“Simple” Corner-Edge Asymptotics

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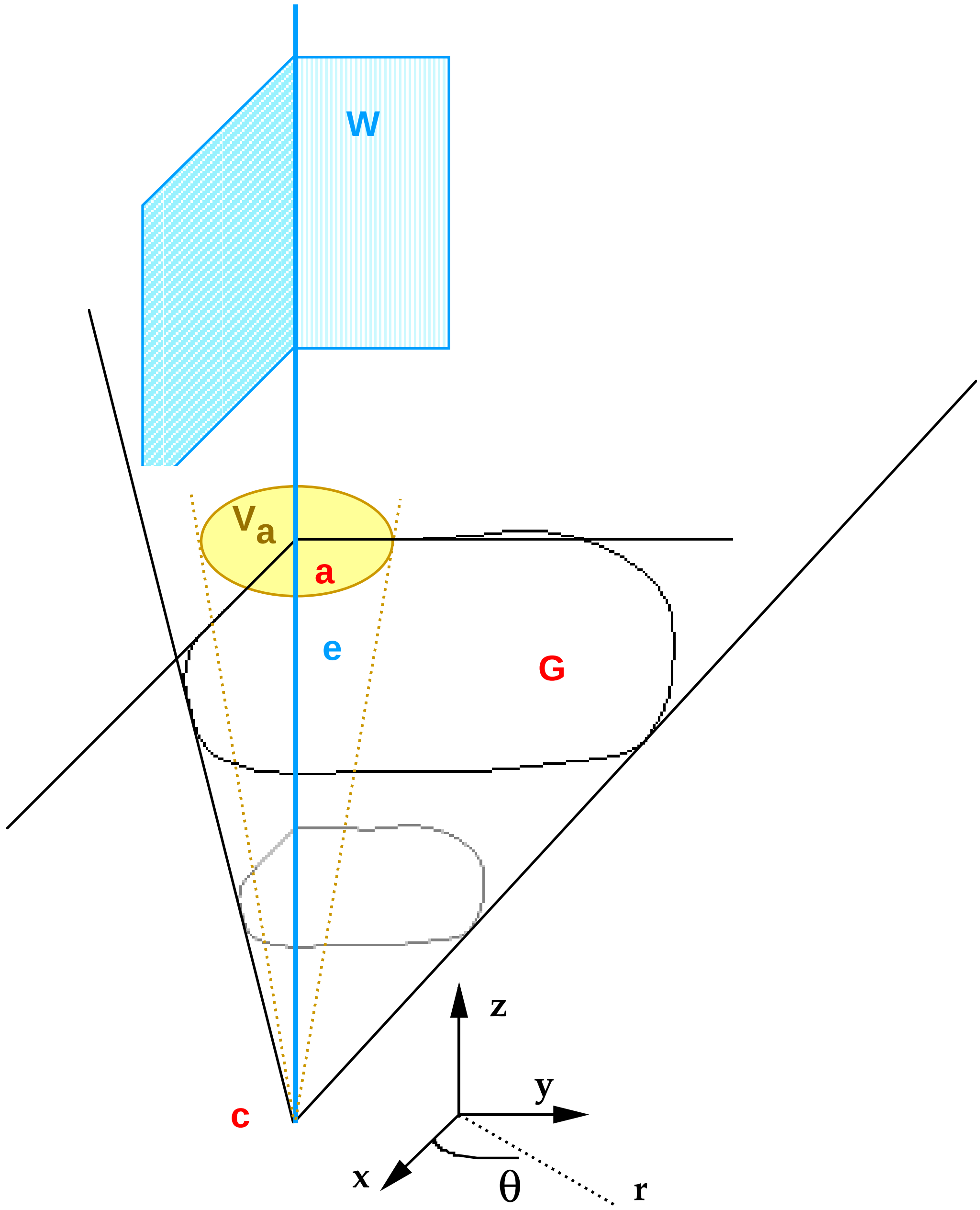
Abstract. *I try to describe the Corner-Edge Asymptotics in a polyhedral cone in the simplest way possible. You may argue that this is still involved, but nature is as it is... The French version of my title could be, with reference to a well known series of computer books and with certain humor, “Les asymptotiques coins-arêtes pour les Nuls”.*

Contents

1. GEOMETRY, COORDINATES



Let $\mathbf{C}$ be a cone in $\mathbb{R}^3$ with corner $\mathbf{c}$. Let $G \subset \mathbb{S}^2$ be the section of $\mathbf{C}$, i.e. the intersection of $\mathbf{C}$ with the unit sphere $\mathbb{S}^2$ centered in $\mathbf{c}$. We assume that the boundary of G is smooth, except in one point, $\mathbf{a}$, that we will call "the angular point" of G. This means that the cone $\mathbf{C}$ has one edge $\mathbf{e}$ which is the ray originating in $\mathbf{c}$ and going through $\mathbf{a}$.

We assume that in a conical neighborhood of $\mathbf{e}$, the cone $\mathbf{C}$ coincides with a wedge* $\mathbf{W}$, which means that there exists a neighborhood $\mathbf{V_a}$ of $\mathbf{a}$ in $\mathbb{S}^2$ such that in the cone $\mathbf{V_e}$ defined as the union of the rays originating in $\mathbf{c}$ and going through any point of $\mathbf{V_a}$, there holds

$$\mathbf{C} \cap \mathbf{V_e} = \mathbf{W} \cap \mathbf{V_e}.$$

Let ω be the opening of the wedge $\mathbf{W}$.

We choose *Cartesian coordinates* $\mathbf{x} = (x, y, z)$ such that:

- the point $(0,0,0)$ is $\mathbf{c}$,
- the half-line $\{(0,0,z),\ z > 0\}$ is the edge $\mathbf{e}$,
- the half-plane $\{(x,0,z),\ x > 0,\ z \in \mathbb{R}\}$ is a side of the wedge $\mathbf{W}$.

The associated *cylindrical coordinates* are (r, θ, z) where $r = \sqrt{x^2+y^2}$ and the angle θ is chosen such that $\theta = 0$ is the equation of the half-line $\{(x,0),\ x > 0\}$ in the (x,y) plane. The wedge $\mathbf{W}$ is described in cylindrical coordinates as

$$\mathbf{W} = \big\{(x,y,z) \mid \ \ r > 0,\ \theta \in (0,\omega),\ z \in \mathbb{R}\big\}.$$

The associated *spherical coordinates* are (ρ, ϑ) where $\rho = \sqrt{x^2+y^2+z^2}$. Here ϑ denotes any coordinate system in $\mathbb{S}^2$. In the neighborhood $\mathbf{V_e}$ of $\mathbf{e}$, we may define the homogeneous (with respect to ρ) coordinates X, Y and R by

$$X = \frac{x}{\rho}, \quad Y = \frac{y}{\rho} \quad \text{and} \quad R = \frac{r}{\rho} = \sqrt{X^2+Y^2}.$$

We note that in the neighborhood $\mathbf{V_a}$ of $\mathbf{a}$, the couple (X, Y) can be taken as coordinate system for ϑ.

The cone $\mathbf{C}$ is described in spherical coordinates as

$$\mathbf{C} = \big\{(x,y,z) \mid \ \ \rho > 0,\ \vartheta \in G\big\}.$$

Finally, let $\kappa > 0$ be such that the cone defined as $\{(x,y,z),\ r < \kappa\rho\}$ is contained in the conical neighborhood $\mathbf{V_e}$.

We end this series of definitions by the notation $\mathrm{B}(\mathbf{c}, T)$ for the ball centered in $\mathbf{c}$ with radius T.

*A wedge, or dihedron, is a three-dimensional domain which coincides with the product of an infinite plane sector by $\mathbb{R}$.

2. THE OPERATOR AND ITS REDUCED FORMS AT EDGE & CORNER

Although any strongly elliptic system L in AGMON-DOUGLIS-NIRENBERG sense with constant coefficients could be considered, we treat for simplicity the case when L is a system of operators of degree 2 without lower order terms. We write

$$L = L(\partial_x, \partial_y, \partial_z)\,.$$

The edge singularities along e will be generated by the non-tangential part of L, i.e. the operator obtained from L by removing the derivatives ∂_z which are tangential along the edge. Thus we define

$$L_{\mathrm{e}} := L(\partial_x, \partial_y, 0)\,.$$

We also need to write L_{e} in polar coordinates (r,θ). There exists an operator $\mathscr{L}_{\mathrm{e}}$ of degree 2 with coefficients depending smoothly on θ only such that

$$L_{\mathrm{e}}(\partial_x, \partial_y) = r^{-2}\mathscr{L}_{\mathrm{e}}(\theta; r\partial_r, \partial_\theta)\,.$$

The associated symbol is $\mathbb{C} \ni \alpha \longmapsto \mathscr{L}_{\mathrm{e}}(\theta; \alpha, \partial_\theta)$, see §3.

The corner singularities are related to the full operator L that we write in spherical coordinates as

$$L(\partial_x, \partial_y, \partial_z) = \rho^{-2}\mathscr{L}_{\mathrm{c}}(\vartheta; \rho\partial_\rho, \partial_\vartheta)\,,$$

with a second order operator $\mathscr{L}_{\mathrm{c}}$ with smooth coefficients depending on $\vartheta \in G$. We have advantage to write $\mathscr{L}_{\mathrm{c}}$ in the special fitted coordinates (ρ, X, Y) in the conical neighborhood V_{e}, which yields:

$$L(\partial_x, \partial_y, \partial_z) = \rho^{-2}\mathscr{L}_{\mathrm{c}}(X, Y; \rho\partial_\rho, \partial_X, \partial_Y)\,.$$

The associated symbol is $\mathbb{C} \ni \lambda \longmapsto \mathscr{L}_{\mathrm{c}}(X, Y; \lambda, \partial_X, \partial_Y)$, see §3.

For any complex number λ, the operator $\mathscr{L}_{\mathrm{c}}(X, Y; \lambda, \partial_X, \partial_Y)$ acts on G, which has the angular point a. The operator M_{a} which will determine its singularities at a is defined as

$$M_{\mathrm{a}}(\partial_X, \partial_Y) = \text{principal part } \big\{\mathscr{L}_{\mathrm{c}}(0,0;0,\partial_X,\partial_Y)\big\}.$$

There holds

Lemma 2.1 *The homogeneous operators of degree 2 with constant coefficients M_{a} and L_{e} coincide with each other.*

The lemma results from the formula

$$(\partial_X, \partial_Y, \rho\partial_\rho) = \rho(\partial_x, \partial_y, \partial_z) + \mathscr{O}(r)\,, \quad r \longrightarrow 0,$$

where $\mathscr{O}(r)$ means operators of first order in ∂_x, ∂_y, ∂_z with coefficients $a_j(\mathrm{x})$ such that $r^{-1+|\beta|}\partial_{\mathrm{x}}^{\beta} a_j(\mathrm{x})$ is bounded for any j and β.

3. Mellin symbols at edge and corner

Let us consider the Dirichlet boundary conditions for L.

The associated Mellin symbol $\mathfrak{M}_{\mathrm{e}}$ at the edge $\mathbf{e}$ is the operator valued function $\mathbb{C} \ni \alpha \longmapsto \mathfrak{M}_{\mathrm{e}}(\alpha)$ where $\mathfrak{M}_{\mathrm{e}}(\alpha)$ is defined as

$$
\begin{array}{rcl}
\mathfrak{M}_{\mathrm{e}}(\alpha)\ :\ \mathrm{H}_0^1(0,\omega) & \longrightarrow & \mathrm{H}^{-1}(0,\omega) \\
\varphi & \longmapsto & \mathscr{L}_{\mathrm{e}}(\theta;\alpha,\partial_\theta)\varphi\,.
\end{array}
\tag{3.1}
$$

The associated Mellin symbol $\mathfrak{M}_{\mathrm{c}}$ at the corner $\mathbf{c}$ is the operator valued function $\mathbb{C} \ni \lambda \longmapsto \mathfrak{M}_{\mathrm{c}}(\lambda)$ where $\mathfrak{M}_{\mathrm{c}}(\lambda)$ is defined as

$$
\begin{array}{rcl}
\mathfrak{M}_{\mathrm{c}}(\lambda)\ :\ \mathrm{H}_0^1(G) & \longrightarrow & \mathrm{H}^{-1}(G) \\
\psi & \longmapsto & \mathscr{L}_{\mathrm{c}}(X,Y;\lambda,\partial_X,\partial_Y)\psi\,.
\end{array}
\tag{3.2}
$$

The operator valued functions $\alpha \mapsto \mathfrak{M}_{\mathrm{e}}(\alpha)^{-1}$ and $\lambda \mapsto \mathfrak{M}_{\mathrm{c}}(\lambda)^{-1}$ are meromorphic in $\mathbb{C}$ with finite dimensional polar part. Let $\mathfrak{A}$ and $\mathfrak{L}$ be respectively the sets of their poles. As a consequence of the ellipticity of L, any strip $\mathrm{Re}\,\alpha \in [\xi_0,\xi_1]$ contains at most a finite number of elements of $\mathfrak{A}$ and any strip $\mathrm{Re}\,\lambda \in [\eta_0,\eta_1]$ contains at most a finite number of elements of $\mathfrak{L}$.

Hypothesis 3.1 *We assume for simplicity that the poles of $\alpha \mapsto \mathfrak{M}_{\mathrm{e}}(\alpha)^{-1}$ and $\lambda \mapsto \mathfrak{M}_{\mathrm{c}}(\lambda)^{-1}$ are simple.*

The index of the operators $\mathfrak{M}_{\mathrm{e}}(\alpha)$ and $\mathfrak{M}_{\mathrm{c}}(\lambda)$ is independent of α and λ, thus is 0. Therefore $\mathfrak{M}_{\mathrm{e}}(\alpha)$ and $\mathfrak{M}_{\mathrm{c}}(\lambda)$ are invertible if and only if their kernels are $\{0\}$.

Definition 3.2 For $\alpha \in \mathfrak{A}$, we call multiplicity of α the dimension of $\ker \mathfrak{M}_{\mathrm{e}}(\alpha)$ and we make the convention that *we repeat α in $\mathfrak{A}$ according to its multiplicity*, which allows for defining bases of $\ker \mathfrak{M}_{\mathrm{e}}(\alpha)$ by simply indexing them by $\alpha \in \mathfrak{A}$, and similarly for $\lambda \in \mathfrak{L}$. We denote these bases

$$
\varphi[\alpha],\ \ \alpha \in \mathfrak{A} \quad \text{and} \quad \psi[\lambda],\ \ \lambda \in \mathfrak{L}\,.
$$

■

4. CORNER ASYMPTOTICS

We are interested in the structure of any u satisfying

$$\forall T>0,\ u\in \mathrm{H}^1(\mathrm{C}\cap \mathrm{B}(0,T)),\quad u\big|_{\partial\mathrm{C}}=0,\quad Lu=f\ \text{ with }\ f\in\mathscr{C}^\infty(\overline{\mathrm{C}})\,. \tag{4.1}$$

Any solution u of (4.1) is $\mathscr{C}^\infty(\mathrm{C})$, i.e. $\mathscr{C}^\infty$ *inside* C, and we are interested in its structure as $\mathbf{x}\longrightarrow\mathbf{c}$, and in particular in the neighborhood of the edge $\mathbf{e}$.

Hypothesis 4.1 *We assume for simplicity that* $f\equiv 0$ *in* $\mathrm{B}(\mathbf{c},1)$.

Under Hypotheses 3.1 and 4.1, there exist coefficients $c_\lambda\in\mathbb{C}$ for any $\lambda\in\mathfrak{L}$ with $\operatorname{Re}\lambda>-\frac12$ such that $u\sim\sum_{-\frac12<\operatorname{Re}\lambda}c_\lambda\,\rho^\lambda\psi[\lambda](\vartheta)$ in the sense of asymptotic expansions as $\rho\longrightarrow 0$, which means that for any $\eta>-\frac12$ there holds

$$\boxed{u\ =\ \sum_{-\frac12<\operatorname{Re}\lambda<\eta} c_\lambda\,\rho^\lambda\psi[\lambda](\vartheta)\ +\ u^\eta_{\mathrm{rem\,,c}}} \tag{4.2}$$

Here the remainder $u^\eta_{\mathrm{rem\,,c}}=u^\eta_{\mathrm{rem\,,c}}(\rho,\vartheta)$ satisfies[†] for all ϑ fixed in G

$$\sup_{\rho\in(0,1]}\rho^{-\eta}\big|u^\eta_{\mathrm{rem\,,c}}(\rho,\vartheta)\big|<\infty\,. \tag{4.3}$$

This means that, roughly, $u^\eta_{\mathrm{rem\,,c}}(\rho,\vartheta)$ is a $\mathscr{O}(\rho^\eta)$ as $\rho\longrightarrow 0$, but be careful about the fact that the $\sup$ in (4.3) depends on $\vartheta\in G$ and may blow up as $\vartheta\longrightarrow\mathbf{a}$.

Thus, expansion (4.2) does not yield any extra regularity along the edge $\mathbf{e}$.

Remark 4.2 *Expansion* (4.2) *has the simplest possible expression due to the homogeneity of the operator* L. *If* L *has lower order terms, the asymptotics contains supplementary "shadow" terms and has the more complicated form:*

$$u\ =\ \sum_{\substack{\operatorname{Re}\lambda>-1/2\\ \operatorname{Re}\lambda+n<\eta}}\ \sum_{n\in\mathbb{N}} c_\lambda\,\rho^{\lambda+n}\psi_n[\lambda](\vartheta)\ +\ u^\eta_{\mathrm{rem\,,c}}. \tag{4.4}$$

Here $\mathbb{N}$ *is the French notation for the set of non-negative integers* $\{0,1,2,\ldots\}$ *and* $\psi_0[\lambda]=\psi[\lambda]$.

[†](4.3) can be differentiated with respect to ρ, i.e. for any integer $k\in\mathbb{N}$, we have

$$\sup_{\rho\in(0,1]}\rho^{-\eta+k}\big|\partial^k_\rho u^\eta_{\mathrm{rem\,,c}}(\rho,\vartheta)\big|<\infty\,.$$

5. Edge asymptotics

First note that we may take ρ as the variable along the edge $\mathbf{e}$, since z coincides with ρ on $\mathbf{e}$. We make the further hypothesis

Hypothesis 5.1 *We assume for simplicity that for all $\alpha \in \mathfrak{A}$ with positive real part and all positive integer $n > 0$, the complex number $\alpha + n$ does not belong to $\mathfrak{A}$.*

Under Hypotheses 3.1, 4.1 and 5.1, there exist functions defined on $\mathbf{e}$, $\rho \mapsto d_\alpha(\rho)$ for any $\alpha \in \mathfrak{A}$ with $\operatorname{Re}\alpha > 0$ such that for any $\xi > 0$ there holds in the conical neighborhood $\mathbf{V}_\mathbf{e}$:

$$\boxed{u \;=\; \sum_{\substack{\operatorname{Re}\alpha>0\\ \operatorname{Re}\alpha+n<\xi}} \; \sum_{n\in\mathbb{N}} \partial_\rho^n d_\alpha(\rho)\, r^{\alpha+n}\varphi_n[\alpha](\theta) \;+\; u^\xi_{\mathrm{rem}\,,\,\mathbf{e}}} \tag{5.1}$$

Here the angular functions $\varphi_0[\alpha]$ are the original $\varphi[\alpha]$ and for $n \geq 1$, the $\varphi_n[\alpha]$ are constructed from the $\varphi[\alpha]$ by recurrence. On the other hand, the remainder $u^\xi_{\mathrm{rem}\,,\,\mathbf{e}}$ satisfies for all z fixed in $(0,1]$ ‡

$$\sup_{\substack{r\in(0,\kappa\rho]\\ \theta\in[0,\omega]}} r^{-\xi}\big|u^\xi_{\mathrm{rem}\,,\,\mathbf{e}}(x,y,z)\big| < \infty\,. \tag{5.2}$$

The positive constant κ is that introduced in §1.

Thus, roughly, $u^\xi_{\mathrm{rem}\,,\,\mathbf{e}}(x,y,z)$ is a $\mathscr{O}\left(r^\xi\right)$ depending on $z \in \mathbf{e}$ as $r \longrightarrow 0$, but the above $\sup$ is a function of z which may blow up as $z \longrightarrow 0$, i.e. as $\mathbf{x} \longrightarrow \mathbf{c}$.

The coefficients d_α are the edge coefficients, or *Stress Intensity Factors*, but the expansion (5.1) does not yield any extra regularity at the corner $\mathbf{c}$.

Remark 5.2 *The structure of the asymptotics* (5.1) *has the same level of complexity as the generalized expansion* (4.4)*, because the operator L is not homogeneous with respect to the only variables (x,y). The “shadow” terms $\varphi_n[\alpha]$ come from the terms containing derivatives ∂_z in L.*

‡(5.2) can be differentiated with respect to (x,y), i.e. for any multi-index $\beta \in \mathbb{N}^2$, we have

$$\sup_{\substack{r\in(0,\kappa\rho]\\ \theta\in[0,\omega]}} r^{-\xi+|\beta|}\big|\partial^\beta_{x,y} u^\xi_{\mathrm{rem}\,,\,\mathbf{e}}(x,y,z)\big| < \infty\,.$$

6. Expansion of the corner spherical singular functions

The corner spherical singular functions $\psi[\lambda]$ are solutions of the Dirichlet problem in G

$$\psi \in \mathrm{H}^1_0(G), \quad \mathscr{L}_{\mathrm{c}}(X,Y;\lambda,\partial_X,\partial_Y)\psi = 0,$$

thus associated with a smooth (!) right hand side. Therefore these functions $\psi[\lambda]$ can be expanded at the angular point a in G: there exist coefficients $\gamma_\alpha[\lambda]$ for any $\alpha \in \mathfrak{A}$ with $\mathrm{Re}\,\alpha > 0$ such that for any $\xi > 0$ there holds

$$\boxed{\psi[\lambda](X,Y) \quad = \quad \sum_{\substack{\mathrm{Re}\,\alpha>0 \\ \mathrm{Re}\,\alpha+n<\xi}} \sum_{n\in\mathbb{N}} \gamma_\alpha[\lambda]\; R^{\alpha+n}\widetilde{\varphi}_n[\alpha](\theta) \quad + \quad \psi^\xi_{\mathrm{rem}\,,\,\mathrm{a}}[\lambda]} \tag{6.1}$$

Here the angular functions $\widetilde{\varphi}_0[\alpha]$ coincide with the original $\varphi[\alpha]$, *cf* Lemma 2.1, and for $n \geq 1$, the $\widetilde{\varphi}_n[\alpha]$ are constructed from the $\varphi[\alpha]$ by recurrence. On the other hand, the remainder $\psi^\xi_{\mathrm{rem}\,,\,\mathrm{a}}[\lambda]$ is such that[§]

$$\sup_{(X,Y)\,\in\,\mathrm{V_a}} R^{-\xi}\big|\psi^\xi_{\mathrm{rem}\,,\,\mathrm{a}}[\lambda]\big| < \infty\,. \tag{6.2}$$

Multiplying (6.1) by ρ^λ and using that $r = \rho R$, we obtain

$$\rho^\lambda\psi[\lambda] \quad = \quad \sum_{\substack{\mathrm{Re}\,\alpha>0 \\ \mathrm{Re}\,\alpha+n<\xi}} \sum_{n\in\mathbb{N}} \rho^{\lambda-\alpha-n}\gamma_\alpha[\lambda]\; r^{\alpha+n}\widetilde{\varphi}_n[\alpha](\theta) \quad + \; \rho^\lambda\psi^\xi_{\mathrm{rem}\,,\,\mathrm{a}}[\lambda]\,. \tag{6.3}$$

But, as a consequence of the equation $\mathfrak{M}_{\mathrm{c}}(\lambda)\psi[\lambda] = 0$, the function $v[\lambda](\mathrm{x}) := \rho^\lambda\psi[\lambda]$ satisfies (4.1) with $f = 0$. Therefore, we have also the edge expansion (5.1) for $v[\lambda]$, i.e.

$$\rho^\lambda\psi[\lambda] \quad = \quad \sum_{\substack{\mathrm{Re}\,\alpha>0 \\ \mathrm{Re}\,\alpha+n<\xi}} \sum_{n\in\mathbb{N}} \partial^n_\rho\delta_\alpha[\lambda](\rho)\; r^{\alpha+n}\varphi_n[\alpha](\theta) \quad + \; v^\xi_{\mathrm{rem}\,,\,\mathrm{e}}[\lambda]\,, \tag{6.4}$$

for some edge coefficients $\delta_\alpha[\lambda]$. For each fixed ρ, the remainders $\rho^\lambda\psi^\xi_{\mathrm{rem}\,,\,\mathrm{a}}[\lambda]$ and $v^\xi_{\mathrm{rem}\,,\,\mathrm{e}}[\lambda]$ are $\mathscr{O}\,(r^\xi)$. Therefore we may identify the asymptotics (6.3) and (6.4) and obtain that

$$\delta_\alpha[\lambda](\rho) = \rho^{\lambda-\alpha}\gamma_\alpha[\lambda]\,. \tag{6.5}$$

[§](6.2) can be differentiated with respect to (X,Y), i.e. for any multi-index $\beta \in \mathbb{N}^2$, we have

$$\sup_{(X,Y)\,\in\,\mathrm{V_a}} R^{-\xi+|\beta|}\big|\partial^\beta_{X,Y}\psi^\xi_{\mathrm{rem}\,,\,\mathrm{a}}[\lambda]\big| < \infty\,.$$

7. Edge expansion of the corner remainder

In the previous section we have obtained the edge expansion of the singular terms $\rho^\lambda\psi[\lambda]$ in the corner expansion (4.2). We may also expand along the edge e the remainder $u^\eta_{\mathrm{rem}\,,\,\mathrm{c}}$. Let $\eta > -\frac{1}{2}$ be fixed.

Under Hypotheses 3.1, 4.1 and 5.1, there exist functions defined on e, $\rho \mapsto d^{(\eta)}_\alpha(\rho)$ for any $\alpha \in \mathfrak{A}$ with $\mathrm{Re}\,\alpha > 0$ such that for any $\xi > 0$ there holds in the conical neighborhood V_e:

$$\boxed{u^\eta_{\mathrm{rem}\,,\,\mathrm{c}} = \sum_{\substack{\mathrm{Re}\,\alpha>0\\ \mathrm{Re}\,\alpha+n<\xi}} \sum_{n\in\mathbb{N}} \partial^n_\rho d^{(\eta)}_\alpha(\rho)\, r^{\alpha+n}\varphi_n[\alpha](\theta) + u^{\eta,\xi}_{\mathrm{rem}\,,\,\mathrm{c}\,,\,\mathrm{e}}} \tag{7.1}$$

The coefficient $d^{(\eta)}_\alpha$ satisfies¶

$$\sup_{\rho\,\in\,(0,1]} \rho^{-\eta+\mathrm{Re}\,\alpha}|d^{(\eta)}_\alpha(\rho)| < \infty\,. \tag{7.2}$$

The remainder $u^{\eta,\xi}_{\mathrm{rem}\,,\,\mathrm{c}\,,\,\mathrm{e}}$ satisfies‖

$$\sup_{\mathrm{x}\,\in\,\mathrm{B(c,1)}} \rho^{-\eta+\xi}r^{-\xi}\big|u^{\eta,\xi}_{\mathrm{rem}\,,\,\mathrm{c}\,,\,\mathrm{e}}(\mathrm{x})\big| < \infty\,. \tag{7.3}$$

Note that the above asymptotics is like expansion (5.1). The supplementary information is that the expansion of a term which is flat at the corner involves terms (edge coefficients and remainder) which are also flat at the corner.

¶(7.2) can be differentiated, i.e. for any integer $k \in \mathbb{N}$, we have

$$\sup_{\rho\,\in\,(0,1]} \rho^{-\eta+\mathrm{Re}\,\alpha+k}\big|\partial^k_\rho d^{(\eta)}_\alpha(\rho)\big| < \infty\,.$$

‖(7.3) can be differentiated in any direction, i.e. for any multi-index $\beta \in \mathbb{N}^3$, we have

$$\sup_{\mathrm{x}\,\in\,\mathrm{B(c,1)}} \rho^{-\eta+\xi}r^{-\xi+|\beta|}\big|\partial^\beta_\mathrm{x} u^{\eta,\xi}_{\mathrm{rem}\,,\,\mathrm{c}\,,\,\mathrm{e}}(\mathrm{x})\big| < \infty\,.$$

8. Corner expansion of the edge coefficients

By (6.3), (6.4) and (6.5) we obtain the edge expansion of the corner singularities in the form

$$\rho^{\lambda}\psi[\lambda] \quad = \sum_{\substack{\operatorname{Re}\alpha>0 \\ \operatorname{Re}\alpha+n<\xi}} \sum_{n\in\mathbb{N}} \gamma_{\alpha}[\lambda]\, \partial_{\rho}^{n}(\rho^{\lambda-\alpha})\; r^{\alpha+n}\varphi_{n}[\alpha](\theta) \quad + \; \rho^{\lambda}\psi^{\xi}_{\mathrm{rem\,,e}}[\lambda]\,. \tag{8.1}$$

Inputting edge expansions (8.1) and (7.1) into the corner expansion (4.2) and identifying with the edge expansion (5.1) we obtain that for any $\alpha \in \mathfrak{A}$ with $\operatorname{Re}\alpha > 0$ the edge coefficient d_{α} expands as $\rho \longrightarrow 0$, for any $\eta > -\frac{1}{2}$:

$$\boxed{d_{\alpha}(\rho) \quad = \sum_{-\frac{1}{2}<\operatorname{Re}\lambda<\eta} c_{\lambda}\, \gamma_{\alpha}[\lambda]\, \rho^{\lambda-\alpha} \quad + \; d_{\alpha}^{(\eta)}(\rho)} \tag{8.2}$$

with $d_{\alpha}^{(\eta)}(\rho) = \mathscr{O}\left(\rho^{\eta-\operatorname{Re}\alpha}\right)$.

9. Corner-edge and edge-corner expansions of solution

Neither the corner expansion (4.2) nor the edge expansion (5.1) have a remainder which is smooth at the corner and the edge at the same time. The "good" remainder is that of (7.1). Combining (4.2) and (7.1) we obtain the corner-edge asymptotics for any $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$:

$$\boxed{\boldsymbol{u} = \sum_{-\frac{1}{2} < \mathrm{Re}\,\boldsymbol{\lambda} < \eta} c_\lambda\, \boldsymbol{\rho}^{\boldsymbol{\lambda}} \boldsymbol{\psi}[\boldsymbol{\lambda}] + \sum_{\substack{\mathrm{Re}\,\boldsymbol{\alpha} > 0 \\ \mathrm{Re}\,\boldsymbol{\alpha}+\boldsymbol{n} < \xi}} \sum_{\boldsymbol{n} \in \mathbb{N}} \partial_\rho^n d_\alpha^{(\eta)}\, r^{\alpha+n} \varphi_n[\alpha] + u^{\eta,\xi}_{\mathrm{rem},\mathrm{c},\mathrm{e}}} \tag{9.1}$$

The edge-corner expansion is obtained by a re-combination of terms in (9.1) using (8.1) and (8.2): for any $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$,

$$\boxed{\boldsymbol{u} = \sum_{\substack{\mathrm{Re}\,\boldsymbol{\alpha} > 0 \\ \mathrm{Re}\,\boldsymbol{\alpha}+\boldsymbol{n} < \xi}} \sum_{\boldsymbol{n} \in \mathbb{N}} \partial_\rho^n d_\alpha\, r^{\alpha+n} \varphi_n[\alpha] + \sum_{-\frac{1}{2} < \mathrm{Re}\,\boldsymbol{\lambda} < \eta} c_\lambda\, \boldsymbol{\rho}^{\boldsymbol{\lambda}} \boldsymbol{\psi}^{\xi}_{\mathrm{rem},\mathrm{e}}[\boldsymbol{\lambda}] + u^{\eta,\xi}_{\mathrm{rem},\mathrm{c},\mathrm{e}}} \tag{9.2}$$

Remark 9.1 *Note that expansion* (9.1) *is "canonical" from the point of view of the Mellin symbols* $\mathfrak{M}_{\mathrm{c}}$ *and* $\mathfrak{M}_{\mathrm{e}}$*, since the original bases* $\psi[\boldsymbol{\lambda}]$ *and* $\varphi[\boldsymbol{\alpha}]$ *are present there. On the contrary, only the remainder terms of the bases* $\psi[\boldsymbol{\lambda}]$ *are present in expansion* (9.2)*. But expansion* (9.2) *has the full edge coefficients, and not only their remainders as in expansion* (9.1)*.*

10. WHAT IS THE STRONGEST SINGULARITY?

This question is somewhat illusive concerning expansions (9.1) and (9.2), since both corner and edge singularity have to be present in order to obtain a regular remainder. But concerning the expansion (8.2) of the edge coefficients, it makes sense to know if $\mathrm{Re}\,\lambda - \mathrm{Re}\,\alpha$ is ≥ 0 or not.

We first remark that the weaker is the edge singularity, the larger is $\mathrm{Re}\,\alpha$ and the smaller is $\mathrm{Re}\,\lambda - \mathrm{Re}\,\alpha$, which is an expression of the loss of regularity of the edge coefficient as $\mathrm{Re}\,\alpha$ increases (in finite regularity theories).

But, in general, one is interested by the edge coefficients associated with the lowest value of $\mathrm{Re}\,\alpha$. We will give examples in crack theory, where the edge coefficients are of particular importance.

For $\gamma \in (0, 2\pi)$ let the cone C_γ be defined as $\mathbb{R}^3 \setminus \Gamma_\gamma$ where Γ_γ is the plane sector of opening γ contained in the plane $y = 0$ and with one of its sides equal to the line $\{(0,0,z),\ z > 0\}$. Let G_γ be the section of C_γ on the sphere $\mathbb{S}^2$. The cone C_γ has two (similar) edges, and the coefficients along each of its edges have a similar asymptotics (8.2).

The edge exponents α are the half-integers (general result from [4]), therefore the first one is $\frac{1}{2}$. Let $\lambda_1(\gamma)$ be the first corner exponent, that is $\lambda \in \mathfrak{L}$ such that $\mathrm{Re}\,\lambda > -\frac{1}{2}$ with least real part.

Case when $L = \Delta$.

Then $\lambda_1(\gamma) = -\frac{1}{2} + \sqrt{\mu(\gamma) + \frac{1}{4}}$ where $\mu(\gamma)$ is the first eigenvalue of the Laplace-Beltrami operator on G_γ with Dirichlet conditions. By the monotonicity of Dirichlet eigenvalues, $\gamma \mapsto \mu(\gamma)$ is an increasing function, therefore $\gamma \mapsto \lambda_1(\gamma)$ is also increasing.

On the other hand, we know the values of $\lambda_1(\gamma)$ for particular values of the opening γ. For $\gamma = \pi$, C_π is the wedge of opening $\omega = 2\pi$, and $\lambda_1(\pi)$ coincides with the least edge exponent, i.e. $\lambda_1(\pi) = \frac{1}{2}$. We remark that, in this case the asymptotics (8.2) is degenerated (it is but a Taylor expansion) since the "corner" is now artificial.

As $\gamma \longrightarrow 0$, the domain G_γ tends to the whole sphere $\mathbb{S}^2$ and $\lambda_1(\gamma)$ tends to 0. As $\gamma \longrightarrow 2\pi$, the domain G_γ tends to the union of two disjoint half-spheres and $\lambda_1(\gamma)$ tends to 1.

As a conclusion,

$$\boxed{\lambda_1(\gamma) - \tfrac{1}{2} \text{ is } < 0 \text{ if } 0 < \gamma < \pi \text{ and is } > 0 \text{ if } \pi < \gamma < 2\pi\,.} \tag{10.1}$$

Case when L is the elasticity operator.

Then, we still have (for similar reasons) that $\lambda_1(\pi) = \frac{1}{2}$, that $\lambda_1(\gamma)$ tends to 0 as $\gamma \longrightarrow 0$ and that $\lambda_1(\gamma)$ tends to 1 as $\gamma \longrightarrow 2\pi$ (the same still holds for λ_2 and λ_3). The monotonicity is known for isotropic materials in the region $-\frac{1}{2} \leq \lambda \leq \frac{1}{2}$.

11. NEUMANN BOUNDARY CONDITIONS

The definitions and results of sections 3.-9. can be extended to any boundary condition which covers the operator L. The Neumann condition is of particular interest in fracture mechanics and has special features.

When Neumann conditions are imposed, the corresponding edge and corner spectra $\mathfrak{A}$ and $\mathfrak{L}$ always contain $\alpha = 0$ and $\lambda = 0$ respectively corresponding to constant $\varphi[\alpha]$ and $\psi[\lambda]$. The associated "singularities" $r^\alpha\varphi[\alpha]$ and $\rho^\lambda\psi[\lambda]$ are then constant too, therefore are regular.

Therefore, we may exclude these $\varphi[\alpha]$ and $\psi[\lambda]$ from the basis of singular functions. Then anything in expansions of §3.-9. goes in the same way *except* concerning the conditions on the different remainders. The Neumann remainders are the sum of a part which satisfies exactly the same "flatness" conditions than in (4.3), (5.2), (6.2) and (7.3), and a smooth part.

We may also discuss, like for Dirichlet conditions, what is the smallest between the first "useful" α and λ. Let us consider the same example of the family of cracked cones C_γ for $\gamma \in (0, 2\pi)$ and $L = \Delta$.

The first edge exponent is still $\frac{1}{2}$. As for the first corner exponents, $\lambda_1(\gamma)$, it is still given by the formula

$$\lambda_1(\gamma) = -\tfrac{1}{2} + \sqrt{\mu(\gamma) + \tfrac{1}{4}}$$

where $\mu(\gamma)$ is now the first *non-zero* eigenvalue of the Laplace-Beltrami operator on G_γ with Neumann conditions.

Thanks to the special structure of the family $\gamma \mapsto G_\gamma$ (the variation of the domains concern zero-measure sets), we still have a monotonicity property for Neumann eigenvalues, but in the converse direction than for Dirichlet: $\gamma \mapsto \mu(\gamma)$ is a decreasing function, therefore $\gamma \mapsto \lambda_1(\gamma)$ is also decreasing.

For the Neumann case, we also have that $\lambda_1(\pi) = \frac{1}{2}$.

As $\gamma \longrightarrow 0$, the domain G_γ tends to the whole sphere $\mathbb{S}^2$ and, as we have to remove the first (zero) Neumann eigenvalue, $\lambda_1(\gamma)$ tends to 1. As $\gamma \longrightarrow 2\pi$, the domain G_γ tends to the union of two disjoint half-spheres. The bottom of the limit spectrum is double zero, corresponding to two (possibly different) constants on each half-sphere. Only one of them has to be removed, corresponding to equal constants. Therefore $\lambda_1(\gamma)$ tends to 0.

The situation is then inverted with respect to Dirichlet conditions:

$$\boxed{\lambda_1(\gamma) - \tfrac{1}{2} \text{ is } < 0 \text{ if } \pi < \gamma < 2\pi \text{ and is } > 0 \text{ if } 0 < \gamma < \pi.} \tag{11.1}$$

12. Without simplifications

1. If we do not assume hypotheses 3.1 and 5.1 any more, then we have to add a level of summation in all asymptotics for logarithmic terms $r^\alpha \log^j r$ and $\rho^\lambda \log^k \rho$.
2. If we assume only that the right hand side f is $\mathscr{C}^\infty(\overline{\mathrm{C}})$ but without being zero near the corner $\mathbf{c}$, then there are minor changes concerning the edge asymptotics (extra $r^\alpha \log^j r$ terms appear for $\alpha \in \mathbb{N} \cap \mathfrak{A}$). As for the corner asymptotics, in general it is necessary to consider $\mathbb{N} \cup \mathfrak{L}$ instead $\mathfrak{L}$, corresponding to homogeneous singular solution $\rho^{|\beta|+2}\psi_\beta$ to problem (4.1) with right hand side x^β for any $\beta \in \mathbb{N}^3$. And the remainders will be the sum of a flat part satisfying (4.3), (5.2), (6.2) and (7.3), and of a smooth part.
3. If f has a finite Sobolev regularity H^s, then the expansions are finite, the edge coefficients d_α have finite Sobolev regularity depending on α and s, and in order to have remainders with the correct regularity H^{s+2}, it is necessary to use special regularization of edge coefficients which act as lifting of traces from the edge to the cone.
4. If L has lower order terms but still constant coefficients, the structure of edge asymptotics requires a new level of summation since the order of derivation n of the edge coefficient d_α is only $\leq$ than the exponent shift (in the power of r) and no more equal. As for corner asymptotics, we have to add "shadow" terms $\rho^{\lambda+n}\psi_n[\lambda]$.
5. If L has a multi-order, the powers of r and ρ are multi-powers.
6. If L has variable coefficients, the corner expansion (4.2) still holds with the introduction of shadow terms, but the edge exponents will depend in general on ρ and they may cross each other or have branching points, which cause changes in the structure in edge asymptotics., see [2, 1, 3].

Endo-references for corner-edge asymptotics [6, Ch.16-17], exposed anew in [5, 7].

See also [9] and [8].

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