{t\in(\log(a),\log(b))\}}$. This function is well defined thanks to Theorem 1.1.

Let $P_i(t)$ the block of $\Pi(t)$ which contains i at time t. Similarly as in Section 1.3, for a homogeneous fragmentation, we define the killed partition

$$\Pi_j^\dagger(t) = \Pi_j(t)\mathbf{1}_{\{\exists i\in\mathbb{N}^*|\ \Pi_j(t)=P_i(t);\ \forall s\le t\ \ |P_i(s)|\in(ae^{-vs},be^{-vs})\}}.$$

When we project the martingale D_t of (1.2.9) on the sub-filtration $(\mathcal{G}_0(t))_{t\ge0}$, we obtain an additive martingale

$$\frac{e^{\rho t}}{h(0)}\sum_{i=1}^{\infty} h(vt+\log(|\Pi_i^\dagger(t)|))\ |\Pi_i^\dagger(t)|\ .$$

As $|\Pi|$ is a ranked fragmentation with dislocation measure ν, this martingale is the same as this of Section 1.3. From now on, we denote this martingale by M_t too.

Observe that the projection (1.6.1) on the sub-filtration $\mathcal{G}_0(t)$ give the identity:

$$d\mathbb{P}_x^\updownarrow|_{\mathcal{G}_0(t)} = M_t d\mathbb{P}_x|_{\mathcal{G}_0(t)}.$$

Like in lemma 8 (ii) [22], with the probability measure $\mathbb{P}^\updownarrow$ we get:

Lemma 1.4. *Under $\mathbb{P}^\updownarrow$, the restriction of w to $\mathbb{R}_+\times\mathcal{P}\times\{2,3,...\}$ has the same distribution as under $\mathbb{P}$ and is independent of the restriction to the fiber $\mathbb{R}_+\times\mathcal{P}\times\{1\}$.*

It follows immediately from Theorem 1.1 that

Remark 1.6. *For $x\in[0,\log(b/a)]$, let $F_x(t) := \mathbf{E}_x\left(\ e^{\rho t}\mathbf{1}_{\{T>t\}}\right)$ for $t\in[0,\infty)$, then $F_x(t)$ converges when $t\to\infty$ to a finite limit, and $F_x(.):[0,\infty)\to[0,\infty)$ is càdlàg. In particular we have*

$$\sup_{x\in[0,\log(b/a)]}\ \sup_{t\ge0}|F_x(t)|<\infty.$$

Remark 1.7. *We have for all $t\ge0$:*

$$|M_t-M_{t-}|\le e^{(\rho-v)t}\frac{b^2}{ah(0)}\sup_{x\in[\log a,\log b]}h(x)\qquad a.s.$$

If $v>\rho$, there exists $0<C'<\infty$ such that

$$\sup_{t\ge0}|M_t-M_{t-}|<C'\quad a.s.$$

Let

$$c_t := \frac{e^{\rho t}}{h(0)}h\left(vt+\log(|\Pi_1^\dagger(t)|)\right)\ |\Pi_1^\dagger(t)|$$

and

$$d_t := \frac{e^{\rho t}}{h(0)}\sum_{i=2}^{\infty}h\left(vt+\log\left(|\Pi_i^\dagger(t)|\right)\right)\ |\Pi_i^\dagger(t)|.$$

Now we have the background that we need to study

$$M_t = c_t + d_t,$$

and we will show that M is bounded in $\mathrm{L}^2(\mathbb{P})$. In order to do that, as $\mathbb{E}(M_t^2) = \mathbb{E}^{\updownarrow}(M_t)$, it is enough to prove that

$$\lim_{t\to\infty} \mathbb{E}^{\updownarrow}(M_t) < \infty \ .$$

1.6.2 The proof of Theorem 1.2.1.

• First we show that $\lim_{t\to\infty} \mathbb{E}^{\updownarrow}(c_t) = 0$.

With the subordinator $\xi(t) := -\log(|\Pi_1(t)|)$, whose Laplace exponent is κ (exactly the same as this defined in Subsection 1.2.3), with the Lévy Process $Y_t = vt - \xi(t) + \log(1/a)$, and $T := T_{\log(b/a)}$ defined in (1.2.7) associated to this Lévy Process, under $\mathbf{P}^{\updownarrow}_{\log(1/a)}$ we get:

$$c_t \ = \ \frac{e^{(\rho-v)t}}{h(0)} W^{(-\rho)}(Y_t)\ e^{Y_t}\ \mathbf{1}_{\{t<T\}}.$$

As a consequence,

$$\mathbf{E}^{\updownarrow}_{\log(1/a)}\left(\frac{W^{(-\rho)}(Y_t)}{h(0)}\ e^{Y_t}\ \mathbf{1}_{\{t<T\}}\right) = \mathbf{E}_{\log(1/a)}\left(\frac{W^{(-\rho)}(Y_t)^2}{h(0)^2}\ e^{Y_t}\ e^{\rho t}\mathbf{1}_{\{t<T\}}\right)$$
$$\leq \sup_{x\in[\log a,\log b]} (h(x))^2\ \frac{b}{ah(0)^2}\ F_{\log(1/a)}(t)$$

which is bounded by a constant independent of t by Remark 1.6, and as $\rho < v$, we have $\lim_{t\to\infty} e^{(\rho-v)t} = 0$. Therefore:

$$\lim_{t\to\infty} \mathbb{E}^{\updownarrow}(c_t) = 0 \ . \tag{1.6.2}$$

• Now we consider d_t. As shown in [22] with $B(r,j) = \{i \geq 2 : \Pi_i(t) \subseteq \pi_j(r) \cap \Pi_1(r-)\}$, we get, for every $r \in [0,t]$ and $j \geq 2$, conditionally on $r \in \mathcal{D}_1$, $\Pi_1(r-)$ and $\pi_j(r)$, the partition $(\Pi_i(t) : i \in B(r,j))$ can be written in the form $\tilde{\Pi}^{(j)}(t-r)_{|\pi_j(r)\cap\Pi_1(r-)}$. Here $(\tilde{\Pi}^{(j)})_{j\in\mathbb{N}}$ is a family of i.i.d. homogeneous fragmentations distributed as Π under $\mathbb{P}$ and independent of the sigma-field $\mathcal{G}_1(t)$. As a consequence:

$$\bigcup_{i\geq 2} \Pi_i(t) = \bigcup_{j\geq 2} \bigcup_{r\in[0,t]\cap\mathcal{D}_1} \tilde{\Pi}^{(j)}(t-r)_{|\pi_j(r)\cap\Pi_1(r-)}.$$

Moreover $|\pi_j(r)||\Pi_1(r-)|$ is $\mathcal{G}_1(t)$ measurable, and we have that for all $i \in \mathbb{N}$

$$|\tilde{\Pi}_i^{(j)}(t-r)_{|\pi_j(r)\cap\Pi_1(r-)}| = |\tilde{\Pi}_i^{(j)}(t-r)||\pi_j(r)||\Pi_1(r-)|$$

so that we get:

$$\begin{aligned}
\mathbb{E}^{\updownarrow}(d_t|\mathcal{G}_1(t)) &\leq \frac{e^{\rho t}}{h(0)} C_8 \sum_{r\in[0,t]\cap\mathcal{D}_1} \sum_{j=2}^{\infty} |\pi_j(r)||\Pi_1^{\dagger}(r-)| \mathbb{1}_{\{a\leq|\pi_j(r)||\Pi_1^{\dagger}(r-)|e^{vr}\leq b\}} \\
&\quad \sum_{i=1}^{\infty} \mathbb{E}^{\updownarrow}\left(|\tilde{\Pi_i}^{(j)}(t-r) \mathbb{1}_{\{a\leq|\tilde{\Pi_i}^{(j)}(t'-r)|e^{v(t'-r)}|\pi_j(r)||\Pi_1^{\dagger}(r-)|e^{vr}\leq b \ \forall t'\in[r,t]\}} \ |\mathcal{G}_1(t)\right) \\
&\leq \frac{e^{\rho t}}{h(0)} C_8 \sum_{r\in[0,t]\cap\mathcal{D}_1} \sum_{j=2}^{\infty} |\pi_j(r)||\Pi_1^{\dagger}(r-)| \mathbb{1}_{\{a\leq|\pi_j(r)||\Pi_1^{\dagger}(r-)|e^{vr}\leq b\}} \\
&\quad \sum_{i=1}^{\infty} \mathbb{E}^{\updownarrow}\left(|\tilde{\Pi_i}^{(j)}(t-r) \mathbb{1}_{\{a/b\leq|\tilde{\Pi_i}^{(j)}(t'-r)|e^{v(t'-r)}\leq b/a \ \forall t'\in[r,t]\}} \ |\mathcal{G}_1(t)\right),
\end{aligned}$$

with C_8 the maximum of $h(t)$ on the compact $[\log(a), \log(b)]$. As $\tilde{\Pi}$ is independent of the sigma-field $\mathcal{G}_1(t)$, $\tilde{\Pi}$ has the same distribution under $\mathbb{P}$ as under $\mathbb{P}^{\updownarrow}$. $\tilde{\Pi}^{(j)}$ is also distributed as Π under $\mathbb{P}$. Thus,

$$\begin{aligned}
&\sum_{i=1}^{\infty} \mathbb{E}^{\updownarrow}\left(|\tilde{\Pi_i}^{(j)}(t-r) \mathbb{1}_{\{a/b\leq|\tilde{\Pi_i}^{(j)}(t'-r)|e^{v(t'-r)}\leq b/a \ \forall t'\in[r,t]\}}| \ \Big| \ \mathcal{G}_1(t)\right) \\
&= \sum_{i=1}^{\infty} \mathbb{E}^{\updownarrow}\left(|\tilde{\Pi_i}^{(j)}(t-r)| \mathbb{1}_{\{a/b\leq|\tilde{\Pi_i}^{(j)}(t'-r)|e^{v(t'-r)}\leq b/a \ \forall t'\in[r,t]\}} \right).
\end{aligned}$$

Now we have by size-biased sampling:

$$\begin{aligned}
&\sum_{i=1}^{\infty} \mathbb{E}\left(e^{\rho(t-r)} |\Pi_i(t-r)| \mathbb{1}_{\{a/b\leq|\Pi_i(t'-r)|e^{v(t'-r)}\leq b/a \ \forall t'\in[r,t]\}}\right) \\
&= \mathbb{E}\left(\ e^{\rho(t-r)} \mathbf{1}_{\{t-r<\inf\{s:\ |\Pi_1(s)|\notin\ (\frac{a}{b}e^{-vs}, \frac{b}{a}e^{-vs})\}\}}\right) \\
&= \mathbf{E}_{\log(1/a)}\left(\ e^{\rho(t-r)} \mathbf{1}_{\{T_{2\log(b/a)}>t-r\}}\right),
\end{aligned}$$

as $\rho_{\cdot}$ is decreasing $\rho_{2\log(b/a)} \leq \rho$, thus

$$\begin{aligned}
&\sum_{i=1}^{\infty} e^{\rho(t-r)} \mathbb{E}^{\updownarrow}\left(|\tilde{\Pi_i}^{(j)}(t-r) \mathbb{1}_{\{a/b\leq|\tilde{\Pi_i}(t'-r)|e^{v(t'-r)}\leq b/a \ \forall t'\in[r,t]\}}| \ \Big| \ \mathcal{G}_1(t)\right) \\
&\leq \mathbf{E}_{\log(1/a)}\left(\ e^{\rho_{2\log(b/a)}(t-r)} \mathbf{1}_{\{T_{2\log(b/a)}>t-r\}}\right).
\end{aligned}$$

Therefore with $F_x^{'}(t) := \mathbf{E}_x\left(\ e^{\rho_{2\log(b/a)}t} \mathbf{1}_{\{T_{2\log(b/a)}>t\}}\right)$ and since $|\Pi_1^{\dagger}(r-)| = ae^{Y_{r-}-vr}\mathbb{1}_{\{r-<T\}}$ under $\mathbb{P}^{\updownarrow}_{\log(1/a)}$, we get:

$$\mathbb{E}^{\updownarrow}(d_t|\mathcal{G}_1(t)) \leq a\frac{e^{(\rho-v)r}C_8}{h(0)} \sum_{r\in[0,t]\cap\mathcal{D}_1} \sum_{j=2}^{\infty} |\pi_j(r)| e^{Y_{r-}} \sup_{x\in[0,\log(b/a)]} \sup_{t\geq 0} |F_x^{'}(t)|.$$

Moreover we have by definition $e^{Y_{r-}}\mathbb{1}_{\{r-<T\}} \leq b/a$. We let

$$C_9 := \sup_{x\in[0,\log(b/a)]} \sup_{t\geq 0} |F_x'(t)| bC_8/h(0)$$

according to Remark 1.6 we have $C_9 < \infty$. Thus

$$\mathbb{E}^{\updownarrow}(d_t|\mathcal{G}_1(t)) \;\leq\; C_9 \sum_{r\in[0,t]\cap\mathcal{D}_1} \sum_{j=2}^{\infty} e^{(\rho-v)r}|\pi_j(r)|.$$

Under $\mathbb{P}$, the $\mathcal{G}_0(t-)$ predictable compensator of

$$A_t := \sum_{r\in[0,t]\cap\mathcal{D}_1} \sum_{j=2}^{\infty} e^{(\rho-v)r}|\pi_j(r)|$$

is

$$N_t := \int_0^t dr \int_{\mathcal{P}} \mu(ds) e^{(\rho-v)r} \sum_{j=2}^{\infty} |\pi_j|.$$

Additionally

$$\int_{\mathcal{P}} \mu(ds) \sum_{j=2}^{\infty} |\pi_j| = \int_{\mathcal{S}^*} \nu(ds) \sum_{i=1}^{\infty} s_i \left[(\sum_{j=1}^{\infty} s_j) - s_i \right].$$

As $\sum_{j=1}^{\infty} s_j = 1$ $\nu - a.s.$, we achieve:

$$\int_{\mathcal{P}} \mu(ds) \sum_{j=2}^{\infty} |\pi_j| \leq \int_{\mathcal{S}^*} \nu(ds) 2(1-s_1),$$

which is finite by (1.2.1). Moreover as $\rho < v$, the term $e^{(\rho-v)r}$ is integrable on $[0,\infty)$, so that we have $\lim_{t\to\infty} N_t < \infty$.

As both $X_t := A_t - N_t$ and M_t are martingales, by Theorem 4.50 of [43], we get that

$$XM - [X,M] \quad \text{is a local martingale.}$$

A sequence $(\tau_n = (T(m,n))_{m\in\mathbb{N}})_{n\in\mathbb{N}}$ of adapted subdivisions is called a Riemann sequence if $\sup_{m\in\mathbb{N}}[T(m+1,n)\wedge t - T(m,n)\wedge t] \to 0$ for all $t \in \mathbb{R}_+$. By Theorem 4.47 of [43], for any Riemann sequence $\{\tau_n = (T(m,n))_{m\in\mathbb{N}}\}_{n\in\mathbb{N}}$ of adapted subdivisions, the processes $S_{\tau_n}(X,M)$ defined by

$$S_{\tau_n}(X,M)_t := \sum_{m\in\mathbb{N}} (X_{T(m+1,n)\wedge t} - X_{T(m,n)\wedge t})(M_{T(m+1,n)\wedge t} - M_{T(m,n)\wedge t})$$

converge to the process $[X,M]$, in measure, uniformly on every compact interval.

We will now bound $S_{\tau_n}(X,M)_t$ uniformly in t. As

$$S_{\tau_n}(X,M)_t \leq \sup_{l\in\mathbb{N}} |M_{T(l+1,n)\wedge t} - M_{T(l,n)\wedge t}| \sum_{m\in\mathbb{N}} |X_{T(m+1,n)\wedge t} - X_{T(m,n)\wedge t}| \qquad (1.6.3)$$

we will first focus on $\sum_{m\in\mathbb{N}} |X_{T(m+1,n)\wedge t} - X_{T(m,n)\wedge t}|$:

$$\begin{aligned}
&\sum_{m\in\mathbb{N}} |X_{(m+1)/n\wedge t} - X_{T(m,n)\wedge t}| \\
&\leq \sum_{m\in\mathbb{N}} \left(\sum_{r\in[T(m,n)\wedge t, T(m+1,n)\wedge t]\cap\mathcal{D}_1} \sum_{j=2}^{\infty} e^{(\rho-v)r}|\pi_j(r)| + \int_{T(m,n)\wedge t}^{T(m+1,n)\wedge t} dr \int_{\mathcal{P}} \mu(ds) e^{(\rho-v)r} \sum_{j=2}^{\infty} |\pi_j| \right) \\
&\leq \sum_{r\in[0,t]\cap\mathcal{D}_1} \sum_{j=2}^{\infty} e^{(\rho-v)r}|\pi_j(r)| + \int_0^{\infty} dr \int_{\mathcal{P}} \mu(ds) e^{(\rho-v)r} \sum_{j=2}^{\infty} |\pi_j|.
\end{aligned}$$

Therefore by the previous study of A_t and N_t we get that there exist $C_{10} < \infty$ independent of t such that:

$$\lim_{n\to\infty} \mathbb{E}\left(\sum_{m\in\mathbb{N}} |X_{T(m+1,n)\wedge t} - X_{T(m,n)\wedge t}| \right) \leq C_{10} \quad \text{for all} \quad t.$$

Moreover

$$\lim_{n\to\infty} \sup_{l\in\mathbb{N}} |M_{T(l+1,n)\wedge t} - M_{T(l,n)\wedge t}| \leq \sup_{r\leq t} |M_r - M_{r-}|$$

is a.s. bounded by C' (see remark 1.7) independently of t. Consequently by (1.6.3)

$$\lim_{t\to\infty} \mathbb{E}([X,M]_t) < \infty.$$

Thus as $XM - [X,M]$ is a local martingale, we get that $\lim_{t\to\infty} \mathbb{E}^{\updownarrow}(d_t) < \infty$.

Finally according to (1.6.2), we get

$$\lim_{t\to\infty} \mathbb{E}^{\updownarrow}(M_t) = \lim_{t\to\infty} \mathbb{E}^{\updownarrow}(\mathbb{E}^{\updownarrow}(d_t + c_t|\mathcal{G}_1(t))) < \infty.$$

Chapter 2

Self-similar branching Markov chains

The main purpose of this work is to study self-similar branching Markov chains. First we will construct such a process. Then we will establish certain Limit Theorems using the theory of self-similar Markov processes.

Contents

The material of this chapter is an article to appear in "Séminaire de probabilités".

2.1 Introduction.

This work is a contribution to the study of a special type of branching Markov chains. We will construct a continuous time branching chain $\mathbf{X}$ which has a self-similar property and which takes its values in the space of finite point measures of $\mathbb{R}_+^*$. This type of process is a generalization of a self-similar fragmentation (see [17]), which may apply to cases where the size models non additive quantities as e.g. surface energy in aerosols. We will focus on the case where the index of self-similarity α is non-negative, which means that the bigger individuals will reproduce faster than the smaller ones. There is no loss of generality by considering this model, as the map $x \to x^{-1}$ on atoms in $\mathbb{R}_+^*$ transforms a self-similar process with index α into another one with index $-\alpha$ (and preserves the Markov property).

In this article we choose to construct the process by bare hand. We extend the method used in [17] to deal with more general processes where we allow an individual to have a mass bigger than that of its parent. We will explain in the sequel, which difficulties this new set-up entails. There exists closely related articles about branching processes, like among others [53], [54] from Kyprianou and [30], [31] from Chauvin. However notice that the time of splitting of the process depends on the size of the atoms of the process.

More precisely we will first introduce a branching Markov chains as a marked tree and we will obtain a process, indexed by generations (it is simply a random mark on the tree of generation, see Section 2.2). Thanks to a martingale which is associated to the latter and the theory of random stopping lines on a tree of generation, we will define the process indexed by time. After having constructed the process, we will study the evolution of the randomly chosen branch of the chain, from which we shall deduce some Limit Theorems, relying on the theory of self-similar Markov processes. In an appendix we will consider the intrinsic process and give some properties in the spirit of the article of Jagers [44]. By the way we will show properties about the earlier martingale.

2.2 The marked tree.

In this part we will introduce a branching Markov chain as a marked tree, which gives a genealogic description of the process that we will construct. This terminology comes from Neveu in [62] even if here the marked tree we consider is slightly different. First we introduce some notations and definitions.

A finite point measure on $\mathbb{R}_+^*$ is a finite sum of Dirac point masses $\mathbf{s} = \sum_{i=1}^{n} \delta_{s_i}$, where the s_i are called the atoms of $\mathbf{s}$ and $n \geq 0$ is an arbitrary integer. We shall

often write $\sharp\mathbf{s} = n = \mathbf{s}(\mathbb{R}_+^*)$ for the number of atoms of $\mathbf{s}$, and $\mathcal{M}_p(\mathbb{R}_+^*)$ for the space of finite point measures on $\mathbb{R}_+^*$. We also define for $f : \mathbb{R}_+^* \to \mathbb{R}$ measurable function and $\mathbf{s} \in \mathcal{M}_p(\mathbb{R}_+^*)$

$$\langle f, \mathbf{s}\rangle := \sum_{i=1}^{\sharp s} f(s_i),$$

by taking the sum over the atoms of $\mathbf{s}$ repeated according to their multiplicity and we will sometimes use the slight abuse of notation

$$\langle f(x), \mathbf{s}\rangle := \sum_{i=1}^{\sharp s} f(s_i)$$

when f is defined as a function depending on the variable x. We endow the space $\mathcal{M}_p(\mathbb{R}_+^*)$ with the topology of weak convergence, which means that $\mathbf{s}_n$ converge to $\mathbf{s}$ if and only if $\langle f, \mathbf{s}_n\rangle$ converge to $\langle f, \mathbf{s}\rangle$ for all continuous bounded functions f.

Let $\alpha \geq 0$ be an index of self-similarity and ν be some **probability measure** on $\mathcal{M}_p(\mathbb{R}_+^*)$.The aim of this work is to construct a branching Markov chain $\mathbf{X} = ((\sum_{i=1}^{\sharp\mathbf{X}(t)} \delta_{X_i(t)})_{t\geq 0})$ with values in $\mathcal{M}_p(\mathbb{R}_+^*)$, which is self-similar with index α and has reproduction law ν. The index of self-similarity will play a part in the rate at which an individual will reproduce and the reproduction law ν will specify the distribution of the offspring. We stress that our setting includes the case when

$$\nu(\exists i \ : \ s_i > 1) > 0, \tag{2.2.1}$$

which means that with a positive probability the size of a daughter can exceed that of her mother.

To do that, exactly as described in Chapter 1 section 1.2.1 of [17] , we will construct a marked tree.

We consider the Ulam Harris labelling system

$$\mathbb{T} := \cup_{n=0}^{\infty} \mathbb{N}^n,$$

with the notation $\mathbb{N} = \{1, 2, ...\}$ and $\mathbb{N}^0 = \{\emptyset\}$. In the sequel the elements of $\mathbb{T}$ are called nodes (or sometimes also individuals) and the distinguished node $\emptyset$ the root. For each $u = (u_1, ..., u_n) \in \mathbb{T}$, we call n the *generation* of u and write $|u| = n$, with the obvious convention $|\emptyset| = 0$. When $n \geq 0$, $u = (u_1, ..., u_n) \in \mathbb{N}^n$ and $i \in \mathbb{N}$, we write $ui = (u_1, ..., u_n, i) \in \mathbb{N}^{n+1}$ for the i-th child of u. We also define for $u = (u_1, ..., u_n)$ with $n \geq 2$,

$$mu = (u_1, ..., u_{n-1})$$

the *mother* of u, $mu = \emptyset$ if $u \in \mathbb{N}$. If $v = m^n u$ for some $n \geq 0$ we write $v \preceq u$ and say that u *stems* from v. Additionally for M a set of $\mathbb{T}$, $M \preceq v$ means that $u \preceq v$ for some $u \in M$. Generally we write $M \preceq L$ if all $x \in L$ stem from M.

Here it will be convenient to identify the point measure $\mathbf{s}$ with the infinite se-

quence $(s_1, ..., s_n, 0, ...)$ obtained by aggregation of infinitely many 0's to the finite sequence of the atoms of $\mathbf{s}$.

In particular we say that a random infinite sequence $(\xi_i, i \in \mathbb{N})$ has the law ν, if there is a (random) index n such that $\xi_i = 0 \Leftrightarrow i > n$ and the finite point measure $\sum_{i=1}^{n} \delta_{\xi_i}$ has the law ν.

Definition 2.1. *Let two independent families of i.i.d. variables be indexed by the nodes of the tree, $(\overline{\xi}_u, u \in \mathbb{T})$ and $(\mathbf{e}_u, u \in \mathbb{T})$, where for each $u \in \mathbb{T}$, $\overline{\xi}_u = (\widetilde{\xi}_{ui})_{i\in\mathbb{N}}$ is distributed according to the law ν, and $(\mathbf{e}_{ui})_{i\in\mathbb{N}}$ is a sequence of i.i.d. exponential variables with parameter 1. We define recursively for some fixed $x > 0$*

$$\xi_\emptyset := x, \qquad a_\emptyset := 0, \qquad \zeta_\emptyset := x^{-\alpha}\mathbf{e}_\emptyset,$$

and for $u \in \mathbb{T}$ and $i \in \mathbb{N}$:

$$\xi_{ui} := \widetilde{\xi}_{ui}\xi_u, \qquad a_{ui} := a_u + \zeta_u, \qquad \zeta_{ui} := \xi_{ui}^{-\alpha}\mathbf{e}_{ui}.$$

To each node u of the tree $\mathbb{T}$, we associate the mark (ξ_u, a_u, ζ_u) where ξ_u is the size, a_u the birth-time and ζ_u the lifetime of the individual with label u. We call

$$T_x = ((\xi_u, a_u, \zeta_u)_{u\in\mathbb{T}})$$

a marked tree with root of size x, and the law associated is denoted by $\mathbb{P}_x$. Let $\bar{\Omega}$ be the set of all the possible marked trees.

The size of the individuals $(\xi_u, u \in \mathbb{T})$ defines a multiplicative cascade (see the references in Section 3 of [18]). However the latter is not sufficient to construct the process $\mathbf{X}$, in fact we also need the information given by $((a_u, \zeta_u), u \in \mathbb{T})$.

Another useful concept is that of *line*. A subset $L \subset \mathbb{T}$ is a line if for every $u, v \in L$, $u \preceq v \Rightarrow u = v$. The *pre-L-sigma algebra* is

$$\mathcal{H}_L := \sigma(\widetilde{\xi}_u, \mathbf{e}_u; \exists l \in L : u \preceq l \text{ or } \nexists l \in L : l \preceq u).$$

A random set of individuals

$$\mathcal{J} : \bar{\Omega} \to \mathcal{P}(\mathbb{T})$$

is *optional* if $\{\mathcal{J} \preceq L\} \in \mathcal{H}_L$ for all line $L \subset \mathbb{T}$, where $\mathcal{P}(\mathbb{T})$ is the power set of $\mathbb{T}$. An *optional line* is a random line which is optional. For any optional set $\mathcal{J}$ we define the pre-$\mathcal{J}$-algebra by:

$$A \in \mathcal{H}_\mathcal{J} \Leftrightarrow \forall L \text{ line } \subset \mathbb{T} : A \cap \{\mathcal{J} \preceq L\} \in \mathcal{H}_L.$$

The first result is:

Lemma 2.1. *The marked tree constructed in Definition 2.1 satisfies the strong Markov branching property: for $\mathcal{J}$ an optional line and $\varphi_u : \bar{\Omega} \to [0, 1]$, $u \in \mathbb{T}$,*

measurable functions, we get that,

$$\mathbb{E}_1\left(\prod_{u\in\mathcal{J}}\varphi_u\circ T^{\xi_u}\middle|\mathcal{H}_{\mathcal{J}}\right)=\prod_{u\in\mathcal{J}}\mathbb{E}_{\xi_u}(\varphi_u),$$

where T^{ξ_u} is the marked tree extracted from T_1 at the node (ξ_u,a_u,ζ_u). More precisely

$$T^{\xi_u}=((\xi_{uv},a_{uv}-a_u,\zeta_{uv})_{v\in\mathbb{T}}).$$

Proof. Thanks to the i.i.d properties of the random variables $(\widetilde{\xi}_u,u\in\mathbb{T})$ and $(\mathbf{e}_u,u\in\mathbb{T})$, the Markov property for lines is of course easily checked. In order to get the result for a more general optional line, we use Theorem 4.14 of [44]. Indeed, the tree we have constructed is a special case of the tree constructed by Jagers in [44]. In our case the Jagers's notation ρ_u, τ_u and σ_u are such that the type ρ_u of $u\in\mathbb{T}$, is the mass of u: ξ_u, the birth time σ_u is a_u and τ_u is here equal to ζ_{mu} (because the mother dies when she gives birth to her daughters). We notice that all the sisters have the same birth time, which means that for all $u\in\mathbb{T}$ and all $i\in\mathbb{N}$, we have that τ_{ui} is here equal to ζ_u. □

2.3 Malthusian hypotheses and the intrinsic martingale.

We introduce some notation to formulate the fundamental assumptions of this work:

$$\underline{p}:=\inf\left\{p\in\mathbb{R}:\int_{\mathcal{M}_p(\mathbb{R}_+^*)}\langle x^p,\mathbf{s}\rangle\nu(d\mathbf{s})<\infty\right\},$$

and

$$p_\infty:=\inf\left\{p>\underline{p}:\int_{\mathcal{M}_p(\mathbb{R}_+^*)}\langle x^p,\mathbf{s}\rangle\nu(d\mathbf{s})=\infty\right\}$$

(with the convention $\inf\emptyset=\infty$) and then for every $p\in(\underline{p},p_\infty)$:

$$\kappa(p):=\int_{\mathcal{M}_p(\mathbb{R}_+^*)}(1-\langle x^p,\mathbf{s}\rangle)\,\nu(d\mathbf{s}).$$

Note that κ is a continuous and concave function (but not necessarily a strictly increasing function) on $(\underline{p},p_\infty)$, as $p\to\int_{\mathcal{M}_p(\mathbb{R}_+^*)}\langle x^p,\mathbf{s}\rangle\nu(d\mathbf{s})$ is a convex application. By concavity, the equation $\kappa(p)=0$ has at most two solutions on $(\underline{p},p_\infty)$. When a solution exists, we denote by $p_0:=\inf\{p\in(\underline{p},p_\infty):\kappa(p)=0\}$ the smallest, and call p_0 the Malthusian exponent.

We now make the fundamental:

Malthusian Hypotheses. *We suppose that the Malthusian exponent p_0 exists, that $p_0 > 0$, and that*

$$\kappa(p) > 0 \quad \textit{for some} \quad p > p_0. \tag{2.3.1}$$

Furthermore we suppose that the integral

$$\int_{\mathcal{M}_p(\mathbb{R}_+^*)} (\langle x^{p_0}, \mathbf{s}\rangle)^p \, \nu(d\mathbf{s}) \tag{2.3.2}$$

is finite for some $p > 1$.

Throughout the rest of this article, these hypotheses will always be taken for granted.

Note that (2.3.1) always holds when $\nu(s_i \leq 1 \text{ for all } i) = 1$ (fragmentation case). We stress that κ may not be strictly increasing, and may not be negative when p is sufficiently large (see Subsection 2.6.1 for a consequence of this fact.)

We will give one example based on the Dirichlet process (see the book Kingman [45]). Fix $n \geq 2$, $(\upsilon_1, ..., \upsilon_n)$ n positive real numbers and $\upsilon = \sum_{i=1}^n \upsilon_i$. We define the simplex Δ_n by

$$\Delta_n := \left\{ (p_1, p_2, ..., p_n) \in \mathbb{R}_+^n, \ \sum_{j=1}^n p_i = 1 \right\}.$$

The Dirichlet distribution of parameter $(\upsilon_1, ..., \upsilon_n)$ over the simplex Δ_n has the density (with respect to the $(n-1)$-dimensional Lebesgue measure on Δ_n):

$$f(p_1, ..., p_n) = \frac{\Gamma(\upsilon)}{\Gamma(\upsilon_1)...\Gamma(\upsilon_n)} p_1^{\upsilon_1 - 1} ... p_n^{\upsilon_n - 1}.$$

Let $a := \upsilon(\upsilon+1)/(\sum_{i=1}^n \upsilon_i(\upsilon_i+1))$. Note that a is strictly larger than 1. Let the reproduction measure be the law of $(aX_1, ..., aX_n)$, where $(X_1, ..., X_n)$ is a random vector with Dirichlet distribution of parameter $(\upsilon_1, ..., \upsilon_n)$. Therefore

$$\kappa(p) = a^p \frac{\Gamma(\upsilon)}{\Gamma(\upsilon + p)} \sum_{i=1}^n \frac{\Gamma(p + \upsilon_i)}{\Gamma(\upsilon_i)},$$

$\underline{p} = -\upsilon$, $p_0 = 1$ and the Malthusian hypotheses are verified.

In this article we will call *extinction* the event that for some $n \in \mathbb{N}$, all nodes u at the n-th generation have zero size, and *non-extinction* the complementary event. We see that the probability of extinction is always strictly positive whenever $\nu(s_1 = 0) > 0$, and equals zero if and only if $\nu(s_1 = 0) = 0$ (as we have suppose (2.3.2); see p.28 [17]).

After these definitions, we introduce a fundamental martingale associated to $(\xi_u, u \in \mathbb{T})$.

Theorem 2.1. *The process*

$$M_n := \sum_{|u|=n} \xi_u^{p_0}, \quad n \in \mathbb{N}$$

is a martingale in the filtration $(\mathcal{H}_{L_n})$*, with* L_n *the line associated to the* n*-th generation (i.e.* $L_n := \{u \in \mathbb{T} : \ |u| = n\}$*). This martingale is bounded in* $\mathrm{L}^p(\mathbb{P})$ *for some* $p > 1$*, and in particular is uniformly integrable.*

Moreover, conditionally on non-extinction the terminal value M_∞ *is strictly positive a.s.*

Remark 2.1. *As* κ *is concave the equation* $\kappa(p) = 0$ *may have a second root* $p_+ := \inf\{p > p_0, \ \kappa(p) = 0\}$*). This second root is less interesting: even though*

$$M_n^+ := \sum_{|u|=n} \xi_u^{p_+}, \quad n \in \mathbb{N},$$

is also a martingale, it is easy to check that for all $p > 1$ *the* p*-variation of* M_n^+ *is infinite, i.e.* $\mathbb{E}\left(\sum_{n=0}^{\infty} |M_{n+1} - M_n|^p\right) = \infty$*).*

We can notice that for all $p \in (p_0, p_+)$ $(M_n^{(p)})_{n\in\mathbb{N}} := (\sum_{|u|=n} \xi_u^p)_{n\in\mathbb{N}}$ *is a supermartingale.*

The assumption (2.3.2) means actually that $\mathbb{E}(M_1^p) < \infty$*.*

Proof. • We will use the fact that the empirical measure of the logarithm of the sizes of fragments

$$Z^{(n)} := \sum_{|u|=n} \delta_{\log \xi_u} \tag{2.3.3}$$

can be viewed as a branching random walk (see the article of Biggins [25]) and use Theorem 1 of [25]. In order to do that we first introduce some notation: for $\theta > \underline{p}$, we define

$$m(\theta) := \mathbb{E}\left(\int e^{\theta x} Z^{(1)}(dx)\right) = \mathbb{E}\left(\sum_{|u|=1} \xi_u^\theta\right) = 1 - \kappa(\theta)$$

and

$$W^{(n)}(\theta) := m(\theta)^{-n} \int e^{\theta x} Z^{(n)}(dx) = (1 - \kappa(\theta))^{-n} \sum_{|u|=n} \xi_u^\theta.$$

We notice that $M_n = W^{(n)}(p_0)$. Therefore in order to apply Theorem 1 of [25] and to get the convergence almost surely and in pth mean for some $p > 1$, it is enough to show that

$$\mathbb{E}(W^{(1)}(p_0)^\gamma) < \infty$$

for some $\gamma \in (1, 2]$ and

$$m(pp_0)/|m(p_0)|^p < 1$$

for some $p \in (1, \gamma]$. The first condition is a consequence of the Malthusian assump-

tion. Moreover the second follows from the identities

$$m(pp_0)/|m(p_0)|^p = (1-\kappa(pp_0))/|1-\kappa(p_0)|^p = 1-\kappa(pp_0)$$

which, by the definition of p_0, is smaller than 1 for $p > 1$ well chosen.

• Finally, let us now check that $M_\infty > 0$ *a.s.* conditionally on non-extinction. Define $q = \mathbb{P}(M_\infty = 0)$, therefore as $\mathbb{E}(M_\infty) = 1$ we get that $q < 1$. Moreover, an application of the branching property yields

$$\mathbb{E}(q^{Z_n}) = q,$$

where Z_n is the number of individuals with positive size at the n-th generation. Notice that $Z_n = \langle Z^{(n)}, 1\rangle$. By the construction of the marked tree and as ν is a probability measure: $(Z_n, n \in \mathbb{N})$ is of course a Galton-Watson process and it follows that q is its probability of extinction. Since $M_\infty = 0$ conditionally on the extinction, the two events coincide a.s. □

2.4 Evolution of the process in continuous time.

After having defined the process indexed by generation and having shown that the martingale M_n is $\mathrm{L}^p(\mathbb{P})$ bounded, we are now able to define properly the main objet of this paper. In order to do this, when an individual labelled by u has a positive size, $\xi_u > 0$, let $I_u := [a_u, a_u + \zeta_u)$ be the interval of times during which this individual is alive. Otherwise, i.e. when $\xi_u = 0$, we decide that $I_u = \emptyset$. With this definition, we set:

Definition 2.2. *We define the process* $\mathbf{X} = (\mathbf{X}(t), t \geq 0)$ *by*

$$\mathbf{X}(t) = \sum_{u\in\mathbb{T}} \mathbb{1}_{\{t\in I_u\}}\delta_{\xi_u}, t \geq 0. \tag{2.4.1}$$

In particular we have for $f : \mathbb{R}_+ \to \mathbb{R}$ *measurable function*

$$\langle f, \mathbf{X}(t)\rangle = \sum_{u\in\mathbb{T}} f(\xi_u)\mathbb{1}_{\{t\in I_u\}}.$$

For every $x > 0$, let $\mathbb{P}_x$ be the law of the process $\mathbf{X}$ starting from a single individual with size x. And for simplification, we denote $\mathbb{P}$ for $\mathbb{P}_1$, and let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of the process $(\mathbf{X}(t), t \geq 0)$. We use the notation $(X_1(t), ..., X_{\sharp\mathbf{X}(t)}(t))$ for the sequence of atoms of $\mathbf{X}(t)$. In the following we will show that this sequence is almost surely finite. Of course the set $(X_1(t), ..., X_{\sharp\mathbf{X}(t)}(t))$ is the same as the set $((\xi_u); t \in I_u)$; but sometimes it will be clearer to use the notation $(X_i(t))$.

We define for $u \in \mathbb{R}_+$:

$$F(u) := \int_{\mathcal{M}_p(\mathbb{R}_+^*)} u^{\sharp \mathbf{s}} \nu(d\mathbf{s}).$$

We notice that $F(u)$ is the generating function of the Galton-Watson process $(Z_n, n \geq 0) = (\sharp\{u \in \mathbb{T} : \ \xi_u > 0 \text{ and } \ |u| = n\}, n \geq 0)$.

From now on, we will suppose that for every $\epsilon > 0$

$$\int_{1-\epsilon}^{1} \frac{du}{F(u) - u} = \infty. \tag{2.4.2}$$

Of course if $F'(1) = \mathbb{E}(Z_1) < \infty$ this last assumption is fulfilled. Therefore we get the first theorem about the continous time process:

Theorem 2.2. *The process $\mathbf{X}$ takes its values in the set $\mathcal{M}_p(\mathbb{R}_+^*)$. It is a branching Markov chain, more precisely the conditional distribution of $\mathbf{X}(t+r)$ given that $\mathbf{X}(r) = \mathbf{s}$ is the same as that of the sum $\sum \mathbf{X}^{(i)}(t)$, where for each index i, $\mathbf{X}^{(i)}(t)$ is distributed as $\mathbf{X}(t)$ under $\mathbb{P}_{s_i}$ and the variables $\mathbf{X}^{(i)}(t)$ are independent.*

The process $\mathbf{X}$ also has the scaling property, namely for every $c > 0$, the distribution of the rescaled process $(c\mathbf{X}(c^\alpha t), t \geq 0)$ under $\mathbb{P}_1$ is $\mathbb{P}_c$.

In the fragmentation case, the fact that the size of the fragments decreases with time entails that the process of the fragments of size larger than or equal to ϵ is Markovian, and which leads easily to Theorem 2.2. This property is lost in the present case.

Proof. • First we will check that for all $t \geq 0$, $\mathbf{X}(t)$ is a (random) finite point measure. By Theorem 2.1 and the Doob's L^p-inequality we get that for some $p > 1$:

$$\sup_{n \in \mathbb{N}} M_n = \sup_{n \in \mathbb{N}} \sum_{|u|=n} \xi_u^{p_0} \in \mathrm{L}^p(\mathbb{P}).$$

As a consequence:

$$\sup_{u \in \mathcal{U}} \xi_u^{p_0} \in \mathrm{L}^p(\mathbb{P}) \tag{2.4.3}$$

and then by the definition of the process $\mathbf{X}$, writing $X_1(t), ...$ for the (possibly infinite) sequence of atoms of $\mathbf{X}(t)$

$$\sup_i \sup_{t \in \mathbb{R}_+} X_i(t)^{p_0} \in \mathrm{L}^p(\mathbb{P}).$$

Recall that $p_0 > 0$ by assumption. We fix some arbitrarily large $m > 0$. We now work conditionally on the event that the size of all individuals is bounded by m, and we will show that the number of the individuals alive at time t is almost surely finite for all $t \geq 0$.

As we are conditioning on the event $\{\sup_{u\in\mathbb{T}}\xi_u \leq m\}$, by the construction of the marked tree, we get that the life time of an individual can be stochastically bounded from below by an exponential variable of parameter m^α. Therefore we can bound the number of individuals present at time t by the number of individuals of a continuous time branching process denoted by GW in which each individual lives for a random time whose law is exponential of parameter m^α and the probability distribution of the offspring is the law of $\sharp s \vee 1$ under ν (we have taken the supremum with 1 to ensure the absence of death). For the Markov branching process GW, we are in the temporally homogeneous case and, we notice that

$$\int_{\mathcal{M}_p(\mathbb{R}_+^*)} u^{(n_{\mathbf{s}})\vee 1}\nu(d\mathbf{s}) = (f(u)-u)\nu(n_{\mathbf{s}}\neq 0)+u,$$

therefore as we have supposed (2.4.2), we can use Theorem 1 p.105 of the book of Athreya and Ney [6] (proved in Theorem 9 p.107 of the book of Harris [40]) and get that we are in the non-explosive case for the GW. As the number of the individuals is bounded by that of GW we get that the number of individuals at time t is a.s. finite.

Therefore conditioning on the event $\{\sup_{u\in\mathbb{T}}\xi_u \leq m\}$, we have that for all $t\geq 0$, the number of individuals at time t is a.s. finite, i.e. $\mathbf{X}(t)$ is a finite point measure.

• Second we will show the Markov property. Fix $r\in\mathbb{R}_+$. Let τ_r be equal to $\{u\in\mathbb{T}: \quad r\in I_u\}$. We notice that τ_r is an optional line. In fact for all lines $L\subset\mathbb{T}$ we have that

$$\{\tau_r \preceq L\} = \{r < a_u+\zeta_u \ \ \forall u\in L\}\in\mathcal{H}_L.$$

By definition, we have the identity

$$\sum_{i=1}^{\sharp\mathbf{X}(t+r)} \mathbb{1}_{\{X_j(t+r)>0\}}\delta_{X_j(t+r)} = \sum_{u\in\mathbb{T}}\mathbb{1}_{\{t+r\in I_u\}}\delta_{\xi_u}.$$

Let $\mathbf{X}(r)=\sum_{i=1}^n\delta_{\xi_{vn}}\in\mathcal{M}_p(\mathbb{R}_+^*)$ with $n=\sharp\mathbf{X}(r)$ and $(v_1,...,v_n)$ the nodes of $\mathbb{T}$. Define for all $i\leq n$,

$$\tilde{T}^{(i)} := ((\xi_{v_iu}, a_{v_iu}-a_{v_i}, \zeta_{v_iu}-\mathbb{1}_{\{u=\emptyset\}}(r-a_{v_i}))_{u\in\mathbb{T}}) = ((\tilde{\xi}_u^{(i)},\tilde{a}_u^{(i)},\tilde{\zeta}_u^{(i)})_{u\in\mathbb{T}}),$$

$\tilde{I}_u^{(i)} := [\tilde{a}_u^{(i)}, \tilde{a}_u^{(i)}+\tilde{\zeta}_u^{(i)}[$ and

$$\mathbf{X}^{(i)}(t) = \sum_{u\in\mathbb{T}}\mathbb{1}_{\{t\in\tilde{I}_u^{(i)}\}}\delta_{\tilde{\xi}_u^{(i)}}.$$

Then

$$\mathbf{X}(t+r) = \sum_{i=1}^n \mathbf{X}^{(i)}(t).$$

By the lack of memory of the exponential variable, we have that for $u\in\mathbb{T}$, given

$s \in I_u$ the law of the marked tree $\tilde{T}^{(i)}$ is the same as that of

$$T^{\xi_{v_i}} := ((\xi_{v_i u}, a_{v_i u} - a_{v_i}, \zeta_{v_i u})_{u\in\mathbb{T}}) := ((\xi^i_u, a^i_u, \zeta^i_u)_{u\in\mathbb{T}}).$$

Thus we have the equality in law:

$$\sum_{u\in\mathbb{T}} \mathbb{1}_{\{t\in\tilde{I}^{(i)}_u\}}\delta_{\tilde{\xi}^{(i)}_u} \overset{(d)}{=} \sum_{u\in\mathbb{T}} \mathbb{1}_{\{t\in I^i_u\}}\delta_{\xi^i_u},$$

with $I^i_u := [a^i_u, a^i_u + \zeta^i_u[$.

Let $\tau^i_r := \{v_i u \in \mathbb{T} : \quad r \in I^i_u\}$. Moreover for all lines $L \in \mathbb{T}$ we have that

$$\{\tau^i_r \preceq L\} = \{r < a_{v_i u} + \zeta_{v_i u} \quad \forall v_i u \in L\} \in \mathcal{H}_L.$$

Therefore τ^i_r is an optional line and by applying Lemma 2.1 for the optional line τ^i_s, we have that the condition distribution of the point measure

$$\sum_{u\in\mathbb{T}} \mathbb{1}_{\{t+r\in I^i_u\}}\delta_{\xi^i_u}$$

given $\mathcal{H}_{\tau_r}$ is the law of $\mathbf{X}(t)$ under $\mathbb{P}_{x_i}$. We notice that $\mathcal{H}_{\tau_s} = \sigma(\tilde{\xi}_u, \mathbf{e}_u : a_u \leq s)$ is the same filtration as $\mathcal{F}_s = \sigma(\mathbf{X}(s') : \quad s' \leq s)$. Therefore $(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, ..., \mathbf{X}^{(n)})$ is a sequence of independent random processes, where for each i, $\mathbf{X}^{(i)}(t)$ is distributed as $\mathbf{X}(t)$ under $\mathbb{P}_{x_i}$. We then have proven the Markovian property.

• The scaling property is an easy consequence of the definition of the tree T_x. □

Remark 2.2. *For every measurable function $g : \mathbb{R}^*_+ \to \mathbb{R}^*_+$, define a multiplicative functional such that for every $\mathbf{s} = \sum_{i=1}^{\sharp s} \delta_{s_i} \in \mathcal{M}_p(\mathbb{R}^*_+)$:*

$$\varphi_g(\mathbf{s}) := \exp(-\langle g, \mathbf{s}\rangle) = \exp(-\sum_{i=1}^{\sharp s} g(s_i)).$$

Then the generator G of the Markov process $\mathbf{X}(t)$ fulfills for every $\mathbf{y} = \sum_{i=1}^{\sharp \mathbf{y}} \delta_{y_i} \in \mathcal{M}_p(\mathbb{R}^*_+)$:

$$G\varphi_g(\mathbf{y}) = \sum y_i^\alpha e^{-\sum_{j\neq i} g(y_j)} \int_{\mathcal{M}_p(\mathbb{R}^*_+)} (e^{-\langle g(xy_i),\mathbf{s}\rangle} - e^{-g(y_i)})\nu(d\mathbf{s}). \tag{2.4.4}$$

The intrinsic martingale M_n is indexed by the generations; it will also be convenient to consider its analogue in continuous time, i.e

$$M(t) := \langle x^{p_0}, \mathbf{X}(t)\rangle = \sum_{u\in\mathbb{T}} \mathbb{1}_{\{t\in I_u\}}\xi^{p_0}_u.$$

It is straightforward to check that $(M(t), t \geq 0)$ is again a martingale in the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of the process $(\mathbf{X}(t), t \geq 0)$; and more precisely, the argument

Proposition 1.5 in [17] gives:

Corollary 2.1. *The process $(M(t), t \geq 0)$ is a martingale, and more precisely*

$$M(t) = \mathbb{E}(M_\infty | \mathcal{F}_t),$$

where M_∞ is the terminal value of the intrinsic martingale $(M_n, n \in \mathbb{N})$. In particular $M(t)$ converges in $\mathrm{L}^p(\mathbb{P})$ to M_∞ for some $p > 1$.

Proof. We will use the same argument as in the proof of Proposition 1.5 of [17] . Netherless, we have to deal here with the fact that $\sup_{u\in\mathcal{U}} \xi_u$ may be larger than 1. Therefore we will have to condition. We know that M_n converges in $\mathrm{L}^p(\mathbb{P})$ to M_∞ as n tends to ∞, so

$$\mathbb{E}(M_\infty | \mathcal{F}_t) = \lim_{n\to\infty} \mathbb{E}(M_n | \mathcal{F}_t).$$

By Theorem 2.1 as already seen in (2.4.3), we have $\sup_{u\in\mathcal{U}} \xi_u^{p_0} \in \mathrm{L}^p(\mathbb{P})$, so that we fix $m > 0$ and we now work on the event $B_m := \{\sup_{u\in\mathcal{U}} \xi_u \leq m\}$.

By applying the Markov property at time t we easily get that

$$\mathbb{E}(M_n | \mathcal{F}_t) = \sum_{i=1}^{\sharp\mathbf{X}(t)} X_i^{p_0}(t) \mathbb{1}_{\{\varrho(X_i(t))\leq n\}} + \sum_{|u|=n} \xi_u^{p_0} \mathbb{1}_{\{a_u+\zeta_u<t\}} \tag{2.4.5}$$

where $\varrho(\xi_v)$ stands for the generation of the individual v (i.e. $\varrho(\xi_v) = |v|$), and $a_u + \zeta_u$ is the instant when the individual corresponding to the node u reproduces. We can rewrite the latter as

$$a_u + \zeta_u = \xi_{m^{|u|}u}^{-\alpha} \mathbf{e}_0 + \xi_{m^{|u|-1}u}^{-\alpha} \mathbf{e}_1 + ... + \xi_u^{-\alpha} \mathbf{e}_{|u|}$$

where $\mathbf{e}_0$,... is a sequence of independent exponential variables with parameter 1, which is also independent of ξ_u. We can remark that in the first term of sum (2.4.5) we sum over the sizes of the individuals which belong to the n-th generation and are alive at time t, and in the second term we sum over those belonging to the n-th generation and are dead at time t.

As α is nonnegative, and as we are working on the event B_m: $\xi_{m^i u}^{-\alpha} \geq m^{-\alpha}$ we have that for each fixed node $u \in \mathbb{T}$, $a_u + \zeta_u$ is bounded from below by the sum of $|u|+1$ independent exponential variables with parameter m^α which are independent of ξ_u. Thus

$$\lim_{n\to\infty} \mathbb{E}\left(\sum_{|u|=n} \xi_u^{p_0} \mathbb{1}_{\{a_u+\zeta_u<t\}} \mathbb{1}_{\{B_m\}} \right) = 0,$$

and therefore by (2.4.5) on the event $\{B_m\}$, we get that for all $m > 0$: $\mathbb{E}(M_\infty | \mathcal{F}_t)\mathbb{1}_{\{B_m\}} = M(t)\mathbb{1}_{\{B_m\}}$, and then by letting m tend to ∞ we get the result. □

2.5 A randomly tagged leaf.

We will here (as in [17]) define what a tagged individual is by using a tagged leaf.

We call *leaf* of the tree $\mathbb{T}$ an infinite sequence of integers $l = (u_1, ...)$. For each n, $l^n := (u_1, ..., u_n)$ is the ancestor of l at the generation n. We enrich the probabilistic structure by adding the information about a so called tagged leaf, chosen at random as follows. Let H_n be the space of bounded functionals Φ which depend on the mark M and of the leaf l up to the n-th first generation, i.e. such that $\Phi(M, l) = \Phi(M', l')$ if $l^n = l^{n'}$ and $M(u) = M'(u)$ whenever $|u| \leq n$. For such functionals, we use the slightly abusing notation $\Phi(M, l) = \Phi(M, l^n)$. As in [17] for a pair (M, λ) where $M : \mathbb{T} \to [0, 1] \times \mathbb{R}_+ \times \mathbb{R}_+$ is a random mark on the tree and λ is a random leaf of $\mathbb{T}$, the joint distribution denoted by $\mathbb{P}^*$ (and by $\mathbb{P}^*_x$ if the size of the first mark is x instead of 1) can be defined unambiguously by

$$\mathbb{E}^*(\Phi(M, \lambda)) = \mathbb{E}\left(\sum_{|u|=n} \Phi(M, u)\xi_u^{p_0}\right), \qquad \Phi \in \mathrm{H}_n.$$

Moreover since the intrinsic martingale $(M_n, n \in \mathbb{Z}_+)$ is uniformly integrable (cf. Theorem 2.1), the first marginal of $\mathbb{P}^*$ is absolutely continuous with respect to the law of the random mark M under $\mathbb{P}$, with density M_∞.

Let λ_n be the node of the tagged leaf at the n-th generation. We denote $\chi_n := \xi_{\lambda_n}$ for the size of the individual corresponding to the node λ_n and $\chi(t)$ for the size of the tagged individual alive at time t, viz.

$$\chi(t) := \chi_n \quad \text{ if } \ a_{\lambda_n} \leq t < a_{\lambda_n} + \zeta_{\lambda_n},$$

because in the case considered $\sup_{n\in\mathbb{N}} a_{\lambda_n} = \infty$. We stress that, in general the process $\chi(t)$ is not monotonic. However as in [17] , Lemma 1.4 there becomes:

Lemma 2.2. *Let $k : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function such that $k(0) = 0$. Then we have for every $n \in \mathbb{N}$*

$$\mathbb{E}^*(k(\chi_n)) = \mathbb{E}\left(\sum_{|u|=n} \xi_u^{p_0} k(\xi_u)\right),$$

and for every $t \geq 0$

$$\mathbb{E}^*(k(\chi(t))) = \mathbb{E}\left(\langle x^{p_0}k(x), X(t)\rangle\right).$$

Proposition 1.6 of [17] becomes:

Proposition 2.1. *Under $\mathbb{P}^*$,*

$$S_n := \ln \chi_n, \quad n \in \mathbb{Z}_+$$

is a random walk on $\mathbb{R}$ *with step distribution*

$$\mathbb{P}(\ln \chi_n - \ln \chi_{n+1} \in dy) = \widetilde{\nu}(dy),$$

where the probability measure $\widetilde{\nu}$ *is defined by*

$$\int_{]0,\infty[} k(y)\widetilde{\nu}(dy) = \int_{\mathcal{M}_p(\mathbb{R}_+^*)} \langle x^{p_0} k(\ln(x)), \mathbf{s}\rangle \nu(d\mathbf{s}).$$

Equivalently, the Laplace transform of the step distribution is given by

$$\mathbb{E}^*(e^{pS_1}) = \mathbb{E}^*(\chi_1^p) = 1 - \kappa(p + p_0), \quad p \geq 0.$$

Moreover, conditionally on $(\chi_n, n \in \mathbb{Z}_+)$ *the sequence of the lifetimes* $(\zeta_{\lambda_0}, \zeta_{\lambda_1}, ...)$ *along the tagged leaf is a sequence of independent exponential variables with respective parameters* $\chi_0^\alpha, \chi_1^\alpha, ...$

We now see that we can use this proposition to obtain the description of $\chi(t)$ using a Lamperti transformation. Let

$$\eta_t := S \circ N_t, \quad t \geq 0,$$

with N a Poisson process with parameter 1 which is independent of the random walk S; for probabilities and expectations related to η we use the notation P and E. The process $(\chi(t), t \geq 0)$ is Markovian and enjoys a scaling property. More precisely under $\mathbb{P}_x^*$ we get that

$$\chi(t) \overset{(d)}{=} \exp(\eta_{\tau(tx^{-\alpha})}), \quad t \geq 0, \tag{2.5.1}$$

where η is the compound Poisson defined above and τ the time-change defined implicitly by

$$t = \int_0^{\tau(t)} \exp(\alpha\eta_s)ds, \quad t \geq 0. \tag{2.5.2}$$

2.6 Asymptotic behaviors.

2.6.1 The convergence of the size of a tagged individual.

Let

$$\kappa'(p_0) = -\int_{\mathcal{M}_p(\mathbb{R}_+^*)} \langle x^{p_0} \ln(x), \mathbf{s}\rangle \nu(d\mathbf{s})$$

denote the derivative of κ at the Malthusian parameter p_0.

In this part we focus on the asymptotic behavior of the size of a tagged individual. In this direction, the quantity $\varpi_t = e^{\alpha\eta_t}$ plays an important role, as it appears at the time change of the Lamperti transformation (see (2.5.2)), as we see in the next proposition:

Proposition 2.2. *Suppose that $\alpha > 0$, that the support of ν is not a discrete subgroup $r\mathbb{Z}$ for any $r > 0$ and that $0 < \kappa'(p_0) < \infty$. Then for every $y > 0$, under $\mathbb{P}^*_y$, $t^{1/\alpha}\chi(t)$ converges in law as $t \to \infty$ to a random variable Y whose law is specified by*

$$\mathbb{E}(k(Y^\alpha)) = \frac{1}{\alpha m_1} E(k(I)I^{-1}),$$

for every measurable function $k : \mathbb{R}_+ \to \mathbb{R}_+$, with $I := \int_0^\infty \exp(\alpha\eta_s)ds$ and $m_1 := E(\eta_1) = -\kappa'(p_0)$.

Proof. As $-\kappa'(p_0)$ is the mean of the step distribution of the random walk S_n (see Proposition 2.1), therefore $\kappa'(p_0) > 0$ imply that $E(-\eta_1) > 0$ thus the assumption of Theorem 1 in the works of Bertoin and Yor [23] is fulfilled by the self-similar Markov process $\chi(t)^{-1}$, which gives the result. □

We could also try to use the same method as the one used in [20] for which we need Proposition 1.7 [17] . But in this latter we needed $\mathbb{E}(\langle x^p, X(t)\rangle)$ to be finite when p is large, and its derivative to be completely monotone. But here neither of these requirements is necessarily true as κ is not necessarily positive when p is large. This explains why we have to use a different method.

Remark 2.3. *In the case $\kappa'(p_0) = 0$ we can extend this proposition. More precisely if $\int_{\mathcal{M}_p(\mathbb{R}^*_+)} \langle x^{p_0} |\ln(x)|, \mathbf{s}\rangle \nu(d\mathbf{s}) < \infty$,*

$$J := \int_1^\infty \frac{x\nu^-((x,\infty))dx}{1 + \int_0^x dy \int_y^\infty \nu^-((-\infty,-z))dz} < \infty,$$

(where ν^- is the image of $\widetilde{\nu}$ by the map $u \to -u$ and $\widetilde{\nu}$ is defined in Proposition 2.1) and $E\left(\log^+ \int_0^{T_1} e^{-\eta_s} ds\right) < \infty$ (with $T_z := \inf\{t : -\eta_t \geq z\}$) hold then, for any $y > 0$ under $\mathbb{P}^*_y$, $t^{1/\alpha}\chi(t)$ converge in law as $t \to \infty$, to a random variable $\tilde{Y}$ whose law is specified by

for any bounded and continuous function k and for $t > 0$:

$$\mathbb{E}(k(\tilde{Y}^\alpha)) = \lim_{\lambda \to 0} \frac{1}{\lambda} E(I_\lambda^{-1} k(I_\lambda)),$$

where

$$I_\lambda = \int_0^\infty \exp(\alpha\eta_s - \lambda s)ds. \tag{2.6.1}$$

The proof is the same as the previous one using Theorem 1 and Theorem 2 from the works of Caballero and Chaumont [28] instead of [23].

2.6.2 Convergence of the mean measure and L^p-convergence.

We encode the configuration of masses $X(t) = \{(X_i(t))_{1\leq i\leq \sharp \mathbf{X}(t)}\}$ by the weighted empirical measure

$$\sigma_t := \sum_{i=1}^{\sharp \mathbf{X}(t)} X_i^{p_0}(t)\delta_{t^{1/\alpha}X_i(t)}$$

which has total mass $M(t)$.

The associated mean measure σ_t^* is defined by the formula

$$\int_0^\infty k(x)\sigma_t^*(dx) = \mathbb{E}\left(\int_0^\infty k(x)\sigma_t(dx)\right)$$

which is required to hold for all compactly supported continuous functions k. Since $M(t)$ is a martingale, σ_t^* is a probability measure. We are interested in the convergence of this measure. This convergence was already established in the case of binary conservative fragmentation (see the results of Brennan and Durrett [26] and [27]). A very useful tool for this is the renewal theorem, for which they needed the fact that the process $\chi(t)$ is decreasing; here we no longer have such a monotonicity property. See also Theorem 2 and 5 of [20], Theorem 1.3 of [17] and Proposition 4 of [52] for Theorems about empirical measures which have the property $\nu(s_i \leq 1\ \forall i \in \mathbb{N}) = 1$.

Nonetheless, with Proposition 2.2 and Lemma 2.2, we easily get:

Corollary 2.2. *With the assumptions of Proposition 2.2 we get:*

1. *The measures σ_t^* converge weakly, as $t \to \infty$, to the distribution of Y i.e. for any continuous bounded function $k : \mathbb{R}_+ \to \mathbb{R}_+$, we have:*

$$\mathbb{E}\left(\langle x^{p_0}k(t^{1/\alpha}x), X(t)\rangle\right) \underset{t\to\infty}{\to} \mathbb{E}(k(Y)).$$

2. *For all $p_+ > p > p_0$:*

$$t^{(p-p_0)/\alpha}\mathbb{E}\left(\langle x^p, X(t)\rangle\right) \underset{t\to\infty}{\to} \mathbb{E}(Y^{p-p_0}).$$

We now formulate a more precise result concerning the convergence of the empirical measure:

Theorem 2.3. *Under the same assumptions as in Proposition 2.2 we get that for every bounded continuous function k:*

$$L^p - \lim_{t\to\infty} \int_0^\infty k(x)\sigma_t(dx) = M_\infty \mathbb{E}(k(Y)) = \frac{M_\infty}{\alpha m} E(k(I)I^{-1}),$$

for some $p > 1$.

Remark 2.4. *A slightly different version of Corollary 2.2 and Theorem 2.3 exists also under the assumptions in Remark 2.3.*

See also Asmussen and Kaplan [4] and [5] for a closely related result.

Proof. We follow the same method as Section 1.4. in [17] and in this direction we use Lemma 1.5 there: for $(\lambda(t))_{t\geq 0} = (\lambda_i(t), i \in \mathbb{N})_{t\geq 0}$ a sequence of non-negative random variables such that for fixed $p > 1$

$$\sup_{t\geq 0} \mathbb{E}\left(\left(\sum_{i=1}^{\infty} \lambda_i(t)\right)^p\right) < \infty \quad \text{and} \quad \lim_{t\to\infty} \mathbb{E}\left(\sum_{i=1}^{\infty} \lambda_i(t)\right) = 0,$$

and for $(Y_i(t), i \in \mathbb{N})$ a sequence of random variables which are independent conditionally on $\lambda(t)$, we assume that there exists a sequence $(\bar{Y}_i, i \in \mathbb{N})$ of i.i.d variables in $\mathrm{L}^p(\mathbb{P})$, which is independent of $\lambda(t)$ for each fixed t, and such that $|Y_i(t)| \leq \bar{Y}_i$ for all $i \in \mathbb{N}$ and $t \geq 0$.

Then we know from Lemma 1.5 in [17] that

$$\lim_{t\to\infty} \sum_{i=1}^{\infty} \lambda_i(t)(Y_i(t) - \mathbb{E}\left(Y_i(t)|\lambda(t)\right)) = 0. \tag{2.6.2}$$

Now, let k be a continuous function bounded by 1 and let

$$A_t := \langle x^{p_0} k(t^{1/\alpha} x), X(t)\rangle.$$

By application of the Markov property at time t for A_{t+s} and the self-similarity property of the process $\mathbf{X}$ we can rewrite A_{t+s} as

$$\sum_{i=1}^{\sharp\mathbf{X}(t)} \lambda_i(t) Y_i(t,s)$$

where $\lambda_i(t) := X_i^{p_0}(t)$ and

$$Y_i(t,s) := \langle x^{p_0} k((t+s)^{1/\alpha} X_i(t) x), \mathbf{X}_{i,.}(s)\rangle,$$

with $\mathbf{X}_{1,.}$, $\mathbf{X}_{2,.}$, ... a sequence of i.i.d. copies of $\mathbf{X}$ which is independent of $\mathbf{X}(t)$.

By Theorem 2.1 we get that

$$\sup_{t\geq 0} \mathbb{E}\left(\left(\sum_{i=1}^{\sharp\mathbf{X}(t)} \lambda_i(t)\right)^p\right) < \infty.$$

By the last corollary we also obtain that

$$\mathbb{E}\left(\sum_{i=1}^{\sharp\mathbf{X}(t)} \lambda_i^p(t)\right) \sim t^{-(p-1)p_0} \mathbb{E}(\chi^{(p-1)p_0}(1)) \to 0,$$

as $t \to \infty$.
Moreover the variables $Y_i(t,s)$ are uniformly bounded by

$$Y_i = \sup_{s\geq 0} \langle x^{p_0}, \mathbf{X}_{i,\cdot}(s)\rangle,$$

which are i.i.d. variables and also bounded in $\mathrm{L}^p(\mathbb{P})$ thanks to Doob's inequality (as $\langle x_{p_0}, \mathbf{X}_{i,\cdot}(s)\rangle$ is a martingale bounded in $\mathrm{L}^p(\mathbb{P})$).

Thus we may apply (2.6.2), which reduces the study to that of the asymptotic behavior of:

$$\sum_{i=1}^{\sharp\mathbf{X}(t)} \lambda_i(t)\mathbb{E}(Y_i(t,s)|\mathbf{X}(t)),$$

as t tends to ∞. On the event $\{X_i(t) = y\}$, we get

$$\mathbb{E}(Y_i(t,s)|\mathbf{X}(t)) = \mathbb{E}\left(\langle x^{p_0} k((t+s)^{1/\alpha} yx), \mathbf{X}(s)\rangle\right).$$

Then by Lemma 2.2:

$$\mathbb{E}\left(\langle x^{p_0} k((t+s)^{1/\alpha} yx), \mathbf{X}_{i,\cdot}(s)\rangle\right) = \mathbb{E}^*\left(k\left((t+s)^{1/\alpha} y\chi(s)\right)\right).$$

With Proposition 2.2, we obtain

$$\lim_{t\to\infty} \mathbb{E}^*\left(k\left((t+s)^{1/\alpha} y\chi(s)\right)\right) = \mathbb{E}\left(k\left(Y\right)\right).$$

Moreover recall from Corollary 2.1 that $\sum_{i=1}^{\sharp\mathbf{X}(t)} \lambda_i(t)$ converges to M_∞ in $\mathrm{L}^p(\mathbb{P})$. Therefore we finally get that when t goes to infinity:

$$\sum_{i=1}^{\sharp\mathbf{X}(t)} \lambda_i(t)\mathbb{E}(Y_i(t,s)|X(t)) \sim \mathbb{E}\left(k\left(Y\right)\right) \sum_{i=1}^{\sharp\mathbf{X}(t)} \lambda_i(t) \sim \mathbb{E}\left(k\left(Y\right)\right) M_\infty.$$

□

2.7 Appendix: Further results about the intrinsic process

We will give more general properties about the intrinsic process $\{M_Q,\ Q \subset \mathbb{T}\}$, $M_Q = \sum_{u\in M} \xi_u^{p_0}$. For a line Q, $\{M_Q\}$ is adapted to the filtration $\{\mathcal{H}_L\}$. We use the abuse of notation that M_n stand for the process M_{L_n}, with $L_n = \{u \in \mathbb{T} :\ |u| = n\}$ the labels of the n-th generation. We introduce new definitions, we say that a line Q *covers* L, if $Q \succeq L$ and any individual stemming from L either stems from Q or has progeny in Q. If Q covers the ancestor it may simply be called *covering*. Let $\mathcal{C}_0$ be the class of covering lines with finite maximal generation. We denote the generation of Q: $|Q| = \sup_{u\in Q} |u|$. The origin of the intrinsic martingale comes from real time

martingale of Nerman [60].

Also for $r \in \mathbb{R}_+^*$, let ϑ_r be the structural measure:

$$\vartheta_r(B) := \mathbb{E}_r(\sharp\{u \in \mathbb{T} : \xi_u \in B\}) = \sum_{i=1}^{\infty} \nu(rs_i \in B) \quad \text{for } B \subset \mathcal{B},$$

where $\mathcal{B}$ is the Borel algebra on $\mathbb{R}_+^*$. Let the reproduction measure μ on the sigma-field $\mathcal{B} \otimes \mathcal{B}$ be such that for every $r \geq 0$:

$$\mu(r, dv \times du) := r^\alpha \exp(-r^\alpha u) du \vartheta_r(dv)$$

and for any $\lambda \in \mathbb{R}$

$$\mu_\lambda(r, dv \times du) := e^{-\lambda u} \mu(r, du \times dv).$$

The composition operation $*$ denotes the Markov transition on the size space $\mathbb{R}_+$ and convolution on the time space $\mathbb{R}_+$, so that: for all $A \in \mathcal{B}$ and $B \in \mathcal{B}$,

$$\mu^{*2}(s, A \times B) = \mu * \mu(s, A \times B) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} \mu(r, A \times (B - u)) \mu(s, dr \times du).$$

With the convention that the $*$-power 0 is $\mathbb{1}_{\{A \times B\}}(s, 0)$ which gives all the mass to $(s, 0)$. We define the renewal measure as

$$\psi_\lambda := \sum_0^\infty \mu_\lambda^{*n}.$$

Let

$$\alpha' := \inf\{\lambda \ : \ \psi_\lambda(r, \mathbb{R}_+ \times \mathbb{R}_+) < \infty \ \text{ for some } \ r \in \mathbb{R}_+\}.$$

Moreover as

$$\mu_\lambda(r, \mathbb{R}_+ \times \mathbb{R}_+) = \begin{cases} mr^\alpha/(r^\alpha + \lambda) & \text{if } \ \lambda > -r^\alpha \\ \infty & \text{else,} \end{cases}$$

thus

$$\psi_\lambda(r, \mathbb{R}_+ \times \mathbb{R}_+) < \infty \ \text{ if and only if } \lambda < (r/(m-1))^{1/\alpha}$$

therefore we get $\alpha' = 0$. For $A \in \mathcal{B}$, let

$$\pi(A) := \lim_{n \to \infty} \mu^{*n}(1, A \times \mathbb{R}_+) \tag{2.7.1}$$

which is well defined as $\mu^{*n}(1, A \times \mathbb{R}_+)$ is a decreasing function in n and nonnegative. Let $h(s) := s^{p_0}$ for all $s \in \mathbb{R}_+$ and $\beta := 1$. These objects correspond to those defined in [44].

Recall that the Galton-Watson process $(Z_n, n \geq 0))$ is equal to $(\sharp\{u \in \mathbb{T} : \ \xi_u > 0 \text{ and } \ |u| = n\}, n \geq 0)$.

We suppose that

$$m := \mathbb{E}(Z_1) < \infty,$$

i.e. $\int_{\mathcal{M}_p(\mathbb{R}_+^*)} \sharp s\nu(d\mathbf{s}) < \infty$ this assumption is slightly stronger than (2.4.2), therefore we get that:

Proposition 2.3. *1. If $L \preceq Q$ are lines, then*

$$\mathbb{E}(M_Q|\mathcal{H}_L) \leq M_L.$$

If Q verifies $|Q| < \infty$ and covers L, then

$$\mathbb{E}(M_Q|\mathcal{H}_L) = M_L.$$

2. For all $s > 0$, $\{M_L;\ L \in \mathcal{C}_0\}$ is uniformly $\mathbb{P}_s$-integrable.

3. There is a random variable $M \geq 0$ such that for π-almost all $s > 0$

$$M_L = \mathbb{E}_s(M|\mathcal{H}_L)$$

and $M_L \overset{L^1(\mathbb{P}_s)}{\to} M$, as $L \in \mathcal{C}_0$ filters ($\preceq$). If $\varsigma_n \preceq \varsigma_{n+1} \in \mathcal{C}_0$ and to any $x \in \mathbb{T}$ there is an ς_n such that x has progeny in ς_n, $M_{\varsigma_n} \to M$, as $n \to \infty$, also a.s. $\mathbb{P}_s$.

A consequence of the first and second points applied for $L_n = \{u \in \mathbb{T} :\ |u| = n\}$ and $L_m = \{u \in \mathbb{T} :\ |u| = m\}$ with $m \geq n \geq 0$, is that M_n is a martingale and the uniform $\mathbb{P}_s$-integrability of this martingale. The third point applied for the lines τ_t gives the convergence of $M(t)$ in $\mathrm{L}^1(\mathbb{P}_s)$ and almost surely.

Proof. First the conditions of Malthusian population, as defined by Jagers in [44] , are fulfilled, thus by Theorem 5.1 therein we get the first point.

Let $\overline{\xi} := \int_{\mathbb{R}_+\times\mathbb{R}_+} h(s)r^\alpha e^{-tr^\alpha} dt\vartheta_1(ds) = \sum_{|u|=1} \xi_u^{p_0}$ and $\mathbb{E}_\pi$ be the expectation with respect to $\int_{\mathbb{R}_+} \mathbb{P}_s(dw)\pi(ds)$. Therefore,

$$\mathbb{E}_\pi(\overline{\xi}\log^+\overline{\xi}) = \int_{\mathbb{R}_+} \mathbb{E}_x\left(\sum_{i=1}^\infty \xi_i^{p_0}\left(\log^+\sum_{j=1}^\infty \xi_j^{p_0}\right)\right)\pi(dx),$$

and it follows readily from the Malthusian hypotheses and the fact that $\sum_{|u|=n} \xi_u^{pp_0}$ is a supermartingale, that this quantity is finite. Therefore the assumption of Theorem 6.1 of [44] are check, which gives by Theorem 6.1 of [44] the second point and by Theorem 6.3 of [44] we get the third point. □

Chapter 3

Statistical analysis of self-similar conservative fragmentation chains

We address statistical inference in self-similar conservative fragmentation chains, when only (aproximate) observations of the size of the fragments below a given threshold are available.

This framework, introduced by Bertoin and Martinez, is motivated by mineral crushing in mining industry.

The underlying estimated object is the step distribution of the random walk associated to a randomly tagged fragment that evolves along the genealogical tree representation of the fragmentation process. We compute upper and lower rates of estimation in a parametric framework, and show that in the non-parametric case, the difficulty of the estimation is comparable to ill-posed linear inverse problems of order 1 in signal denoising.

Contents

The material of this chapter is an article writing with Marc Hoffmann and submitted.

3.1 Introduction

3.1.1 Motivation

Random fragmentation models, commonly used in a variety of physical models, lay their theoretical roots in the works of Kolmogorov [46] and Filippov [35] (see also [2, 17, 48, 49] and the references therein). Informally, we imagine an object that falls apart randomly as time passes. The resulting particles break independently of each other in an independent and self-similar way. A thorough account on random fragmentation processes and chains is given in the book by Bertoin [17], a key reference for the paper.

In this work, we take the perspective of statistical inference. We focus on the quite specific class of *self-similar fragmentation chains*. The law of the fragmentation is then entirely determined by its dislocation measure and its index of self-similarity, which govern the way and the rate at which the fragments split. If one is allowed to observe the whole fragmentation process up to some fixed time, then the statistical problem is somehow degenerate[1]. We postulate a more realistic observation scheme, motivated by mining industry, where the goal is to separate metal from non valued components in large mineral blocks by a series of blasting, crushing and grinding. In this setting, one rather observes approximatively the fragments arising from an initial block of size m only when they reach a size smaller than some screening threshold, say $\eta > 0$. Asymptotics are taken as the ratio $\varepsilon := \eta/m$ vanishes. See Bertoin and Martinez [21] and the references therein.

3.1.2 Organization and of the paper

In Section 2, we recall some well known facts about the construction of conservative fragmentation chains, following closely the book by Bertoin [17]. For statistical purposes, our main tool is the empirical measure $\mathcal{E}_\varepsilon$ of the size of fragments when they reach a size smaller than a threshold ε in the limit $\varepsilon \to 0$. We highlight the fact that $\mathcal{E}_\varepsilon$ captures information about the dislocation measure through the Lévy measure π of a randomly tagged fragment associated to the fragmentation process.

In Section 3, we give a rate of convergence for the empirical measure $\mathcal{E}_\varepsilon$ toward its limit in Theorem 3.1, extending former results (under more stringent assumptions) of Bertoin and Martinez [21]. The rate is of the form $\varepsilon^{1/2-\ell(\pi)}$, where $\ell(\pi) > 0$ can be made arbitrarily small under adequate exponential moment conditions for π. We add-up the more realistic framework of observations with limited accuracy, where each fragment is actually known up to a systematical stochastic error of order $\sigma \ll \varepsilon$.

In Section 3, we construct estimators related to functionals of π in the absolute continuous case. In the parametric case (Theorem 3.3), we establish that the best

[1] in the sense that it can be mapped into relatively standard equivalent inference problems such as probability distribution estimation from independent observations, see Section 3.4.4

achievable rate is $\varepsilon^{1/2}$ in the particular case of binary fragmentations, where a particle splits into two blocks at each step exactly. We construct a convergent estimator in a general setting (Theorem 3.2) with an error of order $\varepsilon^{1/2-\ell'(\pi)}$, for another $\ell'(\pi) > 0$ that can be made arbitrarily small under appropriate assumptions on the density of π near 0 and $+\infty$. In the non-parametric case, we construct an estimator that achieves (Theorem 3.4) a rate of the form $(\varepsilon^{1-\ell''(\pi)})^{s/(2s+3)}$, where $s > 0$ is the local smoothness of the density of π, up to appropriate rescaling. Except for the factor $\ell''(\pi) > 0$, we obtain the same rate as for ill-posed linear inverse problems of degree 1. We suggest a simple interpretation of this result in terms of the asymptotic form of $\mathcal{E}_\varepsilon$ in the discussion Section 4, appended with further remarks about Theorems 3.1, 3.2 and 3.4.

An appendix (Section 6) recalls sharp results on the key renewal theorem from Sgibnev [65] that are used to derive Theorem 3.1.

3.2 Statistical model

3.2.1 Fragmentation chains

Let $X = \big(X(t), t \geq 0\big)$ be a fragmentation chain with state space

$$\mathcal{S}^{\downarrow} := \big\{\mathbf{s} = (s_1, s_2, \ldots),\ s_1 \geq s_2 \geq \ldots \geq 0,\ \sum_{i=1}^{\infty} s_i \leq 1\big\}.$$

We assume that X has self-similar parameter $\alpha \geq 0$. For well-definiteness, see *e.g.* Bertoin [17], the following mild assumptions on the dislocation measure $\nu(d\mathbf{s})$ of X are in force throughout the paper:

Assumption A. *We have* $\nu\big((1, 0, \ldots)\big) = 0$ *and* $\nu\big(s_1 \in (0, 1)\big) > 0$. *Moreover, for every* $\varepsilon > 0$*:* $\int_{\mathcal{S}^{\downarrow}} \sum_{i=1}^{\infty} 1_{\{s_i > \varepsilon\}} \nu(d\mathbf{s}) < \infty$.

We denote by $\mathbb{P}_m$ the law of X started from the initial configuration $(m, 0, \ldots)$ with $m \in (0, 1]$. Under $\mathbb{P}_m$, X is a Markov process and its evolution can be described as follows: a fragment with size x lives for an exponential time with parameter $x^\alpha \nu(\mathcal{S}^{\downarrow})$ and then splits and gives rise to a family of smaller fragments distributed as $x\xi$, where ξ is distributed according to $\nu(\bullet)/\nu(\mathcal{S}^{\downarrow})$. Under $\mathbb{P}_m$, the law of X is entirely determined by α and $\nu(\bullet)$.

We will repeatedly use the representation of fragmentation chains as random infinite marked trees. Let

$$\mathcal{U} := \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

denote the infinite genealogical tree (with $\mathbb{N}^0 := \{\emptyset\}$) associated to X as follows: to each node $u \in \mathcal{U}$, we set a mark

$$(\xi_u, a_u, \zeta_u),$$

where ξ_u is the size of the fragment labelled by u, a_u is its birthtime and ζ_u is its lifetime. We have the following identity between point measures on $(0, +\infty)$:

$$\sum_{i=1}^{\infty} 1_{\left\{X_i(t)>0\right\}} \delta_{X_i(t)} = \sum_{u\in\mathcal{U}} 1_{\left\{t\in[a_u, a_u+\zeta_u)\right\}} \delta_{\xi_u}, \quad t \geq 0,$$

with $X(t) = \big(X_1(t), X_2(t), \dots\big)$, and where δ_x denotes the Dirac mass at x. Finally, X has the following branching property: for every fragment $\mathbf{s} = (s_1, \dots) \in \mathcal{S}^{\downarrow}$ and every $t \geq 0$, the distribution of $X(t)$ given $X(0) = \mathbf{s}$ is the same as the decreasing rearrangement of the terms of independent random sequences $X^{(1)}(t)$, $X^{(2)}(t), \dots$ where, for each i, $X^{(i)}(t)$ is distributed as $X(t)$ under $\mathbb{P}_{s_i}$.

3.2.2 Observation scheme

For simplicity, we assume from now on that $\nu(\mathcal{S}^{\downarrow}) = 1$. Keeping in mind the motivation of mineral crushing, we consider the fragmentation under[2] $\mathbb{P} := \mathbb{P}_1$, initiated with a unique block of size $m = 1$ and we observe the process stopped at the time when all the fragments become smaller than some given threshold $\varepsilon > 0$, so we have data ξ_u, for every $u \in \mathcal{U}_\varepsilon$, with

$$\mathcal{U}_\varepsilon := \big\{u \in \mathcal{U}, \ \ \xi_{u-} \geq \varepsilon, \ \xi_u < \varepsilon\big\},$$

where we denote by $u-$ the parent of the fragment labelled by u. We will further assume that the total mass of the fragments remains constant through time:

Assumption B. *(Conservative property). We have:* $\nu\big(\sum_{i=1}^{\infty} s_i = 1\big) = 1$.

We next consider the empirical measure

$$\mathcal{E}_\varepsilon(g) := \sum_{u\in\mathcal{U}_\varepsilon} \xi_u \, g(\xi_u/\varepsilon),$$

where $g(\bullet)$ is a test function. Indeed, under Assumption B, we have

$$\sum_{u\in\mathcal{U}_\varepsilon} \xi_u = 1 \quad \mathbb{P}-\text{almost surely}, \tag{3.2.1}$$

so $\mathcal{E}_\varepsilon(g)$ appears as a weighted empirical version of $g(\bullet)$. Bertoin and Martinez show in [21] that under mild assumptions on $\nu(\bullet)$, the measure $\mathcal{E}_\varepsilon(g)$ converges to

$$\mathcal{E}(g) := \frac{1}{c(\nu)} \int_0^1 \frac{g(a)}{a} \int_{\mathcal{S}^{\downarrow}} \sum_{i=1}^{\infty} s_i \, 1_{\{s_i<a\}} \nu(d\mathbf{s}) da$$

[2]We often need to accomodate further random variables independent of X. Abusing notation slightly, we will still use the notation $\mathbb{P}$ without further notice, working tacitly on an appropriate enlargement of the original probability space.

in probability, as $\varepsilon \to 0$, with $c(\nu) = -\int_{\mathcal{S}^\downarrow} \sum_{i=1}^{\infty} s_i \log s_i \, \nu(d\mathbf{s})$, tacitly assumed to be well-defined. This suggests a strategy for recovering information about $\nu(\bullet)$ by picking suitable test functions $g(\bullet)$.

3.2.3 First estimates

From now on, we assume we have data

$$X_\varepsilon := \big(\xi_u, \ \ u \in \mathcal{U}_\varepsilon\big) \tag{3.2.2}$$

and we specialize in the estimation of $\nu(\bullet)$. Clearly, the data gives no information about the self-similar parameter α that we consider as a nuisance parameter[3]. Assumptions A and B are in force. At this stage, we can relate $\mathcal{E}(g)$ to a more appropriate quantity by means of the so-called *tagged frament* approach.

The randomly tagged fragment. Let us first consider the homogenous case $\alpha = 0$. Assume we can "tag" a point at random –according to a uniform distribution– on the initial fragment and imagine we can follow the evolution of the fragment that contains this point.

Let us denote by $\big(\chi(t), t \geq 0\big)$ the process of the size of the fragment that contains the randomly chosen point. This fragment is a typical observation in our data set X_ε, and it appears at time

$$T_\varepsilon := \inf\big\{t \geq 0, \ \ \chi(t) < \varepsilon\big\}.$$

Bertoin [17] shows that the process $\xi(t) := -\log \chi(t)$ is a subordinator, with Lévy measure:

$$\pi(dx) := e^{-x} \sum_{i=1}^{\infty} \nu(-\log s_i \in dx). \tag{3.2.3}$$

We can anticipate that the information we get from X_ε is actually information about the Lévy measure $\pi(dx)$ of $\xi(t)$ throughout $\xi(T_\varepsilon)$. The dislocation measure $\nu(d\mathbf{s})$ and $\pi(dx)$ are related by (3.2.3) which reads

$$\int_{\mathcal{S}^\downarrow} \sum_{i=1}^{\infty} s_i f(s_i) \nu(d\mathbf{s}) = \int_{(0,+\infty)} f(e^{-x}) \pi(dx), \tag{3.2.4}$$

for any suitable $f(\bullet) : [0,1] \to [0,+\infty)$. In particular, by Assumption B and the fact that $\nu(\mathcal{S}^\downarrow) = 1$, $\pi(dx)$ is a probability measure hence $\xi(t)$ is a compound Poisson process. Informally, a typical observation takes the form $\xi(T_\varepsilon)$, which is the value of a subordinator with Lévy measure $\pi(dx)$ at its first passage time strictly above $-\log \varepsilon$. The case $\alpha \neq 0$ is a bit more involved and reduces to the homogenous case by a time change.

[3] See however Section 3.4.4 for auxiliary results about the inference on α.

In terms of the limit of the empirical measure $\mathcal{E}_\varepsilon(g)$, we equivalently have

$$\mathcal{E}(g) = \frac{1}{c(\pi)} \int_0^1 \frac{g(a)}{a} \pi(-\log a, +\infty)\, da = \frac{1}{c(\pi)} \int_0^{+\infty} g(e^{-x}) \pi(x, +\infty)\, dx,$$

with $c(\pi) = \int_{(0,+\infty)} x\, \pi(dx)$, the two representations being useful either way. This approach will prove technically convenient and will be detailed later on. Except in the binary case (a particular case of interest, see Section 3.4.1), the knowledge of $\pi(\bullet)$ does not allow to recover $\nu(\bullet)$ in general.

Measurements with limited accuracy. It is unrealistic to assume that we can observe exactly the size ξ_u of the fragments. This becomes even more striking if the dislocation splits at a given time into infinitely many fragments of non zero size, a situation that we do not discard in principle. Therefore, we replace (3.2.2) by the more realistic observation scheme $X_{\varepsilon,\sigma} := \big(\xi_u^{(\sigma)},\ u \in \mathcal{U}_{\varepsilon,\sigma}\big)$ with

$$\mathcal{U}_{\varepsilon,\sigma} := \big\{u \in \mathcal{U},\ \xi_{u-}^{(\sigma)} \geq \varepsilon,\ \xi_u^{(\sigma)} < \varepsilon\big\},$$

and

$$\xi_u^{(\sigma)} := \xi_u + \sigma U_u. \tag{3.2.5}$$

The random variables $(U_u, u \in \mathcal{U})$ are identically distributed, and account for a systematic experimental noise in the measurement of X_ε, independent of X_ε. We assume furthermore that, for every $u \in \mathcal{U}$,

$$|U_u| \leq 1 \ \text{ and } \ \mathbb{E}[U_u] = 0.$$

The noise level $0 \leq \sigma = \sigma(\varepsilon) \ll \varepsilon$ is assumed to be known and represents the accuracy level of the statistician.

The observations $\xi_u + \sigma U_u$ are further discarded below a threshold $\sigma \leq t_\varepsilon \leq \varepsilon$ beyond which they become irrelevant, leading to the modified empirical measure

$$\mathcal{E}_{\varepsilon,\sigma}(g) := \sum_{u \in \mathcal{U}_{\varepsilon,\sigma}} 1_{\{\xi_u^{(\sigma)} \geq t_\varepsilon\}}\, \xi_u^{(\sigma)}\, g\big(\xi_u^{(\sigma)}/\varepsilon\big).$$

In the sequel, we take $t_\varepsilon = \gamma_0 \varepsilon$ for some (arbitrary) $0 < \gamma_0 < 1$.

3.3 Main results

We first exhibit explicit rates in the convergence $\mathcal{E}_\varepsilon(g) \to \mathcal{E}(g)$ as $\varepsilon \to 0$, extending Proposition 1.12 in Bertoin[4] [17]. We then turn to the estimation of $\pi(\bullet)$.

[4]See also Bertoin and Martinez [21].

3.3.1 A rate of convergence for the empirical measure

For $\kappa > 0$, we say that a spread-out[5] probability measure $\pi(dx)$ defined on $[0, +\infty)$ belongs to $\Pi(\kappa)$ if

$$\int_{[0,+\infty)} e^{\kappa x}\, \pi(dx) < +\infty,$$

appended with $\Pi(\infty) := \bigcap_{\kappa>0} \Pi(\kappa)$. For $m > 0$, define the class of continuous functions

$$\mathcal{C}(m) := \big\{ g(\bullet) : [0,1] \to \mathbb{R}, \;\; \|g\|_\infty := \sup_x |g(x)| \leq m \big\},$$

and $\mathcal{C}'(m)$ the class of continuously differentiable functions $g(\bullet)$ such that $g' \in \mathcal{C}(m)$.

Theorem 3.1. *Grant Assumptions A and B. Let $0 < \kappa \leq \infty$ and assume that $\pi \in \Pi(\kappa)$.*

- *For every $m > 0$ and $0 < \mu < \kappa$, we have*

$$\sup_{g \in \mathcal{C}(m)} \mathbb{E}\left[\big(\mathcal{E}_\varepsilon(g) - \mathcal{E}(g) \big)^2 \right] = o\big(\varepsilon^{\mu/(\mu+1)}\big). \tag{3.3.1}$$

- *The convergence* (3.3.1) *remains valid if we replace $\mathcal{E}_\varepsilon(\bullet)$ by $\mathcal{E}_{\varepsilon,\sigma}(\bullet)$ and $\mathcal{C}(m)$ by $\mathcal{C}'(m)$, up to an additional error term:*

$$\sup_{g \in \mathcal{C}'(m)} \mathbb{E}\left[\big(\mathcal{E}_{\varepsilon,\sigma}(g) - \mathcal{E}_\varepsilon(g) \big)^2 \right] = \mathcal{O}(\sigma^2 \varepsilon^{-2}). \tag{3.3.2}$$

3.3.2 Statistical estimation

We study the estimation of $\pi(\bullet)$ by constructing estimators based on $\mathcal{E}_\varepsilon(\bullet)$ or rather $\mathcal{E}_{\varepsilon,\sigma}(\bullet)$. We need the following regularity assumption:

Assumption C. *The probability measure $\pi(dx)$ is absolutely continuous.*

We denote by $x \rightsquigarrow \pi(x)$ its density function. We distinguish two cases: the *parametric case*, where we estimate a linear functional of $\pi(\bullet)$ of the form

$$m_k(\pi) := \int_0^{+\infty} x^k \pi(x) dx, \;\; \text{for some } k \geq 1,$$

and the *non-parametric case*, where we estimate the function $x \rightsquigarrow \pi(x)$ pointwise. In that latter case, it will prove convenient to assess the local smoothness properties of $\pi(\bullet)$ on a logarithmic scale. Henceforth, we consider the mapping

$$a \rightsquigarrow \beta(a) := a^{-1} \pi(-\log a), \;\; a \in (0,1). \tag{3.3.3}$$

[5] We recall some properties on spread-out measures in the Appendix.

In the non-parametric case, we estimate $\beta(a)$ for every $a \in (0,1)$.

3.3.3 The parametric case

For $k \geq 1$, we estimate

$$m_k(\pi) := \int_0^{+\infty} x^k \, \pi(x)dx = \int_0^1 \log(1/a)^k \beta(a)da$$

by the correspondence (3.3.3) and implicitly assumed to be well-defined. We first focus on the case $k = 1$. Pick a sufficiently smooth test function $f(\bullet) : [0,1] \to \mathbb{R}$ such that $f(1) = 0$ and let $g(a) := -af'(a)$. Plainly

$$\begin{aligned}\mathcal{E}(g) &= \frac{1}{c(\pi)} \int_0^1 \frac{g(a)}{a} \pi(-\log a, +\infty)\, da \\ &= -\frac{1}{m_1(\pi)} \int_0^1 f'(a) \int_0^a \beta(u)du\, da = \frac{1}{m_1(\pi)} \int_0^1 f(a)\beta(a)da.\end{aligned}$$

Formally, taking $f(\bullet) \equiv 1$ would identify $1/m_1(\pi)$ since $\beta(\bullet)$ integrates to one, but this choice is forbidden by the boundary condition $f(1) = 0$. We then consider instead the following approximation. Let $f_\gamma(\bullet) : [0,1] \to \mathbb{R}$ with $0 < \gamma < 1$ be a smooth function such that

- $f_\gamma(a) = 1$ for $a \leq 1 - \gamma$ and $f_\gamma(1) = 0$.
- $\|f_\gamma\|_\infty = 1$ and $\|f'_\gamma\|_\infty \leq c\,\gamma^{-1}$, for some $c > 0$,

a choice which is obviously possible. For a parametrization $\gamma := \gamma_\varepsilon \to 0$, we set $g_{\gamma_\varepsilon}(a) := -af'_{\gamma_\varepsilon}(a)$ and define

$$\widehat{m}_{1,\varepsilon} := \frac{1}{\mathcal{E}_{\varepsilon,\sigma}\big(g_{\gamma_\varepsilon}\big)}. \tag{3.3.4}$$

More generally, for $k > 1$, we define successive moment estimators as follows. Set $h_{\gamma_\varepsilon}(a) := f_{\gamma_\varepsilon}(1-a)\log(1/a)^k$ and $\widetilde{g}_{\gamma_\varepsilon}(a) := -ah'_{\gamma_\varepsilon}(a)$. Let

$$\widehat{m}_{k,\varepsilon} := \frac{\mathcal{E}_{\varepsilon,\sigma}\big(\widetilde{g}_{\gamma_\varepsilon}\big)}{\mathcal{E}_{\varepsilon,\sigma}\big(g_{\gamma_\varepsilon}\big)}. \tag{3.3.5}$$

We can describe the performances of $\widehat{m}_{k,\varepsilon}$ under an additional decay condition of $\pi(\bullet)$ near the origin. For $\kappa > 0$, we say[6] that the probability $\pi(\bullet)$ belongs to the class $\mathcal{R}(\kappa)$ if

$$\limsup_{x \to 0} x^{-\kappa+1}\pi(x) < +\infty$$

[6]In the notation, we identify the probability measure $\pi(dx)$ and its density function $x \rightsquigarrow \pi(x)$ when no confusion is possible.

appended with $\mathcal{R}(\infty) := \bigcap_{\kappa>0} \mathcal{R}(\kappa)$. We obtain the following upper bound:

Theorem 3.2. *Grant Assumptions A, B and C. Let $0 < \kappa_1, \kappa_2 \leq \infty$ with $\kappa_1 > \max\{1, \kappa_2\}$.*

For $1 \leq \mu < \kappa_1$, let $\widehat{m}_{k,\varepsilon}$ be specified by $\gamma_\varepsilon := \varepsilon^{\mu/(\mu+1)(2\kappa_2+1)}$. The family

$$\varepsilon^{-\mu\,\kappa_2/(\mu+1)(2\kappa_2+1)}\big(\widehat{m}_{k,\varepsilon} - m_k(\pi)\big)$$

is tight under $\mathbb{P}_1$ as soon as

$$\pi \in \Pi(\kappa_1) \cap \mathcal{R}(\kappa_2)$$

and $\sigma\varepsilon^{-3}$ remains bounded.

Some remarks: the convergence of $\widehat{m}_{k,\varepsilon}$ to $m_k(\pi)$ is of course no surprise by (3.3.1). However, the dependence in ε in the test function $g_\varepsilon(\bullet)$ (in particular $g_\varepsilon(\bullet)$ is unbounded as $\varepsilon \to 0$) requires a slight improvement of Theorem 3.1. This can be done thanks to Assumption C, see Proposition 3.1 in the proof Section 3.5.2. The requirement $\sigma\varepsilon^{-3} = \mathcal{O}(1)$ ensures that the additional term coming from the approximation of $\mathcal{E}_\varepsilon(\bullet)$ by $\mathcal{E}_{\sigma,\varepsilon}(\bullet)$ is negligible. This condition is probably not optimal, see Section 4.

Our next result shows that the exponent $\mu\kappa_2/(\mu+1)(2\kappa_2+1) \leq 1/2$ in the rate of convergence is nearly optimal, to within an arbitrarily small polynomial order.

Definition 3.1. *Let $\pi_0(\bullet)$ satisfy the assumptions of Theorem 3.2. The rate $0 < v_\varepsilon \to 0$ is a lower rate of convergence for estimating $m_k(\pi_0)$ if there exists a family $\pi_\varepsilon(\bullet)$ satisfying the assumptions of Theorem 3.2 and a constant $c > 0$ such that*

$$\liminf_{\varepsilon\to 0} \inf_{F_\varepsilon} \max_{\pi\in\{\pi_0,\pi_\varepsilon\}} \mathbb{P}\left[v_\varepsilon^{-1}|F_\varepsilon - m_k(\pi)| \geq c\right] > 0, \tag{3.3.6}$$

where the infimum is taken (for every ε) over all estimators constructed with $X_{\varepsilon,\sigma}$ at stage ε.

Definition 3.1 expresses a kind of local min-max information bound: given $\pi_0(\bullet)$, one can find an opponent $\pi_\varepsilon(\bullet)$ such that no estimator can discriminate between $\pi_0(\bullet)$ and $\pi_\varepsilon(\bullet)$ at a rate faster than v_ε.

We further restrict our attention to binary fragmentations, see Section 3.4.1. In that case, the dislocation measure satisfies $\nu(s_1 + s_2 \neq 1) = 0$, and, because of the conservation Assumption B, can be represented as

$$\nu(d\mathbf{s}) = \rho(ds_1)\delta_{1-s_1}(ds_2), \tag{3.3.7}$$

where $\rho(\bullet)$ is a probability measure on $[1/2, 1]$.

Assumption D. *(Binary case.) The probability measure $\rho(\bullet)$ associated to $\pi(\bullet)$ is absolutely continuous and its density function is bounded away from zero.*

Theorem 3.3. *Assume that the fragmentation is binary and grant Assumption D. In the same setting as in Theorem 3.2, the rate $\varepsilon^{1/2}$ is a lower rate of convergence for estimating $m_k(\pi)$.*

The restriction to the binary case is made for technical reason and is inessential. Theorem 3.3 presumably holds in a more general setting.

3.3.4 The non-parametric case

Under local smoothness assumptions on the parameter $\beta(\bullet)$, we estimate $\beta(a)$ for every $a \in (0,1)$. Given $s > 0$, we say that $\beta(\bullet)$ belongs to the Hölder class $\Sigma(s)$ if there exists a constant $c > 0$ such that

$$\big|\beta^{(n)}(y) - \beta^{(n)}(x)\big| \leq c|y-x|^{\{s\}},$$

where $s = n + \{s\}$, with n a non-negative integer and $\{s\} \in (0,1]$. We also need to relate $\beta(\bullet)$ to the decay of its corresponding Lévy measure $\pi(\bullet)$. Abusing again notation, we identify $\Pi(\kappa)$ with the set of $\beta(\bullet)$ such that $e^x\beta(e^{-x})dx \in \Pi(\kappa)$, thanks to the inverse of (3.3.3). Likewise for $\mathcal{R}(\kappa)$.

We construct an estimator of $\beta(\bullet)$ as follows: for $a \in (0,1)$ and a normalizing factor $0 < \gamma_\varepsilon \to 0$, set

$$\varphi_{\gamma_\varepsilon,a}(\bullet) := \gamma_\varepsilon^{-1}\varphi\big((\bullet - a)/\gamma_\varepsilon\big),$$

where $\varphi(\bullet)$ is a smooth function with support in $(0,1)$ that satisfies the following oscillating property: for some integer $N \geq 1$,

$$\int_0^1 \varphi(a)da = 1, \quad \int_0^1 a^k\varphi(a)da = 0, \;\; k = 1,\ldots,N. \tag{3.3.8}$$

Our estimator then takes the form

$$\widehat{\beta}_\varepsilon(a) := \widehat{m}_{1,\varepsilon}\,\mathcal{E}_{\varepsilon,\sigma}\big(-\bullet\varphi'_{\gamma_\varepsilon,a}(\bullet)\big) \;\; a \in (0,1), \tag{3.3.9}$$

where $\widehat{m}_{1,\varepsilon}$ is the estimator of $m_1(\pi)$ defined in (3.3.4). We then have the following

Theorem 3.4. *Grant Assumptions A, B and C. Let $1 < \kappa_1, \kappa_2 \leq \infty$.*

For $1 \leq \mu < \min\{1, \kappa_1/2\}$, let $\widehat{\beta}_\varepsilon(\bullet)$ be specified by $\gamma_\varepsilon := \varepsilon^{\mu/(\mu+1)(2s+3)}$. For every $a \in (0,1)$, the family

$$\varepsilon^{-\mu s/(\mu+1)(2s+3)}\big(\widehat{\beta}_\varepsilon(a) - \beta(a)\big)$$

is tight under $\mathbb{P}_1$, as soon as

$$\beta \in \Sigma(s) \cap \Pi(\kappa_1) \cap \mathcal{R}(\kappa_2)$$

for $s < \max\{N, \kappa_2 - 1\}$ and $\sigma\varepsilon^{-3}$ remains bounded.

A proof of the (near)-optimality in the sense of the lower bound Definition 3.1 and in the spirit of Theorem 3.3 is presumably a delicate problem that lies beyond the scope of the paper. More in Section 3.4.3.

3.4 Discussion

3.4.1 Binary fragmentations

The case of binary fragmentations is the simplest, yet an important model of random fragmentation, where a particle splits into two blocs at each step (see *e.g.* [35], [27]). By using representation (3.3.7), if we assume further that $\rho(ds_1) = \rho(s_1)ds_1$ is absolutely continuous, so is $\pi(dx) = \pi(x)dx$ and we have

$$\pi(x) = e^{-2x}\big(\rho(e^{-x})1_{[0,\log 2]}(x) + \rho(1-e^{-x})1_{(\log 2,+\infty)}(x)\big), \tag{3.4.1}$$

for $x \in [0,+\infty)$ and

$$\beta(a) = a\big(\rho(a)1_{[1/2,1]}(a) + \rho(1-a)1_{[0,1/2)}(a)\big), \;\; a \in [0,1].$$

In particular, the regularity properties of $\beta(\bullet)$ are readily obtained from the local smoothness of $\rho(\bullet)$ and its behaviour near $1/2$. For instance, if $\rho(a+1/2) = \mathcal{O}(a^{\kappa-1})$ near the origin, for some $\kappa > 0$, then

$$\pi \in \Pi(\kappa) \cap \mathcal{R}(\kappa).$$

3.4.2 Concerning Theorem 3.1

Optimal rate of convergence. First, Theorem 3.1 readily extends to error measurements of the form $\mathbb{E}\left[\left|\mathcal{E}_\varepsilon(g) - \mathcal{E}(g)\right|^p\right]$ with $1 \le p \le 2$. The rate becomes $\varepsilon^{-\mu p/2(\mu+1)}$ in (3.3.1) and $\sigma^p\varepsilon^{-p}$ in (3.3.2) under the less stringent condition $\mu < \kappa/2p$.

Generally speaking, we obtain in (3.3.1) the (normalized) rate $\varepsilon^{\mu/2(\mu+1)}$, for any $\mu < \kappa$. Intuitively, we have a number of observations that should be of order ε^{-1}, so the expected rate would rather be $\varepsilon^{1/2}$. Why cannot we obtain the rate $\varepsilon^{1/2}$ or simply $\varepsilon^{\kappa/2(\kappa+1)}$? The proof in Section 3.5.1 shows that we loose quite much information when applying Sgibnev's result (see Proposition 3.2 in Appendix) on the kew renewal theorem for a random walk with step distribution $\pi(\bullet)$ in the limit $\log(1/\varepsilon) \to +\infty$.

Proposition 3.2 ensures that if $\pi(\bullet)$ has exponential moments up to order κ, then we can guarantee in the renewal theorem the rate $o(\varepsilon^\mu)$ for any $\mu < \kappa$ with some uniformity in the test function, a crucial point for the subsequent statistical applications. It is presumably possible to improve this rate to $\mathcal{O}(\varepsilon^\kappa)$ by accomodating Ney's result [63]. However, a careful glance at the proof of Theorem 3.1 shows that we would then loose an extra logarithmic term when replacing $\varepsilon^{\mu/2(\mu+1)}$ by $\varepsilon^{\kappa/(2\kappa+1)}$. More generally, exhibiting exact rates of convergence in Theorem

3.1 remains a delicate issue: the kew renewal theorem is sensitive to a modification of the distribution outside a neighbourhood of $+\infty$, see *e.g.* Asmussen [3], p.196.

Uniformity in $\pi(\bullet)$. A slightly annoying fact is that convergence (3.3.1) is not uniform in $\pi(\bullet)$, which can become a methodological issue for the statistical applications of the subsequent Theorems 3.2 and 3.4, in particular if min-max results are sought. An inspection of the proof in Section 3.5.1 shows that we loose information about the uniformity in $\pi(\bullet)$ when applying Proposition 3.2 again. A glance at the proof of Sgibnev's result suggest that uniform results in $\pi(\bullet)$ could presumably be obtained over classes of $\pi(\bullet)$ defined in terms of appropriate bounds on their Stone decomposition [66].

The non-conservative case. If Assumption B is dropped, we define $p_- = \inf\left\{p > 0, \ \int_{\mathcal{S}^\downarrow} \sum_{i=1}^\infty s_i^p \nu(d\mathbf{s}) < +\infty\right\}$ and make the so-called Malthusian hypothesis: there exists a (unique) solution $p^\star \geq p_-$ to the equation

$$\int_{\mathcal{S}^\downarrow} \big(1 - \sum_{i=1}^\infty s_i^{p^\star}\big)\nu(d\mathbf{s}) = 1.$$

The empirical measure now becomes

$$\mathcal{E}_\varepsilon^{(p^\star)}(g) := \sum_{u\in\mathcal{U}_\varepsilon} \xi_u^{p^\star}\, g(\xi_u/\varepsilon).$$

The choice of the weights $\xi_u^{p^\star}$ is motivated by the fact that the process $\big(\sum_{|u|=n} \xi_u^{p^\star}, n \geq 0\big)$ is a positive martingale. We denote by $\mathcal{M}_\infty$ its terminal value. Note that under Assumption B, we always have $p^\star = 1$ and $\mathcal{M}_\infty = 1$. Bertoin and Martinez [21] prove the convergence of $\mathcal{E}_\varepsilon^{(p^\star)}(g)$ to

$$\mathcal{E}^{(p^\star)}(g) := \frac{\mathcal{M}_\infty}{c(\nu)} \int_0^1 \frac{g(a)}{a} \int_{\mathcal{S}^\downarrow} \sum_{i=1}^\infty s_i^{p^\star}\, 1_{\{s_i<a\}} \nu(d\mathbf{s})da$$

in probability, as $\varepsilon \to 0$, with now $c(\nu) := -\int_{\mathcal{S}^\downarrow} \sum_{i=1}^\infty s_i^{p^\star} \log s_i \, \nu(d\mathbf{s})$. In this setting, Theorem 3.1 becomes

Corollary 3.1. *Grant Assumptions A, C and the Malthusian hypothesis. Let $0 < \kappa \leq \infty$ and $m > 0$. For every $0 < \mu < \kappa$, we have*

$$\sup_{g\in\mathcal{C}(m)} \mathbb{E}\left[\big(\mathcal{E}_\varepsilon^{(p^\star)}(g) - \mathcal{E}^{(p^\star)}(g)\big)^2\right] = o\big(\varepsilon^{\mu/(\mu+p^\star)}\big).$$

3.4.3 Concerning Theorems 3.2 and 3.4

The parametric case. We obtain the rate

$$\left(\varepsilon^{\mu/(\mu+1)}\right)^{\kappa_2/(2\kappa_2+2)}, \text{ for all } \mu < \kappa_1$$

which can be made arbitrary close to the lower bound $\varepsilon^{1/2}$ by assuming κ_1 and κ_2 large enough. The factor $\mu/(\mu+1)$ comes from Theorem 3.1 whereas the factor $\kappa_2/(2\kappa_2+1)$ arises when using the technical assumption $\pi \in \mathcal{R}(\kappa_2)$. We do not know how to improve it.

Also, the condition $\sigma\varepsilon^{-3} = \mathcal{O}(1)$ is fairly restrictive, and can be readily improved by assuming that κ_2 is large. Indeed, if $\kappa_2 \geq 1$, which only amounts to require that $\pi(\bullet)$ is bounded near the origin, a glance at the error term (3.5.10) in the proof Section 3.5.2 shows that the condition drops to $\sigma\varepsilon^{-2} = \mathcal{O}(1)$. In the limit $\kappa_2 \to \infty$, we obain $\sigma\varepsilon^{-3/2} = \mathcal{O}(1)$.

The non-parametric case. The situation is a bit different than in the parametric case: we obtain now the rate

$$\left(\varepsilon^{\mu/(\mu+1)}\right)^{s/(2s+3)}, \text{ for all } \mu < \kappa_1$$

for the estimation of $\beta(a)$ for any $a \in (0,1)$. In the limit $\kappa_1 \to +\infty$ it becomes $\varepsilon^{s/(2s+3)}$, which can be related to more classical models: consider the apparently different problem of recovering a function $\beta(\bullet)$ in the integral white noise model

$$dY_a = K\beta(a)da + \varepsilon^{1/2}dW_a, \quad a \in [0,1], \tag{3.4.2}$$

from the observation of $(Y_a, a \in [0,1])$. Here, $(W_a, a \in [0,1])$ is a standard Brownian motion and $K\beta(a) := \int_0^a \beta(u)du$ is the integration operator. Model (3.4.2) serves as a toy representation for the problem of recovering a signal in white noise at level $\varepsilon^{1/2}$, when the observation is obtained from the action of a smoothing linear operator with unbounded inverse (here K). The difficulty of the problem is quantified by the degree of ill-posedness of the operator (equal to ν for ν-fold integration; here $\nu = 1$). The well-known optimal rate (see *e.g.* [47]) of pointwise recovery for a function $\beta \in \Sigma(s)$ is

$$\varepsilon^{s/(2s+2\nu+1)} = \varepsilon^{s/(2s+3)}.$$

The factor 2ν is a futher penalty in the rate of recovery quantifying the smoothing action of K. The same phenomenon seems to occur in the setting of fragmentation chain. Put $\sigma := 0$ here for simplicity. For a test function $g(\bullet)$, we can form the observation

$$\mathcal{E}_\varepsilon(g) \approx \mathcal{E}(g) = \frac{1}{m_1(\pi)} \int_0^1 g(u)K\beta(u)du$$

up to an error of (near)-order $\varepsilon^{1/2}$. If we discard the pre-factor $m_1(\pi)$ (which can

be estimated at a fast rate when κ_2 is large) we obtain the same kind of statistics in Model (3.4.2) by considering

$$\int_0^1 g(a)dY_a = \int_0^1 g(u)K\beta(u)du + \varepsilon^{1/2}\mathcal{N}(g),$$

where $\mathcal{N}(g)$ is centred Gaussian with variance $\int_0^1 g(a)^2da$. Note that the order of the variance in the noise term $\mathcal{N}(g)$ is consistent with the improvement obtained in Proposition 3.1 in the proof Section 3.5.2: if $g \in \widetilde{\mathcal{C}}_{b_\varepsilon}(m)$, we have $\int_0^1 g(a)^2da \lesssim b_\varepsilon$.

This suggests the (near)-optimality of the result in the sense of Definition 3.1 but a complete proof lies beyond the scope of the paper.

3.4.4 Other statistical issues

Observation of the whole path of X. Suppose we observe continuously in time the sample path of X up to some fixed time $T > 0$. Asymptotics are taken as $T \to \infty$. Equivalently, we observe all triples (ξ_u, a_u, ζ_u) for every u in the random set

$$\mathcal{U}(T) := \big\{u \in \mathcal{U},\ \ a_u \le T\big\}$$

with the restriction that ζ_u is set to $T - a_u$ when $a_u + \zeta_u > T$. We denote by $\mathcal{U}(T_-)$ the subset of $\mathcal{U}(T)$ such that $a_u + \zeta_u \le T$. In this setting, statistical inference about the the self-similar parameter $\alpha \ge 0$ and the dislocation measure $\nu(d\mathbf{s})$ is relatively straightforward:

Estimation of α: conditional on $\big(\xi_u, u \in \mathcal{U}(T_-)\big)$ the sequence of random variables $\big(\zeta_u, u \in \mathcal{U}(T)\big)$ are independent and follow exponential distributions with parameters ξ_u^α. Conditional on $\mathrm{Card}\,\mathcal{U}(T) = n$ and since the law of the $(\xi_u, u \in \mathcal{U})$ does not depend on α we are in the setting of estimating a one-dimensional parameter from n independent random variables with explicit likelihood ratio

$$\alpha \rightsquigarrow \prod_{i=1}^{n} \xi_{u_i}^\alpha \exp\left(\xi_{u_i}^\alpha\, \zeta_{u_i}\right), \tag{3.4.3}$$

where the u_i range through $\mathcal{U}(T)$. The main difficulty remains that the law of $\mathrm{Card}\,\mathcal{U}(T)$ usually depends on α.

Estimation of $\nu(\bullet)$: for $u- \in \mathcal{U}(T_-)$, when the fragment of size ξ_u splits, conditional on $\xi_u = x$, it gives rize to the observation of the rescaled size of its offsprings $(x^{-1}\,\xi_{ui}, i \in \mathbb{N})$ which is a realization of the law $\nu(d\mathbf{s})$. As a consequence, conditional on $\mathrm{Card}\,\mathcal{U}(T) = n$, we observe a sequence of n independent and identically distributed random variables with law $\nu(d\mathbf{s})$. We are back to the classical problem of estimating a probability distribution from an n-sample.

More about estimating α. We cannot estimate the index of self-similarity α from the data X_ε. However, if we add the possibility to "tag" a point at random

on the initial fragment[7] and if we can observe the random time T_ε when the tagged fragment becomes smaller then ε, then identifying α from the sole observation of T_ε is possible.

In the case $\alpha > 0$, if $\chi(t)$ denotes the size of the tagged fragment at time t, then

$$T_\varepsilon = \big\{t > 0, \;\; \chi(t) \leq \varepsilon\big\}.$$

Applying Proposition 3 of [19], the distribution of $\varepsilon^\alpha T_\varepsilon$ under $\mathbb{P}_1$ is tight as $\varepsilon \to 0$. Therefore, the rate $-1/\log\varepsilon$ is achievable for estimating α and it is attained by the estimator $\log(T_\varepsilon)/\log\varepsilon$. More precise results about limit laws can be obtained from [19].

3.5 Proofs

We will repeatedly use the convenient notation $a_\varepsilon \lesssim b_\varepsilon$ if $0 < a_\varepsilon \leq c\, b_\varepsilon$ for some constant $c > 0$ which may depend on $\pi(\bullet)$ and m only, any other dependence on other ancillary quantities being obvious from the context. A function $g \in \mathcal{C}(m)$ is tacitly defined on the whole real line by setting $g(a) = 0$ for $a \notin [0, 1]$.

3.5.1 Proof of Theorem 3.1

Step 1: A preliminary decomposition. We first use the fact that for $\eta > \varepsilon$, during the fragmentation process, the unobserved state X_η necessarily anticipates the state X_ε. The choice $\eta = \eta(\varepsilon)$ will follow later. This yields the following representation:

$$\mathcal{E}_\varepsilon(g) = \sum_{v\in\mathcal{U}_\eta} \xi_v \sum_{w\in\mathcal{U}} 1_{\left\{\xi_v\tilde{\xi}^{(v)}_{w-} \geq \varepsilon,\, \xi_v\tilde{\xi}^{(v)}_w < \varepsilon\right\}} \tilde{\xi}^{(v)}_w g\big(\xi_v\tilde{\xi}^{(v)}_w/\varepsilon\big),$$

where, for each label $v \in \mathcal{U}_\eta$ and conditional on X_η, a new independent fragmentation chain $(\tilde{\xi}^{(v)}_w, w \in \mathcal{U})$ is started thanks to the branching property, see Section 3.2.1. Define now

$$\lambda_\eta(v) := 1_{\{\xi_{v-}\geq\eta, \xi_v<\eta\}}\xi_v$$

and

$$Y_\varepsilon(v, g) := \sum_{w\in\mathcal{U}} 1_{\left\{\xi_v\tilde{\xi}^{(v)}_{w-} \geq \varepsilon,\, \xi_v\tilde{\xi}^{(v)}_w < \varepsilon\right\}} \tilde{\xi}^{(v)}_w g\big(\xi_v\tilde{\xi}^{(v)}_w/\varepsilon\big).$$

We obtain the decomposition of $\mathcal{E}_\varepsilon(g) - \mathcal{E}(g)$ as a sum of a centred and a bias term:

$$\mathcal{E}_\varepsilon(g) - \mathcal{E}(g) = M_{\varepsilon,\eta}(g) + B_{\varepsilon,\eta},$$

[7] in physical terms, we must be able to identify the mass or length of the fragment.

with

$$M_{\varepsilon,\eta}(g) := \sum_{v\in\mathcal{U}} \lambda_\eta(v)\big(Y_\varepsilon(v,g) - \mathbb{E}\left[Y_\varepsilon(v,g)\,|\,\lambda_\eta(v)\right]\big)$$

and

$$B_{\varepsilon,\eta}(g) := \sum_{v\in\mathcal{U}} \lambda_\eta(v)\big(\mathbb{E}\left[Y_\varepsilon(v,g)\,|\,\lambda_\eta(v)\right] - \mathcal{E}(g)\big),$$

where we used (3.2.1) in order to incoporate the limit term $\mathcal{E}(g)$ within the sum in v.

Step 2: The term $M_{\varepsilon,\eta}(g)$. Conditional on the sigma-field generated by $\big(1_{\{\xi_{u-}\geq\eta\}}\xi_u, u- \in \mathcal{U}\big)$, the variables $(Y_\varepsilon(v,g), v\in\mathcal{U})$ are independent. Therefore

$$\mathbb{E}\left[M_{\varepsilon,\eta}(g)^2\right] \leq \sum_{v\in\mathcal{U}} \mathbb{E}\left[\lambda_\eta(v)^2\,\mathbb{E}[Y_\varepsilon(v,g)^2\,|\,\lambda_\eta(v)]\right], \tag{3.5.1}$$

thus we first need to control the conditional variance of $Y_\varepsilon(v,g)^2$ given $\lambda_\eta(v)=u$, for $0\leq u\leq\eta$, since $\mathbb{P}$-almost surely, $\lambda_\eta(v)\leq\eta$. Moreover, we have $Y_\varepsilon(v,g)=0$ on $\{\lambda_\eta(v)<\varepsilon\}$, hence we may assume $\varepsilon\leq u\leq\eta$.

To this end, we will use the following representation property:

Lemma 3.1. *Let $f(\bullet):[0,+\infty)\to[0,+\infty)$. Then*

$$\mathbb{E}\Big[\sum_{v\in\mathcal{U}_\eta}\xi_v\,f(\xi_v)\Big] = \mathbb{E}^\star\left[f\big(\chi(T_\eta)\big)\right], \tag{3.5.2}$$

where $\chi(t)=\exp\big(-\xi(t)\big)$ and $\big(\xi(t),t\geq0\big)$ is a subordinator with Lévy measure $\pi(\bullet)$ defined on an appropriate probability space $(\Omega^\star,\mathbb{P}^\star)$, and

$$T_\eta := \inf\big\{t\geq0,\ \xi(t)>-\log\eta\big\}.$$

The proof is given in Appendix 3.6.1. In order to remain self-contained, we elaborate in particular on the construction of the randomly tagged fragment following the presentation of Bertoin [17].

We now plan to bound the right-hand side of (3.5.1) thanks to Lemma 3.1. For $0<\varepsilon\leq u\leq\eta$, we have

$$\begin{aligned}\mathbb{E}\left[Y_\varepsilon(v,g)^2\,|\,\lambda_\eta(v)=u\right] &= \mathbb{E}\Big[\Big(\sum_{w\in\mathcal{U}_{\varepsilon/u}}\widetilde{\xi}_w^{(v)}g\big(\varepsilon u^{-1}\widetilde{\xi}_w^{(v)}\big)\Big)^2\,\Big|\,\lambda_\eta(v)=u\Big]\\ &\leq \mathbb{E}\big[\sum_{w\in\mathcal{U}_{\varepsilon/u}}\widetilde{\xi}_w^{(v)}g\big(\varepsilon u^{-1}\widetilde{\xi}_w^{(v)}\big)^2\,\big|\,\lambda_\eta(v)=u\big]\end{aligned}$$

where we used Jensen's inequality combined with (3.2.1). Applying Lemma 3.1 we derive

$$\mathbb{E}\left[Y_\varepsilon(v,g)^2\,|\,\lambda_\eta(v)=u\right] \leq \mathbb{E}^\star\left[g\big(u\varepsilon^{-1}e^{-\xi(T_{\varepsilon/u})}\big)^2\right]. \tag{3.5.3}$$

Let $U(\bullet)$ denote the renewal function associated with the subordinator $(\xi(t),t\geq0)$.

By Proposition 2, Ch. III in [10], the right-hand side of (3.5.3) is equal to

$$\begin{aligned}
&\int_{\left[0,-\log(\varepsilon/u)\right)} dU(s) \int_{(-\log(\varepsilon/u)-s,+\infty)} g\big(u\,\varepsilon^{-1}e^{-x-s}\big)^2\pi(dx),\\
&= \int_{\left[0,-\log(\varepsilon/u)\right)} dU(s) \int_{\mathcal{S}^\downarrow} \sum_{i=1}^{\infty} s_i\, 1_{\{s_i<\varepsilon u^{-1}\,e^s\}}\, g\big(s_i u\,\varepsilon^{-1}e^{-s}\big)^2\nu(d\mathbf{s})\\
&\lesssim \frac{1}{c(\pi)}\|g\|_\infty^2\log(u/\varepsilon),
\end{aligned}$$

where we successively used the representation (3.2.4) and the upper bound $U(s)\lesssim s/c(\pi)$, see for instance Proposition 1, Ch. I in [10]. Therefore, for $\varepsilon\le u\le\eta$,

$$\mathbb{E}\left[Y_\varepsilon(v,g)^2\,|\,\lambda_\eta(v)=u\right]\lesssim\frac{1}{c(\pi)}\|g\|_\infty^2\log(\eta/\varepsilon).$$

Going back to (3.5.1), since $\lambda_\eta(v)^2\le\eta\,\lambda_\eta(v)$ and using (3.2.1) again, we readily derive

$$\mathbb{E}\left[M_{\varepsilon,\eta}(g)^2\right]\lesssim\frac{1}{c(\pi)}\|g\|_\infty^2\eta\log(\eta/\varepsilon)\lesssim\eta\log(\eta/\varepsilon). \tag{3.5.4}$$

Step 3: The bias term $B_{\varepsilon,\eta}(g)$. Note first that

$$\mathbb{E}\left[Y_\varepsilon(v,g)\,|\,\lambda_\eta(v)\right]=\xi_v^{-1}\,\mathbb{E}_{\xi_v}\left[\mathcal{E}_\varepsilon(g)\right],$$

$\mathbb{P}$-almost surely, henceforth

$$B_{\varepsilon,\eta}(g)=\sum_{v\in\mathcal{U}}\lambda_\eta(v)\Big(\xi_v^{-1}\,\mathbb{E}_{\xi_v}\left[\mathcal{E}_\varepsilon(g)\right]-\mathcal{E}(g)\Big). \tag{3.5.5}$$

Conditioning on the mark of the parent $u-=\omega$ of u and applying the branching property, we get that $\mathbb{E}_{\xi_v}\left[\mathcal{E}_\varepsilon(g)\right]$ can be written as

$$\mathbb{E}_{\xi_v}\Big[\sum_{\omega\in\mathcal{U}}1_{\left\{\widehat{\xi}_\omega\ge\varepsilon\right\}}\widehat{\xi}_\omega\int_{\mathcal{S}^\downarrow}\sum_{i=1}^{\infty}1_{\left\{\widehat{\xi}_\omega\,s_i<\varepsilon\right\}}s_i\,g\big(\widehat{\xi}_\omega\,s_i\,\varepsilon^{-1}\big)\Big].$$

where the $(\widehat{\xi}_w,w\in\mathcal{U})$ are the sizes of the marked fragments of a fragmentation chain with same dislocation measure $\nu(\bullet)$, independent of $(\xi_v,v\in\mathcal{U})$. Set

$$H_g(a):=\int_{\mathcal{S}^\downarrow}\sum_{i=1}^{\infty}1_{\left\{s_i<e^{-a}\right\}}s_i g\big(s_i e^a\big)\nu(d\mathbf{s}),\quad a\ge0.$$

It follows that $\mathbb{E}_{\xi_v}\left[\mathcal{E}_\varepsilon(g)\right]$ is equal to

$$\mathbb{E}_{\xi_v}\Big[\sum_{n=0}^{\infty}\sum_{|\omega|=n}1_{\left\{\log\widehat{\xi}_\omega\ge\log\varepsilon\right\}}\widehat{\xi}_\omega H_g(\log\widehat{\xi}_\omega-\log\varepsilon)\Big]$$

$$= \xi_v \, \mathbb{E}\Big[\sum_{n=0}^{\infty} \sum_{|\omega|=n} 1_{\left\{ \log \widehat{\xi}_\omega \geq \log(\varepsilon/\rho) \right\}} H_g\big(\log \widehat{\xi}_\omega - \log(\varepsilon/\rho) \big) \Big]_{\rho=\xi_v}$$

by self-similarity, with the notation $|\omega| = n$ if $\omega = (\omega_1, \ldots, \omega_n) \in \mathcal{U}$. Using Lemma 1.4 in [17], we finally obtain

$$\mathbb{E}_{\xi_v}\big[\mathcal{E}_\varepsilon(g)\big] = \xi_v \sum_{n=0}^{\infty} \mathbb{E}\left[1_{\left\{ S_n \leq \log(\rho/\varepsilon) \right\}} H_g\big(\log(\rho/\varepsilon) - S_n \big) \right]_{\rho=\xi_v},$$

where S_n is a random walk with step distribution $\pi(dx)$. We plan to apply a version of the renewal theorem with explicit rate of convergence as given in Sgibnev [65], see Proposition 3.2 in Appendix 3.6.2, with rate function $\varphi(a) := \exp(\mu a)$ for some arbitrary $\mu < \kappa/2$ and dominating function $r(a) := e^{-\kappa|a|}$. Indeed, for $a < 0$:

$$H_g(-a) = 1_{\left\{ a \leq 0 \right\}} \int_{(-a, +\infty)} g\big(e^{-x-a}\big) \pi(dx),$$

by (3.2.4). Since $g(\bullet)$ has support in $[0,1]$ and $\pi \in \Pi(\kappa)$,

$$\big| H_g(-a) \big| \leq \int_{(-a, +\infty)} \big| g\big(e^{-x-a}\big) \big| \pi(dx) \lesssim e^{\kappa a},$$

Therefore $|1_{\{a \leq 0\}} H_g(-a)| \lesssim r(a)$ for all $a \in \mathbb{R}$.

Since $\kappa > 2\mu$, Assumption F of Proposition 3.2 is readily checked. Let $A > 0$ (depending on κ, m and $\pi(\bullet)$ only) such that, if $\log(\xi_v/\varepsilon) \geq A$, then

$$\Big| \xi_v^{-1} \, \mathbb{E}_{\xi_v}\big[\mathcal{E}_\varepsilon(g)\big] - \frac{1}{\mathbb{E}^\star\big[S_1\big]} \int_0^{+\infty} H_g(a) da \Big| \leq \Big(\frac{\varepsilon}{\xi_v} \Big)^{\mu}. \tag{3.5.6}$$

We next note that

$$\frac{1}{\mathbb{E}^\star\big[S_1\big]} \int_0^{+\infty} H_g(a) da = \mathcal{E}(g).$$

Introducing the family of events $\big\{ \log(\xi_v/\varepsilon) \geq A \big\}$ in the sum (3.5.5), we obtain the following decomposition:

$$B_{\varepsilon,\eta}(g)^2 \lesssim I + II,$$

with

$$I := \sum_{v \in \mathcal{U}_\eta} \xi_v 1_{\left\{ \log(\xi_v/\varepsilon) > A \right\}} \big(\xi_v^{-1} \, \mathbb{E}_{\xi_v}\big[\mathcal{E}_\varepsilon(g)\big] - \mathcal{E}(g) \big)^2,$$

and

$$II := \sum_{v \in \mathcal{U}_\eta} \xi_v 1_{\left\{ \log(\xi_v/\varepsilon) \leq A \right\}} \big(\xi_v^{-1} \, \mathbb{E}_{\xi_v}\big[\mathcal{E}_\varepsilon(g)\big] - \mathcal{E}(g) \big)^2.$$

By (3.5.6), we have

$$I \leq \varepsilon^{2\mu} \sum_{v \in \mathcal{U}_\eta} 1_{\left\{-\log \xi_v < -A + \log(1/\varepsilon)\right\}} \xi_v \exp\left(2\mu(-\log \xi_v)\right).$$

Integrating w.r.t. $\mathbb{P}$ and applying Lemma 3.1 and in the same way as in Step 2, we have

$$\begin{aligned}
\mathbb{E}\left[I\right] &\leq \varepsilon^{2\mu}\, \mathbb{E}^\star\left[e^{2\mu\xi(T_\eta)}\right] \\
&= \varepsilon^{2\mu} \int_{[0,-\log\eta)} dU(s) \int_{(-\log\eta - s, +\infty)} e^{2\mu(s+x)} \pi(dx) \\
&\leq \varepsilon^{2\mu} \int_{[0,-\log\eta)} e^{2\mu s} dU(s) \lesssim \left(\varepsilon\eta^{-1}\right)^{2\mu} \log(1/\eta)
\end{aligned}$$

for small enough ε and where we used $\pi \in \Pi(\kappa)$ with $2\mu < \kappa$. For the term II, we first notice that by (3.2.1) and self-similarity,

$$\mathbb{E}_{\xi_v}\left[\sum_{u \in \mathcal{U}_\varepsilon} \widehat{\xi}_u\right] = \xi_v, \;\; \mathbb{P}_{\xi_v}\text{−almost surely,}$$

hence

$$\left(\xi_v^{-1}\, \mathbb{E}_{\xi_v}\left[\mathcal{E}_\varepsilon(g)\right] - \mathcal{E}(g)\right)^2 \leq 4\|g\|_\infty^2, \;\; \mathbb{P}_{\xi_v}\text{−almost surely.}$$

In the same way as for the term I, we derive

$$\begin{aligned}
\mathbb{E}\left[II\right] &\lesssim \mathbb{E}\left[\sum_{v \in \mathcal{U}_\eta} \xi_v 1_{\left\{-\log\xi_v \geq -A + \log(1/\varepsilon)\right\}}\right] \\
&= \mathbb{P}^\star\left[\xi(T_\eta) \geq -A + \log(1/\varepsilon)\right] \\
&\leq \int_{[0,-\log\eta)} dU(s) \int_{(-A+\log(1/\varepsilon)-s, +\infty)} \pi(dx) \\
&\lesssim \varepsilon^\mu \log(1/\eta)
\end{aligned}$$

for small enough ε. Putting all the estimates together, we conclude

$$\mathbb{E}\left[B_{\varepsilon,\eta}(g)^2\right] \lesssim \left(\varepsilon^\mu + (\varepsilon\eta^{-1})^{2\mu}\right) \log(1/\eta). \tag{3.5.7}$$

Step 4: Proof of (3.3.1)**.** Putting the estimates (3.5.4) and (3.5.7), we have

$$\begin{aligned}
\mathbb{E}\left[\left(\mathcal{E}_\varepsilon(g) - \mathcal{E}(g)\right)^2\right] &\lesssim \mathbb{E}\left[M_{\varepsilon,\eta}(g)^2\right] + \mathbb{E}\left[B_{\varepsilon,\eta}(g)^2\right] \\
&\lesssim \eta \log(\eta/\varepsilon) + \left(\varepsilon\eta^{-1}\right)^{2\mu} \log(1/\eta).
\end{aligned}$$

The choice $\eta(\varepsilon) := \varepsilon^{2\mu/(2\mu+1)}$ yields the rate $\varepsilon^{2\mu/(2\mu+1)} \log(1/\varepsilon)$. Since $\mu < \kappa/2$ is arbitrary, the conclusion follows.

Step 5: Proof of (3.3.2)**.** We plan to use the following decomposition:

$$\mathcal{E}_{\varepsilon,\sigma}(g) - \mathcal{E}_{\varepsilon}(g) = I + II,$$

with

$$I := \sum_{u\in\mathcal{U}} \Big(1_{\{\xi^{(\sigma)}_{u-}\geq\varepsilon,\, \xi^{(\sigma)}_{u}<\varepsilon\}} - 1_{\{\xi_{u-}\geq\varepsilon,\, \xi_u<\varepsilon\}}\Big)\widetilde{\xi}^{(\sigma)}_u\, g\big(\xi^{(\sigma)}_u/\varepsilon\big),$$

and

$$II := \sum_{u\in\mathcal{U}_\varepsilon} \Big(\widetilde{\xi}^{(\sigma)}_u\, g\big(\xi^{(\sigma)}_u/\varepsilon\big) - \xi_u\, g\big(\xi_u/\varepsilon\big)\Big),$$

where we have set $\widetilde{\xi}^{(\sigma)}_u := \xi^{(\sigma)}_u 1_{\{\xi^{(\sigma)}_u\geq t_\varepsilon\}}$. Clearly,

$$\begin{aligned}\Big|1_{\{\xi^{(\sigma)}_{u-}\geq\varepsilon,\, \xi^{(\sigma)}_{u}<\varepsilon\}} - 1_{\{\xi_{u-}\geq\varepsilon,\, \xi_u<\varepsilon\}}\Big| \leq & 1_{\{\xi^{(\sigma)}_{u-}\geq\varepsilon,\, \xi_{u-}<\varepsilon\}} + 1_{\{\xi^{(\sigma)}_{u}<\varepsilon,\, \xi_u\geq\varepsilon\}} \\ & + 1_{\{\xi_{u-}\geq\varepsilon,\, \xi^{(\sigma)}_{u-}<\varepsilon\}} + 1_{\{\xi_{u}<\varepsilon,\, \xi^{(\sigma)}_{u}\geq\varepsilon\}}.\end{aligned}$$

Let $\delta > \sigma/\varepsilon$ and $\omega = u$ or $u-$. Since $|U_\omega| \leq 1$ for every ω, we readily check that

$$\big\{\xi^{(\sigma)}_\omega \geq \varepsilon,\, \xi_\omega < \varepsilon\big\} \subset \big\{(1-\delta)\varepsilon \leq \xi_\omega < \varepsilon\big\}$$

and

$$\big\{\xi_\omega \geq \varepsilon,\, \xi^{(\sigma)}_\omega < \varepsilon\big\} \subset \big\{\varepsilon \leq \xi_\omega < (1+\delta)\varepsilon\big\}.$$

It follows that $|I| \leq III + IV$, with

$$III := \sum_{u\in\mathcal{U}} 1_{\{(1-\delta)\varepsilon\leq\xi_{u-}\leq\varepsilon(1+\delta)\}}\big|\widetilde{\xi}^{(\sigma)}_u\, g\big(\xi^{(\sigma)}_u/\varepsilon\big)\big|$$

and

$$IV := \sum_{u\in\mathcal{U}} 1_{\{(1-\delta)\varepsilon\leq\xi_{u}\leq(1+\delta)\varepsilon\}}\big|\widetilde{\xi}^{(\sigma)}_u\, g\big(\xi^{(\sigma)}_u/\varepsilon\big)\big|.$$

By picking δ is small enough, we may (and will) assume that $\widetilde{\xi}^{(\sigma)}_u \lesssim \xi_u$. By (3.2.1), conditioning on the mark of the parent $u- = \omega$ of u and applying the branching property, $\mathbb{E}\left[III^2\right]$ is less than

$$\begin{aligned}&\mathbb{E}\Big[\sum_{v\in\mathcal{U}} 1_{\{(1-\delta)\varepsilon\leq\xi_v\leq\varepsilon(1+\delta)\}}\xi_v \int_{\mathcal{S}^\downarrow}\sum_{i=1}^\infty s_i\, g\big(\varepsilon^{-1}(\xi_v s_i + \sigma U_v)\big)^2\nu(d\mathbf{s})\Big] \\ =&\,\mathbb{E}\Big[\sum_{\omega\in\mathcal{U}} 1_{\{(1-\delta)\varepsilon\leq\xi_\omega\leq\varepsilon(1+\delta)\}}\xi_\omega G_1(\xi_\omega)\Big],\end{aligned}$$

with

$$G_1(a) := \int_{\mathcal{S}^\downarrow}\sum_{i=1}^\infty s_i\, \mathbb{E}\left[g\big(\varepsilon^{-1}(as_i + \sigma U)\big)^2\right]\nu(d\mathbf{s})$$

and U distributed as the U_ω. Likewise,

$$\mathbb{E}\left[IV^2\right] \lesssim \mathbb{E}\Big[\sum_{u\in\mathcal{U}} 1_{\{(1-\delta)\varepsilon\le\xi_u\le\varepsilon(1+\delta)\}}\xi_u G_2(\xi_u)\Big],$$

with

$$G_2(a) := \mathbb{E}\left[g\big(\varepsilon^{-1}(a+\sigma U)\big)^2\right].$$

For $i = 1, 2$, the crude bound $|G_i(a)| \le \|g\|_\infty^2$ and the genealogical representation argument used in Step 3 enables to bound either $\mathbb{E}[III^2]$ or $\mathbb{E}[IV^2]$ by

$$\|g\|_\infty^2 \sum_{n=0}^{\infty} \mathbb{P}^\star\big[-\log(1+\delta) \le S_n - \log(1/\varepsilon) \le -\log(1-\delta)\big]$$

where S_n is a random walk with step distribution $\pi(\bullet)$. We proceed as in Step 3 and apply Proposition 3.2. The above term converges to

$$m_1(\pi)^{-1}\log\left(\frac{1+\delta}{1-\delta}\right) \lesssim \delta$$

uniformly in δ, as soon as δ is bounded, at rate ε^μ for any $0 < \mu < \kappa$, and thus is of order $\delta + \varepsilon^\mu$.

We now turn to the term II. We have $II := V + VI + VII$, with

$$V := \sum_{u\in\mathcal{U}_\varepsilon} \xi_u\Big(g\big(\xi_u^{(\sigma)}/\varepsilon\big) - g\big(\xi_u/\varepsilon\big)\Big),$$
$$VI := \sigma\sum_{u\in\mathcal{U}_\varepsilon} U_u 1_{\{\xi_u^{(\sigma)}\ge t_\varepsilon\}} g\big(\xi_u^{(\sigma)}/\varepsilon\big),$$
$$VII := -\sum_{u\in\mathcal{U}_\varepsilon} \xi_u 1_{\{\xi_u^{(\sigma)}< t_\varepsilon\}} g\big(\xi_u^{(\sigma)}/\varepsilon\big).$$

From $g \in \mathcal{C}'(m)$, (3.2.1) and Jensen's inequality we derive

$$\mathbb{E}[V^2] \le \|g'\|_\infty^2\sigma^2\varepsilon^{-2}.$$

From $|U_u| \le 1$ and the inclusion $\{\xi_u^{(\sigma)} \ge t_\varepsilon\} \subset \{\xi_u \ge t_\varepsilon - \sigma\}$ we derive

$$\mathbb{E}\left[VI^2\right] \le \|g\|_\infty^2 \frac{\sigma^2}{(t_\varepsilon-\sigma)^2}\,\mathbb{E}\Big[\big(\sum_{u\in\mathcal{U}_\varepsilon}\xi_u\big)^2\Big] \lesssim \frac{\sigma^2}{\varepsilon^2},$$

where we used that $t_\varepsilon = \gamma_0\varepsilon$ with $0 < \gamma_0 < 1$. Likewise, the inclusion $\{\xi_u^{(\sigma)} < t_\varepsilon\} \subset \{\xi_u \le t_\varepsilon + \sigma\}$ and Lemma 3.1 yield

$$\mathbb{E}\left[VII^2\right] \le \|g\|_\infty^2\,\mathbb{P}^\star\left[-\log\chi(T_\varepsilon) > -\log(t_\varepsilon+\sigma)\right] \lesssim \varepsilon^\mu\log(1/\varepsilon)$$

for any $\mu < \kappa$, in the same line as for the bound of the right-hand side of (3.5.3) in Step 2.

Putting all the estimates together with, for instance, $\delta := \sigma/2\varepsilon$ we obtain (3.3.1). The proof of Theorem 3.1 is complete.

3.5.2 Proof of Theorem 3.2

Preliminaries. Let $0 < b_\varepsilon \to 0$ as $\varepsilon \to 0$. For $m > 0$, define the class

$$\widetilde{\mathcal{C}}_{b_\varepsilon}(m) := \big\{ g \in \mathcal{C}(m),\ \big|\mathrm{supp}\big(g(\bullet)\big)\big| \le m b_\varepsilon \big\}.$$

We have the following extension of Theorem 3.1.

Proposition 3.1. *Grant Assumptions A, B and C. In the same setting as Theorem 3.1, if in addition, we assume $\kappa > 1$, then, for every $\mu < \kappa$*

$$\sup_{g \in \widetilde{\mathcal{C}}_{b_\varepsilon}(m)} \mathbb{E}\left[\big(\mathcal{E}_\varepsilon(g) - \mathcal{E}(g)\big)^2\right] = o\big(\varepsilon^{\mu/(\mu+1)} b_\varepsilon\big).$$

Proof. We revisit carefully Steps 2 to 4 of the proof of Theorem 3.1 under the additional Assumption C, and we write $g(\bullet) = g_\varepsilon(\bullet)$ to emphasize that $g(\bullet)$ may now depend on the asymptotics.

In Step 2, the right-hand side of (3.5.3) is now bounded by the following chain of arguments:

$$\begin{aligned}
&\int_0^{-\log(\varepsilon/u)} dU(s) \int_{-\log(\varepsilon/u)-s}^{+\infty} g_\varepsilon\big(u\,\varepsilon^{-1} e^{-x-s}\big)^2 \pi(x) dx \\
&= \int_0^{-\log(\varepsilon/u)} dU(s) \int_0^{\varepsilon u^{-1} e^s} g_\varepsilon\big(xu\,\varepsilon^{-1} e^{-s}\big)^2 \beta(x)\, dx \\
&\le \|\beta\|_\infty\, u^{-1} \varepsilon \int_{[0,-\log(\varepsilon/u))} e^s dU(s) \int_0^1 g_\varepsilon\big(x\big)^2 dx \lesssim b_\varepsilon \log(u/\varepsilon)
\end{aligned}$$

where we used that $|\,\mathrm{supp}(g_\varepsilon)| \lesssim b_\varepsilon$ and $U(s) \lesssim s/c(\pi)$ again. Note that $\|\beta\|_\infty \lesssim 1$ since $\kappa_1 > 1$ and $\kappa_2 > 1$. Therefore

$$\mathbb{E}\left[Y_\varepsilon(v,g)^2 \,|\, \lambda_\eta(v) = u\right] \lesssim b_\varepsilon,$$

Hence

$$\mathbb{E}\left[M_{\varepsilon,\eta}(g)^2\right] \lesssim b_\varepsilon \eta.$$

In Step 3, we replace $g(\bullet)$ by $g_\varepsilon(\bullet)$ in $\mathcal{E}_\varepsilon(g)$ and $\mathcal{E}(g)$. We have, for any $0 < \mu < \kappa$,

$$\big|\mathcal{E}(g_\varepsilon)\big| \le \frac{1}{c(\pi)} \int_0^1 \frac{|g_\varepsilon(a)|}{a} \int_{\log(1/a)}^{+\infty} \pi(x) dx\, da$$

$$\lesssim \int_0^1 |g_\varepsilon(a)| a^{\mu-1} da \lesssim b_\varepsilon$$

for $\mu \geq 1$ and since $\pi \in \Pi(\kappa)$ with $\kappa > 1$. By Cauchy-Schwarz, for $a < 0$,

$$\begin{aligned}\big|H_{g_\varepsilon}(-a)\big| \leq & \Big(\int_{-a}^{+\infty} g_\varepsilon\big(e^{-x-a}\big)^2 \pi(x)dx \Big)^{1/2} \Big(\int_{-a}^{+\infty} \pi(x)dx \Big)^{1/2} \\ \lesssim & e^{a/2} \Big(\int_0^1 g_\varepsilon\big(y\big)^2 \beta(ye^a)dy \Big)^{1/2} e^{\,\kappa a/2} \\ \lesssim & b_\varepsilon^{1/2} e^{a(1+\kappa)/2},\end{aligned}$$

using again that $\|\beta\|_\infty \lesssim 1$. Therefore $\xi_v^{-1}\,\mathbb{E}_{\xi_v}\big[\mathcal{E}_\varepsilon(g_\varepsilon)\big] \lesssim b_\varepsilon^{1/2}$, and we can apply Proposition 3.2 with rate function $\varphi(a) = \exp(\mu a)$ and dominating function $r(a) := e^{-(1+\kappa)|a|/2}$.

The terms I and II are bounded in the same way. We obtain

$$\mathbb{E}\big[B_{\varepsilon,\eta}(g)^2\big] \lesssim b_\varepsilon\big(\varepsilon\eta^{-1}\big)^{2\mu} \log(1/\eta)$$

for $\mu < \min\{\kappa/2, (1+\kappa)/2\} = \kappa/2$, uniformly over the class $\widetilde{\mathcal{C}}_{b_\varepsilon}(m)$. The trade-off between $M_{\varepsilon,\eta}(g_\varepsilon)$ and $B_{\varepsilon,\eta}(g_\varepsilon)$ yields the result. □

Completion of proof of Theorem 3.2. We first write

$$\mathcal{E}(g_{\gamma_\varepsilon}) - m_1(\pi)^{-1} = \frac{1}{m_1(\pi)} \int_{1-\gamma_\varepsilon}^1 (f_{\gamma_\varepsilon}(a) - 1)\beta(a)da.$$

We have

$$\Big| \int_{1-\gamma_\varepsilon}^1 \big(f_{\gamma_\varepsilon}(a) - 1\big)\beta(a)da \Big| \leq 2 \int_0^{-\log(1-\gamma_\varepsilon)} \pi(x)dx \lesssim \gamma_\varepsilon^{\kappa_2}$$

since $\pi \in \mathcal{R}(\kappa_2)$ and $-\log(1-x) \lesssim x$ for small enough $x \geq 0$. We derive

$$\big|\mathcal{E}(g_{\gamma_\varepsilon}) - m_1(\pi)^{-1}\big| \lesssim \gamma_\varepsilon^{\kappa_2}. \tag{3.5.8}$$

Next, by construction, $\gamma_\varepsilon^2 g_{\gamma_\varepsilon} \in \widetilde{\mathcal{C}}_{\gamma_\varepsilon}(1)$, hence for any $0 < \mu < \kappa_1$, Proposition 3.1 entails

$$\mathbb{E}\big[\big|\mathcal{E}_\varepsilon(g_{\gamma_\varepsilon}) - \mathcal{E}(g_{\gamma_\varepsilon})\big|\big] \lesssim \gamma_\varepsilon^{-1/2} \varepsilon^{\mu/(2\mu+2)} \tag{3.5.9}$$

Moreover, since $\gamma_\varepsilon g_{\gamma_\varepsilon} \in \mathcal{C}'(1)$, we have, by (3.3.2) in Theorem 3.1

$$\mathbb{E}\big[\big|\mathcal{E}_\varepsilon(g_{\gamma_\varepsilon}) - \mathcal{E}_{\varepsilon,\sigma}(g_{\gamma_\varepsilon})\big|\big] \lesssim \gamma_\varepsilon^{-2}\sigma\varepsilon^{-1}. \tag{3.5.10}$$

The specification $\gamma_\varepsilon = \varepsilon^{\mu/(\mu+1)(2\kappa_2+1)}$ yields the correct rate for (3.5.8) and (3.5.9). The assumption that $\sigma\varepsilon^{-3}$ is bounded ensures that the right-hand side in (3.5.10) is asymptotically negligible. The conclusion readily follows for $\widehat{m}_{1,\varepsilon}$.

We now turn to higher moment estimators. Thanks to the proof for the case

$k = 1$, it suffices to show that

$$m_1(\pi)\mathcal{E}_\varepsilon\big(\widetilde{g}_{\gamma_\varepsilon}\big) \to \int_0^1 \log(1/a)^k \beta(a)da$$

in probability with the correct rate as $\varepsilon \to 0$. Note first that

$$\widetilde{g}_{\gamma_\varepsilon}(a) = -a\log(1/a)^k f'_{\gamma_\varepsilon}(1-a) + k f_{\gamma_\varepsilon}(1-a)\log(1/a)^{k-1}$$

is a sum of a function in $\widetilde{\mathcal{C}}_{\gamma_\varepsilon}(c_1)$ and a function in $\mathcal{C}'(c_2)$, for some positive c_1 and c_2, hence we may apply Proposition 3.1 and Theorem 3.1 to each term respectively. Next

$$\begin{aligned} c(\pi)\mathcal{E}\big(\widetilde{g}_{\gamma_\varepsilon}\big) &= -\int_0^1 \big(f_{\gamma_\varepsilon}(1-a)\log(1/a)^k\big)' \int_0^a \beta(u)du\, da \\ &= \int_0^1 f_{\gamma_\varepsilon}(1-a)\log(1/a)^k \beta(a)da, \end{aligned}$$

since $k > 1$ and $\pi \in \Pi(\kappa_1)$ so that the boundary terms vanish when integrating by part. We conclude by noticing that by Hölder's inequality, for any $\tau > 0$, we have

$$\begin{aligned} &\Big| \int_0^1 \big(1 - f_{\gamma_\varepsilon}(1-a)\big)\log(1/a)^k \beta(a)da\Big| \\ \le& 2\Big(\int_{-\log\gamma_\varepsilon}^{+\infty} \pi(x)dx\Big)^{1-\tau}\Big(\int_0^{+\infty} x^{k/\tau}\pi(x)dx\Big)^{\tau} \lesssim \gamma_\varepsilon^{\kappa_1(1-\tau)}. \end{aligned}$$

This term also has the right order since the choice of τ is free and $\kappa_1 > \kappa_2$ by assumption. The proof of Theorem 3.2 is complete.

3.5.3 Proof of Theorem 3.3

With no loss of generality, we consider the homogeneous case with $\alpha = 0$. We may also assume that $\sigma = 0$, since adding experimental noise to the observation of the fragments only increases the error bounds.

Step 1: An augmented experiment. In the binary case, the dislocation measure $\nu(d\mathbf{s})$ is equivalently mapped by a probability measure on $[1/2, 1]$ with density function $a \rightsquigarrow \rho(a)$, see (3.3.7).

We prove a lower bound in the augmented experiment where one can observe all the sizes $\widetilde{X}_\varepsilon$ of the fragments until they become smaller than ε, namely

$$\widetilde{X}_\varepsilon := \big\{\xi_u,\, \xi_{u-} \ge \varepsilon\big\} \cup \big\{\xi_u,\, u \in \mathcal{U}_\varepsilon\big\}$$

Clearly, taking the infimum over all estimators based on $\widetilde{X}_\varepsilon$ instead of $X_\varepsilon = X_{\varepsilon,0}$ only reduces the lower bound.

For every $u \in \mathcal{U}_\varepsilon$, we have $\xi_{u-} \ge \varepsilon$. By the conservative Assumption B, there are

at most ε^{-1} such ξ_{u-} so $\mathrm{Card}\,\mathcal{U}_\varepsilon \leq 2\varepsilon^{-1}$. For every node $u \in \mathcal{U}$, the fragmentation process gives rize to two offsprings with size $\xi_u U$ and $\xi_u(1-U)$, where U is a random variable independent of ξ_u with density function $\rho(\bullet)$. Therefore, the process of the sizes of the fragments in the enlarged experiment can be realized by less than

$$2\varepsilon^{-1}\big(1+\tfrac{1}{2}+\ldots+\tfrac{1}{2^{k(\varepsilon)}}\big) \leq \lfloor 4\varepsilon^{-1}\rfloor + 1 =: n(\varepsilon)$$

independent realizations of the law $\rho(\bullet)$, where $k(\varepsilon) := \log_2(2/\varepsilon)$, assumed to be integer with no loss of generality.

In turn, Theorem 3.3 reduces to proving that $\varepsilon^{1/2}$ is a lower rate of convergence for estimating $m_k(\pi)$ based on the observation of a $n(\varepsilon)$-sample of the law $\rho(\bullet)$. The one-to-one correspondence between $\rho(\bullet)$ and $\pi(\bullet)$ is given in (3.4.1).

Step 2: Construction of π_ε. We write $\rho_\pi(\bullet)$ to emphasize the dependence upon $\pi(\bullet)$. Let

$$\varphi_k(a) := a(\log(1/a)^k + (1-a)\log\big(1/(1-a)\big)^k, \;\; a \in [1/2,1].$$

From (3.4.1), we have

$$m_k(\pi_0) = \int_{1/2}^1 \varphi_k(a)\rho_{\pi_0}(a)da.$$

Let $0<\tau<1$. Pick a function $\psi_k(\bullet) : [1/2,1] \to \mathbb{R}$ such that

$$\|\psi_k\|_\infty \leq \tau \inf_a \rho_{\pi_0}(a), \;\; \int_{1/2}^1 \psi_k(a)da = 0, \;\; r(k) := \int_{1/2}^1 \varphi_k(a)\psi_k(a)da \neq 0,$$

a choice which is obviously possible thanks to Assumption D. For $\varepsilon > 0$, define

$$\rho_{\pi_\varepsilon}(a) := \rho_{\pi_0}(a) + \varepsilon^{1/2}\psi_k(a), \;\; a \in [1/2,1].$$

(And so (3.4.1) defines $\pi_\varepsilon(\bullet)$ unambiguously.) By construction, $\rho_{\pi_\varepsilon}(\bullet)$ is a density function on $[1/2,1]$ and has a corresponding binary fragmentation with Lévy measure given by $\pi_\varepsilon(\bullet)$. Moreover,

$$m_k(\pi_\varepsilon) = m_k(\pi_0) + r(k)\varepsilon^{1/2}.$$

Step 3: A two-point lower bound. The following chain of arguments is fairly classical. We denote by $\widetilde{\mathbb{P}}_\pi$ the law of the independent random variables $\big(U_i, i=1,\ldots,n(\varepsilon)\big)$ with common density $\rho_\pi(\bullet)$ that we use to realize the augmented experiment.

Let F_ε be an arbitrary estimator based on $\widetilde{X}_\varepsilon$. Put $c := |r(k)|/2$. We have

$$\max_{\pi\in\{\pi_0,\pi_\varepsilon\}} \widetilde{\mathbb{P}}_\pi\big[\varepsilon^{-1/2}|F_\varepsilon - m_k(\pi)| \geq c\big]$$

$$\geq \tfrac{1}{2}\left(\widetilde{\mathbb{P}}_{\pi_0}\big[\varepsilon^{-1/2}|F_\varepsilon - m_k(\pi_0)| \geq c\big] + \widetilde{\mathbb{P}}_{\pi_\varepsilon}\big[\varepsilon^{-1/2}|F_\varepsilon - m_k(\pi_\varepsilon)| \geq c\big]\right)$$
$$\geq \tfrac{1}{2}\widetilde{\mathbb{E}}_{\pi_0}\left[1_{\{\varepsilon^{-1/2}|F_\varepsilon - m_k(\pi_0)|\geq c\}} + 1_{\{\varepsilon^{-1/2}|F_\varepsilon - m_k(\pi_\varepsilon)|\geq c\}}\right] - \tfrac{1}{2}\|\widetilde{\mathbb{P}}_{\pi_0} - \widetilde{\mathbb{P}}_{\pi_\varepsilon}\|_{TV},$$

where $\|\bullet\|_{TV}$ denotes the total variation distance between probability measures. By the triangle inequality, we have

$$\varepsilon^{-1/2}\big(|F_\varepsilon - m_k(\pi_0)| + |F_\varepsilon - m_k(\pi_\varepsilon)|\big) \geq |r(k)| = 2c,$$

so one of the two indicators within the expectation above must be equal to one with full $\widetilde{\mathbb{P}}_{\pi_0}$-probability. Therefore

$$\max_{\pi\in\{\pi_0,\pi_\varepsilon\}} \widetilde{\mathbb{P}}_{\pi}\big[\varepsilon^{-1/2}|F_\varepsilon - m_k(\pi)| \geq c\big] \geq \tfrac{1}{2}(1 - \|\widetilde{\mathbb{P}}_{\pi_0} - \widetilde{\mathbb{P}}_{\pi_\varepsilon}\|_{TV}),$$

and Theorem 3.3 is proved if

$$\limsup_{\varepsilon\to 0} \|\widetilde{\mathbb{P}}_{\pi_0} - \widetilde{\mathbb{P}}_{\pi_\varepsilon}\|_{TV} < 1. \tag{3.5.11}$$

By Pinsker's inequality

$$\|\widetilde{\mathbb{P}}_{\pi_0} - \widetilde{\mathbb{P}}_{\pi_\varepsilon}\|_{TV} \leq \frac{\sqrt{2}}{2}\left(\widetilde{\mathbb{E}}_{\pi_0}\Big[\log\frac{d\,\widetilde{\mathbb{P}}_{\pi_0}}{d\,\widetilde{\mathbb{P}}_{\pi_\varepsilon}}\Big]\right)^{1/2},$$

and

$$\widetilde{\mathbb{E}}_{\pi_0}\Big[\log\frac{d\,\widetilde{\mathbb{P}}_{\pi_0}}{d\,\widetilde{\mathbb{P}}_{\pi_\varepsilon}}\Big] = -\sum_{i=1}^{n(\varepsilon)}\widetilde{\mathbb{E}}_{\pi_0}\left[\log\frac{\rho_{\pi_\varepsilon}(U_i)}{\rho_{\pi_0}(U_i)}\right]$$
$$= -\sum_{i=1}^{n(\varepsilon)}\widetilde{\mathbb{E}}_{\pi_0}\left[\log\left(1+\varepsilon^{1/2}\psi_k(U_i)\rho_{\pi_0}(U_i)^{-1}\right) - \varepsilon^{1/2}\psi_k(U_i)\rho_{\pi_0}(U_i)^{-1}\right],$$

where we used

$$\widetilde{\mathbb{E}}_{\pi_0}\left[\psi_k(U_i)\rho_{\pi_0}(U_i)^{-1}\right] = \int_{1/2}^{1}\psi_k(a)da = 0.$$

We also have $\varepsilon^{1/2}|\psi_k(U_i)\rho_{\pi_0}(U_i)^{-1}| \leq \tau\varepsilon^{1/2}$ hence for small enough τ,

$$\left|-\log\left(1+\varepsilon^{1/2}\psi_k(U_i)\rho_{\pi_0}(U_i)^{-1}\right) + \varepsilon^{1/2}\psi_k(U_i)\rho_{\pi_0}(U_i)^{-1}\right| \leq \tau^2\varepsilon.$$

Therefore $\|\widetilde{\mathbb{P}}_{\pi_0} - \widetilde{\mathbb{P}}_{\pi_\varepsilon}\|_{TV} \leq \frac{\sqrt{2}}{2}\tau\varepsilon^{1/2}n(\varepsilon)^{1/2} < 1$ by picking τ small enough, and (3.5.11) follows. The proof of Theorem 3.3 is complete.

3.5.4 Proof of Theorem 3.4

We plan to use the following decomposition

$$\widehat{\beta}(a) - \beta(a) = \widehat{m}_{1,\varepsilon}\, \mathcal{E}_{\varepsilon,\sigma}\big(- \bullet\, \varphi'_{\gamma_\varepsilon,a}(\bullet)\big) - \beta(a) = I + II + III + IV,$$

with

$$\begin{aligned} I &:= \widehat{m}_{1,\varepsilon}\Big(\mathcal{E}_{\varepsilon,\sigma}\big(- \bullet\, \varphi'_{\gamma_\varepsilon,a}(\bullet)\big) - \mathcal{E}_{\varepsilon}\big(- \bullet\, \varphi'_{\gamma_\varepsilon,a}(\bullet)\big)\Big) \\ II &:= \widehat{m}_{1,\varepsilon}\Big(\mathcal{E}_{\varepsilon}\big(- \bullet\, \varphi'_{\gamma_\varepsilon,a}(\bullet)\big) - \mathcal{E}\big(- \bullet\, \varphi'_{\gamma_\varepsilon,a}(\bullet)\big)\Big), \\ III &:= \big(\widehat{m}_{1,\varepsilon} - m_1(\pi)\big)\, \mathcal{E}\big(- \bullet\, \varphi'_{\gamma_\varepsilon,a}(\bullet)\big), \\ IV &:= m_1(\pi)\, \mathcal{E}\big(- \bullet\, \varphi'_{\gamma_\varepsilon,a}(\bullet)\big) - \beta(a). \end{aligned}$$

Considering I and II, the term $\widehat{m}_{1,\varepsilon}$ is bounded in probability by Theorem 3.2. By (3.3.2) in Theorem 3.1 together with the fact that $\gamma_\varepsilon^3 \varphi'_{\gamma_\varepsilon} \in \mathcal{C}'(\|\varphi'\|_\infty)$, we have

$$\mathbb{E}\left[\left|\mathcal{E}_{\varepsilon}\big(- \bullet\, \varphi'_{\gamma_\varepsilon,a}(\bullet)\big) - \mathcal{E}_{\varepsilon,\sigma}\big(- \bullet\, \varphi'_{\gamma_\varepsilon,a}(\bullet)\big)\right|\right] \lesssim \gamma_\varepsilon^{-3}\sigma\varepsilon^{-1}.$$

By construction, we have $\gamma_\varepsilon^2 \bullet\, \varphi'_{\gamma_\varepsilon,a}(\bullet) \in \widetilde{\mathcal{C}}_{\gamma_\varepsilon}(\|\varphi\|_\infty)$, therefore, by Proposition 3.1

$$\mathbb{E}\left[\Big(\mathcal{E}_{\varepsilon}\big(- \bullet\, \varphi'_{\gamma_\varepsilon,a}(\bullet)\big) - \mathcal{E}\big(- \bullet\, \varphi'_{\gamma_\varepsilon,a}(\bullet)\big)\Big)^2\right] \lesssim \gamma_\varepsilon^{-3}\varepsilon^{\mu/(\mu+1)}. \tag{3.5.12}$$

Considering III, using (3.5.8), we have $\left|\mathcal{E}\big(- \bullet\, \varphi'_{\gamma_\varepsilon,a}(\bullet)\big)\right| \lesssim \gamma_\varepsilon^{-1}$. By Theorem 3.2, we conclude that III^2 has order

$$\gamma_\varepsilon^{-2}\varepsilon^{2\mu\kappa_2/(\mu+1)(2\kappa_2+1)} \tag{3.5.13}$$

in probability. For IV, we first notice that

$$m_1(\pi)\, \mathcal{E}\big(- \bullet\, \varphi'_{\gamma_\varepsilon,a}(\bullet)\big) = \int_0^1 \varphi_{\gamma_\varepsilon,a}(u)\beta(u)du,$$

hence

$$IV^2 = \Big(\int_0^1 \varphi_{\gamma_\varepsilon,a}(u)\beta(u)du - \beta(a)\Big)^2 \lesssim \gamma_\varepsilon^{2s} \tag{3.5.14}$$

by a Taylor expansion and using that the terms up to order $s-1$ vanish by the cancellation property (3.3.8) of $\varphi(\bullet)$ since $s < N$.

Putting together (3.5.12), (3.5.13) and (3.5.14), we see that the specification $\gamma_\varepsilon = \varepsilon^{\mu/(\mu+1)(2s+3)}$ yields the correct rate for II and IV, as well as for III as soon as $\kappa_2 \geq s+1$. Finally, the term I proves asymptotically negligible in the same way as (3.5.10) thanks to the assumption that $\sigma\varepsilon^{-3}$ is bounded. The proof of Theorem 3.4 is complete.

3.6 Appendix

3.6.1 Proof of Lemma 3.1

First, we enrich the structure of the genealogical tree representation of Section 3.2 by adding a random mark $M : \mathcal{U} \to [0,1] \times [0,+\infty) \times [0,+\infty)$ on the tree together with a random branch[8] β of $\mathcal{U}$ and define unambiguously the law $\mathbb{P}^\star$ of (M,β) by setting

$$\mathbb{E}^\star\big[\Phi_n(M,\beta)\big] := \mathbb{E}\big[\sum_{|u|=n} \Phi_n(M,u)\xi_u\big], \quad n \geq 1,$$

where Φ_n is a bounded functional which depends on the mark M and the branch β up to the n-th generation only. If β_n is the node of the random branch at the n-th generation, we set $\chi_n := \xi_{\beta_n}$ and $\chi(t)$ for the size of the tagged particle at time t:

$$\chi(t) := \begin{cases} \chi_n & \text{if} \quad a_{\beta_n} \leq t < a_{\beta_n} + \zeta_{\beta_n}, \\ 0 & \text{if} \quad t \geq \lim_{n\to\infty} a_{\beta_n}, \end{cases}$$

where a_{β_n} and ζ_{β_n} denote respectively the birth-time and lifetime of the particle labeled by the tagged node β_n. We then have

$$\begin{aligned} \mathbb{E}\big[\sum_{v\in\mathcal{U}_\varepsilon} \xi_v f(\xi_v)\big] &= \mathbb{E}\big[\sum_{n=0}^{\infty}\sum_{|v|=n} 1_{\{\xi_{v-}\geq\eta,\, \xi_v<\eta\}}\xi_v f(\xi_v)\big] \\ &= \sum_{n=0}^{\infty} \mathbb{E}^\star\big[1_{\{\chi_{n-1}\geq\eta,\, \chi_n<\eta\}} f(\chi_n)\big]. \end{aligned}$$

By Proposition 1.6 in [17], $-\log\chi_n$ is a random walk under $\mathbb{P}^\star$ with step distribution $\pi(dx)$. In particular, the last term above is independent of α.

Consider now a homogeneous fragmentation process with same dislocation measure $\nu(\bullet)$ living on the same (possibly enlarged) probability space for simplicity. Applying the same construction above, we obtain a process $\big(\chi^{(0)}(t), t \geq 0\big)$ that can be expressed in the form $\chi^{(0)}(t) = \exp\big(-\xi^{(0)}(t)\big)$ where $\big(\xi^{(0)}(t), t \geq 0\big)$ is a compound Poisson process with jump intensity 1 and jump distribution $\pi(\bullet)$. By construction, we have $\chi^{(0)}\big(T^{(0)}_\eta\big) = \chi^{(0)}_n$ on the event $\{\chi^{(0)}_{n-1} \geq \eta,\, \chi^{(0)}_n < \eta\}$ therefore

$$\mathbb{E}^\star\big[f\big(\chi^{(0)}(T^{(0)}_\eta)\big)\big] = \sum_{n=0}^{\infty} \mathbb{E}^\star\big[1_{\{\chi^{(0)}_{n-1}\geq\eta,\, \chi^{(0)}_n<\eta\}} f\big(\chi^{(0)}_n\big)\big].$$

The conclusion follows by identifying the right-hand sides of the last two equalities.

[8] A branch is an infinite sequence of positive integers which we can think of as the line of ancestors of some leaf of the tree.

3.6.2 Rates of convergence in the key renewal theorem

We give a version of Sgibnev's result [65] on uniform rates of convergence in the key renewal theorem, which is proved in a more general setting.

Let $F(dx)$ be a probability distribution with positive mean m and renewal function $H = \sum_{n=0}^{\infty} F^{n\star}$, with $F^{0\star} := \delta_0$, $F^{1\star} := F$ and $F^{(n+1)\star} := F \star F^{n\star}$, $n \geq 0$.

We assume that F is *spread-out*, that is, for some $n \geq 1$, F^{*n} has a non-zero absolutely continuous component. Stone [66] shows that then there exists a decomposition $H = H_1 + H_2$, where H_2 is finite measure and H_1 is absolutely continuous with bounded continuous density function $h(\bullet)$ such that $\lim_{x\to+\infty} h(x) = m^{-1}$ and $\lim_{x\to-\infty} h(x) = 0$.

We denote by $T(F)$ the σ-finite measure with density function

$$\int_{(x,+\infty)} F(du)1_{[0,+\infty)}(x) - \int_{(-\infty,x]} F(du)1_{(-\infty,0)}(x).$$

and $T^2(F) := T\big(T(F)\big)$. Let $\varphi(\bullet) : \mathbb{R} \to [0,+\infty)$ be a submultiplicative function, *i.e.* such that $\varphi(0) = 1$, $\varphi(x+y) \leq \varphi(x)\varphi(y)$. Then we have (see for instance [41], Section 6)

$$-\infty < r_1 := \lim_{x\to-\infty} \frac{\log\varphi(x)}{x} \leq \lim_{x\to+\infty} \frac{\log\varphi(x)}{x} =: r_2 < +\infty,$$

Assumption E. *We have $r_1 \leq 0 \leq r_2$ and there exists $r(\bullet) : \mathbb{R} \to \mathbb{R}$ an integrable function and such that the following conditions are fulfilled:*

$$\sup_x |r(x)|\varphi(x) < +\infty \quad , \quad \lim_{|x|\to\infty} r(x)\varphi(x) = 0,$$

$$\lim_{x\to+\infty} \varphi(x) \int_{[x,+\infty)} r(u)du = \lim_{x\to-\infty} \varphi(x) \int_{(-\infty,x]} r(u)du = 0,$$

and

$$\int_{\mathbb{R}} \varphi(x)T^2(F)(dx) < \infty.$$

Sgibnev's result takes the form:

Proposition 3.2. *(Theorem 5.1 in [65]). Grant Assumption E. Then*

$$\lim_{|t|\to\infty} \varphi(t) \sup_{g,|g(x)|\leq|r(x)|} \left| g \star H(t) - m^{-1} \int_{\mathbb{R}} g(x)dx \right| = 0.$$

We call $\varphi(\bullet)$ a rate function and $r(\bullet)$ a dominating function.

Chapter 4

On the concatenation of two fragmentation processes

We establish the base for the study of a problem emanated from the mining industry. We estimate the energy used to reduce a unit-size fragment into fragments of size not larger than a fixed threshold $\eta' \in (0, 1]$, by consecutively using two different fragmentation processes, representing two different machines or "crushers". More precisely, each mass fragment evolves in the first fragmentation process until the instant it first becomes smaller than or equal to $\eta \in (\eta', 1]$, and then immediately enters the second fragmentation process, where it evolves until it first becomes smaller that or equal to η', when it exists the system. Assume that in both fragmentation processes, each sudden dislocation has a given (known) energy cost, which depends on the size of the fragment that breaks, and on the mass partition it gives raise to.

Contents

The material of this chapter is an article en progress with Joaquim Fontbona and Servet Martinez.

4.1 Introduction

Consider an homogeneous fragmentation process as introduced by Bertoin in [17]. This is a homogeneous Markov process $X = (X(t, \mathbf{x}))_{t\geq 0}$ with values in

$$\mathcal{S}^{\downarrow} := \left\{ x = (x_1, x_2, ...) \mid x_1 \geq x_2 \geq ... \geq 0 \ , \sum_{i=1}^{\infty} x_i \leq 1 \right\}$$

which satisfies the two fundamental properties of homogeneity and branching.

The parameter $\mathbf{x} = (x_1, x_2, ...)$ is an element of $\mathcal{S}^{\downarrow}$ standing for the initial condition: $X(0, \mathbf{x}) = \mathbf{x}$ *a.s.*, and if $\mathbf{x} = (1, 0, \dots)$ we shall simply write

$$X(t) = X(t, \mathbf{x}), \ t \geq 0.$$

It is known that in this case, the process X is entirely characterized by an erosion coefficient $c \geq 0$ and a dislocation measure ν, which is a measure on $\mathcal{S}^{\downarrow}$ satisfying the conditions

$$\nu(\{1, 0, 0, ...\}) = 0,$$

and

$$\int_{\mathcal{S}^{\downarrow}} (1 - s_1)\nu(d\mathbf{s}) < \infty, \quad \text{with} \quad \mathbf{s} = (s_1, s_2, ...) \tag{4.1.1}$$

We shall suppose moreover that we are in the dissipative case,

$$\sum_{i=1}^{\infty} s_i \leq 1 \quad a.s.$$

and we assume absence of erosion ($c = 0$).

Inspired by a problem arising in the mining industry, we are interested in the following question:

Suppose we need to reduce a unit-size fragment into fragments of size not larger than a fixed threshold $\eta' \in (0, 1]$, by using consecutively two different fragmentation processes, representing two different machines or "crushers". More precisely, each mass fragment evolves in the first fragmentation process until the instant it first becomes smaller than or equal to $\eta \in (\eta', 1]$, and then immediately enters the second fragmentation process, where it evolves until it first becomes smaller that or equal to η', when it exists the system. Assume that in both fragmentation processes, each sudden dislocation has a given (known) energy cost, which depends on the size of the fragment that breaks, and on the mass partition it gives raise to. **Can we determine the optimal $\eta \in (\eta', 1]$, in order to minimize the expected cost of reducing the unit-size fragment into fragments of size not larger than η' ?**

4.2 Preliminaries

We recall the construction of a homogeneous fragmentation process in terms the atoms of a Poisson point process (see [8]). Let ν be a dislocation measure fulfilling conditions (4.1.1). Let $K = ((\Delta(t), k(t)), t \geq 0)$ be a Poisson point process with values in $\mathcal{S}^{\downarrow} \times \mathbb{N}$, and with intensity measure $\nu \otimes \sharp$, where $\sharp$ is the counting measure on $\mathbb{N}$. As in [8], we can construct a unique $\mathcal{S}^{\downarrow}$-valued process $X = (X(t, \mathbf{x}), t \geq 0)$ started from $\mathbf{x}$ with paths that jumps only at instants $t \geq 0$ at which a point $(\Delta(t) = (\Delta_1, \Delta_2, ...), k(t))$ occurs. Plainly, $X(t, \mathbf{x})$ is obtained by replacing the $k(t)$-term $X(t-, \mathbf{x})$ by the decreasing rearrangement of the sequence $X_1(t-, \mathbf{x}), ..., X_{k-1}(t-, \mathbf{x}), X_k(t-, \mathbf{x})\Delta_1, X_k(t-, \mathbf{x})\Delta_2, ..., X_{k+1}(t-, \mathbf{x}), ...$

Define

$$\underline{p} := \inf \left\{ p \in \mathbb{R} : \int_{\mathcal{S}^{\downarrow}} \sum_{j=2}^{\infty} s_j^p \nu(d\mathbf{s}) < \infty \right\}$$

and for every $q \in (\underline{p}, \infty)$,

$$\kappa(q) := \int_{\mathcal{S}^{\downarrow}} \left(1 - \sum_{j=1}^{\infty} s_j^q \right) \nu(d\mathbf{s}).$$

In all the sequel, we assume **the Malthusian hypothesis**, that is

$$\textbf{there exists a unique } \alpha \geq \underline{p} \quad \textbf{such that } \kappa(\alpha) = 0,$$

which is called the Malthusian exponent.

Following Bertoin and Martinez [21] , we shall assume that the energy needed to split a fragment of size $x \in [0, 1]$ into a sequence $x_1 \geq x_2 \geq \dots$ is given by the formula

$$x^{\beta} \varphi \left(\frac{x_1}{x}, \frac{x_2}{x}, \dots \right),$$

where $\beta > 0$ is a fixed constant and $\varphi : \mathcal{S} \to \mathbb{R}$ is a measurable "cost function" such that $\varphi((1, 0, ...)) = 0$.

We are interested in the total energy $\epsilon^{\mathbf{x}}(\eta)$ used in splitting the fragments until each of them has reached, for the first time, a size that is smaller than or equal to η. That is, the quantity given by

$$\epsilon^{(\mathbf{x})}(\eta) = \sum_{t \geq 0} \mathbb{1}_{X_{k(t),\mathbf{x}}(t-) \geq \eta} X^{\beta}_{k(t)}(t-, \mathbf{x}) \varphi(\Delta(t)).$$

We shall simply write $\epsilon^{(1,0,\dots)}(\eta) = \epsilon(\eta)$

The following consequence of the homogeneity property will be useful:

Lemma 4.1. *Let* $\mathbf{x} = (x_1, x_2, ...) \in \mathcal{S}^{\downarrow}$ *and* $\eta \in [0, 1]$. *We have*

$$\epsilon^{(\mathbf{x})}(\eta) =^{(law)} \sum_i \mathbb{1}_{x_i > \eta} x_i^{\beta} \epsilon_i(\eta / x_i), \tag{4.2.1}$$

where for each $i \geq 1$, $\epsilon_i(\bullet)$ *is the energy of a fragmentation process* X^i *issued from* $(1, 0, \dots)$ *with same characteristics as* X *and with the* $(X^i)_{i \geq 1}$ *being independent.*

Proof. Let $(X^{(x_i)})_{i \geq 1}$ be independent homogeneous fragmentation processes with same characteristics a X, each of which starting from $(x_i, 0, \cdots)$, and $((k_i(t), \Delta_{(i)}(t)), t \geq 0, i \geq 0)$ be the Poison point processes giving raise to each of them. From the branching property of X, we have the identity

$$\epsilon^{(\mathbf{x})}(\eta) \stackrel{(d)}{=} \sum_i \sum_{t \geq 0} \mathbb{1}_{x_i > \eta} \mathbb{1}_{X^{(x_i)}_{k_i(t)}(t_-) \geq \eta} (X^{(x_i)}_{k_i(t)})^{\beta}(t_-) \varphi(\Delta_i(t)).$$

By homogeneity we then get

$$\epsilon^{(\mathbf{x})}(\eta) \stackrel{(d)}{=} \sum_i \sum_{t \geq 0} \mathbb{1}_{x_i > \eta} \mathbb{1}_{x_i X^i_{k^i(t)}(t_-) \geq \eta} x_i^{\beta} (X^i_{k^i(t)})^{\beta}(t_-) \varphi(\Delta^i(t)),$$

and the statement follows. □

To formulate our problem, introduce a second Poisson point process $K' = ((\Delta'(t), k'(t)), t \geq 0)$ with values in $\mathcal{S}^{\downarrow} \times \mathbb{N}$, and with intensity measure $\nu' \otimes \sharp$, where ν' is a dislocation measure satisfying the same type of assumptions as ν. We can then simultaneously define a family of fragmentation processes $(X'(t, \mathbf{x}))_{t \geq 0}$ independent of X, indexed by the initial condition $\mathbf{x} = (x_1, x_2, ...)$.

We denote by α' the Malthus coefficient of ν'. The energy used in the second fragmentation process is assumed to take the same form as for the first, in terms of (possibly different) parameters β' and φ'.

For each $\eta \in (0, 1]$ let now $\mathbf{x}^{\eta} \in \mathcal{S}^{\downarrow}$ be the mass partition given by the "output" of X when each of the fragments reaches for the first time a size smaller that or equal to η. More precisely, each fragment is "frozen" at that time, while other (larger than η) fragments continue their independent evolutions. We write

$$\mathbf{x}^{\eta} = (x_1^{\eta}, x_2^{\eta}, \cdots)$$

for the decreasing rearrangement of the (random) sizes of all fragments once each of them has been "frozen".

We assume in all the sequel that X is issued from the unitary fragment $(1, 0, \dots)$. Let $1 \geq \eta > \eta' > 0$. Recall that we let each mass fragment evolve in the first fragmentation process until the instant it first becomes smaller than or equal to $\eta \in (\eta', 1]$. Then it immediately enters the second fragmentation process, and then it evolves until it first becomes smaller that or equal to η'. By the homogeneity

and branching properties, if $E(\eta, \eta')$ denotes the total energy spent in reducing the unit-size fragment by these procedure, we have the identity

$$E(\eta, \eta') \stackrel{(d)}{:=} \epsilon'^{(\mathbf{x}_\eta)}(\eta') + \epsilon(\eta)$$

with $\epsilon'^{(\mathbf{x})}(\bullet)$ the energy of the second fragmentation X' when starting from $\mathbf{x}$.

4.3 The expected total energy

We next compute the expectation of $E(\eta, \eta')$. A key tool will be the tagged fragment associated with the fragmentation processes X. For the precise definition, we refer the reader to Bertoin [14]. We shall keep the following facts in mind: the tagged fragment is a process defined by

$$\chi(t) := X_{K(t)}(t)$$

where $K(t)$ is a random integer such that, conditioned on $X(t)$, $\mathbb{P}(K(t) = i|X(t)) = X_i(t)$ for all $i \geq 1$, and $\mathbb{P}(K(t) = 0|X(t)) = 1 - \sum_{i=1}^{\infty} X_i(t)$. By assumption, we have $X_0 = 0$.

Is is shown by Bertoin (Theorem 3 in [14]) that

$$\xi(t) = -\log \chi(t)$$

is a subordinator. Moreover, its Laplace exponent φ given by $\varphi(q) := \kappa(q+1)$ for $q > \underline{p} - 1$.

Since $\varphi(\alpha - 1) = 0$, $\exp((1-\alpha)\xi(t))$ is a nonnegative martingale, and we can then define a probability measure $\tilde{\mathbb{P}}$ on the path space by

$$d\tilde{\mathbb{P}}_{|\mathcal{F}_t} = \exp((1-\alpha)\xi(t))d\mathbb{P}_{|\mathcal{F}_t},$$

where $(\mathcal{F}_t)_{t\geq 0}$ denotes the natural filtration of ξ. It is well known that under this "tilted" law, ξ is a subordinator with Laplace exponent $\tilde{\varphi}(q) = \varphi(q + \alpha + 1)$.

The notation ξ', $\tilde{\mathbb{P}}'$, and so on, will be used for the analogous objects associated with the fragmentation process X'.

Theorem 4.1. *Assume that $\varphi \in L^1(\nu)$ and $\varphi' \in L^1(\nu')$ We have*

$$\begin{aligned}
&\mathbb{E}(E(\eta, \eta')) \\
&= C' \int_0^{\log(1/\eta)} \left[\int_{\log(1/\eta)}^{\log(1/\eta')} e^{(\alpha-\beta')z} \int_0^{\log(1/\eta')+z} e^{(\alpha'-\beta')x} \tilde{U}'(dx)\tilde{\Pi}(dz-y) \right] \tilde{U}(dy) \\
&+ C \int_0^{\log(1/\eta)} e^{(\alpha-\beta)y} \tilde{U}(dy),
\end{aligned}$$

where: $C = \int_S \varphi(\mathbf{s})\nu(d\mathbf{s})$, $C' = \int_S \varphi'(\mathbf{s})\nu'(d\mathbf{s})$, $\tilde{\Pi}$ *and* $\tilde{U}$ *respectively stand for the Lévy measure and the renewal measure of* $\xi(t)$ *under* $\tilde{\mathbb{P}}$, *and* $\tilde{U}'$ *is the renewal measure of* ξ' *under* $\tilde{\mathbb{P}}'$.

Proof. By the compensation formula for the Poisson point process $(\Delta(u), k(u))$ associated with the first fragmentation process X, we get that

$$\begin{aligned}\mathbb{E}(\epsilon(\eta)) &= \mathbb{E}\left(\int_0^\infty \mathbb{1}_{\chi(t)>\eta}(\chi(t))^{\beta-1}dt\right)\int_S \varphi(\mathbf{s})\nu(d\mathbf{s}) \\ &= C\ \mathbb{E}\left(\int_0^\infty \mathbb{1}_{\xi(t)<\log(1/\eta)}e^{(1-\beta)\xi(t)}dt\right)\end{aligned} \tag{4.3.1}$$

Thus

$$\begin{aligned}\mathbb{E}(\epsilon(\eta)) &= C\tilde{\mathbb{E}}\left(\int_0^\infty \mathbb{1}_{\xi(t)<\log(1/\eta)}e^{(\alpha-\beta)\xi(t)}dt\right) \\ &= C\int_0^{\log(1/\eta)} e^{(\alpha-\beta)y}\tilde{U}(dy),\end{aligned}$$

For the second term we have, with similar notations as before,

$$\begin{aligned}\mathbb{E}(\epsilon'^{(\mathbf{x}_\eta)}(\eta')) &= \mathbb{E}\left(\sum_i \mathbb{1}_{x_{\eta,i}>\eta'}x_{\eta,i}^{\beta'}\epsilon'^{i}(\eta'/x_{\eta,i})\right) \\ &= \mathbb{E}\left(\sum_i \mathbb{1}_{x_{\eta,i}>\eta'}x_{\eta,i}^{\beta'}\mathbb{E}(\epsilon'^{i}(\eta'/x_{\eta,i})|x_{\eta,i})\right) \\ &= \mathbb{E}\left(\mathbb{1}_{\chi(L_\eta)>\eta'}(\chi(L_\eta))^{\beta'}\mathbb{E}'(\epsilon'(\eta'/y))|_{y=\chi(L_\eta)}\right),\end{aligned}$$

where $L_\eta := \inf\{t \geq 0 : \chi(t) < \eta\}$ and $\epsilon'(.)$ is the energy of the second fragmentation which is independent of the first one. Then with $T_\eta := \inf\{t \geq 0 : \xi(t) > \log(1/\eta)\}$, it holds

$$\mathbb{E}(\epsilon^{(\mathbf{x}_\eta)}(\eta')) = \tilde{\mathbb{E}}\left(\mathbb{1}_{\xi(T_\eta)<\log(1/\eta')}e^{(\alpha-\beta')\xi(T_\eta)}\mathbb{E}'(\epsilon'(\eta'e^{-z}))|_{z=\xi(T_\eta)}\right).$$

According to Lemma 1.10 of [13] the distribution of $\xi(T_\eta)$ under $\tilde{\mathbb{P}}$ is given by

$$\tilde{\mathbb{P}}(\xi(T_\eta) \in dz) = \int_0^{\log(1/\eta)} \mathbb{1}_{z>\log(1/\eta)}\tilde{\Pi}(dz-y)d\tilde{U}(y).$$

Therefore

$$\mathbb{E}(\epsilon^{(\mathbf{x}_\eta)}(\eta')) = \int \mathbb{1}_{z<\log(1/\eta')}e^{(\alpha-\beta')z}\mathbb{E}(\epsilon'(\eta'/e^z))\ \int_0^{\log(1/\eta)} \mathbb{1}_{z>\log(1/\eta)}\tilde{\Pi}(dz-y)d\tilde{U}(y).$$

By similar arguments as in the first part of the proof we finally get:

$$\mathbb{E}(\epsilon^{(\mathbf{x}_\eta)}(\eta^{'}))$$
$$= C^{'} \int_0^{\log(1/\eta)} \left[\int_{\log(1/\eta)}^{\log(1/\eta^{'})} e^{(\alpha-\beta^{'})z} \int_0^{\log(1/\eta^{'})+z} e^{(\alpha^{'}-\beta^{'})x} U^{'}(dx)\tilde{\Pi}(dz-y) \right] d\tilde{U}(y).$$

□

4.4 Some trails to continue.

4.4.1 A Lemma

Knowing the localization of the minima of the total energy would allow us to compute the total energy cost. The following lemma is a first step towards this goal. Denote

$$F(z) = e^{(\alpha-\beta^{'})z}\mathbb{E}^{'}(\epsilon^{'}(\eta^{'}e^{-z})) = C' e^{(\alpha-\beta^{'})z} \int_0^{\log(1/\eta')+z} e^{(\alpha-\beta)y}\tilde{U}'(dy).$$

Lemma 4.2. *Consider* $1 \geq \eta_1 > \eta > \eta' > 0$. *We have*

$$\begin{aligned}\mathbb{E}(E(\eta,\eta^{'}) - E(\eta_1,\eta^{'})) =& C \int_{\log(1/\eta_1)}^{\log(1/\eta)} e^{(\alpha-\beta)y}\tilde{U}(dy) \\ &+ \int_{\log(1/\eta_1)}^{\log(1/\eta)} \int_{\log(1/\eta)-y}^{\log(1/\eta')-y} F(y+z)\tilde{\Pi}(dz)\tilde{U}(dy) \\ &- \int_0^{\log(1/\eta_1)} \int_{\log(1/\eta_1)-y}^{\log(1/\eta)-y} F(y+z)\tilde{\Pi}(dz)\tilde{U}(dy)\end{aligned}$$

Proof. Since $\mathbb{E}(E(\eta,\eta^{'}) = \mathbb{E}(\epsilon(\eta))) + \mathbb{E}(\epsilon^{'(\mathbf{x}_\eta)}(\eta^{'}))$, the first integral corresponds to $\mathbb{E}(\epsilon(\eta) - \epsilon(\eta_1))$ and its value follows from the computations in the proof of the previous result. Now, we have

$$\begin{aligned}&\mathbb{E}(\epsilon^{'(\mathbf{x}_\eta)}(\eta^{'}) - \epsilon^{'(\mathbf{x}_{\eta_1})}(\eta^{'})) \\ &= \tilde{\mathbb{E}}(\mathbb{1}_{\xi(T_\eta)<\log(1/\eta')} F(\xi(T_\eta))) - \tilde{\mathbb{E}}(\mathbb{1}_{\xi_{T_{\eta_1}}<\log(1/\eta')} F(\xi_{T_{\eta_1}})) \\ &= \tilde{\mathbb{E}}(\mathbb{1}_{\xi(T_\eta)<\log(1/\eta')} F(\xi(T_\eta))\mathbb{1}_{T_\eta>T_{\eta_1}}) - \tilde{\mathbb{E}}(\mathbb{1}_{\xi(T_{\eta_1})<\log(1/\eta')} F(\xi(T_{\eta_1}))\mathbb{1}_{T_\eta>T_{\eta_1}})\end{aligned}$$

since $T_\eta \geq T_{\eta_1}$ a.s. Observe also that

$$\begin{aligned}\{T_\eta > T_{\eta_1}\} &= \{\xi(T_\eta-) > \log(1/\eta_1)\} = \{e^{-\xi(T_\eta-)} < \eta_1\} \\ &= \{\xi(T_{\eta_1}) \leq \log(1/\eta)\} = \{e^{-\xi(T_{\eta_1})} \geq \eta\}.\end{aligned}$$

We deduce that

$$\begin{aligned}\mathbb{E}(\epsilon^{'(\mathbf{x}_\eta)}(\eta^{'}) - \epsilon^{'(\mathbf{x}_{\eta_1})}(\eta^{'})) =& \tilde{\mathbb{E}}(\mathbb{1}_{<\log(1/\eta_1)<\xi_{T_\eta-}\leq\xi(T_\eta)<\log(1/\eta^{'})}F(\xi(T_\eta))) \\ &- \tilde{\mathbb{E}}(\mathbb{1}_{\xi(T_{\eta_1})\leq\log(1/\eta)}F(\xi(T_{\eta_1})))\end{aligned}$$

The statement follows from Lemma 1.10 of [13], which implies that

$$\tilde{\mathbb{P}}(\xi(T_\varepsilon)\in dz, \xi(T_\varepsilon-)\in dy) = \mathbb{1}_{0\leq y\leq\varepsilon\leq z+y}\tilde{\Pi}(dz)\tilde{U}(dy)$$

for all $\varepsilon > 0$. □

4.4.2 The limit of the energy when η goes to zero.

By (4.3.1), we get that

$$\mathbb{E}(\epsilon(\eta)) = C\mathbb{E}\left(\int_0^\infty \mathbb{1}_{\xi(t)<\log(1/\eta)}e^{(1-\beta)\xi(t)}dt\right).$$

Therefore, by the theorem of monotone convergence we get that

$$\lim_{\eta\to 0}\mathbb{E}(\epsilon(\eta)) = C\mathbb{E}\left(\int_0^\infty e^{(1-\beta)\xi(t)}dt\right) = C\int_0^\infty e^{t\varphi(1-\beta)}dt = C/\kappa(2-\beta),$$

by the definition of the Laplace exponent of the subordinator of $\xi(t)$.

4.4.3 A particular case

As the general problem seen to be complicate, we will consider a special case: the stable case. In the case where we consider a fragmentation process for which there exists $0 < a < 1$

$$\varphi(q) = (q-a-1)^a.$$

Therefore $\xi(t)$ under $\tilde{\mathbb{P}}$ is a stable subordinator with parameter a. Thus by using Exercice 5.6 P.135 of the book of Kyprianou [55], we get that

$$\Pi(dx) = \frac{x^{-(1+a)}}{-\Gamma(-a)}dx \quad \text{and} \quad U(dx) = \frac{x^{a-1}}{\Gamma(a)}dx.$$

First we recall the generalised gamma function for $b > 0$ and $x \leq 0$

$$\gamma(b,x) = \int_0^x t^{b-1}e^{-t}dt.$$

Using this notation, it is obvious that;

$$\mathbb{E}(\epsilon(\eta)) = C \int_0^{\log(1/\eta)} e^{(\alpha-\beta)y} \frac{y^{a-1}}{\Gamma(a)} dy = C\gamma\left(a, \log(1/\eta)(\beta-\alpha)\right)/\Gamma(a)$$

which goes to 0 when η goes to 1.

By doing the same we get that

$$F(z) = e^{(\alpha-\beta)z} C^{'} \gamma\left(a^{'}, (\log(1/\eta^{'} + z)(\beta^{'} - \alpha^{'})\right),$$

and thus

$$F'(0) := \frac{(\alpha-\beta)}{\Gamma(a^{'})} C^{'} \gamma(a^{'}, \log(1/\eta^{'})(\beta^{'} - \alpha^{'})) + \frac{C^{'}(\eta^{'})^{\alpha^{'}-\beta^{'}}}{\Gamma(a^{'})} (\beta^{'} - \alpha^{'})^{a^{'}+1} (\log(1/\eta^{'}))^{a^{'}}.$$

If $\varphi(s_1, s_2, ...) = \sum_i s_i^{\beta} - 1$, then

$$C = \int_{\mathcal{S}^{\downarrow}} \varphi(\mathbf{s}) \nu(d\mathbf{s}) = \beta^{\alpha}.$$

If $\varphi^{'}(s_1, s_2, ...) = \sum_i s_i^{\beta^{'}} - 1$, then

$$C^{'} = \int_{\mathcal{S}^{\downarrow}} \varphi^{'}(\mathbf{s}) \nu^{'}(d\mathbf{s}) = (\beta^{'})^{\alpha^{'}}.$$

Bibliography

[1] R. Abraham and J.-F. Delmas (2007). Asymptotics for the small fragments of the fragmentation at nodes. *Bernoulli*, **Vol. 13**, Issue 1, 211-228.

[2] D.J. Aldous (1999). Deterministic and stochastic models for coalescence (aggregation, coagulation): a review of the mean-field theory for probabilists. *Bernoulli* **5**, 3-48.

[3] S. Asmussen (2003). *Applied probability and queues*, 2nd ed., Springer-Verlag, New York, 2003.

[4] S. Asmussen and N. Kaplan (1976). Branching random walks. I. *Stochastic Process. Appl.* **4** , no. 1, 1-13.

[5] S. Asmussen and N. Kaplan (1976). Branching random walks. II. *Stochastic Process. Appl.* **4** , no. 1, 15-31.

[6] K. B. Athreya and P. E. Ney (1972). *Branching processes*. Springer-Verlag Berlin Heidelberg.

[7] A.-L. Basdevant (2006). Fragmentation of ordered partitions and intervals. *Elect. J. Probab.* **11** no. 16, 394-417.

[8] J. Berestycki (2002). Ranked fragmentations. *ESAIM Prob. Stat.,* **6**, 157-176.

[9] J. Berestycki (2003). Multifractal spectra of fragmentation processes. *J. Statist. Phys,* **113**, no. 3-4, 411-430.

[10] J. Bertoin (1996). *Lévy processes.* Cambridge Univ. Press.

[11] J. Bertoin (1996). On the first exit time of a completely asymmetric stable process from a finite interval. *Bull. London Math. Soc.* **28** , no. 5, 514-520.

[12] J. Bertoin (1997). Exponential decay and ergodicity of completely asymmetric Lévy processes in a finite interval. *Ann. Appl. Probab,* **7**, no. 1, 156-169.

[13] J. Bertoin (1999). Subordinators: examples and applications. *In lectures on Probability Theory and Statistics (St-Flour, 1997; Lecture Notes Math. **1717**). Springer. Berlin,* 1-91.

[14] J. Bertoin (2001). Homogeneous fragmentation processes. *Probab. Theory Relat. Fields,* **121**, 301-318.

[15] J. Bertoin (2002). Self similar fragmentations. *Ann. Inst. Henri poincaré. Probabilités et Statistiques* **38**, 319–340.

[16] J. Bertoin (2003). The asymptotic behavior of fragmentation processes. *J. Europ. Math. Soc. ,* **14**, 395-416.

[17] J. Bertoin (2006). *Random fragmentation and coagulation processes.* Cambridge Univ. Pr.

[18] J. Bertoin (2006). Different aspects of a random fragmentation model. *Stochastic Process. Appl.* **116**, 345-369.

[19] J. Bertoin and M.-E. Caballero (2002). Entrance from 0+ for increasing semi-stable Markov processes. *Bernoulli* **8**, no. 2, 195–205.

[20] J. Bertoin and A. V. Gnedin (2004). Asymptotic laws for nonconservative self-similar fragmentations. *Electron. J. Probab.* **9** no. 19, 575-593.

[21] J. Bertoin and S. Martinez (2005). Fragmentation energy. *Adv. Appl. Probab.* **37** 553-570.

[22] J. Bertoin and A. Rouault (2005). Discretization methods for homogeneous fragmentations. *J. London Math. Soc.* **72**, 91-109.

[23] J. Bertoin and M. Yor (2002). The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. *Potential Analysis* **17** 389-400.

[24] D. Beysens, X. Campi and E. Peffekorn (editors) (1995). Proceedings of the workshop : Fragmentation phenomena. *Les Houches Series, World Scientific.*

[25] J. D. Biggins (1992). Uniform convergence of martingales in the branching random walk. *Ann. Probab.* **20**, No. 1, 131-151.

[26] M. D. Brennan and R. Durrett (1986). Splitting intervals. *Ann. Probab.* **14**, No. 3, 1024-1036.

[27] M. D. Brennan and R. Durrett (1987). Splitting intervals. II. Limit laws for lengths. *Probab. Theory Related Fields* **75** No. 1, 109-127.

[28] M.-E. Caballero and L. Chaumont (2006). Weak convergence of positive self-similar Markov processes and overshoots of Lévy processes. *Ann. Probab.* **34**, No. 3, 1012-1034.

[29] T. Chan and A. E. Kyprianou Smoothness of scale function for spectrally negative Lévy processes. *Preprint*.

[30] B. Chauvin (1986). Arbres et processus de Bellman-Harris. *Ann. Inst. Henri Poincaré.* **22**, No. 2, 209-232.

[31] B. Chauvin (1991). Product martingales and stopping lines for branching Brownian motion. *Ann. Probab.* **19**, No. 3, 1195-1205.

[32] J. Engländer, S. Harris and A. E. Kyprianou (2007). Strong law of large numbers for branching diffusions. *Preprint*.

[33] J. Engländer and A. E. Kyprianou (2004). Local extinction versus local exponential growth for spatial branching processes. *Ann. Probab.* **32**, no 1A, 78-99.

[34] K. Falconer (1997). *Fractal geometry.* John Wiley and Sons.

[35] A. F. Filippov (1961). On the distribution of the sizes of particles which undergo splitting. *Th. Probab. Appl.* **6**, 275-293.

[36] M. Ghorbel and T. Huillet (2006). On two fragmentation schemes with algebraic splitting probability. *Appl. Math. (Warsaw)* **33**, no. 1, 95-110.

[37] A. Gnedin and J. Pitman (2007). Poisson representation of a Ewens fragmentation process. *Combin. Probab. Comput.* **16**, no. 6, 819-827.

[38] B. Haas (2003). Loss of mass in deterministic and random fragmentations *Stoch. Proc. App.* **106**, 245-277.

[39] T. E. Harris (1948). Branching processes. *Ann. Math. Statist,* **41**, 474-494.

[40] T. E. Harris (1963). *The theory of branching processes.* Springer

[41] E. Hille and R. S. Phillips (1957). *Functional Analysis and Semi-groups.* Amer. Math. Soc. Colloquium Publications, vol. 31, Providence.

[42] M. Hoffmann and N. Krell (2008). Self-similar branching Markov chains *submitt*

[43] J. Jacod and A. N. Shiryaev (2003). *Limit theorems for stochastic processes.* Second edition. Grundlehren der Mathematischen Wissenschaften, 288. Springer-Verlag, Berlin.

[44] P. Jagers (1989). General branching processes as Markov fields. *Stochastic Process. Appl.* **32**, 183-212.

[45] J. F. C. Kingman (1993). *Poisson processes.* Oxford Studies in Probability, 3. Oxford Science Publications. The Clarendon Press, Oxford University Press.

[46] A. N. Kolmogorov (1941). Über das logaritmisch normale Verteinlungsgesetz der Dimensionen des Teilchen bei Zerstückelung. *C.R. (Doklady) Acad. Sci. USSR* **31**, 99–101.

[47] A. P. Korostelev and A.B. Tsybakov (1993). *Minimax theory of image reconstruction*. Lecture notes in Statistics, vol. 82, Springer, Berlin.

[48] P. L. Krapivsky and E. Ben-Naim (1994). Scaling and multiscaling in models of fragmentation. *Phys. Rev. E* **50**, 3502–3507.

[49] P. L. Krapivsky, E. Ben-Naim and I. Grosse (2004). Stable distributions in stochastic fragmentation. *J. Phys. A* **37**, 2863–2880.

[50] P. L. Krapivsky and S. N. Majumdar (2000). Traveling waves, front selection, and exact nontrivial exponents in random fragmentation problem. *Phys. Rev. Lett.* **85**, 5492-5495.

[51] N. Krell (2007).Self-similar branching Markov chains. *To appear in Séminaire de Probabilités*.

[52] N. Krell (2008). Multifractal spectra and precise rates of decay in homogeneous fragmentations. *Stochastic Process. Appl.* **118**, 897-916.

[53] A. E. Kyprianou (1999). A note on branching Lévy processes. *Stochastic Process. Appl.* **82** , no. 1, 1-14.

[54] A. E. Kyprianou (2000). Martingale convergence and the stopped branching random walk. *Probab. Theory Related Fields* **116** , no. 3, 405-419.

[55] A. E. Kyprianou (2006). *Introductory lectures on fluctuations of Lévy processes with applications*. Universitext. Springer-Verlag, Berlin.

[56] A. Lambert (2000). Completely asymmetric Lévy processes confined in a finite interval. *Ann. Inst. H. Poincaré Probab. Statist,* **36** , no. 2, 251-274.

[57] R. Lyons (1997). 'A simple path to Biggins' martingale convergence for branching random walk', *eds K.B Athreya and P. Jagers, Springer, New York,* 217-221.

[58] R. Lyons, R. Pemantle and Y. Peres (1995). Conceptual proofs of $L\log(L)$ citeria for mean behaviour of branching processes , *Ann. Probab,* **23**, 1125-1138.

[59] G. Miermont and J. Schweinsberg (2003). Self-similar fragmentations and stable subordinators. *Séminaire de Probabilités XXXVII, Lecture Notes in Math., 1832, Springer, Berlin,* 333-359.

[60] O. Nerman (1984). *The growth and composition of supercritical branching populations on general type spaces*. Technical report, Dept. Mathematics, Chalmers Univ. Technology and Goteborg Univ.

[61] J. Neveu (1986). Arbres et processus de Galton-Watson. *Ann. Inst. H. Poincaré Probab. Statist.* **22**, no. 2, 199-207.

[62] J. Neveu (1987). Multiplicative martingales for spatial branching processes, *Seminar on Stochastic Processes, Progr. Probab. Statist.*, **15**, 223-242.

[63] P. Ney (1981). A refinement of the coupling method in renewal theory. *Stoch. Processes Appl.*, **11**, 11–26.

[64] D. Revuz and M. Yor (1999). *Continuous martingales and Brownian motion,* 3nd edn. Springer, Berlin. **85**.

[65] M. S. Sgibnev (2002). Stone's decomposition of the renewal measure via Banach-algebraic techniques. *Proc. of the American Math. Soc.*, **130**, 2425–2430.

[66] C. Stone (1966). On absolutey continuous components and renewal theory. *Ann. Math. Statist.* **37**, 271–275.

[67] H. G. Tucker (1962). Absolute continuity of infinitely divisible distributions. *Pacific J. Math.* **12** 1125-1129.

Premier trimestre 2008

Quelques développements récents en théorie des fragmentations.

Résumé : Le sujet principal de cette thèse de doctorat est l'étude de diverses quantités reliées aux processus de fragmentation. Ces processus sont destinés à modéliser un objet de masse unité se fragmentant au cours du temps.

Ce travail comporte quatre chapitres. Le premier chapitre est consacré à l'étude de la dimension de Hausdorff de l'ensemble des points ayant une décroissance exponentielle dans une fragmentation homogène en intervalles. Dans le deuxième chapitre, on construit un processus de Markov auto-similaire qui généralise les fragmentations classiques autorisant en particulier la taille des descendants à être plus grande que celle de leurs parents. On établit ensuite certains théorèmes limites en utilisant la théorie des processus auto-similaires. Dans le troisième chapitre, on s'intéresse à un problème statistique provenant de l'industrie minière avec l'estimation statistique de la mesure de Lévy du subordinateur classiquement associé à la fragmentation. Plus précisément, on observe les fragments seulement à l'instant où ils atteignent une taille inférieure à un seuil fixé. Enfin, dans un quatrième chapitre on étudie le coût énergétique d'une succession de fragmentations.

Mots-clés : Processus de branchement, Chaîne de Markov à paramètre continue, L^p Théorème limite, Processus auto-similaire, Echangeabilité, Estimation, Processus de Markov: estimation.

Some recent developments in the theory of fragmentation

Abstract: The main subject of this PHD thesis is the study of various quantities related to fragmentation processes. These processes are designed to modelize a unit mass object which fragments with time.

This work is composed of four chapters. The aim of the first one is to study the Hausdorff dimension of the set of locations having exactly an exponential decay. In the second chapter, we construct a self-similar Markov process which generalizes the classical fragmentation by allowing in particular the size of the descendants to be bigger than the one of their parents. Then we show some Limit Theorems using the theory of self-similar Markov processes. In the third chapter, we are interested by the statistical estimation of the Lévy measure of the classical subordinator associated to the fragmentation. More precisely, we observe the fragments only when their size reach a size smaller than a given threshold. Finally, in the fourth chapter, we study the energy cost of a succession of fragmentations.

Keywords: Branching processes, Markov chains with continuous parameter, L^p-limit theorems, Self-similar processes, Exchangeability, Estimation, Markov processes: estimation

---

AMS Classification: Primary: 60J80. Secondary: 60G18, 60F25, 60J27 60G09,62G05, 62M05.