On singularities in direct and inverse scattering problems

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Introduction

Direct and Inverse Scattering Problems

Edge and Corner Singularities

Corner Singularities around a Polygonal Obstacle (2d)
Edge Singularities around a Polyhedral Obstacle (3d)
Combined Corner and Edge Singularities

High-Frequency Scattering

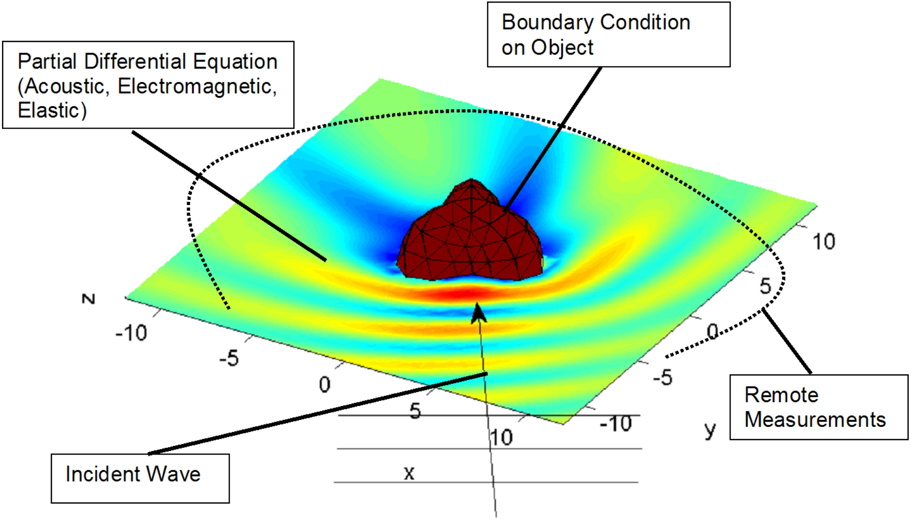
Effect of Singularities - Far Field Behaviour
Rigorous High Frequency Asymptotics for Whole Bounded Scatterers

Inverse Scattering

Field Reconstructions via Point Source Method
Shape reconstruction via the singular sources method
Reduce need of data via the no response test (NRT)

Introduction

Setting for reconstruction problem



Acoustic Scattering

Given some *smooth* incident field u^i and bounded obstacle D, find a scattered field u^s, governed by

- A Differential Equation, *for example* time-harmonic Helmholtz equation

$$\triangle u^s + \kappa^2 u^s = 0 \quad \text{in} \quad \Omega := \mathbb{R}^3 \setminus D$$

- A Boundary Condition, *for example* Dirichlet BC

$$u|_{\partial D} = 0$$

  for the total field

$$u = u^i + u^s$$

- A Radiation Condition, for example Sommerfeld RC (3d)

$$\left(\frac{\partial}{\partial r} - i\kappa r\right) u^s \to 0, \quad r \to \infty.$$

Electromagnetic Scattering

- Bounded scatterer D in three dimensions with piecewise smooth boundary, incident field E^i
- Scattered field E^s solves Maxwell equations

$$\mathrm{curl} E^s - i\kappa H^s = 0 \qquad \mathrm{curl} H^s + i\kappa E^s = 0 \tag{1}$$

in $\mathbb{R}^3 \setminus \overline{D}$ and satisfies the Silver-Müller radiation condition

$$E^s \times x + rH^s \to 0, \quad r = |x| \to \infty. \tag{2}$$

- On the boundary $\Gamma := \partial D$ the tangential component of the total field $E = E^i + E^s$ vanishes, i.e. we have the perfect conductor boundary condition

$$\nu \times E|_\Gamma = 0 \tag{3}$$

Solution Techniques

- There are several techniques to prove uniqueness for the forward problem.
- Existence can be shown by variational methods, integral equation methods or several further special techniques.
- For numerical calculations groups work on FEM, BEM, FDM, Spectral Methods, FIT ...
- The scattered field has the asymptotic behaviour

$$u^s(x) = \frac{e^{i\kappa|x|}}{|x|}\left\{u^\infty(\hat{x}) + O\left(\frac{1}{x}\right)\right\}, \quad \hat{x} := x/|x|$$

Measured data

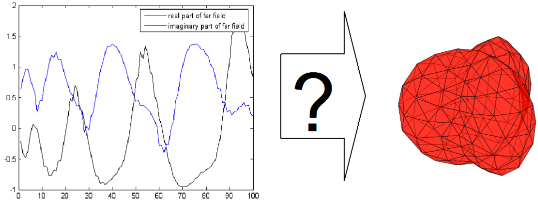
Measured data are either the scattered field E^s on some surface Λ or the far field pattern E^∞ defined by

$$E^s(x) = \frac{e^{i\kappa|x|}}{|x|}\left\{E^\infty(\hat{x}) + O\left(\frac{1}{x}\right)\right\}, \quad \hat{x} := x/|x| \qquad (4)$$

uniformly on $\mathbb{S}$ for $|x| \to \infty$.

For the above scattering problem the far field pattern is calculated via integral equations of the second kind.

The inverse problem



Task: *reconstruct the shape and properties of the unknown scatterer!*

Edge and Corner Singularities

Corner Singularities in 2d

Polygonal Obstacle D : Corners c, apertures ω'_c
Exterior Domain $\Omega = \mathbb{R}^2 \setminus D$: Corners c, apertures $\omega_c = 2\pi - \omega'_c$

- Typical Laplace-Dirichlet singularity :

$$d\, r^{k\pi/\omega} \sin\left(\frac{k\pi\theta}{\omega}\right), \quad d \in \mathbb{R}, \quad k \in \mathbb{N},$$

  with polar coordinates (r, θ) centered at the corner.

- Splitting in regular and singular parts

$$u = u_{\mathrm{reg}} + \sum_{c \,|\, \omega_c > \pi} d_c\, r_c^{\pi/\omega_c} \sin\left(\frac{\pi\theta_c}{\omega_c}\right) \chi(r_c)\,,$$

  - with regular part $u_{\mathrm{reg}} \in H^2(B_R \cap \Omega), \quad \forall R > 0$
  - coefficients d_c depending on the incident field.
  - and cut-off function χ.

References for corner asymptotics

V. A. Kondrat'ev.
Boundary-value problems for elliptic equations in domains with conical or angular points.
Trans. Moscow Math. Soc. **16** (1967) 227–313.

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Boundary Value Problems in Non-Smooth Domains.
Pitman, London 1985.

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Polygonal interface problems.
Methoden und Verfahren Math. Phys., **39** 1993.

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Elliptic boundary value problems in domains with point singularities.
Mathematical Surveys and Monographs **52** 1997.

Edge Singularities in 3d – closed surface

Obstacle $D \subset \mathbb{R}^3$ with edges : Edges e, apertures $e \ni z \mapsto \omega'_e(z)$

Domain $\Omega = \mathbb{R}^3 \setminus D$: Edges e, apertures $\omega_e(z) = 2\pi - \omega'_e(z)$

- Typical Laplace-Dirichlet edge singularity :

$$d(z)\, r^{k\pi/\omega} \sin\left(\frac{k\pi\theta}{\omega}\right), \quad z \mapsto d(z) \in \mathbb{R}, \quad k \in \mathbb{N},$$

  cylindrical coordinates (r, θ, z) with axis on the edge.

- Splitting in regular and singular parts

$$u = u_{\text{reg}} + \sum_{e \,|\, \omega_e > \pi} d_e(z_e)\, r_e^{\pi/\omega_e(z_e)} \sin\left(\frac{\pi\theta_e}{\omega_e(z_e)}\right) \chi(r_e),$$

  - with regular part $u_{\text{reg}} \in H^2(B_R \cap \Omega)$, $\forall R > 0$
  - smooth edge coefficients $e \ni z_e \mapsto d_e(z_e)$.
  - and cut-off function χ.

Edge Singularities in 3d – open surface

2d obstacle D with smooth boundary (screen surface).
Exterior Domain $\Omega = \mathbb{R}^3 \setminus D$.
Edge e of Ω = connected component of ∂D.

- As before, with $\omega_e \equiv 2\pi$.
- Splitting in regular and singular parts

$$u = u_{\text{reg}} + \sum_e d_e(z_e)\, r_e^{1/2} \sin\left(\frac{\theta_e}{2}\right) \chi(r_e),$$

- with regular part $u_{\text{reg}} \in H^2(B_R \cap \Omega), \quad \forall R > 0$
- smooth edge coefficients $e \ni z_e \mapsto d_e(z_e)$.
- and cut-off function χ.

References for edge asymptotics

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Asymptotics without logarithmic terms...
Comm. P. D. E. **28**, no 5 & 6 (2003) 869–926.

Corner Singularities in 3d

Obstacle D with conical points : Corners c, solid angles G'_c
Domain $\Omega = \mathbb{R}^3 \setminus D$: Corners c, solid angles $G_c = \mathbb{S}^2 \setminus G'_c$

- Typical corner singularity : $\rho^\lambda \Phi(\vartheta)$, with

  $$(\Phi, \lambda(\lambda+1)) \;\; \textit{Laplace-Beltrami eigenpair} \text{ on } H^1_0(G_c)$$

  and polar coordinates (ρ, ϑ) centered at the corner.
- Splitting in regular and singular parts

  $$(\star) \qquad u = u_{\text{reg}} + \sum_{c \,|\, \lambda_c \le \frac{1}{2}} d_c \, \rho_c^{\lambda_c} \Phi_c(\vartheta_c) \, \chi(\rho_c),$$

  - with regular part $u_{\text{reg}} \in H^2(B_R \cap \Omega), \; \forall R > 0,$ **provided**

    $$\Phi_c \in H^2(G_c), \quad \text{for all occurrence in } (\star)$$

  - coefficients d_c depending on the incident field.
  - and cut-off function χ.

Polyhedral Edge-Corner Singularities

Obstacle D = polyhedron : edges e, corners c.
Exterior Domain $\Omega = \mathbb{R}^3 \setminus D$: same edges and corners.
Splitting in regular and singular parts

$$u = u_{\mathrm{reg}} + \sum_{c \,|\, \lambda_c \leq \frac{1}{2}} u_c + \sum_{e \,|\, \omega_e > \pi} u_e$$

- with regular part $u_{\mathrm{reg}} \in H^2(B_R \cap \Omega)$, $\forall R > 0$,
- corner singularities $u_c = d_c\, \rho_c^{\lambda_c} \Phi_c(\vartheta_c) \chi(\rho_c)$ as in $(\star)$
- residual edge singularities

$$u_e = d_e(z_e)\, \rho_e^{\pi/\omega_e} \sin\left(\frac{\pi\theta_e}{\omega_e}\right) \chi(\rho_e) \quad \text{with}$$

  - $\rho_e = r_e/\delta_e$, with a *smooth distance function* δ_e to both ends of e
  - Edge coeff. d_e satisfies: $\forall \alpha \in \mathbb{N}$, $\delta_e^{\alpha-1} \partial_{z_e}^{\alpha} d_e \in L^2(e)$.

Comments on Edge & Corner Singularities

- We assume that *incident field u_i is smooth*, which makes the edge coefficients d_e smooth inside each edge e. We do not need any smoothing (or lifting) operator for edge coefficients.
- The results above are also valid for the exterior domain Ω of a polygonal screen D. The edge exponents are then $\frac{1}{2}$.
- We look for a *regular part in H^2*. Then
  - There is one edge singularity per edge, and at most one corner singularity per corner
  - The wave number κ has no influence at this level
  - The possible curvature of boundary has not much influence on the structure of singularities.

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Elliptic Boundary Value Problems in Corner Domains – Smoothness and Asymptotics of Solutions.
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High-Frequency Scattering

Effect of Singularities - Far Field Behaviour

The above theory of edge/corner singularities tells us what happens near the corner for Helmholtz/Maxwell, e.g. within one wavelength.

The edges/corners also influence the field strongly globally, especially at high frequency, by generating diffracted wave fields

The Geometrical Theory of Diffraction

see Keller and Lewis, 1995. It is a partly heuristic, semi-rigorous theory, whose principles are:

- At high frequency a ray model is appropriate
- The paths of rays are determined by Fermat's principle, i.e. rays take the quickest route
- Phase of the field on a ray is determined by distance along the ray, i.e. $u(x) = A(x)\mathrm{e}^{iks}$, s distance along ray, where A is not oscillatory or oscillates only slowly.
- Localization: interaction with obstacles depends only on the geometry local to the point where the ray hits the obstacle, and so can be determined by solving canonical scattering problems

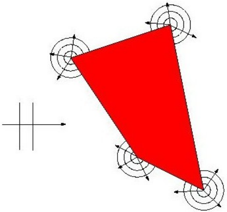
Example 1: Polygon.

If obstacle has corners then rays are reflected from sides but also diffracted from corners. Each diffracted ray (in 2D) has the form:

$$u^{\mathrm{diff}}(x) = u^i(x_c)D(\theta,\theta_0)\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{kr}},$$

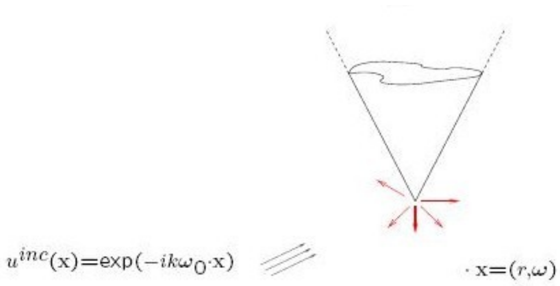
as $kr \to \infty$, where x_c is the corner, (r,θ) are polar coordinates of x relative to the corner (i.e. of $x - x_c$), θ_0 is the angle of incidence and $D(\theta,\theta_0)$ is a diffraction coefficient

which depends on the local geometry.



Example 2: Polyhedron.

The local problems are:
(i) reflection by an infinite plane (for reflection at a side);
(ii) reflection by an infinite 2D wedge (for diffraction at each edge);
(iii) conical diffraction problems (for diffraction at the corners).



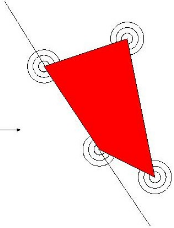
Example 2: The Conical Diffraction Problem.

The total field consists of waves specularly reflected from the surface plus a tip diffracted wave which behaves like

$$u^{\mathrm{diff}}(x) = \frac{e^{ikr}}{kr} f(\omega, \omega_0) + O((kr)^{-2}),$$

as $kr \to \infty$. Here (r, ω) are the spherical coordinates of x and ω_0 is the incident direction. The function $f(\omega, \omega_0)$ is computed by solving BVPs for the Laplace-Beltrami operator on that part of the surface of the unit sphere lying outside the cone. See Bonner, Graham Smyshlyaev, 2005.

Rigorous High Frequency Asymptotics for Whole Bounded Scatterers



Where s is distance along γ,

$$\frac{\partial u}{\partial \nu}(s) \quad = \quad 2\frac{\partial u^i}{\partial \nu}(s) + \mathrm{e}^{\mathrm{i}ks}v_+(s) + \mathrm{e}^{-\mathrm{i}ks}v_-(s)$$

where (rigorous h.f. asymptotics)

$$k^{-n}|v_+^{(n)}(s)| \leq C_n(ks)^{-1/2-n}, \quad |ks| \geq 1,$$

and (from the corner singularity theory),

$$k^{-n}|v_+^{(n)}(s)| \leq C_n(ks)^{-\alpha-n}, \quad 0 < ks \leq 1.$$

where $\alpha < 1/2$ depends on the exterior corner angle ω_c $(\alpha = 1 - \pi/\omega_c)$.

A Numerical Scheme for the Convex Polygon

... which uses this precise understanding of solution behaviour

$$\frac{\partial u}{\partial \nu}(s) \quad = \quad 2\frac{\partial u^i}{\partial \nu}(s) + \mathrm{e}^{\mathrm{i}ks} v_+(s) + \mathrm{e}^{-\mathrm{i}ks} v_-(s)$$

where

$$k^{-n} |v_+^{(n)}(s)| \leq C_n (ks)^{-1/2-n}, \quad |ks| \geq 1,$$

and

$$k^{-n} |v_+^{(n)}(s)| \leq C_n (ks)^{-\alpha-n}, \quad 0 < ks \leq 1.$$

where $\alpha < 1/2$ depends on the exterior corner angle.

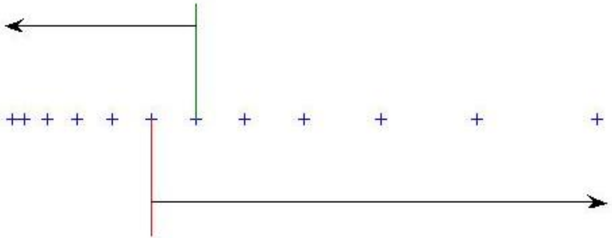
Thus approximate

$$\frac{\partial u}{\partial \nu}(s) \approx 2\frac{\partial u^i}{\partial \nu}(s) + \mathrm{e}^{\mathrm{i}ks}V_+(s) + \mathrm{e}^{-\mathrm{i}ks}V_-(s),$$

where V_+ and V_- are finite element functions,

using meshes optimally designed given this behaviour!

$$k^{-n}|v_+^{(n)}(s)| \leq \begin{cases} C_n(ks)^{-\alpha-n}, & 0 < ks < 1. \\ C_n(ks)^{-1/2-n}, & ks \geq 1, \end{cases}$$



$s = 0$ $\qquad t_m = \left(\frac{m}{N}\right)^q \lambda \qquad t_m = cr^m$

$$\frac{\partial u}{\partial \nu}(s) \approx 2\frac{\partial u^i}{\partial \nu}(s) + \mathrm{e}^{\mathrm{i}ks}V_+(s) + \mathrm{e}^{-\mathrm{i}ks}V_-(s),$$

where V_+ and V_- are pth order finite elements on graded meshes.

Theorem (C-W & Langdon, 2007)

Where $\phi = \frac{\partial u}{\partial \nu}$, ϕ_M is the best L_2 approximation to ϕ from the approximation space, J is the number of sides, M the degrees of freedom, p the polynomial degree, and L the total arc-length,

$$k^{-1/2}||\phi - \phi_M||_2 \leq C \sup_{x \in D} |u(x)| \frac{[J(1 + \log(kL/J))]^{p+3/2}}{M^{p+1}},$$

where C depends (only) on the corner angles and p.

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Inverse Scattering

Iterative Methods

1. Least Squares Method
2. Newton's Method, Local Newton
3. Gradient or Landweber Methods
4. Level Set Methods
5. Reciprocity Gap Methods
6. Evolutionary Method,
   Evolutionary Newton Method,
   Evolutionary Gradient Method
7. Successive Iteration Schemes (Kress-Rundell)
8. Hybrid Method

Decomposition Methods

1. Series Expansion Methods: Spherical Waves, Bessel functions
2. Fourier - Plane Wave Expansion Methods
3. Kirsch-Kress Method or Potential Method
4. Point Source Method: Greens Representation and Point Source Approximation

When the field is reconstructed, singularities can be used to find corners and reconstruct polygonal scatterers

Sampling and Probe Methods

1. Linear sampling method (Colton-Kirsch)
2. Factorization method (Kirsch)
3. Singular sources method (P.)
4. Probe method (Ikehata)
5. Enclosure method (Ikehata)
6. No response test (Luke/P.)
7. Range test (Kusiak/P./Sylvester)
8. Orthogonality Sampling (P./Nakamura)

Decomposition Methods: Point source method

1. Green's formula (using Dirichlet boundary condition)

$$u^s(x) = \int_\Gamma \Phi(x,y)\frac{\partial u}{\partial \nu}(y)ds(y) \tag{5}$$

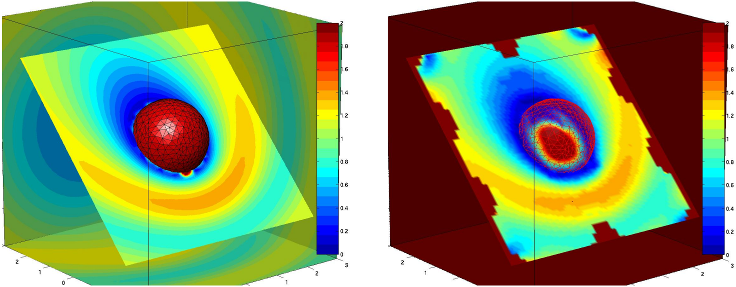
2. Approximation of the point source (on test domain)

$$\Phi(x,y) \approx \int_{\mathbb{S}} e^{i\kappa y\cdot d} g_x(d)ds(d) \tag{6}$$

3. Insert and exchange order of integration:

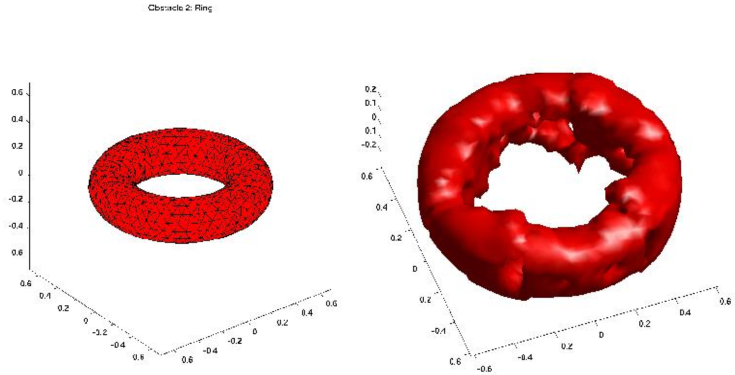
$$\begin{aligned} u^s(x) &\approx \int_\Gamma \left(\int_{\mathbb{S}} e^{i\kappa y\cdot d} g_x(d)ds(d)\right)\frac{\partial u}{\partial \nu}(y)ds(y) \\ &= \int_{\mathbb{S}} \left(\int_\Gamma e^{i\kappa y\cdot d}\frac{\partial u}{\partial \nu}(y)ds(y)\right) g_x(d)ds(d) \\ &= 4\pi \int_{\mathbb{S}} u^\infty(-d)\; g_x(d)ds(d). \end{aligned} \tag{7}$$

PSM: acoustics 3d, field reconstruction



PSM: Field reconstruction by Ben Hassen, Erhard and P. 2005

PSM: acoustics 3d, shape reconstruction



PSM: Shape reconstruction by Ben Hassen, Erhard and P. 2005

PSM: electromagnetics 3d, field reconstruction

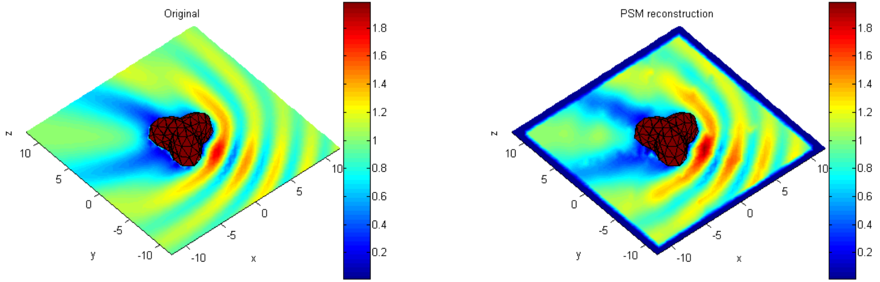
Point source method, electromagnetics
(Stratton-Chu formula and approximation of the point source.)

$$E^s(x) = \operatorname{curl} \int_\Gamma \nu(y) \times E^s(y)\Phi(x,y)\, ds(y) - \frac{1}{i\kappa}\operatorname{curl}\operatorname{curl} \int_\Gamma \nu(y) \times H^s(y)\Phi(x,y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \overline{D}, \quad (8)$$

Approximation of Point source by superposition, exchange of the order of integration $\Rightarrow$ Reconstruction formula

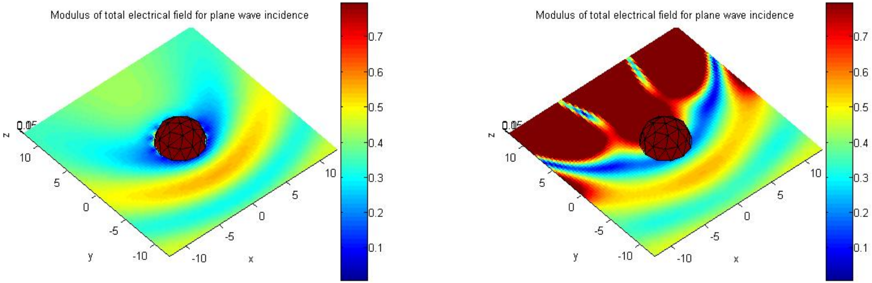
$$E^s(x) \approx 4\pi \int_{\mathbb{S}} E^\infty(-d) g_x(d)\, ds(d)\,, \quad x \in \mathbb{R}^3 \setminus \overline{D} \qquad (9)$$

PSM: electromagnetics 3d, field reconstruction



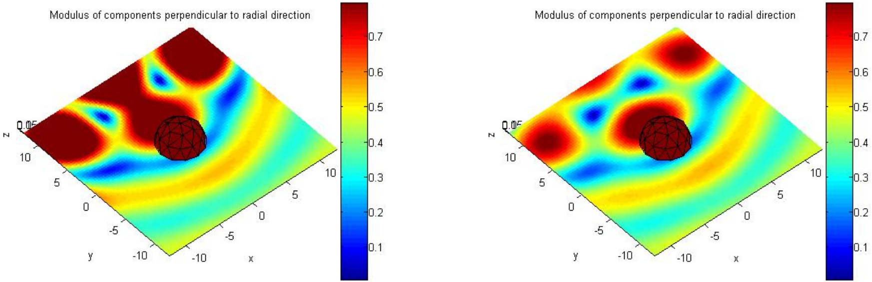
Original and reconstruction of the modulus of the total electromagnetic field.

PSM: electromagnetics 3d, field reconstruction



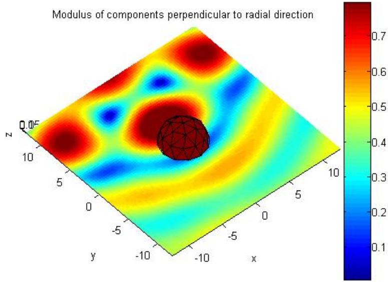
Reconstruction with data error of 1%.

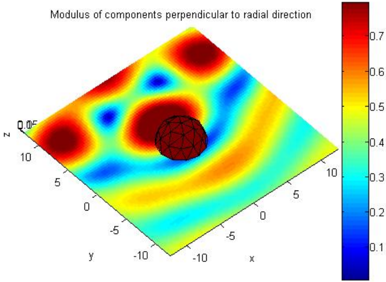
PSM: electromagnetics 3d, field reconstruction



Reconstruction with data error of 10% and 20%.

PSM: electromagnetics 3d, field reconstruction





Reconstruction with data error of 33% and 53%.

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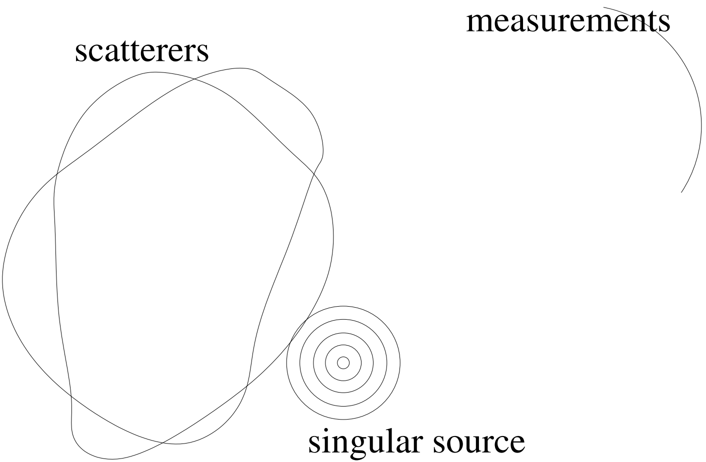
Singular sources method, idea



Singular sources method, idea

Basic property used for the singular sources method

$$|E^s(z,z)| \to \infty \text{ for } z \to \Gamma. \tag{10}$$

1. Use PSM to reconstruct $E^s(z,z)$ on a sampling grid from the knowledge of $E^\infty(\hat{x},d)$ for $\hat{x},d \in \mathbb{S}$.
2. Use the blow-up (10) to find the unknown shape as level curves of $|E^s(z,z)|$.

Singular sources method, derivation

Approximation of the point source (on test domain)

$$\Phi(x, y) \approx \int_{\mathbb{S}} e^{i\kappa x \cdot d} g_y(d) ds(d) \tag{11}$$

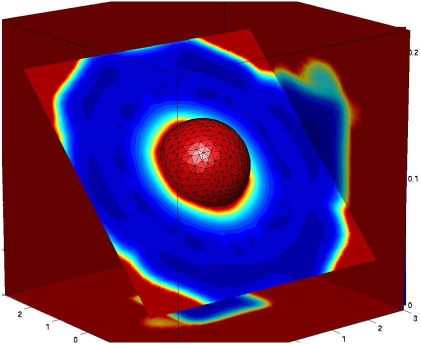
leads to the far field approximation

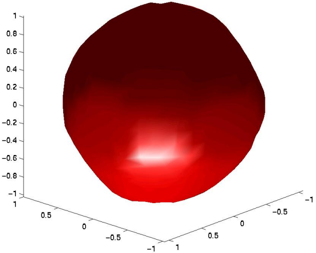
$$E^\infty(\hat{x}, y) \approx \int_{\mathbb{S}} E^\infty(\hat{x}, d) g_y(d) ds(d) \tag{12}$$

Using the point source method to reconstruct $E^s(\cdot, y)$ from $E^\infty(\cdot, y)$ leads to the formula

$$E^s(x, y) \approx 4\pi \int_{\mathbb{S}} \int_{\mathbb{S}} E^\infty(-\hat{x}, d) g_y(d) ds(d) g_x(\hat{x}) ds(\hat{x}) \tag{13}$$

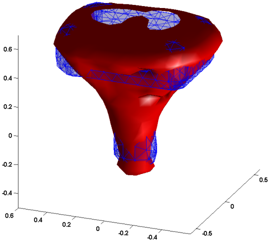
Singular sources method

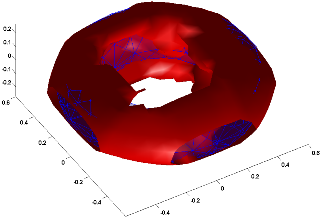




Neumann BC, Indicator function and Reconstruction.
Image by Erhard, Ben Hassen, P. 2006.

Singular sources method: T shape

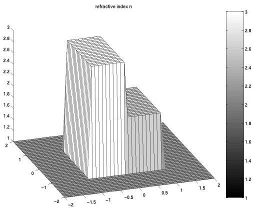


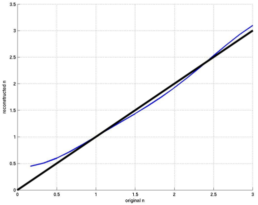


Dirichlet BC. Image by Erhard, Ben Hassen, P. 2006.

Note:

The singular sources method (SSM) can also be used to detect piecewise constant inhomogeneous media! The strength of the blow-up of the scattered field of multipoles is proportional to the jump in the refractive index.





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  Using fundamental solutions in inverse scattering.
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  Point-sources and Multipoles in Inverse Scattering.
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PSM and SSM revisited

- PSM uses measurements for *one* wave, but needs to know the boundary condition for shape reconstruction (only impenetrable scatterers)
- SSM uses measurements for many incident waves, does not need to know the boundary condition

Is there a one-wave method which does not need to know the boundary condition?

The idea of the no response test

Setting. Consider scattering of time-harmonic acoustic waves by some sound-soft obstacle (Dirichlet boundary condition).

Preparation step I: The far field pattern can be expressed as:

$$u^{\infty}(\hat{x}) = \frac{1}{4\pi} \int_{\Gamma} e^{-i\kappa \hat{x}\cdot y} \frac{\partial u}{\partial \nu}(y) ds(y) \qquad (14)$$

for $\hat{x} \in \mathbb{S}$.

The idea of the no response test

Preparation step II:
Consider a superposition of plane waves (a Herglotz wave function)

$$v[g](y) := \int_{\mathbb{S}} e^{-i\kappa y\cdot\hat{x}} g(\hat{x}) ds(\hat{x}), \quad y \in \mathbb{R}^m \tag{15}$$

with density $g \in L^2(\mathbb{S})$.

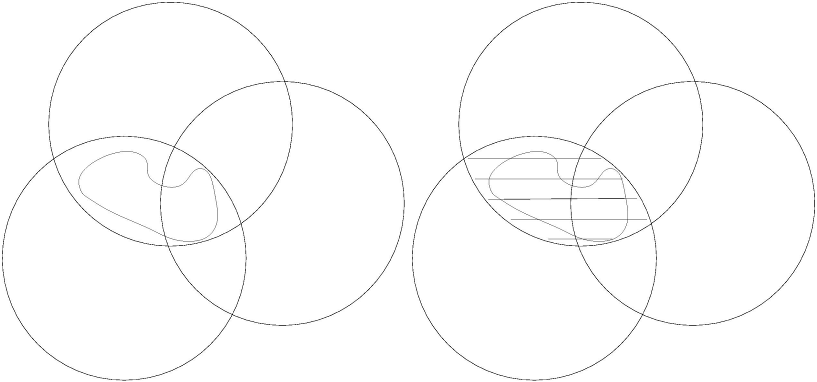
Choosing appropriate densities g we can make $v[g]$ arbitrarily small on some given test domain $\overline{G}$ where on compact subsets of $\mathbb{R}^m \setminus \overline{G}$ the function $v[g]$ becomes arbitrarily large.

The idea of the no response test

Basic derivation step:
Multiply u^∞ by $g \in L^2(\mathbb{S})$, integrate over $\mathbb{S}$ and exchange the order of integration to obtain

$$\begin{aligned}
\mu(g) &:= \int_{\mathbb{S}} g(\hat{x}) u^\infty(\hat{x}) ds(\hat{x}) \\
&= \int_{\mathbb{S}} g(\hat{x}) \left(4\pi \int_\Gamma e^{-i\kappa \hat{x}\cdot y} \frac{\partial u}{\partial \nu}(y) ds(y) \right) ds(\hat{x}) \\
&= 4\pi \int_\Gamma \left(\int_{\mathbb{S}} e^{-i\kappa \hat{x}\cdot y} g(\hat{x}) ds(\hat{x}) \right) \frac{\partial u}{\partial \nu}(y) ds(y) \\
&= 4\pi \underbrace{\int_\Gamma v[g](y) \frac{\partial u}{\partial \nu}(y) ds(y)}_{response}.
\end{aligned}$$

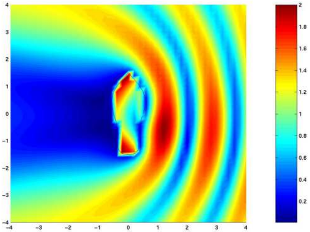
NRT sampling

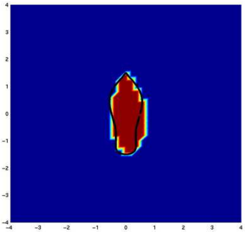


Sampling for the no response test

Reconstruction example from inverse scattering

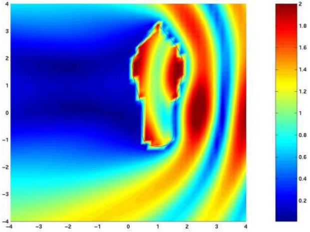
Boat-like sound-soft obstacle:

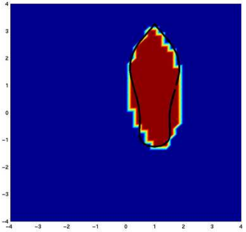




Numerical proof of concept

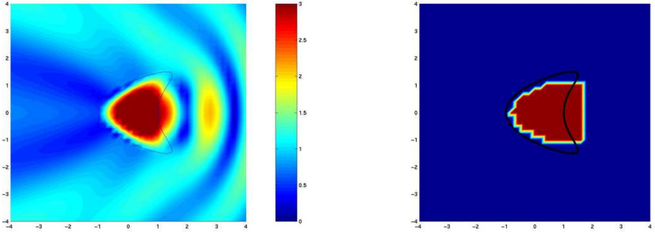
Boat-like sound-soft obstacle2:





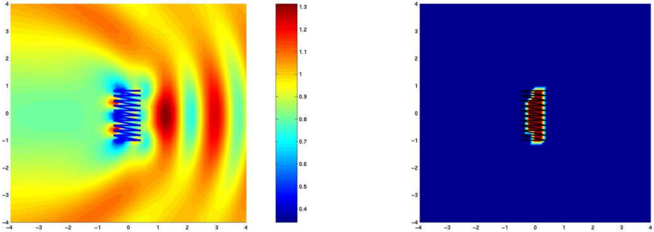
Numerical proof of concept

Boat-like sound-hard obstacle:

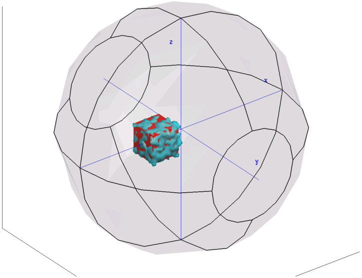


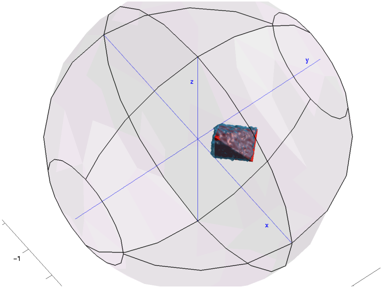
Numerical proof of concept

Rectangular penetrable obstacle:



Magnetic tomography





Images by L. Kühn, Dissertation Thesis Göttingen 2005

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