Selfsimilar perturbation near a corner: matching and multiscale expansions

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Abstract

In this paper we consider the Laplace-Dirichlet equation in a polygonal domain perturbed at the scale ε near one of its vertices. We assume that this perturbation is self-similar, that is, derives from the same pattern for all values of ε. We construct and validate asymptotic expansions of the solution in powers of ε via two different techniques, namely the method of matched asymptotic expansions and the method of multiscale expansions. Then we show how the terms of each expansion can be split into a finite number of sub-terms in order to reconstruct the other expansion. Compared with the fairly general approach of [13] relying on multiscale expansions, the novelty of our paper is the rigorous validation of the method of matched asymptotic expansions, and the comparison of its result with that of the multiscale method. The consideration of a model problem allows to simplify the exposition of these rather complicated two techniques.

Introduction

The solutions of elliptic problems in corner domains are known to be singular near the vertices, see [10, 7, 5]. In engineering applications, however, it often happens that the corners are rounded in a small region near the vertices (fillet), so that the resulting domain is at the same time smooth and close to a corner domain ω. On the one hand, the solution is in principle smooth, but keeps information of singularities which would be present in the limit corner domain ω. On the other hand, from the numerical approximation point of view, meshing the corner domain ω may be easier than meshing small rounded regions. For these reasons, the question of comparing solutions in a rounded corner domain and a true corner domain, is interesting and important for applications.

The investigation of small perturbations of a domain can be done by an asymptotic analysis, considering families of *self-similar* domains $\left(\omega_\varepsilon\right)_{\varepsilon\geq 0}$ indexed by the small parameter ε which represents the order of magnitude of the perturbed region. Namely, the domains ω_ε coincide with a fixed pattern Ω shrunk at the scale ε in regions of size ε and coincide with ω_0 outside. Thus, in a certain sense, ω_ε tends to ω_0 as $\varepsilon \to 0$.

Such a self-similar family is a typical example of *singular perturbation* of domains, and may include numerous different situations besides rounded corners: Let us mention

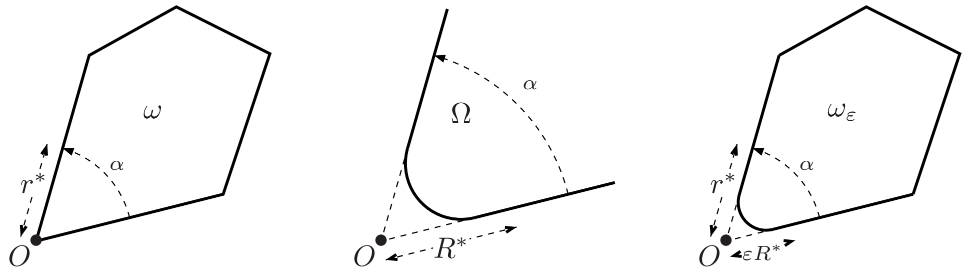

Figure 1: Rounded corner: Domains ω, Ω and ω_ε.

small cracks or small holes, originating from an interior or a boundary point of the limiting domain, see Fig.3 and 5, and also small junctions between several connected components of ω, see Fig.6 and 7 p.26.

The relation between a fixed perturbed domain $\tilde{\omega}$ and its limit domain ω can be formalized by embedding $\tilde{\omega}$ and ω in such a family so that the following two conditions are satisfied

$$\textit{(i)}\ \ \omega = \omega_0 \quad \text{and} \quad \textit{(ii)}\ \exists \varepsilon_0 > 0 \ \text{ so that } \ \tilde{\omega} = \omega_{\varepsilon_0}.$$

In this paper, as a model situation, we mainly investigate the solutions u_ε of the Dirichlet problem for the Laplace operator set on a family of plane self-similar domains ω_ε. For each fixed ε, the regularity properties of u_ε can be very different from those of their limit u_0 (more or less regular, depending on different configurations, see Fig.1 and 2, respectively). An asymptotic expansion of u_ε as ε tends to 0 is the right way of understanding the mechanism of this transformation.

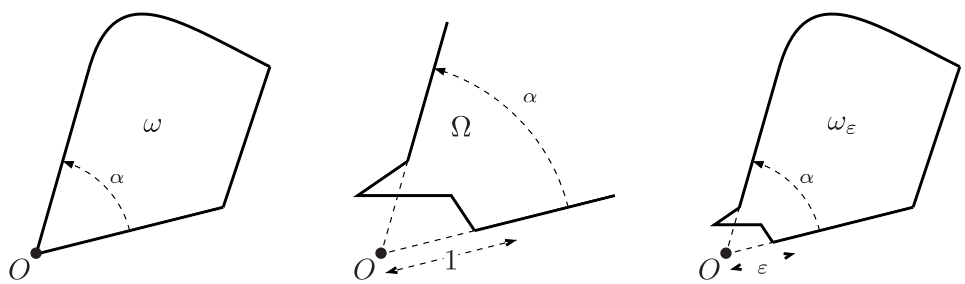

Figure 2: Non-convex corner pattern: Domains ω, Ω and ω_ε.

For such singular perturbation problems, the method of *matched asymptotic expansions* is widely used. This method, spread by [18], consists in constructing two different complete expansions of the solution in different regions and different scalings, and to match them in an intermediate region. It has been used in [11] for situation of Figure 1 (see also [8] for a general framework). Although intuitive, this method is difficult to justify rigorously, see [17, 9] for such a justification for a problem of thin slits. An alternative is given by the *multi-scale expansion* technique, consisting of a superposition of terms via cut-off functions, which involve different scales. A rigorous error analysis can be performed for such a method, which has been used in a very general framework in the monograph [13] for similar problems.

Our aim in this work is twofold

(i) Provide the complete constructions and validations of the two different expansions provided by the two methods of *multi-scale expansion* and *matched asymptotic expansions* for the same simple example, so that everything is made explicit and as clear as possible,

(ii) Compare the two expansions with each other, that is split each term of each expansion into sub-terms, and re-assemble them to reconstruct the terms of the other expansion.

Our paper is organized as follows. In Section 1, we define the families of self-similar domains and the problems under consideration, and next we provide an outline of our results, giving the structure of the first terms of both expansions. In Section 2 we state some preliminary results on limit problems in non-standard spaces, which we call *"super-variational problems"*. In Section 3, we present the method of *matched asymptotic expansions*, with the construction of the terms and matching conditions, and, by the technique of [17, 9], the validation of the expansion by remainder estimates. Section 4 is devoted to the *multiscale approach*, like in [3, 19, 2], where optimal remainder estimates are proved. Sections 3 and 4 may be read independently. We compare the expansions obtained by these two techniques in Section 5, providing formulas for the translation of the terms of each expansion into the terms of the other one. In Section 6 we mention how our results can be generalized to other situations (more general domains, data, operators, etc...). We conclude in Section 7 with comments on more practical applications of our results.

1 Notations, outline of results

1.1 Self-similar perturbations

The families $(\omega_\varepsilon)_{\varepsilon>0}$ under consideration are defined with the help of two domains, ω the *limit* (or unperturbed) domain, and Ω the *pattern* (or profile) of the perturbation. We denote by x and X the Cartesian coordinates in ω and Ω, respectively. Indeed ω and Ω do not "live" in the same world. They only share the origin of coordinates: both systems of coordinates have their origin at O. The x coordinates are the *slow variables* and $X = \frac{x}{\varepsilon}$ are the *fast variables*.

The junction set. The connection between ω and Ω is realized by a plane sector K with vertex O (which is a dilation invariant set). For K, the situation of a half-plane is included. We denote by $\mathcal{B}_R$ and B_R the ball centered at O with radius R in the x and X coordinates, respectively.

The limit domain. Let ω be a bounded domain of $\mathbb{R}^2$, containing the origin O in its boundary $\partial\omega$. We assume that ω is sector-shaped near O (e.g. ω is a polygon with one of its vertices at O): there exists $r^* > 0$ such that

$$\omega \cap \mathcal{B}_{r^*} = K \cap \mathcal{B}_{r^*}.$$

Let α be the opening of K ($0 < \alpha \leqslant 2\pi$), and let $(r, \theta) \in \mathbb{R}^+ \times [0, 2\pi)$ be the polar coordinates such that the sector $(0, +\infty)_r \times (0, \alpha)_\theta$ coincides with K.

The perturbing pattern. Let Ω be an unbounded domain of $\mathbb{R}^2$ such that there exists $R^* > 0$ for which

$$\Omega \cap \complement_{\mathbb{R}^2} B_{R^*} = K \cap \complement_{\mathbb{R}^2} B_{R^*}.$$

Let $(R, \theta) \in \mathbb{R}^+ \times [0, 2\pi)$ be the polar coordinates centered in O such that the sector $(0, +\infty)_R \times (0, \alpha)_\theta$ coincides with K.

The perturbed domains. Let ε_0 such that $\varepsilon_0 R^* = r^*$. For any $\varepsilon < \varepsilon_0$, ω_ε denotes the bounded domain

$$\omega_\varepsilon \;=\; \{x \in \omega \;;\; |x| > \varepsilon R^*\} \cup \{x \in \varepsilon\Omega \;;\; |x| < r^*\}\,. \tag{1.1}$$

The domain ω_ε coincides with the limit domain ω except in an ε-neighborhood of the origin, where its shape is given by the ε-dilation of the domain Ω, see Figures 1-5. In the *intermediate region* $\varepsilon R^* \leqslant |x| \leqslant r^*$, ω_ε coincides with K

$$\omega_\varepsilon \cap \{\varepsilon R^* \leqslant |x| \leqslant r^*\} = K \cap \{\varepsilon R^* \leqslant |x| \leqslant r^*\}. \tag{1.2}$$

Note that Ω is the limit as $\varepsilon \to 0$ of the domain $\omega_\varepsilon/\varepsilon$, whereas ω is the limit of ω_ε.

No regularity assumption is needed for the following work, except the coincidence with the sector K in the matching regions. We only assume the domains ω and Ω to be open subsets of $\mathbb{R}^2$ (bounded and unbounded, respectively).

We have shown in Fig.1 the example of the rounded corner where ω has a corner at O, whereas ω_ε is smooth near O, and in Fig.2 an example where the pattern domain Ω is more singular than the limiting domain ω.

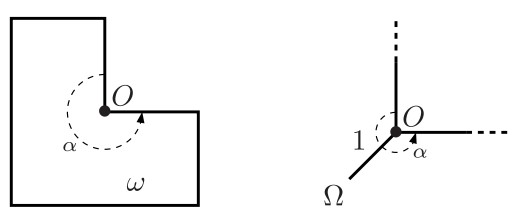

Figure 3: Small crack originating from a non-convex corner.

In Fig.3, ω still has a corner at O while Ω is the infinite sector of same opening, containing a crack at the origin. Thus a small crack of length ε appears in ω_ε.

In Fig.4 and 5 we have the apparition of a hole. At a regular part of the boundary ($\alpha = \pi$), with a corner in Ω in Fig.4, and at a crack tip ($\alpha = 2\pi$) in Fig.5.

Remark 1.1 The family of domains ω_ε is decreasing as ε increases if and only if each ray $\theta = \theta_0$ in Ω is connected and unbounded (this implies in particular that Ω is contained in the sector K). Note that we *do not need* this assumption.

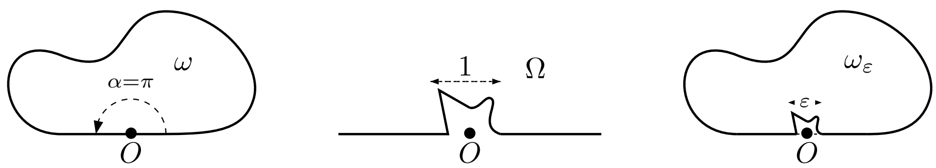

Figure 4: Small hole originating from a smooth boundary point.

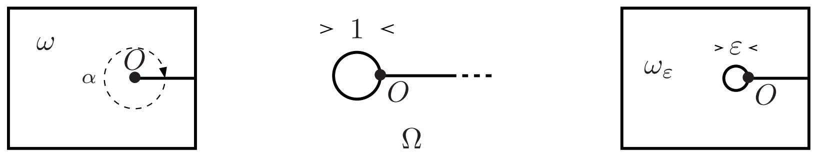

Figure 5: Small bulge at a crack tip.

1.2 The Dirichlet problem and its singularities

As the simplest, and nevertheless typical, example of elliptic boundary value problem on a family (ω_ε) of self-similar domains, we consider the *Laplace-Dirichlet problem*. We are interested in asymptotic expansions with respect to ε of the solution u_ε of the problem

$$\textit{Find } u_\varepsilon \in \mathrm{H}^1_0(\omega_\varepsilon) \textit{ such that } -\Delta u_\varepsilon = f|_{\omega_\varepsilon} \textit{ in } \omega_\varepsilon. \tag{1.3}$$

Here f is a fixed function belonging to $\mathrm{L}^2(\mathbb{R}^2)$. We assume for simplicity that

$$f \equiv 0 \quad \text{in} \quad \mathcal{B}_{r^*}. \tag{1.4}$$

Thus the "active part" of f is $f|_{\omega\setminus\mathcal{B}_{r^*}}$, which is independent of ε. Without risk of misunderstanding, we denote simply by f the right hand side of (1.3). In oder to take a more general situation into account, we have to assume that f has an asymptotics as $r \to 0$ in $\mathcal{B}_{r^*}$, see Section 6.1.2.

When ε tends to 0, we expect the solution u_ε of (1.3) to converge to $u_0 \in \mathrm{H}^1_0(\omega)$ solution of $-\Delta u_0 = f$ in the limit domain ω. In the following, we shall derive the full asymptotic expansion of u_ε into powers of ε. The nature of the terms in this expansion depends on the asymptotics as $r \to 0$ and $R \to \infty$ of solutions to the Dirichlet problem on the limit domain ω and the pattern domain Ω, respectively.

Both are related to the *singular functions* of the Laplace-Dirichlet problem in the sector K, which solve the homogeneous problem

$$\mathfrak{s} = 0 \quad \text{on} \quad \partial K \quad \text{and} \quad -\Delta\mathfrak{s} = 0 \quad \text{in} \quad K. \tag{1.5}$$

From now on, we denote by λ the exponent related to the opening angle α of the sector K

$$\boxed{\lambda = \tfrac{\pi}{\alpha}\,.} \tag{1.6}$$

A generating set for all solutions of (1.5) on the sector K parametrized in polar coordinates by $(0;+\infty)_\rho \times (0;\alpha)_\theta$ is given by (see [10, 7, 5])

$$\boxed{\mathfrak{s}^{p\lambda}(\rho,\theta) = \rho^{p\lambda}\sin(p\lambda\theta), \quad \forall p \in \mathbb{Z}^*.} \tag{1.7}$$

1.3 Outline of results

As a result of our two methods of analysis, this expansion is described by two different formulas, the first terms of which we present now.

- *The powers of* ε appearing in both formulas are the exponents $p\lambda$ of the singularities (1.7).
- *The remainders* in the next formulas are of the form $\mathcal{O}_{\mathrm{H}^1}(\varepsilon^\kappa)$, which means that their norms in $\mathrm{H}^1(\omega_\varepsilon)$ are uniformly bounded by $C\varepsilon^\kappa$ as $\varepsilon \to 0$.

Multi-Scale Expansion (MSE): The MSE method consists in looking for an expansion of u_ε in powers of ε with "coefficients" combining the two scales x and $\frac{x}{\varepsilon}$. Let χ be a smooth cut-off function vanishing at point O and ψ a smooth cut-off function localized near O, cf (4.1). As a result of the MSE method we find for the first terms, see Theorem 4.1

$$u_\varepsilon = \chi(\tfrac{x}{\varepsilon})\, v^0(x) + \psi(x)\, \varepsilon^\lambda V^\lambda(\tfrac{x}{\varepsilon}) + \mathcal{O}_{\mathrm{H}^1}(\varepsilon^{2\lambda}), \tag{1.8}$$

and, next

$$u_\varepsilon = \chi(\tfrac{x}{\varepsilon})\Big(v^0(x) + \varepsilon^{2\lambda} v^{2\lambda}(x)\Big) + \psi(x)\Big(\varepsilon^\lambda V^\lambda(\tfrac{x}{\varepsilon}) + \varepsilon^{2\lambda} V^{2\lambda}(\tfrac{x}{\varepsilon})\Big) + \mathcal{O}_{\mathrm{H}^1}(\varepsilon^{3\lambda}). \tag{1.9}$$

Here, the first term v^0 coincides with the limit u_0. The *profiles* V^λ and $V^{2\lambda}$ are defined in the infinite pattern domain Ω. Thus information concerning the perturbing pattern is contained in the profiles (whose contribution is localized near O), and v^0, $v^{2\lambda}$ carry information corresponding to the bounded domain ω (whose influence does not reach the corner).

In our MSE analysis, all the terms $v(x)$ and all the profiles $V(X)$ are solution of *variational problems* in ω and Ω, respectively, see equations (2.1) and (2.4).

Note that the cut-off functions are used in a different scale than the associated terms of the asymptotic expansion. This allows to consider a wide range of problems, *cf* [13, Ch.4]. This procedure is also similar to the homogenization and asymptotic expansions in periodic structures, see [16].

Matched Asymptotic Expansions (MAEs): The MAEs method consists in constructing two expansions (slow and fast) of u_ε in powers of ε. The coefficients of the slow expansion are function of x, and those of the fast one are function of $\frac{x}{\varepsilon}$. Neither of these two expansions is valid everywhere. They have to be *matched* inside an intermediate region.

In order to have a representation of u_ε everywhere and to optimize remainders, we use a cut-off function at the intermediate scale $\frac{r}{\sqrt{\varepsilon}}$. Let φ be a smooth cut-off function with $\varphi(\rho) = 0$ for $\rho \leqslant 1$ and $\varphi(\rho) = 1$ for $\rho \geq 2$. As a result of the MAEs method we find for the first terms, see Theorem 3.2,

$$u_\varepsilon = \varphi\big(\tfrac{r}{\sqrt{\varepsilon}}\big)\, u^0(x) + \Big(1 - \varphi\big(\tfrac{r}{\sqrt{\varepsilon}}\big)\Big)\, \varepsilon^\lambda U^\lambda(\tfrac{x}{\varepsilon}) + \mathcal{O}_{\mathrm{H}^1}(\varepsilon^\lambda), \tag{1.10}$$

and, next

$$u_\varepsilon = \varphi\big(\tfrac{r}{\sqrt{\varepsilon}}\big)\Big(u^0(x) + \varepsilon^{2\lambda}u^{2\lambda}(x)\Big) \\ + \Big(1 - \varphi\big(\tfrac{r}{\sqrt{\varepsilon}}\big)\Big)\Big(\varepsilon^{\lambda}U^{\lambda}(\tfrac{x}{\varepsilon}) + \varepsilon^{2\lambda}U^{2\lambda}(\tfrac{x}{\varepsilon})\Big) + \mathcal{O}_{\mathrm{H}^1}(\varepsilon^{3\lambda/2}). \quad (1.11)$$

Here, again, the first term u^0 coincides with the limit u_0. The term $u^{2\lambda}$ is defined on ω whereas the *profiles* U^λ and $U^{2\lambda}$ are defined in the infinite domain Ω. The terms $u^{2\lambda}(x)$, $U^\lambda(X)$ and $U^{2\lambda}(X)$ are solution of *"super-variational problems"*, i.e. problems set in spaces larger than the variational spaces, see equations (2.8) and (2.15), and where standard formulations would have non-unique solutions.

Comparison: The terms $v^{2\lambda}$, V^λ, $u^{2\lambda}$ and U^λ exchange with each other via two singular terms of the form $a\,\mathfrak{s}^{-\lambda}$ and $A\,\mathfrak{s}^{\lambda}$, where a and A is are real coefficient and $\mathfrak{s}^{\pm\lambda}$ are the singular functions as defined in (1.7). There holds, see Theorem 5.1

$$\begin{cases} u^{2\lambda}(x) &=& v^{2\lambda}(x) + \psi(x)\,a\,\mathfrak{s}^{-\lambda}(x), \qquad & x\in\omega,\\ U^{\lambda}(X) &=& V^{\lambda}(X) + \chi(X)\,A\,\mathfrak{s}^{\lambda}(X), \qquad & X\in\Omega. \end{cases} \quad (1.12)$$

2 Super-variational problems

All the terms in (1.8)-(1.11) appear as solutions of Dirichlet problems on ω or Ω. We first recall their variational framework before considering their solutions in larger spaces.

2.1 Variational problems

The variational space $V(\omega)$ for the Dirichlet problem on the bounded domain ω is $\mathrm{H}^1_0(\omega)$ and for f in its dual space, the variational formulation is

$$\begin{cases} \textit{Find } \ u\in V(\omega) \textit{ such that}\\ \displaystyle\int_\omega \nabla u(x)\ \nabla v(x)\ dx = \int_\omega f(x)\ v(x)\ dx \quad \forall v\in V(\omega). \end{cases} \quad (2.1)$$

Problem (2.1) has a unique solution. As a classical consequence of an angular Poincaré inequality, we find that the variational space is embedded into a weighted Sobolev space

$$V(\omega) = \mathrm{H}^1_0(\omega) \subset \mathrm{W}^1_0(\omega) := \{u\in\mathrm{H}^1(\omega)\,;\ r^{-1}u\in\mathrm{L}^2(\omega)\}. \quad (2.2)$$

The variational space $V(\Omega)$ for the Dirichlet problem on the unbounded domain Ω is the weighted space

$$V(\Omega) = \{U\in\mathrm{L}^2_{\mathrm{loc}}(\Omega)\,;\ \langle R\rangle^{-1}U\in\mathrm{L}^2(\Omega),\ \nabla U\in\mathrm{L}^2(\Omega),\ U|_{\partial\Omega}=0\}, \quad (2.3)$$

where $\langle R\rangle = \sqrt{R^2+1}$. Then the variational problem below has a unique solution

$$\begin{cases} \textit{Find } \ U\in V(\Omega) \textit{ such that}\\ \displaystyle\int_\Omega \nabla U(X)\ \nabla V(X)\ dX = \int_\Omega f(X)\ V(X)\ dX \quad \forall V\in V(\Omega). \end{cases} \quad (2.4)$$

One can refer for example to [2] for more details.

2.2 Super-variational problems in ω. Behavior at the origin

First, we introduce some functional spaces to specify the behavior near the origin.

Definition 2.1 (i) *Let $V_{\mathrm{loc},0}(\omega)$ be the space of distributions*

$$V_{\mathrm{loc},0}(\omega) = \big\{u \in \mathscr{D}'(\omega)\ ;\ \ \varphi u \in \mathrm{H}^1_0(\omega),\ \forall \varphi \in \mathscr{D}(\mathbb{R}^2 \setminus \{0\})\big\}\,.$$

(ii) *For $m \in \mathbb{N}$ and $s \in \mathbb{R}$ let $\mathrm{W}^m_s(\omega)$ be the weighted Sobolev space*

$$\mathrm{W}^m_s(\omega) = \big\{u \in \mathscr{D}'(\omega)\ ;\ \ r^{|\beta|-s-1}\,\partial^\beta_x u \in \mathrm{L}^2(\omega), \quad \forall \beta,\ |\beta| \leqslant m\big\}.$$

Using these weighted spaces, we particularize the meaning of $\mathcal{O}(r^s)$ as follows:

Notation 2.2 *For $s \in \mathbb{R}$, the function $u : \omega \to \mathbb{R}$ is said to be a $\mathcal{O}_{r\to 0}(r^s)$ and we write $u = \mathcal{O}_{r\to 0}(r^s)$ if there exists a neighborhood $\mathcal{V}$ of O in $\mathbb{R}^2$ such that $u \in \mathrm{W}^m_s(\mathcal{V})$ for all $m \in \mathbb{N}$.*

Note that, combining the change of variables $x \mapsto (t = \log r, \theta)$ with Sobolev embeddings, we can prove that if $u = \mathcal{O}_{r\to 0}(r^s)$, then

$$\forall m, n \in \mathbb{N}, \quad \exists C > 0, \quad |r^m \partial^m_r \partial^n_\theta u| \ \leqslant\ C r^s \ \text{ in } \ \omega \cap \mathcal{V},$$

which motivates the notation $\mathcal{O}_{r\to 0}(r^s)$.

The following result is a consequence of local elliptic a priori estimates.

Lemma 2.3 *Let $u \in V_{\mathrm{loc},0}(\omega)$ such that $\Delta u = 0$ in $\omega \cap \mathcal{V}$ for a neighborhood $\mathcal{V}$ of O. Then for any real number s, we have the equivalence*

$$u \in \mathrm{W}^1_s(\omega) \iff u = \mathcal{O}_{r\to 0}(r^s). \tag{2.5}$$

Proof. The implication $\Longleftarrow$ is obvious. Let us prove the converse implication. Let $u \in \mathrm{W}^1_s(\omega)$ satisfying the assumptions of the lemma. Let $\rho' \in (0, r^*]$ such that the finite sector $\sigma_{\rho'} := \omega \cap \mathcal{B}(O, \rho')$ is contained in $\mathcal{V}$. Let $\rho \in (0, \rho')$, and $m \in \mathbb{N}$ be fixed. Let us prove that u belongs to $\mathrm{W}^{m+2}_s(\sigma_\rho)$, where $\sigma_\rho = \omega \cap \mathcal{B}(O, \rho)$.
For this, we consider two sectorial rings, σ^1 and σ^2, defined as

$$\sigma^1 = \{x \in \omega \ \ \rho_0 < |x| < \rho\} \quad \text{and} \quad \sigma^2 = \{x \in \omega \ \ \rho'_0 < |x| < \rho'\},$$

with $\rho'_0 < \rho_0 < \rho/2$, whence $\sigma^1 \subset \sigma^2$. A standard local elliptic estimate reads, for u satisfying $u \in \mathrm{W}^1_s(\sigma_{\rho'})$, $\Delta u \in \mathrm{W}^m_{s+2}(\sigma_{\rho'})$, and $u = 0$ on $\partial\omega \cap \mathcal{B}(O, \rho')$ – see [1],

$$\|u\|_{\mathrm{H}^{m+2}(\sigma^1)} \leqslant C \left(\|\Delta u\|_{\mathrm{H}^m(\sigma^2)} + \|u\|_{\mathrm{L}^2(\sigma^2)}\right). \tag{2.6}$$

Applying this estimate to the functions $u_k(x) = u(2^{-k}x)$ and summing up over k the obtained inequalities (multiplied by 2^{-sk}), we get the following estimate from dyadic partition equivalence

$$\|u\|_{\mathrm{W}^{m+2}_s(\sigma_\rho)} \leqslant C \left(\|\Delta u\|_{\mathrm{W}^m_{s+2}(\sigma_{\rho'})} + \|u\|_{\mathrm{W}^0_s(\sigma_{\rho'})}\right). \tag{2.7}$$

The conclusion follows from the embedding $\mathrm{W}^1_s(\sigma_{\rho'}) \subset \mathrm{W}^0_s(\sigma_{\rho'})$. ∎

We can now state about the solvability of super-variational problems on ω, that is, in spaces containing some of the *dual singular functions* $\mathfrak{s}^{-p\lambda}$ for $p \geq 1$: If we know the dual singular part of a function $u \in V_{\mathrm{loc},0}(\omega)$ and its Laplacian Δu, then this function is uniquely defined.

Proposition 2.4 *For any data $f \in \mathrm{L}^2(\omega)$, $f \equiv 0$ in a neighborhood of O, and any finite sequence $(a_{p\lambda})_{1\leqslant p\leqslant P}$ of real numbers, there exists a unique solution u to the "super-variational problem"*

$$\begin{cases} \textit{Find } \ u \in V_{\mathrm{loc},0}(\omega) \ \textit{ such that} \\ \quad -\Delta u = f \ \textit{ in } \ \omega \quad \textit{and} \quad u - \displaystyle\sum_{p=1}^{P} a_{p\lambda}\, \mathfrak{s}^{-p\lambda} = \mathcal{O}_{r\to 0}(1). \end{cases} \tag{2.8}$$

Remark 2.5 If the sequence of coefficients $(a_{p\lambda})_p$ is empty, the problem (2.8) is nothing but the variational problem (2.1).

PROOF. Let the smooth cut-off function ψ satisfy $\psi(x) = 1$ for $|x| < \frac{r^*}{2}$ and $\psi(x) = 0$ for $|x| > r^*$. We set $v = \psi \sum_p a_{p\lambda}\, \mathfrak{s}^{-p\lambda}$, which obviously satisfies

$$v \in V_{\mathrm{loc},0}(\omega), \quad \text{and} \quad \Delta v = 0 \ \text{ in } \ \omega \cap \mathcal{B}_{r^*/2}. \tag{2.9}$$

Hence, the problem to find w such that $-\Delta w = f + \Delta v$ in ω admits a unique variational solution $w \in V(\omega) = \mathrm{H}^1_0(\omega)$. Moreover, (2.2) gives that w belongs to $\mathrm{W}^1_0(\omega)$, and by localization near point O, w is a $\mathcal{O}_{r\to 0}(1)$ thanks to (2.5); the function $u = w + v$ meets then the requirements. ■

On the other hand, every solution of the Laplace-Dirichlet equation can be expanded near the corner point O in terms of the singular functions, compare with the results in e.g. [10, 14, 15, 7].

Proposition 2.6 *Let $s \geq 0$ be a real number. We define P as the integer part of $\frac{s}{\lambda}$. For any $u \in V_{\mathrm{loc},0}(\omega)$ for which there is a neighborhood $\mathcal{V}$ of O such that*

$$\Delta u = 0 \ \textit{ in } \ \omega \cap \mathcal{V} \quad \textit{and} \quad u = \mathcal{O}_{r\to 0}(r^{-s}), \tag{2.10}$$

there exist a unique finite sequence $(a_{p\lambda})_{1\leqslant p\leqslant P}$ and a unique sequence $(b_{p\lambda})_{p\in\mathbb{N}^*}$ (generically infinite) such that for all $N \in \mathbb{N}^*$

$$u(x) = \sum_{p=1}^{P} a_{p\lambda}\, \mathfrak{s}^{-p\lambda}(r,\theta) + \sum_{p=1}^{N} b_{p\lambda}\, \mathfrak{s}^{p\lambda}(r,\theta) + \mathcal{O}_{r\to 0}(r^{(N+1)\lambda}). \tag{2.11}$$

Notation 2.7 *In the situation of Proposition* 2.6*, we write*

$$u(x) \underset{r\to 0}{\simeq} \sum_{p=1}^{P} a_{p\lambda}\, \mathfrak{s}^{-p\lambda}(r,\theta) + \sum_{p=1}^{\infty} b_{p\lambda}\, \mathfrak{s}^{p\lambda}(r,\theta). \tag{2.12}$$

PROOF. One can prove this lemma using the Mellin transform, see [10]. In the particular case we are interested in, an argument based on separation of variables via angular Fourier series also leads to the result. ∎

In accordance with the literature on corner asymptotics [15, 6, 4] we can call the sum $\sum a_{p\lambda}\, \mathfrak{s}^{-p\lambda}(r,\theta)$ the *dual singular part* of u, whereas $\sum b_{p\lambda}\, \mathfrak{s}^{p\lambda}(r,\theta)$ represents the asymptotics of the variational part of u and can be called *primal singular part* of u.

In the particular case of an opening angle equal to π, i.e. $\lambda = 1$, the asymptotics of the variational part contains polynomials only – it is a Taylor expansion, but the dual singular part is actually singular. More generally, if α has the form $\frac{\pi}{n}$ with a positive integer n, the asymptotics of the variational part is polynomial and can be regarded as *regular*.

2.3 Super-variational problems in Ω. Behavior at infinity

Solutions of limit problems on ω, which have just been investigated in the previous section, yield a good representation of the solution u_ε in the domain ω_ε, away from the corner point O. On the other hand, the geometries of domains ω and ω_ε differ near that point and the asymptotic expansion of u_ε with respect to ε will be given by terms defined in the pattern domain Ω. The tools needed for their construction are stated hereafter.

As we shall see, the behavior at infinity of Laplace-Dirichlet solutions in Ω, in combination with the behavior at O of solutions in ω, is the key for building and validating the asymptotic expansion of u_ε. Hence, we give similar definitions as in the previous section, $r \to 0$ being replaced with $R \to +\infty$.

Definition 2.8 (i) *Let $V_{\mathrm{loc},\infty}(\Omega)$ be the space of distributions*

$$V_{\mathrm{loc},\infty}(\Omega) = \left\{ U \in \mathscr{D}'(\Omega)\ ;\ \ \varphi U \in \mathrm{H}^1_0(\Omega),\ \forall \varphi \in \mathscr{D}(\mathbb{R}^2) \right\}.$$

(ii) *For $m \in \mathbb{N}$ and $s \in \mathbb{R}$ let $\mathrm{W}^m_s(\Omega)$ be the weighted Sobolev space*

$$\mathrm{W}^m_s(\Omega) = \left\{ U \in \mathscr{D}'(\Omega)\ ;\ \ \langle R\rangle^{|\beta|-s-1}\, \partial^\beta_X U \in \mathrm{L}^2(\omega), \quad \forall \beta,\ |\beta| \leqslant m \right\},$$

where $\langle R\rangle = \sqrt{R^2+1}$.

In the following, we shall say that $\mathcal{W}$ is a neighborhood of infinity if there exists a ball B_R of radius R such that

$$\complement_{\mathbb{R}^2} B_R \ \subset \mathcal{W}. \tag{2.13}$$

We introduce, similarly to Notation 2.2

Notation 2.9 *For $s \in \mathbb{R}$, the function $U : \Omega \to \mathbb{R}$ is said to be a $\mathcal{O}_{R\to\infty}(R^s)$ and we write $U = \mathcal{O}_{R\to\infty}(R^s)$ if there exists a neighborhood $\mathcal{W}$ of infinity such that $U \in \mathrm{W}^m_s(\mathcal{W})$ for all $m \in \mathbb{N}$.*

Then any $U = \mathcal{O}_{R\to\infty}(R^s)$ satisfies

$$\forall m,n \in \mathbb{N}, \quad \exists C > 0, \quad |R^m \partial^m_R \partial^n_\theta U(R,\theta)| \ \leqslant\ CR^s \ \text{ in } \ \Omega \cap \mathcal{W}.$$

Thanks to a similar shift result as for Lemma 2.3, we get

Lemma 2.10 *Let $U \in V_{\mathrm{loc},\infty}(\Omega)$ such that $\Delta U = 0$ in $\Omega \cap \mathcal{W}$ for a neighborhood $\mathcal{W}$ of infinity. Then for any real number s, we have the equivalence*

$$U \in \mathrm{W}^1_s(\Omega) \quad \Longleftrightarrow \quad U = \mathcal{O}_{R\to\infty}(R^s). \tag{2.14}$$

The following two propositions are the counterparts of Propositions 2.4 and 2.6. The dual singular functions at infinity in Ω are now the $\mathfrak{s}^{p\lambda}$ for positive integers p.

Proposition 2.11 *For any $F \in \mathrm{L}^2(\Omega)$ with compact support in $\overline{\Omega}$ and any finite sequence $(A_{p\lambda})_{1\leqslant p\leqslant P}$ of real numbers, there exists a unique solution U to the "super-variational problem"*

$$\begin{cases} \textit{Find } U \in V_{\mathrm{loc},\infty}(\Omega) \textit{ such that} \\ -\Delta U = F \textit{ in } \Omega \quad \textit{and} \quad U - \displaystyle\sum_{p=1}^{P} A_{p\lambda}\,\mathfrak{s}^{p\lambda} = \underset{r\to\infty}{\mathcal{O}}(1). \end{cases} \tag{2.15}$$

Proof. It is very similar to Proposition 2.4, the suitable variational space being here $V(\Omega) = \mathrm{W}^1_0(\Omega)$. ■

Remark 2.12 If the sequence of coefficients $(A_{p\lambda})$ is empty, the problem (2.15) is nothing but the variational problem (2.4).

Proposition 2.13 *Let $s \geq 0$ be a real number. We define P as the integer part of $\frac{s}{\lambda}$. For any $U \in V_{\mathrm{loc},\infty}(\Omega)$ for which there is a neighborhood $\mathcal{W}$ of infinity such that*

$$\Delta U = 0 \textit{ in } \Omega \cap \mathcal{W} \quad \textit{and} \quad U = \mathcal{O}_{R\to\infty}(R^s), \tag{2.16}$$

there exist a unique finite sequence $(A_{p\lambda})_{1\leqslant p\leqslant P}$ and a unique sequence $(B_{p\lambda})_{p\in\mathbb{N}^*}$ (generically infinite) such that for all $N \in \mathbb{N}^*$

$$U(X) = \sum_{p=1}^{P} A_{p\lambda}\,\mathfrak{s}^{p\lambda}(R,\theta) + \sum_{p=1}^{N} B_{p\lambda}\,\mathfrak{s}^{-p\lambda}(R,\theta) + \mathcal{O}_{R\to\infty}(R^{-(N+1)\lambda}). \tag{2.17}$$

Notation 2.14 *In the situation of Proposition 2.13, we write*

$$U(X) \underset{R\to\infty}{\simeq} \sum_{p=1}^{P} A_{p\lambda}\,\mathfrak{s}^{p\lambda}(R,\theta) + \sum_{p=1}^{\infty} B_{p\lambda}\,\mathfrak{s}^{-p\lambda}(R,\theta). \tag{2.18}$$

3 Matching of asymptotic expansions

3.1 Formal derivation of the asymptotic expansions

We will represent the solution u_ε as a *formal series* in each zone of interest, that is the *corner expansion* (or inner expansion) near the origin O and the *outer expansion* away from O. We write these two formal series in the form

$$u_\varepsilon(x) \simeq \sum_{\ell=-\infty}^{+\infty} \varepsilon^{\ell\lambda}\, U^{\ell\lambda}(\tfrac{x}{\varepsilon}) \quad \text{and} \quad u_\varepsilon(x) \simeq \sum_{\ell=-\infty}^{+\infty} \varepsilon^{\ell\lambda}\, u^{\ell\lambda}(x). \tag{3.1}$$

This *Ansatz* is suggested by the homogeneity of the singular functions, see (1.7). We will give a sense to the infinite sums in terms of asymptotic expansions later on.

Since the H^1-norm of u_ε is uniformly bounded with respect to ε, we know that all the $u^{\ell\lambda}$ and $U^{\ell\lambda}$ for $\ell < 0$ are just zero. Moreover, it is clear that the terms of the asymptotic expansions must satisfy

$$\begin{cases} -\Delta u^0 = f \text{ in } \omega \quad \text{and} \quad u^0 = 0 \text{ on } \partial\omega, \\ \forall \ell > 0, \;\; \Delta u^{\ell\lambda} = 0 \text{ in } \omega \quad \text{and} \quad u^{\ell\lambda} = 0 \text{ on } \partial\omega \setminus \{O\}, \\ \forall \ell \geqslant 0, \;\; \Delta U^{\ell\lambda} = 0 \text{ in } \Omega \quad \text{and} \quad U^{\ell\lambda} = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.2}$$

Now we need to ensure the matching of the two formal series in the transition zone

$$\varepsilon \ll r \ll 1. \tag{3.3}$$

To do so, we expand the terms $u^{\ell\lambda}$ and $U^{\ell\lambda}$. Thanks to Propositions 2.6 and 2.13 – note that $r \ll 1$ and $\frac{r}{\varepsilon} \gg 1$ – these expansions read

$$\begin{cases} u^{\ell\lambda}(x) = \displaystyle\sum_{p=1}^{+\infty} \Big(a_{p\lambda}^{\ell\lambda} \, \mathfrak{s}^{-p\lambda}(r,\theta) \; + \; b_{p\lambda}^{\ell\lambda} \, \mathfrak{s}^{p\lambda}(r,\theta) \Big), \\ U^{\ell\lambda}(X) = \displaystyle\sum_{p=1}^{+\infty} \Big(A_{p\lambda}^{\ell\lambda} \, \mathfrak{s}^{p\lambda}(R,\theta) \; + \; B_{p\lambda}^{\ell\lambda} \, \mathfrak{s}^{-p\lambda}(R,\theta) \Big). \end{cases} \tag{3.4}$$

We use the homogeneity of the functions $\mathfrak{s}^{p\lambda}$ and transform the rapid variable $\frac{r}{\varepsilon}$ into the slow one r. Ensuring the equality of the two formal series (3.1), we get

$$\begin{cases} \displaystyle\sum_{\ell} \Big(\varepsilon^{\ell\lambda} \sum_{p} \Big(a_{p\lambda}^{\ell\lambda} \mathfrak{s}^{-p\lambda}(r,\theta) \; + \; b_{p\lambda}^{\ell\lambda} \mathfrak{s}^{p\lambda}(r,\theta) \Big) \Big) \\ \qquad\qquad = \displaystyle\sum_{\ell} \Big(\varepsilon^{\ell\lambda} \sum_{p} \Big(\varepsilon^{-p\lambda} \, A_{p\lambda}^{\ell\lambda} \, \mathfrak{s}^{p\lambda}(r,\theta) \; + \; \varepsilon^{p\lambda} \, B_{p\lambda}^{\ell\lambda} \, \mathfrak{s}^{-p\lambda}(r,\theta) \Big) \Big) \\ \qquad\qquad = \displaystyle\sum_{\ell} \Big(\varepsilon^{\ell\lambda} \sum_{p} \Big(A_{p\lambda}^{(\ell+p)\lambda} \, \mathfrak{s}^{p\lambda}(r,\theta) \; + \; B_{p\lambda}^{(\ell-p)\lambda} \, \mathfrak{s}^{-p\lambda}(r,\theta) \Big) \Big). \end{cases} \tag{3.5}$$

Identifying the terms of the two series leads to

$$b_{p\lambda}^{\ell\lambda} = A_{p\lambda}^{(\ell+p)\lambda} \quad \text{and} \quad a_{p\lambda}^{\ell\lambda} = B_{p\lambda}^{(\ell-p)\lambda}, \tag{3.6}$$

i. e.

$$\begin{cases} a_{p\lambda}^{\ell\lambda} = B_{p\lambda}^{(\ell-p)\lambda} \text{ if } p \leqslant \ell \quad \text{and} \quad a_{p\lambda}^{\ell\lambda} = 0 \text{ if } p > \ell, \\ A_{p\lambda}^{\ell\lambda} = b_{p\lambda}^{(\ell-p)\lambda} \text{ if } p \leqslant \ell \quad \text{and} \quad A_{p\lambda}^{\ell\lambda} = 0 \text{ if } p > \ell, \end{cases} \tag{3.7}$$

knowing that $b_{p\lambda}^{(\ell-p)\lambda} = B_{p\lambda}^{(\ell-p)\lambda} = 0$ for $p > \ell$, since the terms $u^{n\lambda}$ and $U^{n\lambda}$ are 0 for $n < 0$.

Remark 3.1 Here, we have chosen to derive the matching relations without any knowledge of the matched asymptotic technique. However, one can derive the relations (3.7) using the Van Dyke principle, see [18].

3.2 Definition of the asymptotic terms

For $\ell \in \mathbb{N}$, the functions $u^{\ell\lambda}$ and $U^{\ell\lambda}$ are defined inductively. The following algorithm defines step by step $u^{\ell\lambda} : \omega \to \mathbb{R}$, $U^{\ell\lambda} : \Omega \to \mathbb{R}$, $b^{\ell\lambda} = \big(b^{\ell\lambda}_{p\lambda}\big)_{p\in\mathbb{N}^*}$, and $B^{\ell\lambda} = \big(B^{\ell\lambda}_{p\lambda}\big)_{p\in\mathbb{N}^*}$ for $\ell \in \mathbb{N}$.

Step 0. $u^0 \in V_{\mathrm{loc},0}(\omega)$ is defined via Proposition 2.4 (in the particular case of Remark 2.5) as the unique function satisfying

$$\Delta u^0 = -f \ \text{ in } \omega \quad \text{and} \quad u^0 = \mathcal{O}_{r\to 0}(1). \tag{3.8}$$

Moreover, U^0 is chosen to be 0. Let b^0 be the sequence of numbers defined by Proposition 2.6 and B^0 be zero:

$$b^0 = \big(b^0_{p\lambda}\big)_{p\in\mathbb{N}^*} \quad \text{and} \quad B^0 = \big(B^0_{p\lambda}\big)_{p\in\mathbb{N}^*} = 0. \tag{3.9}$$

Step ℓ. We denote by $a^{\ell\lambda} = \big(a^{\ell\lambda}_{p\lambda}\big)_{p\in\mathbb{N}^*}$ and $A^{\ell\lambda} = \big(A^{\ell\lambda}_{p\lambda}\big)_{p\in\mathbb{N}^*}$ the two finite sequences of real numbers such that

$$\begin{cases} a^{\ell\lambda}_{p\lambda} = B^{(\ell-p)\lambda}_{p\lambda} \ \text{ if } 1 \leqslant p \leqslant \ell-1 \quad \text{and} \quad a^{\ell\lambda}_{p\lambda} = 0 \ \text{ if } p \geqslant \ell, \\ A^{\ell\lambda}_{p\lambda} = b^{(\ell-p)\lambda}_{p\lambda} \ \text{ if } 1 \leqslant p \leqslant \ell \quad \text{and} \quad A^{\ell\lambda}_{p\lambda} = 0 \ \text{ if } p \geqslant \ell+1. \end{cases} \tag{3.10}$$

The functions $u^{\ell\lambda}$ and $U^{\ell\lambda}$ are defined via Propositions 2.4 and 2.11 as the unique solutions of the problems

$$\begin{cases} \textit{Find } u^{\ell\lambda} \in V_{\mathrm{loc},0}(\omega) \textit{ such that} \\ \Delta u^{\ell\lambda} = 0 \ \textit{ in } \omega \quad \text{and} \quad u^{\ell\lambda} - \displaystyle\sum_{p=1}^{\ell-1} a^{\ell\lambda}_{p\lambda}\, \mathfrak{s}^{-p\lambda} = \mathcal{O}_{r\to 0}(1), \end{cases} \tag{3.11}$$

and

$$\begin{cases} \textit{Find } U^{\ell\lambda} \in V_{\mathrm{loc},\infty}(\Omega) \textit{ such that} \\ \Delta U^{\ell\lambda} = 0 \ \textit{ in } \Omega \quad \text{and} \quad U^{\ell\lambda} - \displaystyle\sum_{p=1}^{\ell} A^{\ell\lambda}_{p\lambda}\, \mathfrak{s}^{p\lambda} = \mathcal{O}_{R\to \infty}(1). \end{cases} \tag{3.12}$$

Finally, we define the sequences $b^{\ell\lambda}$ and $B^{\ell\lambda}$ associated with $u^{\ell\lambda}$ and $U^{\ell\lambda}$ in Propositions 2.6 and 2.13

$$b^{\ell\lambda} = \big(b^{\ell\lambda}_{p\lambda}\big)_{p\in\mathbb{N}^*} \quad \text{and} \quad B^{\ell\lambda} = \big(B^{\ell\lambda}_{p\lambda}\big)_{p\in\mathbb{N}^*}. \tag{3.13}$$

3.3 Global error estimates

The main idea to prove error estimates is to define a global approximation $\widehat{u}^{\varepsilon}_{n\lambda} \in \mathrm{H}^1_0(\omega_\varepsilon)$ of u_ε by the formula

$$\widehat{u}^{\varepsilon}_{n\lambda}(x) = \varphi\big(\tfrac{r}{\eta(\varepsilon)}\big) \sum_{\ell=0}^{n} \varepsilon^{\ell\lambda}\, u^{\ell\lambda}(x) + \Big(1 - \varphi\big(\tfrac{r}{\eta(\varepsilon)}\big)\Big) \sum_{\ell=1}^{n} \varepsilon^{\ell\lambda}\, U^{\ell\lambda}\big(\tfrac{x}{\varepsilon}\big), \tag{3.14}$$

where φ is a smooth cut-off function with $\varphi(\rho) = 0$ for $\rho < 1$ and $\varphi(\rho) = 1$ for $\rho > 2$ and η is a smooth function of ε such that

$$\lim_{\varepsilon\to 0} \eta(\varepsilon) = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{\eta(\varepsilon)}{\varepsilon} = +\infty. \tag{3.15}$$

Theorem 3.2 *There exists a constant C such that*

$$\|u_\varepsilon - \widehat{u}^\varepsilon_{n\lambda}\|_{\mathrm{H}^1(\omega_\varepsilon)} \;\leqslant\; C\,\Big[\Big(\eta(\varepsilon)\Big)^{(n+1)\lambda} + \Big(\frac{\varepsilon}{\eta(\varepsilon)}\Big)^{(n+1)\lambda}\Big]. \tag{3.16}$$

Remark 3.3 One can optimize the estimate (3.16) by choosing the best η: For $\eta(\varepsilon) = \varepsilon^{1/2}$, there exists a constant C such that

$$\|u_\varepsilon - \widehat{u}^\varepsilon_{n\lambda}\|_{H^1(\omega_\varepsilon)} \;\leqslant\; C\,\varepsilon^{(n+1)\lambda/2}. \tag{3.17}$$

PROOF. First, we denote by $\widehat{e}^\varepsilon_{n\lambda}$ the approximation error at step n

$$\widehat{e}^\varepsilon_{n\lambda}(x) = \widehat{u}^\varepsilon_{n\lambda}(x) - u_\varepsilon(x)$$

and by $\mathcal{E}^\varepsilon_{n\lambda}$ the corresponding matching error

$$\mathcal{E}^\varepsilon_{\lambda n}(x) = \sum_{\ell=0}^{n} \varepsilon^{\ell\lambda}\,\big[u^{\ell\lambda}(x) - U^{\ell\lambda}(\tfrac{x}{\varepsilon})\big].$$

Of course, the matching error makes sense and is small only in the intermediate region; we shall express the H^1-norm of $\widehat{e}^\varepsilon_{n\lambda}$ over ω_ε in terms of $\mathcal{E}^\varepsilon_{n\lambda}$ in this region. By harmonicity of u_ε, $u^{\ell\lambda}$, and $U^{\ell\lambda}$, we obtain

$$\Delta\widehat{e}^\varepsilon_{n\lambda}(x) = \tfrac{2}{\eta(\varepsilon)}\,[\nabla\varphi](\tfrac{r}{\eta(\varepsilon)})\,\nabla\mathcal{E}^\varepsilon_{n\lambda}(x) \;+\; \tfrac{1}{[\eta(\varepsilon)]^2}[\Delta\varphi](\tfrac{r}{\eta(\varepsilon)})\,\mathcal{E}^\varepsilon_{n\lambda}(x).$$

Since $\widehat{e}^\varepsilon_{n\lambda}$ belongs to $\mathrm{H}^1_0(\omega_\varepsilon)$, the Green formula leads to

$$\left\{\begin{aligned} \int_{\omega_\varepsilon} \big(\nabla\widehat{e}^\varepsilon_{n\lambda}\big)^2\,dx \;&=\; \tfrac{2}{\eta(\varepsilon)} \int_{\omega_\varepsilon} [\nabla\varphi](\tfrac{r}{\eta(\varepsilon)})\,\nabla\mathcal{E}^\varepsilon_{n\lambda}\,\widehat{e}^\varepsilon_{n\lambda} \\ &\qquad + \tfrac{1}{[\eta(\varepsilon)]^2} \int_{\omega_\varepsilon} [\Delta\varphi](\tfrac{r}{\eta(\varepsilon)})\,\mathcal{E}^\varepsilon_{n\lambda}\,\widehat{e}^\varepsilon_{n\lambda}\,dx \\ &\leqslant\; \tfrac{C}{[\eta(\varepsilon)]^2}\,\Big[\|\mathcal{E}^\varepsilon_{n\lambda}\|_{\infty,\eta(\varepsilon)} + \eta(\varepsilon)\|\nabla\mathcal{E}^\varepsilon_{n\lambda}\|_{\infty,\eta(\varepsilon)}\Big]\|\widehat{e}^\varepsilon_{n\lambda}\|_{1,\eta(\varepsilon)}\,, \end{aligned}\right.$$

with the notation, for $p \in [1,\infty]$

$$\|u\|_{p,\eta(\varepsilon)} = \|u\|_{\mathrm{L}^p(\,\{x\in\omega\,;\,\eta(\varepsilon)\,\leqslant\, r\,\leqslant\, 2\eta(\varepsilon)\}\,)}. \tag{3.18}$$

Using a Poincaré inequality on ω_ε (uniform with respect to ε), we get

$$\|\widehat{e}^\varepsilon_{n\lambda}\|^2_{\mathrm{H}^1(\omega_\varepsilon)} \;\leqslant\; \frac{C}{(\eta(\varepsilon))^2}\,\Big[\|\mathcal{E}^\varepsilon_{n\lambda}\|_{\infty,\eta(\varepsilon)} + \eta(\varepsilon)\|\nabla\mathcal{E}^\varepsilon_{n\lambda}\|_{\infty,\eta(\varepsilon)}\Big] \times \|\widehat{e}^\varepsilon_{n\lambda}\|_{1,\eta(\varepsilon)}.$$

The conclusion follows from the following two lemmas (proved below). ∎

Lemma 3.4 *There exists a constant C such that for all $u \in \mathrm{H}^1_0(\omega_\varepsilon)$, the norm $\|u\|_{1,\eta(\varepsilon)}$, defined in* (3.18)*, can be estimated as follows*

$$\|u\|_{1,\eta(\varepsilon)} \leqslant C\, [\eta(\varepsilon)]^2\, \|u\|_{\mathrm{H}^1(\omega_\varepsilon)}. \tag{3.19}$$

Lemma 3.5 *There exists a constant C such that – for the definition of the norms, see* (3.18)*,*

$$\|\mathcal{E}^\varepsilon_{n\lambda}\|_{\infty,\eta(\varepsilon)} \leqslant C \left[\Big(\eta(\varepsilon)\Big)^{(n+1)\lambda} + \Big(\frac{\varepsilon}{\eta(\varepsilon)}\Big)^{(n+1)\lambda}\right], \tag{3.20}$$

$$\|\nabla\mathcal{E}^\varepsilon_{n\lambda}\|_{\infty,\eta(\varepsilon)} \leqslant C\, \frac{1}{\eta(\varepsilon)} \left[\Big(\eta(\varepsilon)\Big)^{(n+1)\lambda} + \Big(\frac{\varepsilon}{\eta(\varepsilon)}\Big)^{(n+1)\lambda}\right]. \tag{3.21}$$

PROOF OF LEMMA 3.4. For all $u \in \mathrm{H}^1_0(\omega_\varepsilon)$ and for all $r \in [\eta(\varepsilon), 2\eta(\varepsilon)]$

$$\int_0^\alpha |u(r,\theta)|\, d\theta \leqslant \int_0^\alpha \Big[\int_0^\theta |\frac{\partial u}{\partial\theta}(r,\theta')|\, d\theta'\Big]\, d\theta \leqslant \alpha \int_0^\alpha |\frac{\partial u}{\partial\theta}(r,\theta)|\, r dr\, d\theta.$$

Hence, we have

$$\int_{\eta(\varepsilon)}^{2\eta(\varepsilon)} \int_0^\alpha |u(r,\theta)|\, r dr\, d\theta \leqslant \alpha \int_{\eta(\varepsilon)}^{2\eta(\varepsilon)} \int_0^\alpha |\frac{\partial u}{\partial\theta}(r,\theta)|\, r dr\, d\theta$$

$$\leqslant C\, \eta(\varepsilon)\, \|\nabla u\|_{1,\eta(\varepsilon)}.$$

We conclude using the Cauchy-Schwarz inequality that

$$\|u\|_{1,\eta(\varepsilon)} \leqslant C\, \eta(\varepsilon)\, \|\nabla u\|_{1,\eta(\varepsilon)} \leqslant C\, [\eta(\varepsilon)]^2\, \|\nabla u\|_{2,\eta(\varepsilon)}.$$

■

PROOF OF LEMMA 3.5. We will give the proof of (3.20). Inequality (3.21) can be obtained using the same technique. The first step is to expand the $u^{\ell\lambda}$ and $U^{\ell\lambda}$ using (2.11) and (2.17). By definition of $u^{\ell\lambda}$ and $U^{\ell\lambda}$ – see (3.11) and (3.12), and taking (3.10) into account one finds

$$u^{\ell\lambda}(x) = \sum_{p=1}^{\ell} B^{(\ell-p)\lambda}_{p\lambda}\, \mathfrak{s}^{-p\lambda}(r,\theta) + \sum_{p=1}^{n-\ell} b^{\ell\lambda}_{p\lambda}\, \mathfrak{s}^{p\lambda}(r,\theta) + \mathcal{O}_{r\to 0}(r^{(n+1-\ell)\lambda}),$$
$$U^{\ell\lambda}(X) = \sum_{p=1}^{\ell} b^{(\ell-p)\lambda}_{p\lambda}\, \mathfrak{s}^{p\lambda}(R,\theta) + \sum_{p=1}^{n-\ell} B^{\ell\lambda}_{p\lambda}\, \mathfrak{s}^{-p\lambda}(R,\theta) + \mathcal{O}_{R\to\infty}(R^{(\ell-n-1)\lambda}).$$

Since $\eta(\varepsilon)$ tends to 0 and $\eta(\varepsilon)/\varepsilon$ tends to $+\infty$ when ε tends to 0, one has for $\eta(\varepsilon) \leqslant r \leqslant 2\eta(\varepsilon)$

$$\begin{cases} \Big|u^{\ell\lambda}(x) - \displaystyle\sum_{p=1}^{\ell} B^{(\ell-p)\lambda}_{p\lambda}\, \mathfrak{s}^{-p\lambda}(r,\theta) - \sum_{p=1}^{n-\ell} b^{\ell\lambda}_{p\lambda}\, \mathfrak{s}^{p\lambda}(r,\theta)\Big| \leqslant C\, [\eta(\varepsilon)]^{(n+1-\ell)\lambda}, \\ \Big|U^{\ell\lambda}(\tfrac{x}{\varepsilon}) - \displaystyle\sum_{p=1}^{\ell} b^{(\ell-p)\lambda}_{p\lambda}\, \mathfrak{s}^{p\lambda}(\tfrac{r}{\varepsilon},\theta) - \sum_{p=1}^{n-\ell} B^{\ell\lambda}_{p\lambda}\, \mathfrak{s}^{-p\lambda}(\tfrac{r}{\varepsilon},\theta)\Big| \leqslant C[\tfrac{\varepsilon}{\eta(\varepsilon)}]^{(n+1-\ell)\lambda}. \end{cases} \tag{3.22}$$

Let S be given by

$$
\begin{aligned}
S \;=\; & \sum_{\ell=0}^{n} \varepsilon^{\ell\lambda} \Big(\sum_{p=1}^{\ell} B_{p\lambda}^{(\ell-p)\lambda} \, \mathfrak{s}^{-p\lambda}(r,\theta) + \sum_{p=1}^{n-\ell} b_{p\lambda}^{\ell\lambda} \, \mathfrak{s}^{p\lambda}(r,\theta) \Big) \\
& - \sum_{\ell=0}^{n} \varepsilon^{\ell\lambda} \Big(\sum_{p=1}^{\ell} b_{p\lambda}^{(\ell-p)\lambda} \, \mathfrak{s}^{p\lambda}(\tfrac{r}{\varepsilon},\theta) + \sum_{p=1}^{n-\ell} B_{p\lambda}^{\ell\lambda} \, \mathfrak{s}^{-p\lambda}(\tfrac{r}{\varepsilon},\theta) \Big).
\end{aligned}
\tag{3.23}
$$

From (3.22) and triangle inequalities, we obtain

$$
\begin{aligned}
\| \mathcal{E}^{\varepsilon}_{n\lambda}(r,\theta) - S \|_{\infty,\eta(\varepsilon)} &\leqslant C \left\{ \sum_{\ell=0}^{n} \varepsilon^{\ell\lambda} \, [\eta(\varepsilon)]^{(n+1-\ell)\lambda} + \sum_{\ell=0}^{n} \varepsilon^{\ell\lambda} \, [\tfrac{\varepsilon}{\eta(\varepsilon)}]^{(n+1-\ell)\lambda} \right\} \\
&\leqslant C \left\{ \sum_{\ell=0}^{n} [\tfrac{\varepsilon}{\eta(\varepsilon)}]^{\ell\lambda} [\eta(\varepsilon)]^{(n+1)\lambda} + \sum_{\ell=0}^{n} \eta(\varepsilon)^{\ell\lambda} [\tfrac{\varepsilon}{\eta(\varepsilon)}]^{(n+1)\lambda} \right\} \\
&\leqslant C \left\{ [\eta(\varepsilon)]^{(n+1)\lambda} + [\tfrac{\varepsilon}{\eta(\varepsilon)}]^{(n+1)\lambda} \right\}.
\end{aligned}
$$

Now it remains to show that $S = 0$. By definition – see (1.7) – the singular functions $\mathfrak{s}^{\pm p\lambda}$ satisfy the homogeneity property

$$
\mathfrak{s}^{-p\lambda}(\tfrac{r}{\varepsilon},\theta) = \varepsilon^{p\lambda} \, \mathfrak{s}^{-p\lambda}(r,\theta) \quad \text{and} \quad \mathfrak{s}^{p\lambda}(r,\theta) = \varepsilon^{p\lambda} \, \mathfrak{s}^{p\lambda}(\tfrac{r}{\varepsilon},\theta).
$$

Therefore, S is given by

$$
\begin{aligned}
S \;=\; & \sum_{\ell=0}^{n} \sum_{p=1}^{\ell} \varepsilon^{(\ell-p)\lambda} B_{p\lambda}^{(\ell-p)\lambda} \, \mathfrak{s}^{-p\lambda}(\tfrac{r}{\varepsilon},\theta) \\
& + \sum_{\ell=0}^{n} \sum_{p=1}^{n} \varepsilon^{\ell\lambda} b_{p\lambda}^{\ell\lambda} \, \mathfrak{s}^{p\lambda}(r,\theta) \\
& - \sum_{\ell=0}^{n} \sum_{p=1}^{\ell} \varepsilon^{(\ell-p)\lambda} b_{p\lambda}^{(\ell-p)\lambda} \, \mathfrak{s}^{p\lambda}(r,\theta) \\
& - \sum_{\ell=0}^{n} \sum_{p=1}^{n} \varepsilon^{\ell\lambda} B_{p\lambda}^{\ell\lambda} \, \mathfrak{s}^{-p\lambda}(\tfrac{r}{\varepsilon},\theta).
\end{aligned}
$$

The change of variables $\ell - p \mapsto \ell$ in the first and third terms leads to $S = 0$. ■

3.4 Local error estimates

In this paragraph $\mathcal{B}_r$ will denote the ball of radius r and of center O. Starting from the global error estimates obtained in (3.17), it is easy to get estimates far from the corner and near the corner

Theorem 3.6 *For any $r_0 > 0$, there exists $C > 0$ such that*

$$
\Big\| u_\varepsilon(r,\theta) - \sum_{\ell=0}^{n} \varepsilon^{\ell\lambda} \, u^{\ell\lambda}(r,\theta) \Big\|_{\mathrm{H}^1(\omega \setminus \mathcal{B}_{r_0})} = \mathcal{O}(\varepsilon^{(n+1)\lambda}).
\tag{3.24}
$$

For any $R_0 > 0$, there exists $C > 0$ such that

$$\Big\| u_\varepsilon(\varepsilon R, \theta) - \sum_{\ell=0}^{n} \varepsilon^{\ell\lambda}\, U^{\ell\lambda}(R,\theta) \Big\|_{\mathrm{H}^1(\Omega\cap\mathcal{B}_{R_0})} = \mathcal{O}(\varepsilon^{(n+1)\lambda}). \tag{3.25}$$

PROOF. To prove (3.24), we remark that, for ε small enough, the only contribution comes from the terms $u^{\ell\lambda}$

$$\widehat{u}^{\varepsilon}_{n\lambda} = \sum_{\ell=1}^{n} \varepsilon^{\ell\lambda}\, u^{\ell\lambda} \quad \text{in} \quad \omega_\varepsilon \setminus \mathcal{B}_{r_0} = \omega \setminus \mathcal{B}_{r_0}\,. \tag{3.26}$$

Consequently,

$$\begin{aligned} \|u_\varepsilon - \widehat{u}^{\varepsilon}_{n\lambda}\|_{\mathrm{H}^1(\omega\setminus\mathcal{B}_{r_0})} &\leqslant \|u_\varepsilon - \widehat{u}^{\varepsilon}_{(2n+2)\lambda}\|_{\mathrm{H}^1(\omega\setminus\mathcal{B}_{r_0})} + \|\widehat{u}^{\varepsilon}_{(2n+2)\lambda} - \widehat{u}^{\varepsilon}_{n\lambda}\|_{\mathrm{H}^1(\omega\setminus\mathcal{B}_{r_0})} \\ &\leqslant \|u_\varepsilon - \widehat{u}^{\varepsilon}_{(2n+2)\lambda}\|_{\mathrm{H}^1(\omega_\varepsilon)} + \|\widehat{u}^{\varepsilon}_{(2n+2)\lambda} - \widehat{u}^{\varepsilon}_{n\lambda}\|_{\mathrm{H}^1(\omega\setminus\mathcal{B}_{r_0})}. \end{aligned} \tag{3.27}$$

On the other hand, it follows from (3.26)

$$\widehat{u}^{\varepsilon}_{(2n+2)\lambda} - \widehat{u}^{\varepsilon}_{n\lambda} = \sum_{\ell=n+1}^{2n+2} \varepsilon^{\ell\lambda}\, u^{\ell\lambda} \quad \text{in} \quad \omega \setminus \mathcal{B}_{r_0}\,, \tag{3.28}$$

and, since the $u^{\ell\lambda}$'s do not depend on ε

$$\|\widehat{u}^{\varepsilon}_{(2n+2)\lambda} - \widehat{u}^{\varepsilon}_{n\lambda}\|_{\mathrm{H}^1(\omega\setminus\mathcal{B}_{r_0})} \;\leqslant\; C\, \varepsilon^{(n+1)\lambda}. \tag{3.29}$$

Due to (3.17), one finally has

$$\|u_\varepsilon - \widehat{u}^{\varepsilon}_{(2n+2)\lambda}\|_{\mathrm{H}^1(\omega\setminus\mathcal{B}_{r_0})} \;\leqslant\; C\, \varepsilon^{(n+1)\lambda}. \tag{3.30}$$

The estimate (3.24) follows from (3.26), (3.27), (3.29) and (3.30). The same technique leads to (3.25) as well. A scaling is needed ($R = r/\varepsilon$) to recover a domain independent of ε. ■

Remark 3.7 Due to estimates (3.24) and (3.25), the outer and corner expansions are unique. Moreover, as the remainders are of the same orders as the first neglected term in the outer and corner expansions, these estimates are optimal. The outer and corner expansions can be seen as Taylor expansions of the exact solution expressed in the (r,θ) or $(r/\varepsilon,\theta)$ coordinates.

4 Multiscale technique

4.1 Introduction

The technique of *multiscale expansion* consists in building a global approximation of the solution in the domain ω_ε. The expansion is composed of two different types of terms: the ones involving the original variable x, and the *profiles* appearing in the scaled variable $\frac{x}{\varepsilon}$. They are superposed via cut-off functions

$$u_\varepsilon = \chi(\tfrac{x}{\varepsilon}) \sum_{\ell=0}^{n} \varepsilon^{\ell\lambda} v^{\ell\lambda}(x) + \psi(x) \sum_{\ell=0}^{n} \varepsilon^{\ell\lambda} V^{\ell\lambda}(\tfrac{x}{\varepsilon}) + \mathcal{O}(\varepsilon^{n\lambda}), \tag{4.1}$$

where the functions χ and ψ are smooth and radial, satisfying

$$\begin{cases} \chi(X) = 1 \text{ for } |X| > 2R^* \quad \text{and} \quad \chi(X) = 0 \text{ for } |X| < R^*, \\ \psi(x) = 1 \text{ for } |x| < \dfrac{r^*}{2} \quad \text{and} \quad \psi(x) = 0 \text{ for } |x| > r^*. \end{cases} \tag{4.2}$$

Obviously, the first sum in (4.1) has its support away from an ε-neighborhood of the limit point O and, conversely, the second brings a contribution in a neighborhood of O (independent of ε). Thanks to (1.2), for any $\varepsilon \leqslant \frac{\varepsilon_0}{2}$, the common support of the two sums satisfies

$$\begin{aligned} \omega_\varepsilon \cap \Big(\operatorname{supp} \chi(\frac{\cdot}{\varepsilon}) \cap \operatorname{supp} \psi\Big) &\subset \{x \in \omega_\varepsilon,\ \varepsilon R^* \leqslant |x| \leqslant r^*\} \\ &= \{x \in K,\ \varepsilon R^* \leqslant |x| \leqslant r^*\}, \end{aligned}$$

which means that the *intermediate region* where the two sums have to be simultaneously taken into account is part of the cone K.

The principle of the construction of the terms $v^{\ell\lambda}$ and $V^{\ell\lambda}$ is that they are solutions of variational problems in slow variables in ω and fast variables in Ω, respectively. The cut-off by $\chi(\frac{x}{\varepsilon})$ and $\psi(x)$ introduces an error in fast and slow variables, respectively. These errors can be corrected in fast and slow variables, respectively, with the help of the expansions as $r \to 0$ of the $v^{\ell\lambda}$ and as $R \to \infty$ of the $V^{\ell\lambda}$, respectively. Both expansions in homogeneous terms do make sense in fast *and* slow variables simultaneously, which allow us to bridge the terms in the two sums in (4.1).

4.2 The construction of the terms

We first focus on the construction of the first terms and then give the general algorithm which allows to build the terms $v^{\ell\lambda}$ and $V^{\ell\lambda}$ arising in (4.1).

4.2.1 The first terms

Step 0. Let v^0 be the solution of the limit variational problem

$$\textit{Find } v^0 \in \mathrm{H}^1_0(\omega) \textit{ such that } -\Delta v^0 = f \textit{ in } \omega. \tag{4.3}$$

Then v^0 seems to be a good starting point for the expansion. Nevertheless, it is defined on the domain ω, and not ω_ε. For this reason, we choose to consider the truncated function $\tilde{v}^0 = \chi(\frac{x}{\varepsilon})v^0$. We note that $\tilde{v}^0$ satisfies the Dirichlet boundary condition $\tilde{v}^0 = 0$ on $\partial\omega_\varepsilon$ and belongs to $\mathrm{H}^1_0(\omega_\varepsilon)$. We consider the first remainder r^0_ε defined as

$$u_\varepsilon(x) = \chi(\tfrac{x}{\varepsilon})v^0(x) + r^0_\varepsilon(x).$$

Thus the support of Δr^0_ε is contained in the support of $\nabla\chi(\frac{x}{\varepsilon})$.

Since $f \equiv 0$ in a neighborhood of O, according to Proposition 2.6 (and using Notation 2.7) there exists a sequence[1] $\big(\mathbf{b}^0_{\lambda p}\big)_{p\geq 1}$ such that v^0 expands at O as

$$v^0(x) \underset{r\to 0}{\simeq} \sum_{p=1}^{\infty} \mathbf{b}^0_{p\lambda}\, \mathfrak{s}^{p\lambda}(x). \tag{4.4}$$

Since $f \equiv 0$ near O, equations (4.3) and (4.4) yield that

$$\Delta r^0_\varepsilon(x) = -\big(\big[\Delta, \chi(\tfrac{\cdot}{\varepsilon})\big] v^0\big)(x) \underset{r\to 0}{\simeq} -\sum_{p=1}^{\infty} \mathbf{b}^0_{p\lambda}\, \big[\Delta, \chi(\tfrac{\cdot}{\varepsilon})\big]\, \mathfrak{s}^{\lambda p}(x), \tag{4.5}$$

with the commutator

$$\big[\Delta, \chi(\tfrac{\cdot}{\varepsilon})\big]\, v(x) = \Delta(\chi(\tfrac{x}{\varepsilon}) v(x)) - \chi(\tfrac{x}{\varepsilon})\Delta v(x). \tag{4.6}$$

We are going to consider such terms as *right hand sides* of a problem on Ω in the fast variable $X = \frac{x}{\varepsilon}$. We have the identities for all $p \geq 1$

$$\begin{aligned} \big[\Delta_x, \chi(\tfrac{x}{\varepsilon})\big]\, \mathfrak{s}^{p\lambda}(x) &= 2\nabla_x \mathfrak{s}^{p\lambda}(x)\cdot\nabla_x\big(\chi(\tfrac{x}{\varepsilon})\big) + \mathfrak{s}^{p\lambda}(x)\Delta_x\big(\chi(\tfrac{x}{\varepsilon})\big) \\ &= \varepsilon^{-2}\,\varepsilon^{p\lambda}\big(2\nabla_X\, \mathfrak{s}^{p\lambda}\cdot\nabla_X\,\chi + \mathfrak{s}^{p\lambda}\Delta_X\chi\big)(\tfrac{x}{\varepsilon}) \\ &= \varepsilon^{-2}\,\varepsilon^{p\lambda}\big[\Delta_X, \chi\big]\, \mathfrak{s}^{p\lambda}(\tfrac{x}{\varepsilon}). \end{aligned} \tag{4.7}$$

Thus the remainder (4.5) can be written as

$$\Delta r^0_\varepsilon(x) \underset{r\to 0}{\simeq} -\varepsilon^{-2}\sum_{p=1}^{\infty} \varepsilon^{p\lambda}\, \mathbf{b}^0_{p\lambda}\, \big(\big[\Delta_X, \chi\big]\, \mathfrak{s}^{p\lambda}\big)(\tfrac{x}{\varepsilon}). \tag{4.8}$$

To complete step 0, we set $V^0 = 0$.

Step 1. The first term in the remainder asymptotics (4.8) is nothing but

$$\varepsilon^{-2}\,\varepsilon^{\lambda}\,\mathbf{b}^0_\lambda\,\big(\big[\Delta_X, \chi\big]\,\mathfrak{s}^\lambda\big)(X). \tag{4.9}$$

This function is smooth with compact support. Let V^λ be the solution of the variational problem in Ω,

$$\textit{Find}\ \ V^\lambda \in V(\Omega)\ \ \textit{such that}\ \ -\Delta V^\lambda = \mathbf{b}^0_\lambda\big(\big[\Delta_X, \chi\big]\,\mathfrak{s}^\lambda\big)\ \ \textit{in}\ \ \Omega. \tag{4.10}$$

Then it is clear that $\Delta_x\big(\varepsilon^\lambda V^\lambda(\frac{x}{\varepsilon})\big)$ coincides with the function (4.9). Therefore a better start for the asymptotic expansion of u_ε reads

$$\chi(\tfrac{x}{\varepsilon})v^0(x) + \psi(x)\varepsilon^\lambda V^\lambda(\tfrac{x}{\varepsilon}),$$

[1] Note that we use boldface for the coefficients $\mathbf{b}^0_{p\lambda}$, because we do not yet know whether they coincide with the coefficients $b^0_{p\lambda}$ already defined in Section 3.2. Indeed, the coincidence will be actually shown in Section 5.

which satisfies the Dirichlet boundary conditions on $\partial\omega_\varepsilon$, and the associated remainder r_ε^λ is defined as

$$u_\varepsilon(x) = \chi(\tfrac{x}{\varepsilon})v^0(x) + \psi(x)\varepsilon^\lambda V^\lambda(\tfrac{x}{\varepsilon}) + r_\varepsilon^\lambda(x).$$

Since $\psi \equiv 1$ on the support of the right hand side (for ε small enough) (4.9), we find

$$\Delta r_\varepsilon^\lambda(x) = -\big[\Delta, \chi(\tfrac{x}{\varepsilon})\big]v_\lambda^0(x) - \big[\Delta, \psi\big]\varepsilon^\lambda V^\lambda(\tfrac{x}{\varepsilon}) \tag{4.11}$$

with

$$v_\lambda^0(x) = v^0(x) - \mathbf{b}_\lambda^0\mathfrak{s}^\lambda(x) \underset{R\to\infty}{\simeq} \sum_{p=2}^{\infty}\mathbf{b}_{p\lambda}^0\varepsilon^{p\lambda}\mathbf{b}_{p\lambda}^0\mathfrak{s}^{p\lambda}(\tfrac{x}{\varepsilon}). \tag{4.12}$$

Note that here, the sum starts at $p = 2$ instead of $p = 1$ in (4.8). We have gained one power of ε^λ.

Next, we express the other part of the remainder in slow variables. Thanks to Lemma 2.10, we have $V^\lambda(X) = \mathcal{O}_{R\to\infty}(1)$. Thus Proposition 2.13 yields that V^λ expands at infinity as

$$V^\lambda(X) \underset{R\to\infty}{\simeq} \sum_{p=1}^{\infty}\mathbf{B}_{p\lambda}^\lambda\,\mathfrak{s}^{-p\lambda}(X). \tag{4.13}$$

Since $\Delta\mathfrak{s}^{-p\lambda} = 0$, we find

$$\big[\Delta, \psi\big]\varepsilon^\lambda V^\lambda(\tfrac{x}{\varepsilon}) \underset{\varepsilon\to 0}{\simeq} \sum_{p=1}^{\infty}\varepsilon^{(1+p)\lambda}\mathbf{B}_{p\lambda}^\lambda\,\big[\Delta, \psi\big]\mathfrak{s}^{-p\lambda}(x). \tag{4.14}$$

The terms in (4.14) start with $\varepsilon^{2\lambda}$. They can be compensated by the solution of problems in ω. We set $v^\lambda = 0$.

Step 2. Next we define $v^{2\lambda}$ as the solution of the problem in slow variables in ω

$$\textit{Find } v^{2\lambda} \in \mathrm{H}_0^1(\omega) \textit{ such that } -\Delta v^{2\lambda}(x) = \mathbf{B}_\lambda^\lambda\,\big[\Delta, \psi\big]\,\mathfrak{s}^{-\lambda}(x), \tag{4.15}$$

and $V^{2\lambda}$ as the solution of the problem in fast variables in Ω (compare with (4.10))

$$\textit{Find } V^{2\lambda} \in V(\Omega) \textit{ such that } -\Delta V^{2\lambda}(X) = \mathbf{b}_{2\lambda}^0\,\big[\Delta, \chi\big]\,\mathfrak{s}^{2\lambda}(X). \tag{4.16}$$

Then, the beginning of the asymptotic expansion becomes

$$u_\varepsilon(x) = \chi(\tfrac{x}{\varepsilon})\Big(v^0(x) + \varepsilon^{2\lambda}v^{2\lambda}(x)\Big) + \psi(x)\Big(\varepsilon^\lambda V^\lambda(\tfrac{x}{\varepsilon}) + \varepsilon^{2\lambda}V^{2\lambda}(\tfrac{x}{\varepsilon})\Big) + r_\varepsilon^{2\lambda}(x). \tag{4.17}$$

4.2.2 The general construction

Construction by induction Let us assume the asymptotic expansion built up to order $n-1$, i.e.

$$u_\varepsilon(x) = \chi(\tfrac{x}{\varepsilon})\sum_{\ell=0}^{n-1}\varepsilon^{\ell\lambda}v^{\ell\lambda}(x) + \psi(x)\sum_{\ell=1}^{n-1}\varepsilon^{\ell\lambda}V^{\ell\lambda}(\tfrac{x}{\varepsilon}) + r_\varepsilon^{(n-1)\lambda}(x), \tag{4.18}$$

with $v^{\ell\lambda} \in \mathrm{H}_0^1(\omega)$ and $V^{\ell\lambda} \in V(\Omega)$ (see Definition (2.8)), whose Laplacians vanish in a neighborhood of zero and ∞, respectively. For $\ell = 0, \dots, n-1$, we expand the term $v^{\ell\lambda}$ into singular functions at the corner point (see Proposition 2.6)

$$v^{\ell\lambda}(x) \underset{r\to 0}{\simeq} \sum_{p=1}^{+\infty} \mathbf{b}_{p\lambda}^{\ell\lambda} \mathfrak{s}^{p\lambda}(x), \tag{4.19}$$

and, we also expand the profiles $V^{\ell\lambda}$ into dual singular functions at infinity (see Proposition 2.13)

$$V^{\ell\lambda}(X) \underset{R\to +\infty}{\simeq} \sum_{p=1}^{+\infty} \mathbf{B}_{p\lambda}^{\ell\lambda} \mathfrak{s}^{-p\lambda}(X). \tag{4.20}$$

The definitions for the next terms $v^{n\lambda}$ and $V^{n\lambda}$ follow from the computation of the residual, see thereafter. The function $v^{n\lambda} \in \mathrm{H}_0^1(\omega)$ solves

$$\Delta v^{n\lambda}(x) = -\Delta \left[\psi(x) \sum_{\ell=1}^{n-1} \mathbf{B}_{(n-\ell)\lambda}^{\ell\lambda} \mathfrak{s}^{-(n-\ell)\lambda}(x) \right], \tag{4.21}$$

and, $V^{n\lambda} \in V(\Omega)$ satisfies

$$\Delta V^{n\lambda}(X) = -\Delta \left[\chi(X) \sum_{\ell=0}^{n-1} \mathbf{b}_{(n-\ell)\lambda}^{\ell\lambda} \mathfrak{s}^{(n-\ell)\lambda}(X) \right]. \tag{4.22}$$

Computation of the residual By definition the Laplacian of the remainder is given by

$$\Delta r_\varepsilon^{(n-1)\lambda}(x) = \Delta u^\varepsilon - \sum_{\ell=0}^{n-1} \Big[\varepsilon^{\ell\lambda} \Delta\big(\chi(\tfrac{x}{\varepsilon}) v^{\ell\lambda}(x)\big) + \varepsilon^{\ell\lambda} \Delta\big(\psi(x) V^{\ell\lambda}(\tfrac{x}{\varepsilon})\big) \Big]. \tag{4.23}$$

Next, we expand this relation using (4.19) and (4.20), and we obtain

$$\Delta r_\varepsilon^{(n-1)\lambda} = -\sum_{\ell=0}^{n-1} \Big[\varepsilon^{\ell\lambda} \big([\Delta, \chi(\tfrac{\cdot}{\varepsilon})] v_{(n-1-\ell)\lambda}^{\ell\lambda} \big) + \varepsilon^{\ell\lambda} \big([\Delta, \psi] V_{(n-1-\ell)\lambda}^{\ell\lambda}(\tfrac{\cdot}{\varepsilon}) \big) \Big]. \tag{4.24}$$

with

$$\begin{aligned} v_{k\lambda}^{\ell\lambda}(x) := v^{\ell\lambda}(x) - \sum_{p=1}^{k} \mathbf{b}_{p\lambda}^{\ell\lambda} \mathfrak{s}^{p\lambda}(x) & \underset{r\to 0}{\simeq} \sum_{p=k+1}^{+\infty} \mathbf{b}_{p\lambda}^{\ell\lambda} \mathfrak{s}^{p\lambda}(x) \\ & = \mathcal{O}_{r\to 0}(r^{(k+1)\lambda}), \end{aligned} \tag{4.25}$$

$$\begin{aligned} V_{k\lambda}^{\ell\lambda}(X) := V^{\ell\lambda}(X) - \sum_{p=1}^{k} \mathbf{B}_{p\lambda}^{\ell\lambda} \mathfrak{s}^{-p\lambda}(X) & \underset{R\to \infty}{\simeq} \sum_{p=k+1}^{+\infty} \mathbf{B}_{p\lambda}^{\ell\lambda} \mathfrak{s}^{-p\lambda}(X) \\ & = \mathcal{O}_{R\to +\infty}(R^{-(k+1)\lambda}). \end{aligned} \tag{4.26}$$

The leading term of the remainder $\Delta r_\varepsilon^{(n-1)\lambda}$ corresponds to the first term in the sums (4.25) and (4.26), and, is therefore

$$\Delta\left[\sum_{\ell=0}^{n-1}\varepsilon^{\ell\lambda}\mathbf{b}^{\ell\lambda}_{(n-\ell)\lambda}\mathfrak{s}^{(n-\ell)\lambda}(x)\chi(\tfrac{x}{\varepsilon})\right]+\Delta\left[\sum_{\ell=1}^{n-1}\varepsilon^{\ell\lambda}\mathbf{B}^{\ell\lambda}_{(n-\ell)\lambda}\mathfrak{s}^{-(n-\ell)\lambda}(\tfrac{x}{\varepsilon})\psi(x)\right]$$

which leads after scaling to

$$\varepsilon^{n\lambda}\left(\Delta\left[\sum_{\ell=0}^{n-1}\mathbf{b}^{\ell\lambda}_{(n-\ell)\lambda}\mathfrak{s}^{(n-\ell)\lambda}(\tfrac{x}{\varepsilon})\chi(\tfrac{x}{\varepsilon})\right]+\Delta\left[\sum_{\ell=1}^{n-1}\mathbf{B}^{\ell\lambda}_{(n-\ell)\lambda}\mathfrak{s}^{-(n-\ell)\lambda}(x)\psi(x)\right]\right).$$

These terms are compensated by $v^{n\ell}$ and $V^{n\ell}$, see (4.21) and (4.22).

4.3 Optimal error estimate

Theorem 4.1 *The solution u_ε of problem* (1.3) *admits the following multiscale expansion into powers of ε (λ is connected to the opening angle of ω at O by $\lambda=\pi/\alpha$).*

$$u_\varepsilon(x)=\chi(\tfrac{x}{\varepsilon})\sum_{\ell=0}^{n}\varepsilon^{\ell\lambda}v^{\ell\lambda}(x)+\psi(x)\sum_{\ell=0}^{n}\varepsilon^{\ell\lambda}V^{\ell\lambda}(\tfrac{x}{\varepsilon})+r_\varepsilon^{n\lambda}, \tag{4.27}$$

where the terms $v^{\ell\lambda}$ and $V^{\ell\lambda}$ do not depend on ε, and are defined in ω and Ω by Equations (4.21) *and* (4.22)*, respectively. Moreover, the remainder $r_\varepsilon^{n\lambda}$ satisfies the following estimate*

$$\|r_\varepsilon^{n\lambda}\|_{\mathrm{H}^1(\omega_\varepsilon)}\leqslant C\varepsilon^{(n+1)\lambda}. \tag{4.28}$$

PROOF. A basic idea to estimate the remainder consists in investigating the Laplace-Dirichlet problem it solves. By construction, $r_\varepsilon^{n\lambda}$ satisfies the homogeneous Dirichlet condition on $\partial\omega_\varepsilon$. Moreover, its Laplacian has the expression (4.24) and has to be estimated in the L^2-norm.

For all v, the commutator of Δ and $\chi(\frac{\cdot}{\varepsilon})$ is given by

$$\big([\Delta,\chi(\tfrac{\cdot}{\varepsilon})]v\big)(x)=\frac{2}{\varepsilon}\nabla v(x)\cdot\nabla\chi(\tfrac{x}{\varepsilon})+\frac{1}{\varepsilon^2}v(x)\Delta\chi(\tfrac{x}{\varepsilon}). \tag{4.29}$$

Hence, the support of $[\Delta,\chi(\frac{\cdot}{\varepsilon})]v$ is included in the sector $\{x\in K\mid R^*\varepsilon\leqslant r\leqslant 2R^*\varepsilon\}$. For $v^{\ell\lambda}_{k\lambda}=\mathcal{O}_{r\to0}(r^{(k+1)\lambda})$, one can obtain the L^∞-bound

$$\left|\big([\Delta,\chi(\tfrac{\cdot}{\varepsilon})]v^{\ell\lambda}_{k\lambda}\big)(x)\right|\leqslant C\,\varepsilon^{(k+1)\lambda-2}\qquad\forall x\in\omega_\varepsilon. \tag{4.30}$$

This leads via Hölder inequality to

$$\left\|[\Delta,\chi(\tfrac{\cdot}{\varepsilon})]v^{\ell\lambda}_{k\lambda}\right\|_{\mathrm{L}^2(\omega_\varepsilon)}\leqslant C\,\varepsilon^{(k+1)\lambda-1}. \tag{4.31}$$

In the same way,

$$\left\|[\Delta,\psi]V^{\ell\lambda}_{k\lambda}(\tfrac{\cdot}{\varepsilon})\right\|_{\mathrm{L}^2(\omega_\varepsilon)}\leqslant C\,\varepsilon^{(k+1)\lambda}. \tag{4.32}$$

One deduces immediately from (4.24), (4.31) and (4.32)

$$\|\Delta r_\varepsilon^{n\lambda}\|_{\mathrm{L}^2(\omega_\varepsilon)} \leqslant \sum_{\ell=0}^{n} \Big(\varepsilon^{\ell\lambda} \Big\| [\Delta, \chi(\tfrac{\cdot}{\varepsilon})] v^{\ell\lambda}_{(n+1-\ell)\lambda} \Big\|_{\mathrm{L}^2(\omega_\varepsilon)} + \varepsilon^{\ell\lambda} \Big\| [\Delta, \psi] V^{\ell\lambda}_{(n+1-\ell)\lambda}(\tfrac{\cdot}{\varepsilon}) \Big\|_{\mathrm{L}^2(\omega_\varepsilon)} \Big) \leqslant C\, \varepsilon^{(n+1)\lambda-1}. \quad (4.33)$$

Using an *a priori* estimate (independent of ε) on problem (1.3), we immediately obtain the bound

$$\|r_\varepsilon^{n\lambda}\|_{\mathrm{H}^1(\omega_\varepsilon)} \leqslant C\varepsilon^{(n+1)\lambda-1}. \quad (4.34)$$

To get (4.28), we just need to write the asymptotic expansion at order $n+2$

$$r_\varepsilon^{n\lambda} = r_\varepsilon^{(n+2)\lambda} + \chi(\tfrac{x}{\varepsilon}) \sum_{\ell=n+1}^{n+2} \varepsilon^{\ell\lambda} v^{\ell\lambda}(x) + \psi(x) \sum_{\ell=n+1}^{n+2} \varepsilon^{\ell\lambda} V^{\ell\lambda}(\tfrac{x}{\varepsilon}). \quad (4.35)$$

Indeed, thanks to (4.34), $\|r_\varepsilon^{(n+2)\lambda}\|_{\mathrm{H}^1(\omega_\varepsilon)} = \mathcal{O}(\varepsilon^{(n+3)\lambda-1}) = \mathcal{O}(\varepsilon^{(n+1)\lambda})$ since we have $\lambda \geqslant \frac{1}{2}$. The result will be proven as soon as we show the following energy estimates

$$\|\chi(\tfrac{x}{\varepsilon}) v^{\ell\lambda}(x)\|_{\mathrm{H}^1(\omega_\varepsilon)} = \mathcal{O}(1) \text{ and } \|\psi(x) V^{\ell\lambda}(\tfrac{x}{\varepsilon})\|_{\mathrm{H}^1(\omega_\varepsilon)} = \mathcal{O}(1). \quad (4.36)$$

As the left estimate of (4.36) is easier to obtain than the right one, we will just deal with the latter. We need to use the behavior at infinity of the profile $V^{\ell\lambda}$. Since $V^{\ell\lambda} \in V(\Omega)$, one has

$$\nabla V^{\ell\lambda} \in \mathrm{L}^2(\Omega) \quad \text{and} \quad (1+R)^{-1}\, V^{\ell\lambda} \in \mathrm{L}^2(\Omega). \quad (4.37)$$

Therefore, one has

$$\int_{\omega_\varepsilon} \varepsilon^{-2} |\psi(x) \nabla V^{\ell\lambda}(\tfrac{x}{\varepsilon})|^2 \mathrm{d}x = \int_\Omega |\psi(\varepsilon X) \nabla V^{\ell\lambda}(X)|^2 \mathrm{d}X \leqslant \int_\Omega |\nabla V^{\ell\lambda}(X)|^2 \mathrm{d}X = \mathcal{O}(1).$$

By the same way, we get

$$\int_{\omega_\varepsilon} \big(|\psi(x)|^2 + |\nabla\psi(x)|^2\big) |V^{\ell\lambda}(\tfrac{x}{\varepsilon})|^2 \mathrm{d}x \;\leqslant\; C \int_{\{X\in\Omega; |X| \leqslant \frac{r^*}{\varepsilon}\}} \varepsilon^2 |V^{\ell\lambda}(X)|^2 \mathrm{d}X.$$

As $|X|\varepsilon \;\leqslant\; r^*$ in the last integral, one has

$$\int_{\omega_\varepsilon} \big(|\psi(x)|^2 + |\nabla\psi(x)|^2\big) |V^{\ell\lambda}(\tfrac{x}{\varepsilon})|^2 \mathrm{d}x \;\leqslant\; C' \int_\Omega \frac{|V^{\ell\lambda}(X)|^2}{(1+|X|)^2} \mathrm{d}X = \mathcal{O}(1).$$

Estimates (4.36) follow. ■

5 Comparison of the two expansions

In Section 3, starting from the outer and corner (matched) expansions, we were able to build a global asymptotic expansion for the solution u_ε of problem (1.3), see expression (3.14). Using the multiscale technique, we proved in Section 4 another asymptotic expansion, which is also valid in the whole domain ω_ε. The global error estimates given in Theorems 3.2 and 4.1 allow to compare these expansions.

Theorem 5.1 *The expansions* (3.14) *and* (4.27) *compare in the following way:*
- *The terms $u^{n\lambda}$ and $v^{n\lambda}$ coincide away from the corner point i.e. for $r \geqslant r^*$;*
- *The profiles $U^{n\lambda}$ and $V^{n\lambda}$ coincide in the corner region i.e. for $R \leqslant R^*/2$.*

More precisely, we have the identities

$$\left\{ \begin{array}{rcl} v^{n\lambda}(x) & = & u^{n\lambda}(x) - \psi(x) \displaystyle\sum_{p=1}^{n-1} a^{n\lambda}_{p\lambda}\, \mathfrak{s}^{-p\lambda}(x), \\ V^{n\lambda}(X) & = & U^{n\lambda}(X) - \chi(X) \displaystyle\sum_{p=1}^{n} A^{n\lambda}_{p\lambda}\, \mathfrak{s}^{p\lambda}(X). \end{array} \right. \tag{5.1}$$

where the coefficients $a^{n\lambda}_{p\lambda}$ and $A^{n\lambda}_{p\lambda}$, are those defined in Section 3.2.

PROOF. The first two statements follow directly from the optimal estimates, (3.24), (3.25), (4.27), and (4.28), via localization. To get formulas (5.1), we start from problem (4.22) which defines $V^{n\lambda}$. We set

$$\widetilde{U}^{n\lambda}(X) = V^{n\lambda}(X) + \chi(X) \sum_{\ell=0}^{n-1} \mathbf{b}^{\ell\lambda}_{(n-\ell)\lambda}\, \mathfrak{s}^{(n-\ell)\lambda}(X) \tag{5.2}$$

$$= V^{n\lambda}(X) + \chi(X) \sum_{p=1}^{n} \mathbf{b}^{(n-p)\lambda}_{p\lambda}\, \mathfrak{s}^{p\lambda}(X). \tag{5.3}$$

From the definition of $V^{n\lambda}$ (see (4.22)), $\widetilde{U}^{n\lambda}$ satisfies $\Delta\widetilde{U}^{n\lambda} = 0$ in Ω. Hence, one has

$$\left\{ \begin{array}{l} \widetilde{U}^{n\lambda} - U^{n\lambda} \in C^\infty(\Omega), \\ \Delta[\widetilde{U}^{n\lambda} - U^{n\lambda}] = 0 \ \text{ in } \ \Omega, \\ \widetilde{U}^{n\lambda}(X) - U^{n\lambda}(X) = 0 \ \text{ for } \ R \leqslant R^*/2. \end{array} \right. \tag{5.4}$$

Since $\widetilde{U}^{n\lambda} - U^{n\lambda}$ is harmonic, it is analytic in Ω. Hence, by unique continuation Theorem, $U^{n\lambda} = \widetilde{U}^{n\lambda}$. Moreover, as $V^{n\lambda}$ is a $\mathcal{O}_{R\to\infty}(1)$, one has $A^{n\lambda}_{p\lambda} = \mathbf{b}^{(n-p)\lambda}_{p\lambda}$

$$U^{n\lambda}(X) = V^{n\lambda}(X) + \chi(X) \sum_{p=1}^{n} A^{n\lambda}_{p\lambda}\, \mathfrak{s}^{p\lambda}(X). \tag{5.5}$$

The same argumentation can be done for $u^{n\lambda}$. ■

Remark 5.2 As can be seen in (5.3), another formula linking the two expansions is

$$\left\{\begin{array}{lcl} u^{n\lambda}(x) & = & v^{n\lambda}(x) + \psi(x) \displaystyle\sum_{p=1}^{n-1} \mathbf{B}_{p\lambda}^{(n-p)\lambda}\, \mathfrak{s}^{-p\lambda}(x), \\ U^{n\lambda}(X) & = & V^{n\lambda}(X) + \chi(X) \displaystyle\sum_{p=1}^{n} \mathbf{b}_{p\lambda}^{(n-p)\lambda}\, \mathfrak{s}^{p\lambda}(X). \end{array}\right. \tag{5.6}$$

Moreover, as $A_{p\lambda}^{n\lambda} = \mathbf{b}_{p\lambda}^{(n-p)\lambda}$ and due to the matching condition (3.10), one has

$$\mathbf{B}_{p\lambda}^{\ell\lambda} = B_{p\lambda}^{\ell\lambda} \quad \text{and} \quad \mathbf{b}_{p\lambda}^{\ell\lambda} = b_{p\lambda}^{\ell\lambda}, \quad \forall \ell \in \mathbb{N},\ \forall p \in \mathbb{N}^*. \tag{5.7}$$

Remark 5.3 The mechanism to switch from expansion (4.27) to expansion (3.14) consists in using the homogeneity of the singular functions $\mathfrak{s}^{p\lambda}$ to pass them from fast variables into slow variables

$$\begin{aligned} \psi(x) \sum_{\ell=0}^{n} \varepsilon^{\ell\lambda} V^{\ell\lambda}(\tfrac{x}{\varepsilon}) &= \psi(x) \sum_{\ell=0}^{n} \varepsilon^{\ell\lambda} \Big[U^{\ell\lambda}(\tfrac{x}{\varepsilon}) - \chi(\tfrac{x}{\varepsilon}) \sum_{p=1}^{\ell} A_{p\lambda}^{\ell\lambda} \mathfrak{s}^{p\lambda}(\tfrac{x}{\varepsilon}) \Big] \\ &= \psi(x) \sum_{\ell=0}^{n} \varepsilon^{\ell\lambda} U^{\ell\lambda}(\tfrac{x}{\varepsilon}) - \chi(\tfrac{x}{\varepsilon}) \psi(x) \sum_{\ell=0}^{n} \sum_{p=1}^{\ell} \varepsilon^{(\ell-p)\lambda} A_{p\lambda}^{\ell\lambda} \mathfrak{s}^{p\lambda}(x) \\ &= \psi(x) \sum_{\ell=0}^{n} \varepsilon^{\ell\lambda} U^{\ell\lambda}(\tfrac{x}{\varepsilon}) - \chi(\tfrac{x}{\varepsilon}) \psi(x) \sum_{j=0}^{n} \varepsilon^{j\lambda} \sum_{p=0}^{n-j} \underbrace{A_{p\lambda}^{(p+\ell)\lambda}}_{=b_{p\lambda}^{\ell\lambda}} \mathfrak{s}^{p\lambda}(x). \end{aligned}$$

The second term involves the slow variable and will contribute to the terms $(u^{\ell\lambda})$ in the intermediate region.

Finally, it turns out that it is very easy to obtain one expansion from the other, via formulas (5.1). We emphasize however the particularities of each method

- The matched asymptotic expansions method builds outer and corner terms which are canonical, i.e. they do depend only on the domains ω and Ω, and not on cut-off functions, as it is the case for the multiscale technique;
- The multiscale technique gives a straightforward global approximation of the solution, with optimal estimates of the remainder, whereas more effort is needed in the case of matched asymptotic expansions.

6 Extensions and generalizations

Our results can be more or less easily generalized to other situations of interest. We classify these situations according to

1. Laplace operator with more general data, domains, boundary conditions.
2. Other elliptic operators, homogeneous with constant coefficients or not.

Here we discuss these generalizations within the multiscale approach. Of course, via translations formulas like (5.1), similar extensions apply to matched asymptotic expansions.

6.1 Laplace equation in more general situations

6.1.1 Domains with multiply connected junction sectors

This is the situation where the family of domains (ω_ε) is defined like in Section 1.1, where we relax the assumption on the set K, which was supposed to be a plane sector with opening $\alpha \in (0, 2\pi]$. Our results extend to the situation where K is a finite disjoint union of plane sectors $K_1, \ldots, K_m$ with their vertices at O. Accordingly, we relax the assumption on ω which is still open and bounded, but can be multiply connected. The unbounded open set Ω can also be multiply connected.

The open sets ω_ε have still to be connected. If $m = 2$, this requires that either ω or Ω should be connected. Of course, the interesting case occurs when Ω is connected, see Figures 6 and 7.

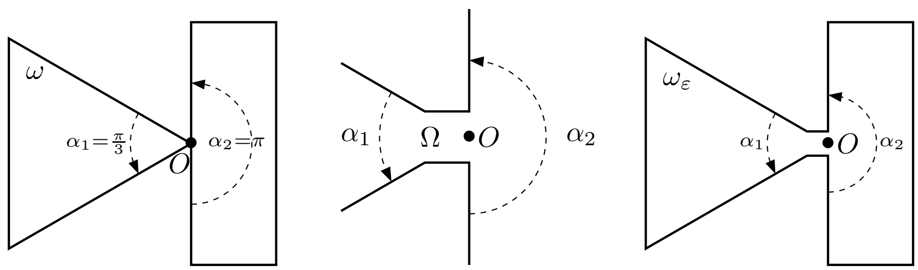

Figure 6: Example of domains ω, Ω and ω_ε in the multiply connected case.

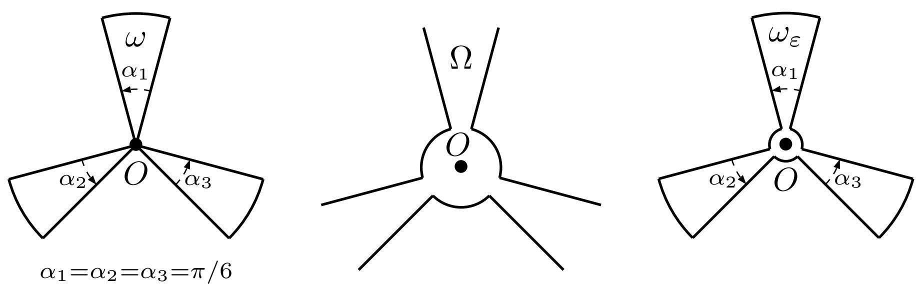

Figure 7: Example of domains ω, Ω and ω_ε in the multiply connected case.

The generalization of our expansions to this situation is straightforward. We denote by $\alpha_1, \ldots, \alpha_m$ the openings of the sectors $K_1, \ldots, K_m$, and set

$$\lambda_1 = \frac{\pi}{\alpha_1}, \ldots, \lambda_m = \frac{\pi}{\alpha_m}.$$

The multiscale expansion of u_ε solution of the Dirichlet problem (1.3) with a right hand side $f \equiv 0$ in a neighborhood of O is as follows. For all real number $s > 0$ there holds

$$u_\varepsilon = \sum_{\substack{p_1,\dots,p_m \in \mathbb{N} \\ p_1\lambda_1+\dots+p_m\lambda_m < s}} \varepsilon^{p_1\lambda_1+\dots+p_m\lambda_m} \Big(\chi(\tfrac{x}{\varepsilon})\, v^{p_1\lambda_1+\dots+p_m\lambda_m}(x) + \psi(x)\, V^{p_1\lambda_1+\dots+p_m\lambda_m}(\tfrac{x}{\varepsilon}) \Big) + \mathcal{O}_{\mathrm{H}^1}(\varepsilon^s). \tag{6.1}$$

Here $V^0 = 0$, and $v^{\lambda_j} = 0$ for $j = 1, \dots, m$.

6.1.2 Smooth data without condition of support

Until now we have assumed that the right hand side f of problem (1.3) is zero in a neighborhood of the limit point O of the ε-perturbation. If we want to relax this assumption, we have to assume that f has an asymptotics as $r \to 0$, e.g. f is the restriction to ω_ε of a $\mathcal{C}^\infty$ function $\bar{f}$ defined on a neighborhood of $\cup_{\varepsilon \leqslant \varepsilon_0} \omega_\varepsilon$

$$f(x) \underset{r\to 0}{\simeq} \sum_{q=0}^{+\infty} f^q(r,\theta) \quad \text{with} \quad f^q(x) = \varepsilon^q\, f^q(\tfrac{x}{\varepsilon}). \tag{6.2}$$

We consider the latter case and revisit the multiscale expansion (in the case when K is a sector of opening $\alpha = \frac{\pi}{\lambda}$). We still define v^0 as the solution of the limit problem (4.3), where the right hand side $f = \bar{f}|_\omega$ belongs to $\mathcal{C}^\infty(\bar{\omega})$. Then instead of the infinite expansion (4.4) we have now

$$v^0(r,\theta) \underset{r\to 0}{\simeq} \sum_{p=1}^{\infty} \mathbf{b}^0_{p\lambda}\, \mathfrak{s}^{p\lambda}(r,\theta) + \sum_{q=1}^{\infty} \mathfrak{T}^q(r,\theta). \tag{6.3}$$

Here, $\mathfrak{T}^q$ is of the form, if $q \notin \lambda\mathbb{N}$,

$$\mathfrak{T}^q(x) = \varepsilon^q\, \mathfrak{T}^q(\tfrac{x}{\varepsilon}), \tag{6.4}$$

and, if $q = p\lambda$, is a linear combination of $\mathfrak{T}^q_1(x)$, homogeneous function of degree q, and $\mathfrak{T}^q_2$, the logarithmic singularity $\mathfrak{t}^q$ defined as $\mathfrak{t}^q(r,\theta) = r^{p\lambda} \log r \sin p\lambda\theta$,

$$\mathfrak{T}^q(x) = \mathfrak{T}^q_1(x) + \mathfrak{T}^q_2(x), \tag{6.5}$$

with

$$\begin{cases} \mathfrak{T}^q_1(x) = \varepsilon^q\, \mathfrak{T}^q_1(\tfrac{x}{\varepsilon}), \\ \mathfrak{T}^q_2(x) = \gamma_q \mathfrak{t}^q, \quad \gamma_q \in \mathbb{R}. \end{cases} \tag{6.6}$$

We still consider the first remainder $r^0_\varepsilon = u_\varepsilon - \chi(\frac{x}{\varepsilon})v^0$.

Case $\lambda \notin \mathbb{Q}$. If $\lambda \notin \mathbb{Q}$, there are no logarithmic terms and now, instead of (4.8), we obtain for the Laplacian of the first remainder

$$\Delta r^0_\varepsilon(x) \underset{r\to 0}{\simeq} -\big(1 - \chi(\tfrac{x}{\varepsilon})\big) f(x) \;-\; \big[\Delta, \chi(\tfrac{x}{\varepsilon})\big] v^0(x). \tag{6.7}$$

This leads to

$$\Delta r^0_\varepsilon \underset{r\to 0}{\simeq} -\frac{1}{\varepsilon^2}\Big[\big(1-\chi\big)\Big(\sum_{q=2}^{+\infty}\varepsilon^q\, f^{q-2}\Big)\Big(\tfrac{x}{\varepsilon}\Big) \\ + \Big(\sum_{p=1}^{\infty}\varepsilon^{p\lambda}\,\mathbf{b}^0_{p\lambda}\,\big[\Delta_X,\chi\big]\,\mathfrak{s}^{p\lambda} + \sum_{q=1}^{\infty}\varepsilon^q\,\big[\Delta_X,\chi\big]\,\mathfrak{T}^q\Big)\Big(\tfrac{x}{\varepsilon}\Big)\Big]. \tag{6.8}$$

Thus, the terms $\big[\Delta_X,\chi\big]\,\mathfrak{T}^q$ enter the construction of fast variable contributions V^q, associated with the power ε^q.

Then, apart this first generation of integer powers of ε, the expansions as $r\to 0$ or $R\to\infty$ of the subsequent terms contain the same functions $\mathfrak{s}^{p\lambda}$ and $\mathfrak{s}^{-p\lambda}$ as previously. We can prove the following result.

Theorem 6.1 *Let $(\omega_\varepsilon)_{\varepsilon<\varepsilon_0}$ be a family of domains of type* (1.1) *with $\lambda=\frac{\pi}{\alpha}\notin\mathbb{Q}$. Let u_ε be the solution of problem* (1.3) *for a smooth right hand side f, see* (6.2). *Then there exist terms $v^\tau\in V(\omega)$ and $V^\tau\in V(\Omega)$ for $\tau=p\lambda+q$ ($p,q\in\mathbb{N}$), such that for all real number $s>0$ there holds*

$$u_\varepsilon = \sum_{\substack{p,\,q\in\mathbb{N}\\ p\lambda+q<s}}\varepsilon^{p\lambda+q}\Big(\chi(\tfrac{x}{\varepsilon})\,v^{p\lambda+q}(x)+\psi(x)\,V^{p\lambda+q}(\tfrac{x}{\varepsilon})\Big)+\mathcal{O}_{\mathrm{H}^1}(\varepsilon^s). \tag{6.9}$$

Here $V^0=0$, $v^\lambda=0$, and, $v^{p\lambda+q}=0$ for all q if $p=0$.

Case $\lambda\in\mathbb{Q}$. If $p\lambda=q$, the logarithmic singularity $\mathfrak{t}^q$ satisfies

$$\mathfrak{t}^q(x)=\varepsilon^q\,\mathfrak{t}^q(\tfrac{x}{\varepsilon})+\varepsilon^q\log\varepsilon\,\mathfrak{s}^q(\tfrac{x}{\varepsilon}),$$

whence the presence of the terms $\varepsilon^q\log\varepsilon$ in Δr^0_ε. Taking this into account, we prove that instead of (6.9) we have an expansion of u_ε containing the terms in (6.9) and, moreover,

$$\sum_{\substack{p,\,q\in\mathbb{N}\,;\,q=p^*\lambda\\ p\lambda+q<s}}\varepsilon^{p\lambda+q}\log\varepsilon\Big(\chi(\tfrac{x}{\varepsilon})\,v_1^{p\lambda+q}(x)+\psi(x)\,V_1^{p\lambda+q}(\tfrac{x}{\varepsilon})\Big), \tag{6.10}$$

with new terms $v_1^{p\lambda+q}$ and $V_1^{p\lambda+q}$ solutions of variational problems in ω and Ω, respectively.

6.1.3 Neumann boundary conditions

Instead of (1.3) let us consider the problem

$$\textit{Find }\ u_\varepsilon\in\mathrm{H}^1(\omega_\varepsilon)\ \textit{ such that }\ \forall v\in\mathrm{H}^1(\omega_\varepsilon),\ \int_{\omega_\varepsilon}\nabla u_\varepsilon\cdot\nabla v\,\mathrm{d}x=\int_{\omega_\varepsilon}f\,v\,\mathrm{d}x. \tag{6.11}$$

Besides we need the compatibility condition

$$\int_{\omega_\varepsilon}f\,\mathrm{d}x=0,\quad \forall\varepsilon<\varepsilon_0. \tag{6.12}$$

We assume that $f \equiv 0$ in a neighborhood of O. Therefore the condition

$$\int_\omega f \,\mathrm{d}x = 0$$

implies condition (6.12) for ε small enough. To ensure uniqueness, we require

$$\int_{\omega_\varepsilon} u_\varepsilon \,\mathrm{d}x = 0, \quad \forall \varepsilon < \varepsilon_0. \tag{6.13}$$

The construction of the multiscale expansion for u_ε relies on the solution of variational Neumann problems in ω and Ω. In the unbounded domain Ω, the variational space is $V(\Omega)$ defined as

$$\{U \in \mathscr{D}'(\Omega)\ ;\ \ \nabla U \in \mathrm{L}^2(\Omega),\ \ (1+R)^{-1}\log^{-1}(2+R)\, U \in \mathrm{L}^2(\Omega)\}. \tag{6.14}$$

The bilinear form

$$(U,V) \longmapsto \int_\Omega \nabla U \cdot \nabla V \,\mathrm{d}x$$

is continuous and coercive on the quotient space $V(\Omega)/\mathbb{R}$, see [2, Prop. 3.22]. Therefore, like in ω, the solution of the Neumann problem in Ω with right hand side F requires the compatibility condition

$$\int_\Omega F \,\mathrm{d}X = 0.$$

Thus new features have to be taken into account in order to deal with the Neumann problem.

1. **Compatibility conditions.** The right hand sides which occur during the construction have the form $\big[\Delta, \psi\big]\, \mathfrak{s}^{-p\lambda}$ in ω and $\big[\Delta_X, \chi\big]\, \mathfrak{s}^{p\lambda}$ in Ω, with the Neumann singularities $\mathfrak{s}^{p\lambda} = \rho^{p\lambda}\cos p\lambda\theta$. Since $\mathfrak{s}^{p\lambda}$ is harmonic, these right hand sides are equal to $\Delta(\psi\, \mathfrak{s}^{-p\lambda})$ and $\Delta_X(\chi\, \mathfrak{s}^{p\lambda})$, respectively. Since $\psi\, \mathfrak{s}^{-p\lambda}$ and $\chi\, \mathfrak{s}^{p\lambda}$ satisfy the Neumann boundary condition on $\partial\omega$ and $\partial\Omega$, respectively, we can show that the compatibility conditions are fulfilled for all integer $p \geq 1$.
2. **The role of constants.** *(i)* The asymptotic expansion of v^0 at O starts with $\mathbf{b}_0^0\mathfrak{s}^0$, which is a constant. The associated problem in fast variables is, cf (4.10)

   $$-\Delta V^0 = \mathbf{b}_0^0 \Delta_X \chi \ \text{ in } \ \Omega \quad \text{and} \quad \partial_n V^0 = 0 \ \text{ on } \ \partial\Omega. \tag{6.15}$$

   We choose the solution $V^0 = \mathbf{b}_0^0(1-\chi)$. Thus, in particular, $\psi(x)V^0(\frac{x}{\varepsilon}) = V^0(\frac{x}{\varepsilon})$: The cut-off by ψ does not introduce any error. Let us notice that

   $$\mathbf{b}_0^0\big(\chi(\tfrac{x}{\varepsilon}) + \psi(x)(1-\chi)(\tfrac{x}{\varepsilon})\big)$$

   represents the extension of a constant from ω to ω_ε.
   (ii) For problems in Ω giving $V^{p\lambda}$, $p \geq 1$, we choose the variational solution which tends to 0 as $R \to \infty$.

3. **The condition for uniqueness** (6.13)**.** By construction the slow variable terms $v^{p\lambda}$ have a zero integral on ω. Using their asymptotics as $r \to 0$ we find that

$$\int_{\omega_\varepsilon} \chi(\tfrac{x}{\varepsilon})\, v^{p\lambda}(x)\, \mathrm{d}x = \beta_p\, \varepsilon^2, \quad \beta_p \in \mathbb{R}.$$

For fast variable terms we find

$$\int_{\omega_\varepsilon} \psi(x)\, V^{p\lambda}(\tfrac{x}{\varepsilon})\, \mathrm{d}x = \beta'_p\, \varepsilon^2, \quad \beta'_p \in \mathbb{R}.$$

We compensate the possible non-zero integral of the multiscale expansion by a series of constant functions – with values $\gamma_{p,n} \in \mathbb{R}$, $p \in \mathbb{N}$, $n \in \mathbb{N}^*$ – associated with the gauge functions $\varepsilon^{p\lambda+2n}$. Taking into account that $\int_{\omega_\varepsilon} \mathrm{d}x = \operatorname{meas}\omega + c\varepsilon^2$. The $\gamma_{p,n}$ are defined by forcing the formal equality

$$\sum_{p=0}^{+\infty} \varepsilon^{p\lambda+2} \left(\beta_p + \beta'_p\right) + \left(\operatorname{meas}\omega + c\varepsilon^2\right) \sum_{p=0}^{+\infty}\sum_{n=1}^{+\infty} \varepsilon^{2n+p\lambda}\gamma_{p,n} = 0. \tag{6.16}$$

In the end we obtain

Theorem 6.2 *Let* $(\omega_\varepsilon)_{\varepsilon<\varepsilon_0}$ *be a family of domains of type* (1.1)*. Let* u_ε *be the solution of problem* (6.11) *with condition* (6.13) *for a right hand side* f *with support away from* O *and satisfying the compatibility condition* (6.12)*. Then there exist terms* $v^{p\lambda} \in V(\omega)$ *and* $V^{p\lambda} \in V(\Omega)$ *for* $p \in \mathbb{N}$*, and constants* $\gamma_{p,n}$ *such that for all real number* $s > 0$ *there holds*

$$u_\varepsilon = \sum_{\substack{p\in\mathbb{N}\\ p\lambda<s}} \varepsilon^{p\lambda}\Big(\chi(\tfrac{x}{\varepsilon})\, v^{p\lambda}(x) + \psi(x)\, V^{p\lambda}(\tfrac{x}{\varepsilon}) + \sum_{\substack{n\in\mathbb{N}^*\\ p\lambda+2n<s}} \gamma_{p,n}\varepsilon^{2n}\Big) + \mathcal{O}_{\mathrm{H}^1}(\varepsilon^s). \tag{6.17}$$

6.1.4 Case when the junction set is the whole plane (small holes)

The set $K = \mathbb{R}^2$ may also be convenient as a junction set. It allows us to consider the case of small holes of size ε inside ω_ε. This is indeed the first case considered in the book [13, sec. 2.4.1], see Figure 8. Let us consider the Dirichlet case. Then we are in a situation which shares common features with the Dirichlet case investigated in the most part of this paper, and the Neumann case considered above.

Indeed, the limit problem in ω is uniquely solvable. But the limit problem in Ω is not coercive on the subspace of $\mathrm{W}^1_0(\Omega)$ with zero trace on $\partial\Omega$. The correct variational space is the subspace of the space (6.14) with zero trace on $\partial\Omega$. Nevertheless, arguments are slightly different from the Neumann case (like in [2], the asymptotic behavior $\log R$ as $R \to \infty$ has to be considered). The outcome of the analysis is the appearance of terms $\log^{-1}\varepsilon$.

In the multiscale expansion, cut-off functions $\chi(\frac{x}{\varepsilon})$ and $\psi(x)$ can be simply omitted since ω_ε is a subset of ω and of $\varepsilon\Omega$.

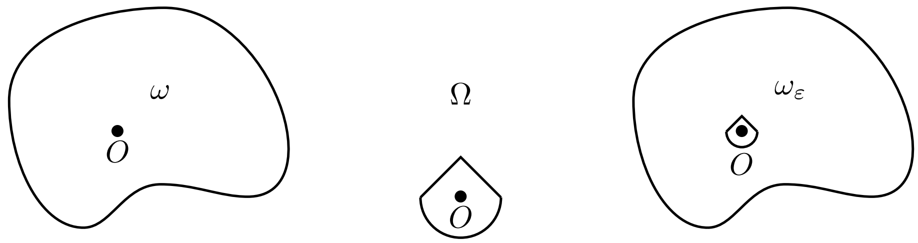

Figure 8: The domains ω, Ω and ω_ε in the case of small holes

6.2 Elliptic operators and systems

For the generalization to other operators and systems, two features are of essential importance:

1. The well-posedness of the limit problems on ω and Ω is essential since we can see from the above case of the Neumann condition for the Laplacian that relaxing this assumption is not easy.
2. If, moreover, the operators are homogeneous with constant coefficients, the extension of the results of the Dirichlet-Laplace case are almost straightforward. In contrast, taking lower order terms (like for the Helmholtz equation) or variable coefficients into account requires much more technicalities.

6.2.1 Coercive homogeneous operators with constant coefficients

Let $L = (L_{ij})$ a $N \times N$ system of homogeneous operators of order 2 with constant coefficients. We assume its coercivity on H^1_0: There exists a ball $\mathcal{B}$ and a constant $c > 0$ such that for all $u \in \mathrm{H}^1_0(\mathcal{B})^N$ there holds

$$\mathrm{Re} \int_{\mathcal{B}} Lu \cdot \bar{u} \, \mathrm{d}x \geq c \|u\|^2_{\mathrm{H}^1_0(\mathcal{B})}.$$

Then the Dirichlet problem

$$\textit{Find } u_\varepsilon \in \mathrm{H}^1_0(\omega_\varepsilon)^N \textit{ such that } Lu_\varepsilon = f|_{\omega_\varepsilon} \textit{ in } \omega_\varepsilon, \tag{6.18}$$

is uniquely solvable for all $f \in \mathrm{L}^2(\mathbb{R}^2)^N$. We assume that $f \equiv 0$ in a neighborhood of O.

The limit problem on ω is uniquely solvable too. Moreover, we can prove that for any $F \in \mathrm{L}^2(\Omega)^N$ with compact support, the following Dirichlet problem in Ω has a unique solution

$$\textit{Find } U \in \mathrm{W}^1_0(\Omega)^N \textit{ such that } LU = F \textit{ in } \Omega \textit{ and } U = 0 \textit{ on } \partial\Omega. \tag{6.19}$$

Then with the help of the Kondrat'ev theory [10] we can prove the analogue of Propositions 2.6 and 2.13. Now, the singularities of L in the sector K with Dirichlet conditions replace the singularities $\mathfrak{s}^{p\lambda}$ of Δ. A generating set for these singularities takes the form

$$\rho^{\mu} \sum_{q=0}^{Q} \log^q \rho \; \varphi_q(\theta), \quad \mu \in \mathfrak{S}(L)$$

where $\mathfrak{S}(L)$ is a discrete set in $\mathbb{C}$, such that any strip of the form $\operatorname{Re}\mu \in [a, b]$ contains a finite number of elements of $\mathfrak{S}(L)$. As a consequence of the coercivity there are no elements of $\mathfrak{S}(L)$ in the line $\operatorname{Re}\mu = 0$. Therefore, we can order the elements of $\mathfrak{S}(L)$ into two sequences

$$\lambda_1^+, \ldots, \lambda_p^+, \ldots \quad \text{with} \quad 0 < \operatorname{Re}\lambda_1^+ \leqslant \ldots \leqslant \operatorname{Re}\lambda_p^+ \leqslant \ldots$$

and

$$-\lambda_1^-, \ldots, -\lambda_p^-, \ldots \quad \text{with} \quad 0 < \operatorname{Re}\lambda_1^- \leqslant \ldots \leqslant \operatorname{Re}\lambda_p^- \leqslant \ldots$$

Then, by the same techniques as for the Laplace equation, we can prove that the asymptotics as $\varepsilon \to 0$ of u_ε solution of (6.18) can be written for all $s > 0$ as the sum of the following terms

- $\chi(\frac{x}{\varepsilon})\, v^0(x)$ with the solution v^0 of the limit problem on ω,
- Slow variable terms of the form

$$\varepsilon^{\lambda_{j_1}^+ + \lambda_{k_1}^- + \cdots + \lambda_{j_m}^+ + \lambda_{k_m}^-} \sum_{q=0}^{Q} \log^q \varepsilon \; \chi(\tfrac{x}{\varepsilon})\, v_q(x)$$

  for any integers $1 \leqslant j_1 \leqslant \ldots \leqslant j_m$ and $1 \leqslant k_1 \leqslant \ldots \leqslant k_m$ such that $\operatorname{Re}\lambda_{j_1}^+ + \operatorname{Re}\lambda_{k_1}^- + \cdots + \operatorname{Re}\lambda_{j_m}^+ + \operatorname{Re}\lambda_{k_m}^- \leqslant s$.
- Fast variable terms of the form

$$\varepsilon^{\lambda_{j_1}^+ + \lambda_{k_1}^- + \cdots + \lambda_{k_{m-1}}^- + \lambda_{j_m}^+} \sum_{q=0}^{Q} \log^q \varepsilon \; \psi(x)\, V_q(\tfrac{x}{\varepsilon})$$

  for any integers $1 \leqslant j_1 \leqslant \ldots \leqslant j_m$ and $1 \leqslant k_1 \leqslant \ldots \leqslant k_{m-1}$ such that $\operatorname{Re}\lambda_{j_1}^+ + \operatorname{Re}\lambda_{k_1}^- + \cdots + \operatorname{Re}\lambda_{k_{m-1}}^- + \operatorname{Re}\lambda_{j_m}^+ \leqslant s$,
- A remainder $\mathcal{O}_{\mathrm{H}^1}(\varepsilon^s)$.

6.2.2 Non homogeneous operators (Helmholtz equation)

The Helmholtz operator is of particular importance in computational physic since it is one of the operator modeling the wave propagation in frequency domain. The multiscale technique can also be used in this case. This is rather more technical since this operator is not self-similar

$$\Delta_x \, + \, \omega^2 = \frac{1}{\varepsilon^2}\Big(\Delta_X \, + \, \varepsilon^2 \, \omega^2\Big). \tag{6.20}$$

The study of the singularities of the fields in rapid variable is particularly difficult since it involves two terms of different orders

$$\Delta_X\, U_i \;+\; \omega^2\, U_{i-2} = 0. \tag{6.21}$$

A second difficulty comes from the loss of ellipticity (for $\omega > 0$). One has consequently to modify the proof of existence and of convergence. One can for example refer to [13, Ch.4] and to [9, 17].

7 Conclusion: A practical two-term expansion

In order to investigate the influence of singular perturbations of the domain on a local functional ϕ_ε acting over the solution u_ε, it is valuable to use a compound version of the asymptotic expansion, in between multiscale and matched asymptotic expansions.

7.1 Compound expansion

Indeed, using (1.8) and the relation (1.12) between the profiles V^λ and U^λ, we get

$$u_\varepsilon = \chi(\tfrac{x}{\varepsilon})u_0(x) + \psi(x)\varepsilon^\lambda \Big[U^\lambda(\tfrac{x}{\varepsilon}) - \chi(\tfrac{x}{\varepsilon})A\mathfrak{s}^\lambda(\tfrac{x}{\varepsilon})\Big] + \mathcal{O}_{\mathrm{H}^1}(\varepsilon^{2\lambda}),$$

which can be written, thanks to the homogeneity of the singular function $\mathfrak{s}^\lambda$

$$u_\varepsilon = \chi(\tfrac{x}{\varepsilon})\Big[u_0(x) - A\psi(x)\mathfrak{s}^\lambda(x)\Big] + \psi(x)\varepsilon^\lambda U^\lambda(\tfrac{x}{\varepsilon}) + \mathcal{O}_{\mathrm{H}^1}(\varepsilon^{2\lambda}).$$

Let us introduce the first "canonical" profile U^λ_Ω as the solution of the super-variational Dirichlet problem on Ω

$$\left\{\begin{array}{l} \textit{Find } U^\lambda_\Omega \in V_{\mathrm{loc},\infty}(\Omega) \textit{ such that} \\ \quad \Delta U^\lambda_\Omega = 0 \textit{ in } \Omega \quad \textit{and} \quad U^\lambda_\Omega - \mathfrak{s}^\lambda = \mathcal{O}_{R\to\infty}(1). \end{array}\right. \tag{7.1}$$

We have $U^\lambda = AU^\lambda_\Omega$ and, hence

$$u_\varepsilon = \chi(\tfrac{x}{\varepsilon})\Big[u_0(x) - A\psi(x)\mathfrak{s}^\lambda(x)\Big] + \psi(x)\varepsilon^\lambda AU^\lambda_\Omega(\tfrac{x}{\varepsilon}) + \mathcal{O}_{\mathrm{H}^1}(\varepsilon^{2\lambda}). \tag{7.2}$$

In (7.2), only canonical objects are involved: the limit term u_0, its first singularity coefficient A, and the first profile U^λ_Ω. The contribution near the corner is fully contained in the profile AU^λ_Ω, whereas the "far-field" information is carried out by $u_0 - A\psi\mathfrak{s}^\lambda$, corresponding to the limit term without its first singularity. In a sense, the strongest singularity of u_0 is "chopped off" for $\varepsilon > 0$ via the cut-off $\chi(\frac{x}{\varepsilon})$, and is replaced with the profile AU^λ_Ω, which connects the local geometry around O with the plane sector of opening α at infinity.

7.2 Application: Coefficients of singularities, Stress Intensity Factors

An interesting application of expansion (7.2) in elasticity is the determination of Stress Intensity Factors at the tip of a short crack emanating from a sharp of a rounded V-notch, see [12]. More generally, the question is the determination of the asymptotic behavior of the coefficients of singularities of u_ε associated with the corners (or cracks) of the domain ω_ε *inside its perturbed region*. The functional $\phi_\varepsilon(u_\varepsilon)$ is then defined as the value of this coefficient of singularity, corresponding to a corner whose position depends on ε.

Indeed, to each corner point (or crack tip) d of the perturbation pattern Ω corresponds a corner point (or crack tip) d_ε of the perturbed domain ω_ε. In the examples shown in the introduction, two such points are involved in Figure 2 (associated with angles equal to $\pi/5$ and $4\pi/3$, respectively), and we have one crack tip in Figure 3.

The solution u_ε of the Laplace-Dirichlet problem (1.3) is singular at point d_ε, with the following first order approximation

$$u_\varepsilon(x) = \gamma_\varepsilon r_\varepsilon^\mu \sin(\mu\theta_\varepsilon) + \mathcal{O}_\varepsilon(r_\varepsilon^\mu), \quad \text{as } \ r_\varepsilon \to 0, \tag{7.3}$$

where $(r_\varepsilon, \theta_\varepsilon)$ denote the polar coordinates around d_ε. The exponent μ is the singular exponent corresponding to d_ε ($\mu = \pi/\vartheta$ for a corner of opening ϑ, $\mu = 1/2$ for a crack).

The functional ϕ_ε is defined as

$$\phi_\varepsilon(u_\varepsilon) = \gamma_\varepsilon.$$

Our results allow to give an asymptotic expansion of the singular coefficient γ_ε as $\varepsilon \to 0$: we still denote by λ the singular exponent associated with the limit problem in ω, see (1.6). Using (7.2), we get

$$u_\varepsilon(x) = \varepsilon^\lambda A U_\Omega^\lambda(\tfrac{x}{\varepsilon}) + \text{higher order profiles}, \quad \text{if } \ |x| \le \varepsilon r_*. \tag{7.4}$$

But the first canonical profile U_Ω^λ has a singularity at point d, associated with exponent μ

$$U_\Omega^\lambda(X) = \Gamma R_d^\mu \sin(\mu\Theta_d) + \mathcal{O}(R_d^\mu), \quad \text{as } \ R_d \to 0, \tag{7.5}$$

where (R_d, Θ_d) are the polar coordinates around point d. We have the relation

$$r_\varepsilon = \varepsilon R_d. \tag{7.6}$$

Back to the variable x, relations (7.4) to (7.5) lead to

$$u_\varepsilon(x) = \varepsilon^{\lambda-\mu} A\Gamma r_\varepsilon^\mu \sin(\mu\theta_\varepsilon) + \mathcal{O}(\varepsilon^{\lambda-\mu} r_\varepsilon^\mu), \quad \text{if } \ |x| \le \varepsilon r_*. \tag{7.7}$$

Putting (7.3) and (7.7) together, we obtain the expression of the singular coefficient γ_ε

$$\gamma_\varepsilon = \varepsilon^{\lambda-\mu} A\Gamma + \mathcal{O}(\varepsilon^{\lambda-\mu}). \tag{7.8}$$

It is worth noticing that a coefficient associated with a stronger singularity than the limit singularity ($\mu < \lambda$) will go to 0 whereas it will blow up to infinity for weaker singularities. It has to be linked to the appearance of singularities discussed above.

Examples. In the case of Figure 2 we have $\alpha = \pi/3$ and hence $\lambda = 3$. The coefficient associated with the corner of opening $\pi/5$ is a $\mathcal{O}(\varepsilon^{3-5}) = \mathcal{O}(\varepsilon^{-2})$, and the coefficient corresponding to the reentrant corner is $\mathcal{O}(\varepsilon^{3-4/3}) = \mathcal{O}(\varepsilon^{5/3})$. For the crack tip in Figure 3, the coefficient behaves like $\mathcal{O}(\varepsilon^{2/3-1/2}) = \mathcal{O}(\varepsilon^{1/6})$.

The above analysis also applies in the framework of elasticity, cf §6.2.1, and is the foundation of the investigation in [12]. We stress that a rigorous derivation of (7.8) with an optimal estimate for the remainder requires more effort in studying the singular-regular expansion of u_ε.

Expansion (7.2) could also be used to investigate the asymptotic behavior of other local functionals $\phi_\varepsilon(u_\varepsilon)$ relating, for example, to the maximal values of the stress tensor in elasticity.

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