Analytic anisotropic regularity in corner domains: A long march to 3D polyhedra.

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WONAPDE 2010

Third Chilean Workshop on Numerical Analysis of PDE

January 11 - 15, 2010

Outline

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Confluence of three theories

- Elliptic operators and systems, covering boundary conditions.
  Regularity in Sobolev spaces and in analytic classes.
  Founding fathers (1957 – 1967):
  AGMON, DOUGLIS, NIRENBERG, MORREY, LIONS, MAGENES...
- Elliptic BVP in corner domains.
  Singularities and regularity in weighted Sobolev spaces.
  Founding fathers (1967 – 1977):
  KONDRAT'EV, MAZ'YA, PLAMENEVSKII, GRISVARD...
- Mathematical theory of finite element method (FEM).
  Convergence analysis, *h*- and *p*-version.
  Founding fathers (1967 – 1977):
  BABUŠKA, STRANG, FIX, BRAMBLE, ZLAMAL, ZIENKIEWICZ, CÉA, RAVIART, CIARLET, ODEN, NÉDÉLEC...

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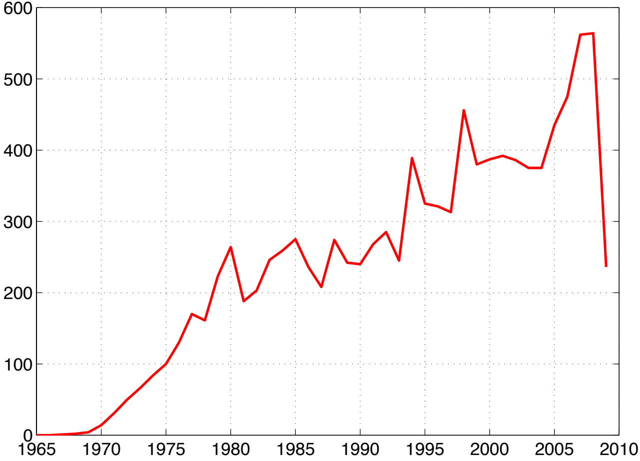


works containing “Finite Element” in their MathSciNet indexation, per year

Outline

Elliptic boundary value problems in smooth domains

Ω: smooth domain in $\mathbb{R}^n$ ($n \geq 2$): bounded and regular boundary.
Example: Ball, Ellipsoid.

L: second order elliptic operator or system with smooth coefficients.
Example: $L = \Delta$ (Laplacian), $L =$ Lamé system (elasticity)

B: operator of order $k = 0$ or 1 with smooth coeff. which "covers" L on $\partial\Omega$
Example: $B = Id$ (Dirichlet, $k = 0$),
$B =$ conormal derivative associated with L (Neumann, $k = 1$)

Problem :

Given f, find u

$$(BVP) \qquad \begin{cases} Lu = f & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

Sobolev Regularity Shift

Sobolev spaces

$$\mathrm{H}^m(\Omega) = \{v \in \mathscr{D}'(\Omega) : \ \partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} v \in \mathrm{L}^2(\Omega), \ |\boldsymbol{\alpha}| \le m\}$$

Theorem: [Agmon-Douglis-Nirenberg 1959, 1964]

Let $m \ge 2$. If $u \in \mathrm{H}^2(\Omega)$ solves (BVP) with

$$f \in \mathrm{H}^{m-2}(\Omega)$$

then $u \in \mathrm{H}^m(\Omega)$ with estimates

$$\|u\|_{\mathrm{H}^m(\Omega)} \le C \left\{ \|f\|_{\mathrm{H}^{m-2}(\Omega)} + \|u\|_{\mathrm{H}^1(\Omega)} \right\}.$$

Remark

If (BVP) has a coercive variational formulation in H^1, the above statement holds for $u \in \mathrm{H}^1(\Omega)$ with estimates (if the solution is unique)

$$\|u\|_{\mathrm{H}^m(\Omega)} \le C \|f\|_{\mathrm{H}^{m-2}(\Omega)}.$$

p-version of FEM (exponential convergence)

In the coercive variational framework.

- $\mathfrak{M}$: mesh, — fixed partition of Ω by a finite number of elements K,

  K affine-equivalent to $\begin{cases} \text{triangle or square in 2D} \\ \text{tetrahedron, cube or pentahedral prism in 3D} \end{cases}$

- $\mathfrak{V}_p$: space of piecewise mapped polynomials of deg. $\leq p$ on each K
- u_p: solution of Galerkin projection on space $\mathfrak{V}_p$

Theorem: [Morrey-Nirenberg 1957] and [Babuška-Guo 1986]

Assume

- $\partial\Omega$ is analytic,
- the coefficients of L and B are analytic,
- the rhs f is analytic,

then u is analytic and there is a $\delta > 0$ independent of u and p such that

$$\|u - u_p\|_{H^1(\Omega)} \leq C\, e^{-\delta p}.$$

Fundamental arguments for exponential convergence

1. *p*-version estimates. Basic estimate in reference interval $\widehat{\Lambda} = (-1, 1)$:

$$\|u - \pi^p u\|^2_{L^2(\widehat{\Lambda})} \leq \frac{(p+1-k)!}{(p+1+k)!} \|u^{(k)}\|^2_{L^2(\widehat{\Lambda})} \qquad 0 \leq k \leq p+1$$

   Here π^p is the orthogonal projection on Legendre polynomials of degree $\leq p$.

2. The proof that f analytic implies u analytic.

   A recent improvement is the proof of *analytic estimates*
   i.e. the analytic control of constants in the “standard” estimate

$$\|u\|_{H^m(\Omega)} \leq C(m) \left\{ \|f\|_{H^{m-2}(\Omega)} + \|u\|_{H^1(\Omega)} \right\}$$

$\cdots / \cdots$

Global analytic estimates

Theorem: [Costabel-Dauge-Nicaise 2010]

Assume

- $\partial\Omega$ is analytic,
- the coefficients of L and B are analytic,
- the rhs $f \in H^{m-2}(\Omega)$ for some $m \geq 2$.

Then u satisfies the a priori estimates of analytic type, $k = 0, 1, \ldots, m$

$$\frac{1}{k!} \sum_{|\boldsymbol{\alpha}|=k} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{L^2(\Omega)} \leq A^{k+1} \left\{ \sum_{\ell=0}^{k-2} \frac{1}{\ell!} \sum_{|\boldsymbol{\alpha}|=\ell} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f\|_{L^2(\Omega)} + \sum_{|\boldsymbol{\alpha}|\leq 1} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{L^2(\Omega)} \right\}$$

with a constant A independent of k, m and u.

Proof

- Nested open sets on model problems
- Faà di Bruno formula for local maps

Local analytic estimates (added after the talk)

With $\mathcal{U}$ and $\mathcal{U}'$ two open sets in $\mathbb{R}^2$ such that $\overline{\mathcal{U}} \subset \mathcal{U}'$, set

$$\mathcal{V} = \mathcal{U} \cap \Omega, \quad \mathcal{V}' = \mathcal{U}' \cap \Omega \quad \text{and} \quad \Gamma := \partial\mathcal{V}' \cap \partial\Omega$$

Assume that each connected component of Γ is an analytic curve in $\partial\Omega$.
We still assume that the coefficients of L and B are analytic.
Then u satisfies the *local a priori estimates* of analytic type, $k = 0, 1, \ldots, m$

$$\frac{1}{k!} \sum_{|\boldsymbol{\alpha}|=k} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{\mathrm{L}^2(\mathcal{V})} \leq A^{k+1} \Big\{ \sum_{\ell=0}^{k-2} \frac{1}{\ell!} \sum_{|\boldsymbol{\alpha}|=\ell} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f\|_{\mathrm{L}^2(\mathcal{V}')} + \sum_{|\boldsymbol{\alpha}|\leq 1} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{\mathrm{L}^2(\mathcal{V}')} \Big\}$$

with a constant A independent of k, m and u.

Outline

Corner domains (definition)

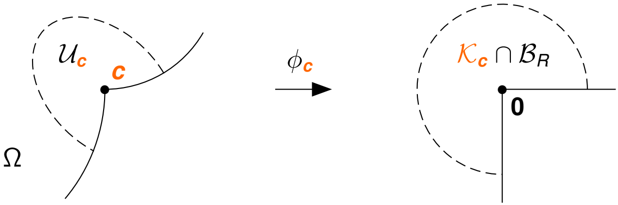


Figure: Corner domain: Local map (made with Fig4TeX)

Ω has a finite set $\mathscr{C}$ of corners c:

- *All corners are points*
- All corners c are in the boundary $\partial\Omega$ of Ω
- Around each boundary point $\boldsymbol{x}_0 \notin \mathscr{C}$, Ω is smooth
- Around each corner point $c \in \mathscr{C}$, Ω is *diffeomorphic to a cone* $\mathcal{K}_c$

Corner singularities

Our boundary value problem,

$$(BVP) \qquad \begin{cases} Lu = f & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

has non-smooth solutions u, even with a very smooth rhs $f \in \mathcal{C}^\infty(\overline{\Omega})$.
Solutions contain singular types at each corner $\boldsymbol{c}$

$$|\boldsymbol{x} - \boldsymbol{c}|^{\lambda_k} \varphi_k(\theta_{\boldsymbol{c}}), \quad k = 1, 2, \ldots$$

Here

- $(|\boldsymbol{x} - \boldsymbol{c}|, \theta_{\boldsymbol{c}})$ are polar coordinates at $\boldsymbol{c}$
- $\lambda_k \in \mathbb{C}$ are singular exponents
- $\varphi_k : \theta_{\boldsymbol{c}} \mapsto \varphi_k(\theta_{\boldsymbol{c}})$ are angular functions

Example : $L = \Delta$, with Dirichlet or Neumann BC's

- $\lambda_k = \dfrac{k\pi}{\omega_{\boldsymbol{c}}}$, $k = 1, 2, \ldots,$
- $\varphi_k(\theta) = \sin \lambda_k \theta$ for Dirichlet, $\varphi_k(\theta) = \cos \lambda_k \theta$ for Neumann.

Weighted Sobolev spaces

- Weight := powers of $r(\boldsymbol{x}) = \min_{\boldsymbol{c}\in\mathscr{C}} |\boldsymbol{x}-\boldsymbol{c}|$
- Weight exponent := $\beta \in \mathbb{R}$
- **Homogeneous weighted Sobolev spaces**
  KONDRAT'EV, MAZ'YA-PLAMENEVSKII, NAZAROV, ROSSMANN

$$\mathrm{K}^m_\beta(\Omega) = \{v \in \mathscr{D}'(\Omega) : \underbrace{r(\boldsymbol{x})^{|\boldsymbol{\alpha}|+\beta}}_{\text{depending on } \boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}}_{\boldsymbol{x}} v \in \mathrm{L}^2(\Omega),\ |\boldsymbol{\alpha}| \le m\}$$

  Solutions (including *singularities*) well described in scale $\mathrm{K}^m_\beta(\Omega)$.

- **Analytic limit**

$$\mathrm{A}_\beta(\Omega) = \Big\{v \in \bigcap_{m\in\mathbb{N}} \mathrm{K}^m_\beta(\Omega) : \sum_{|\boldsymbol{\alpha}|=m} \|r(\boldsymbol{x})^{m+\beta}\partial^{\boldsymbol{\alpha}}_{\boldsymbol{x}} v\|_{\mathrm{L}^2(\Omega)} \le C^{m+1} m!\Big\}$$

Remark

If $S = |\boldsymbol{x}-\boldsymbol{c}|^\lambda \varphi(\theta_{\boldsymbol{c}})$ is a singular function, then φ is *analytic*. Hence

$$\beta + \operatorname{Re}\lambda > -1 \quad\Longrightarrow\quad S \in \mathrm{K}^0_\beta(\Omega) \quad\Longrightarrow\quad S \in \mathrm{A}_\beta(\Omega)$$

Weighted analytic estimates

Theorem: [Costabel-Dauge-Nicaise 2010]

If

- Ω is an analytic corner domain (e.g., a polygon),
- L and B have analytic coefficients (e.g., constant coefficients),
- u solution of (BVP)

there exists a constant $C \geq 1$ indep. of u such that for all $k \in \mathbb{N}$,

$$\frac{1}{k!} \sum_{|\boldsymbol{\alpha}|=k} \| r^{\beta+|\boldsymbol{\alpha}|} \partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u \|_{\Omega} \leq C^{k+1} \Big\{ \sum_{\ell=0}^{k-2} \frac{1}{\ell!} \sum_{|\boldsymbol{\alpha}|=\ell} \| r^{\beta+2+|\boldsymbol{\alpha}|} \partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f \|_{\Omega} + \sum_{|\boldsymbol{\alpha}|\leq 1} \| r^{\beta+|\boldsymbol{\alpha}|} \partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u \|_{\Omega} \Big\}$$

Corollary

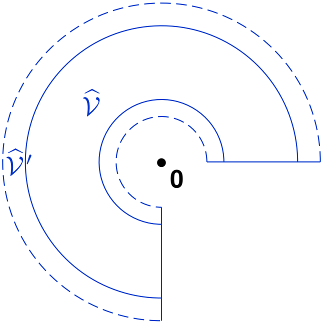
$$u \in \mathrm{K}^1_\beta(\Omega) \;\text{ and }\; f \in \mathrm{A}_{\beta+2}(\Omega) \quad \Longrightarrow \quad u \in \mathrm{A}_\beta(\Omega)$$

Outline

Proof of weighted analytic estimates

1. For simplicity:
   Ω polygon and L, B homogeneous with constant coeff.
2. Localization near a corner $\boldsymbol{c}$. Set $\boldsymbol{c} = \mathbf{0}$. We have $r = r(\boldsymbol{x}) = |\boldsymbol{x}|$
   Proof on a plane sector $\mathcal{K}$.
3. Regular reference configuration

$$\widehat{\mathcal{V}} = \{\boldsymbol{x} \in \mathcal{K},\ \tfrac{1}{2}-\varepsilon < r < 1\} \quad \& \quad \widehat{\mathcal{V}}' = \{\boldsymbol{x} \in \mathcal{K},\ \tfrac{1}{2}-2\varepsilon < r < 1+\varepsilon\}.$$



Proof of weighted analytic estimates

4. Reference estimate

$$\frac{1}{k!}\sum_{|\boldsymbol{\alpha}|=k}\|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{u}\|_{\widehat{\mathcal{V}}} \le A^{k+1}\left\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\sum_{|\boldsymbol{\alpha}|=\ell}\|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{f}\|_{\widehat{\mathcal{V}}'} + \sum_{|\boldsymbol{\alpha}|\le 1}\|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{u}\|_{\widehat{\mathcal{V}}'}\right\}$$

5. Insert the weight ($\widehat{r} \simeq 1$ on $\mathcal{V}'$)

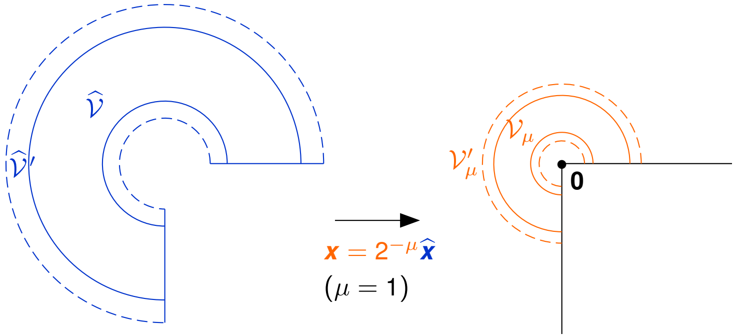
$$\frac{1}{k!}\sum_{|\boldsymbol{\alpha}|=k}\|\widehat{r}^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{u}\|_{\widehat{\mathcal{V}}} \le A^{k+1}\left\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\sum_{|\boldsymbol{\alpha}|=\ell}\|\widehat{r}^{\beta+2+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{f}\|_{\widehat{\mathcal{V}}'} + \sum_{|\boldsymbol{\alpha}|\le 1}\|\widehat{r}^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{u}\|_{\widehat{\mathcal{V}}'}\right\}$$

6. Locally finite covering $\mathcal{V}_\mu = 2^{-\mu}\widehat{\mathcal{V}}$ and $\mathcal{V}'_\mu = 2^{-\mu}\widehat{\mathcal{V}}'$, for $\mu = 1, 2, \ldots$

$$\mathcal{V} := \mathcal{K}\cap\mathcal{B}(\mathbf{0},1) = \bigcup_{\mu\in\mathbb{N}}\mathcal{V}_\mu \quad \text{and} \quad \mathcal{V}' := \mathcal{K}\cap\mathcal{B}(\mathbf{0},1+\varepsilon) = \bigcup_{\mu\in\mathbb{N}}\mathcal{V}'_\mu\,.$$

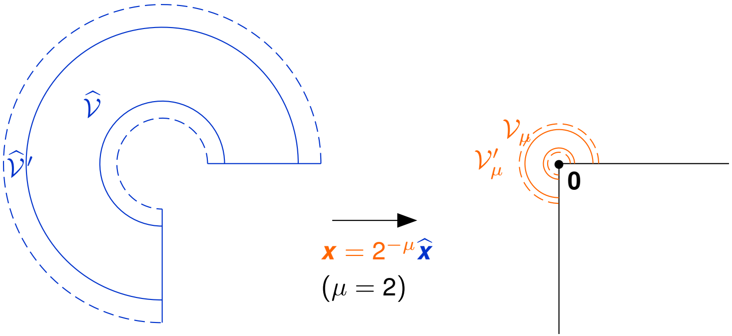
Proof of weighted analytic estimates

7. Scale on $\mathcal{V}_\mu = 2^{-\mu}\mathcal{V}$ and $\mathcal{V}'_\mu = 2^{-\mu}\mathcal{V}'$, for $\mu = 1, \dots$



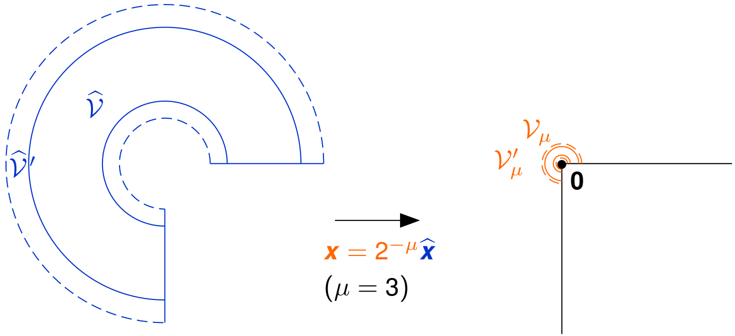
Proof of weighted analytic estimates

7. Scale on $\mathcal{V}_\mu = 2^{-\mu}\mathcal{V}$ and $\mathcal{V}'_\mu = 2^{-\mu}\mathcal{V}'$, for $\mu = 2, \dots$



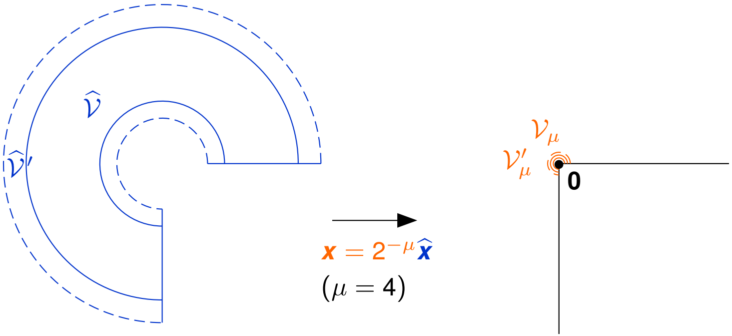
Proof of weighted analytic estimates

7. Scale on $\mathcal{V}_\mu = 2^{-\mu}\mathcal{V}$ and $\mathcal{V}'_\mu = 2^{-\mu}\mathcal{V}'$, for $\mu = 3, \dots$



Proof of weighted analytic estimates

7. Scale on $\mathcal{V}_\mu = 2^{-\mu}\mathcal{V}$ and $\mathcal{V}'_\mu = 2^{-\mu}\mathcal{V}'$, for $\mu = 4, \ldots$



Proof of weighted analytic estimates

8. To estimate u on $\mathcal{V}_\mu$ by $Lu = f$ on $\mathcal{V}'_\mu$ we set

$$\widehat{\boldsymbol{u}}(\widehat{\boldsymbol{x}}) := \boldsymbol{u}(\boldsymbol{x}) \quad \text{and} \quad \widehat{\boldsymbol{f}}(\widehat{\boldsymbol{x}}) := L\widehat{\boldsymbol{u}} \quad \text{which implies} \quad \widehat{\boldsymbol{f}}(\widehat{\boldsymbol{x}}) = 2^{-2\mu}\boldsymbol{f}(\boldsymbol{x}),$$

The reference estimate

$$\frac{1}{k!}\sum_{|\boldsymbol{\alpha}|=k} \|\widehat{r}^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{u}\|_{\widehat{\mathcal{V}}} \le A^{k+1}\Big\{ \sum_{\ell=0}^{k-2}\frac{1}{\ell!}\sum_{|\boldsymbol{\alpha}|=\ell}\|\widehat{r}^{\beta+2+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{f}\|_{\widehat{\mathcal{V}}'} + \sum_{|\boldsymbol{\alpha}|\le 1}\|\widehat{r}^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{u}\|_{\widehat{\mathcal{V}}'}\Big\}$$

becomes

$$\frac{1}{k!}\sum_{|\boldsymbol{\alpha}|=k} 2^{\mu\beta}\|r^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}u\|_{\mathcal{V}_\mu} \le A^{k+1}\Big\{ \sum_{\ell=0}^{k-2}\frac{1}{\ell!}\sum_{|\boldsymbol{\alpha}|=\ell}2^{\mu(\beta+2)}\|r^{\beta+2+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}2^{-2\mu}f\|_{\mathcal{V}'_\mu} + \sum_{|\boldsymbol{\alpha}|\le 1}2^{\mu\beta}\|r^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}u\|_{\mathcal{V}'_\mu}\Big\}$$

Proof of weighted analytic estimates

9. Eliminate the common factor $2^{\mu\beta}$ and square:

$$\left(\frac{1}{k!}\right)^2 \sum_{|\boldsymbol{\alpha}|=k} \|r^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{\mathcal{V}_\mu}^2 \leq A_*^{2k+2}\Big\{ \sum_{\ell=0}^{k-2}\left(\frac{1}{\ell!}\right)^2 \sum_{|\boldsymbol{\alpha}|=\ell} \|r^{\beta+2+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f\|_{\mathcal{V}'_\mu}^2 + \sum_{|\boldsymbol{\alpha}|\leq 1} \|r^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{\mathcal{V}'_\mu}^2 \Big\}$$

10. Sum $\mu \in \mathbb{N}$ and use the finite covering property

$$\left(\frac{1}{k!}\right)^2 \sum_{|\boldsymbol{\alpha}|=k} \|r^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{\mathcal{V}}^2 \leq CA_*^{2k+2}\Big\{ \sum_{\ell=0}^{k-2}\left(\frac{1}{\ell!}\right)^2 \sum_{|\boldsymbol{\alpha}|=\ell} \|r^{\beta+2+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f\|_{\mathcal{V}'}^2 + \sum_{|\boldsymbol{\alpha}|\leq 1} \|r^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{\mathcal{V}'}^2 \Big\}$$

11. QED

Outline

Weighted analytic regularity in polygons (Dirichlet)

- **Assume**: (BVP) has a coercive variational formulation.
- **NB**: Hardy ineq. $\Rightarrow \mathrm{H}^1(\Omega) \subset \mathrm{K}^1_{-1+\varepsilon}(\Omega)\ \forall\varepsilon > 0$ but $\mathrm{H}^1(\Omega) \not\subset \mathrm{K}^1_{-1}(\Omega)$
- **NB**: Poincaré ineq. $\Rightarrow \mathrm{H}^1_0(\Omega) \subset \mathrm{K}^1_{-1}(\Omega)$

Theorem: [Kondrat'ev 1967]

- In the Dirichlet case

there exists $b_{\Omega,L} > 0$ such that the following regularity holds.

$\forall b$, $\boxed{0 \le b < b_{\Omega,L}}$ and $\forall m \ge 1$

$$u \in \mathrm{H}^1_0(\Omega) \quad \text{and} \quad f \in \mathrm{K}^{m-1}_{-b+1}(\Omega) \quad \Longrightarrow \quad u \in \mathrm{K}^{m+1}_{-b-1}(\Omega)$$

Corollary: [Co-Da-Ni 2010]

$\forall b$, $\boxed{0 \le b < b_{\Omega,L}}$

$$u \in \mathrm{H}^1_0(\Omega) \quad \text{and} \quad f \in \mathrm{A}_{-b+1}(\Omega) \quad \Longrightarrow \quad u \in \mathrm{A}_{-b-1}(\Omega)$$

Weighted analytic regularity in polygons (Neumann)

For $-2 < \beta \leq -1$ and $m \geq 1$, replace in the definition of K^m_β and A_β

$$r^\beta u \in \mathrm{L}^2(\Omega) \quad \text{by} \quad r^{\beta+1} u \in \mathrm{L}^2(\Omega)$$

thus defining the new space $\mathrm{J}^m_\beta(\Omega)$ and new analytic class $\mathrm{B}_\beta(\Omega)$.

Theorem: [Maz'ya-Plamenevskii 1984]

There exists $b_{\Omega,L,B} > 0$ such that the following regularity holds.

$\forall b$, $0 < b < b_{\Omega,L,B}$ $\forall m \geq 1$, variational sol. u of (BVP) satisfy

$$f \in \mathrm{J}^{m-1}_{-b+1}(\Omega) \quad \Longrightarrow \quad u \in \mathrm{J}^{m+1}_{-b-1}(\Omega)$$

Theorem: [Co-Da-Ni 2010] *Cf.* [Babuška-Guo 1988, 1989, 1993]

$\forall b$, $0 < b < b_{\Omega,L,B}$ variational sol. u of (BVP) satisfy

$$f \in \mathrm{B}_{-b+1}(\Omega) \quad \Longrightarrow \quad u \in \mathrm{B}_{-b-1}(\Omega)$$

The trick for the proof...

Replace the estimate in the smooth case

u satisfies the a priori estimates of analytic type, $k = 0, 1, 2, \ldots$

$$\frac{1}{k!} \sum_{|\boldsymbol{\alpha}|=k} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{\Omega} \le A^{k+1} \Big\{ \sum_{\ell=0}^{k-2} \frac{1}{\ell!} \sum_{|\boldsymbol{\alpha}|=\ell} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f\|_{\Omega} + \sum_{|\boldsymbol{\alpha}|\le 1} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{\Omega} \Big\}$$

with a constant A independent of k and u.

...

... by

u satisfies the a priori estimates of analytic type, $k = 1, 2, \ldots$

$$\frac{1}{k!} \sum_{|\boldsymbol{\alpha}|=k} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{\Omega} \leq A^{k+1} \Big\{ \sum_{\ell=0}^{k-2} \frac{1}{\ell!} \sum_{|\boldsymbol{\alpha}|=\ell} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f\|_{\Omega} + \sum_{|\boldsymbol{\alpha}|=1} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{\Omega} \Big\}$$

with a constant A independent of k and u.

Mathematical outcome

1. **The proof is much simpler** than in original papers by BABUŠKA-GUO because it clearly separates
   - *the issue of basic regularity* (e.g. in $\mathrm{K}^2_\beta(\Omega)$ or $\mathrm{J}^2_\beta(\Omega)$)
   - *the issue of analytic regularity* (natural regularity shift)

   These two independent modules can be assembled.
2. **The proof can be adapted** without much effort to
   - *homogeneous multi-degree elliptic systems* with constant coeff. e.g. Stokes,
   - *transmission problems* e.g. div $a(\boldsymbol{x})\nabla$, with $\boldsymbol{x} \mapsto a(\boldsymbol{x})$ piecewise constant on a polygonal decomposition of Ω
3. **The generalization** to non-zero boundary conditions, variable (analytic) coefficients, non-homogeneous operators is feasible with the same arguments.

Numerical outcome

1. The regularity in analytic classes A_{-b-1} or B_{-b-1} for a $b > 0$ ensures **exponential convergence** of *hp* version of FEM [BABUŠKA-GUO 1986].
2. *hp* version of FEM consists in, simultaneously
   - *Increase* the degree
   - *Add a layer* of elements with geometrical refinement near corners
3. Next page: Example of *hp* FEM with refinement at the origin (intended for the checker board transmission problem)

Outline

hhp mesh for a 2x2 checker board

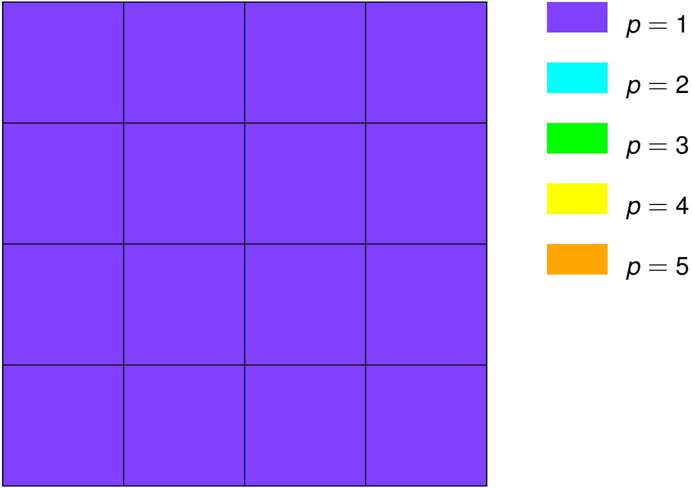


Figure: Level 1

hp mesh for a 2x2 checker board – ratio 2

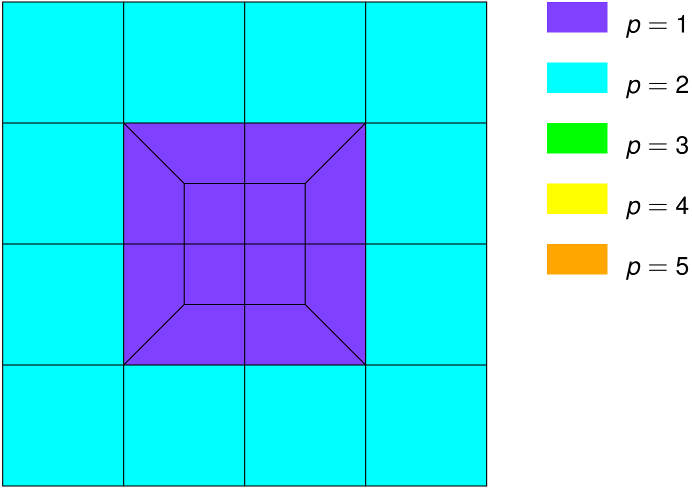


Figure: Level 2

hp mesh for a 2x2 checker board – ratio 2

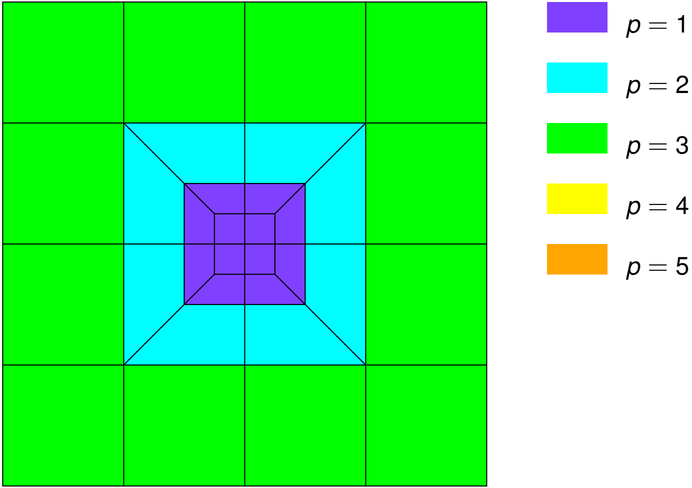


Figure: Level 3

hp mesh for a 2x2 checker board – ratio 2

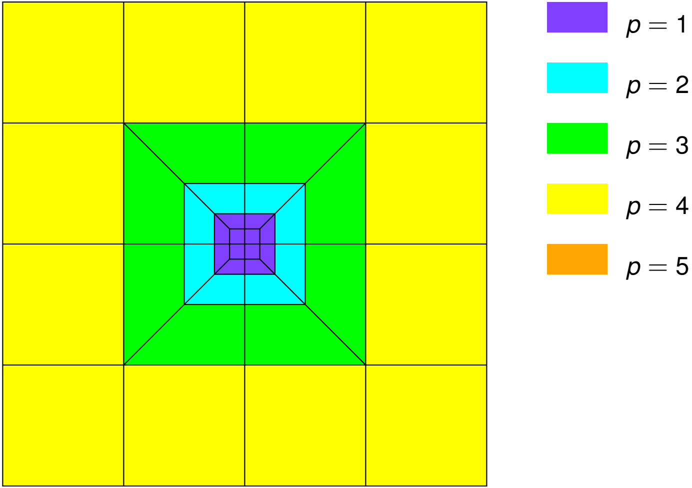


Figure: Level 4

hp mesh for a 2x2 checker board – ratio 2

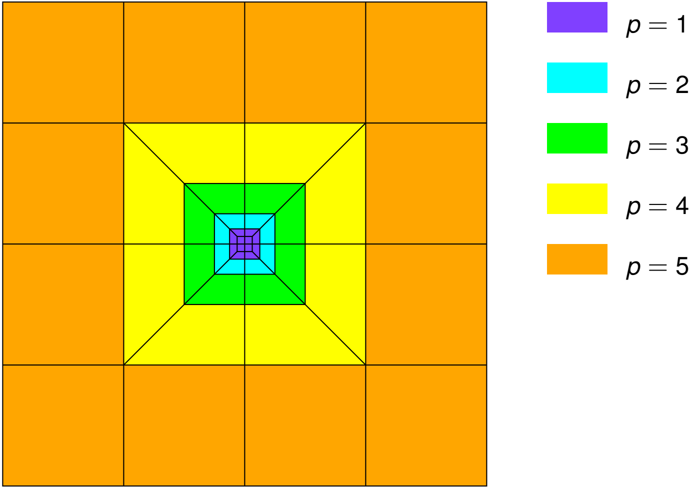


Figure: Level 5

hp mesh for a 2x2 checker board – ratio 4

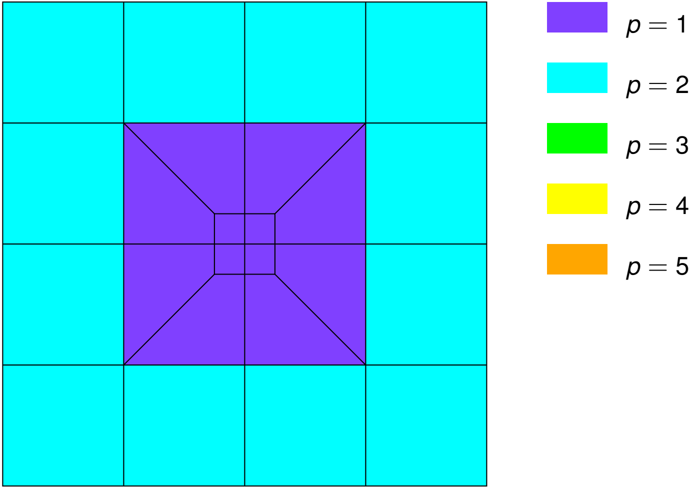


Figure: Level 2

hp mesh for a 2x2 checker board – ratio 4

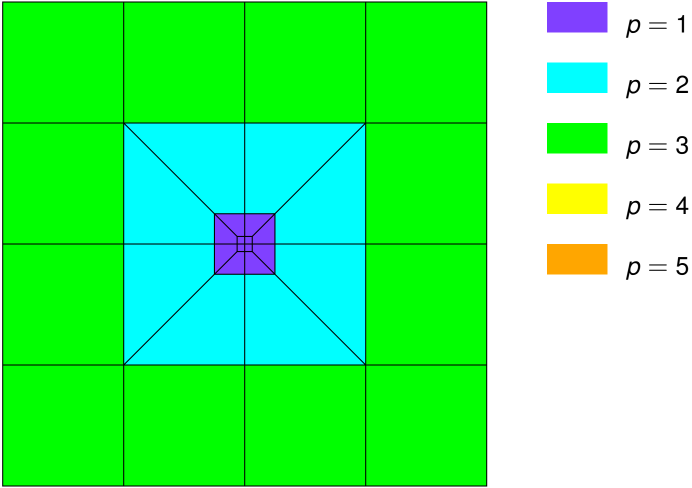


Figure: Level 3

hp mesh for a 2x2 checker board – ratio 8

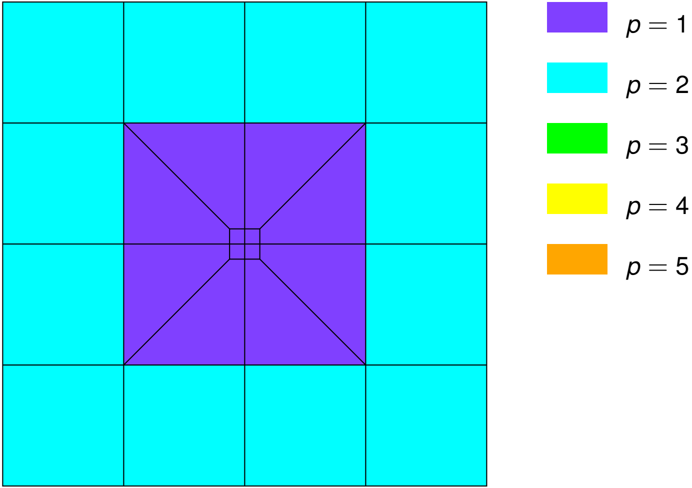


Figure: Level 2

hp mesh for a 2x2 checker board – ratio 8

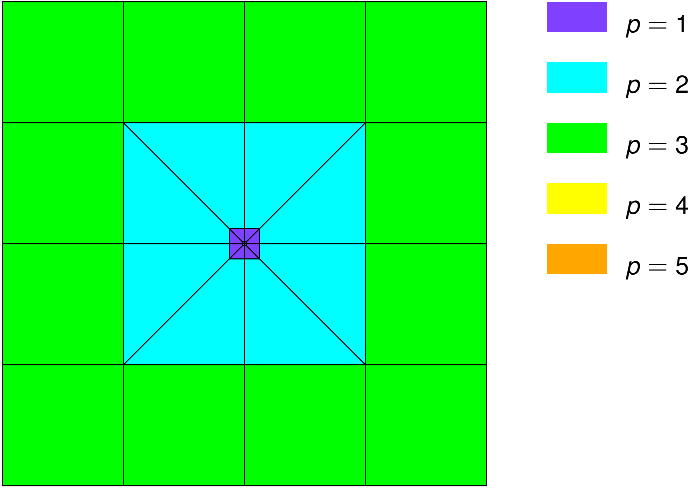


Figure: Level 3

Outline

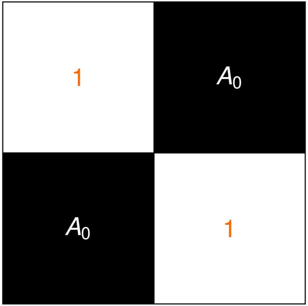
Neumann eigenvalues on a 2x2 checker board

We compute Neumann eigenvalues of $-\operatorname{div} A(\boldsymbol{x})\nabla$ on the square $(-1,1)^2$

$$-\operatorname{div} A(\boldsymbol{x})\nabla u = \lambda\, u$$

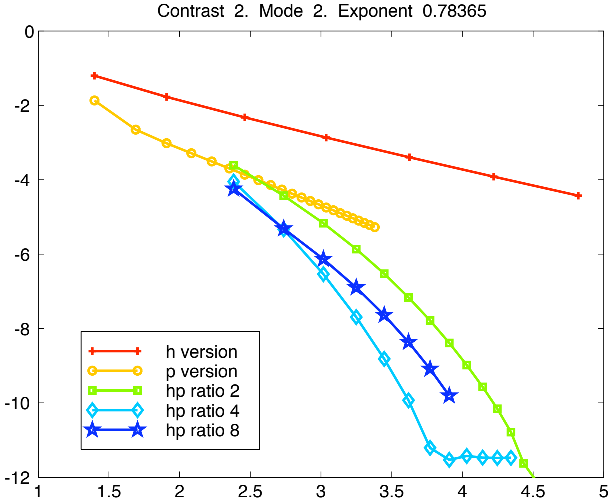
with

$$A \equiv 1 \text{ on } (-1,0)\times(0,1) \cup (0,1)\times(-1,0)$$
$$A \equiv A_0 \text{ on } (-1,0)^2 \cup (0,1)^2$$



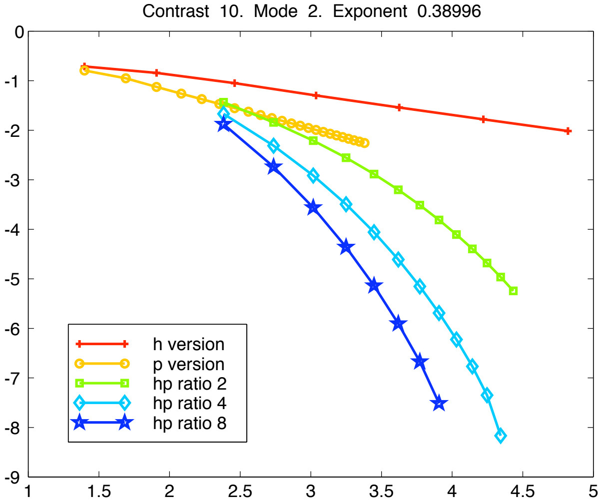
for $A_0 = 2,\ 10,\ 100,\ 10^8$

Error plots for more and more singular eigenvectors



Abcissa = log 10(#DOF) *Ordinate* = log 10(rel. error for eigenvalue)

Error plots for more and more singular eigenvectors



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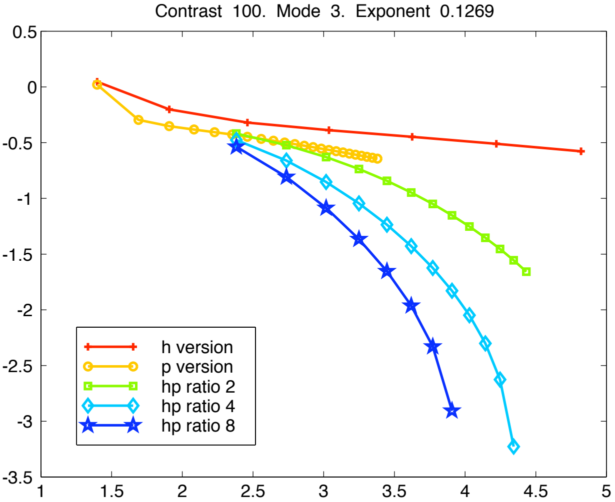
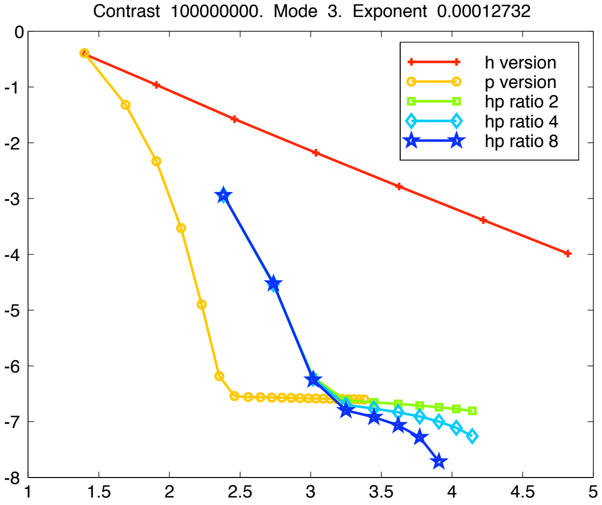
Error plots for more and more singular eigenvectors



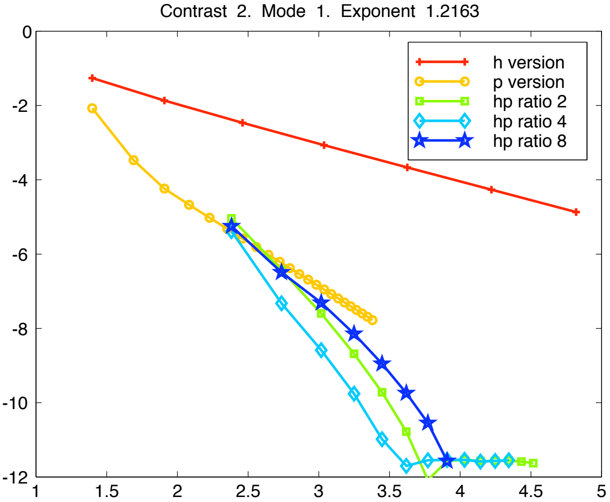
Abcissa = log 10(#DOF)   *Ordinate* = log 10(rel. error for eigenvalue)

Error plots for more and more singular eigenvectors



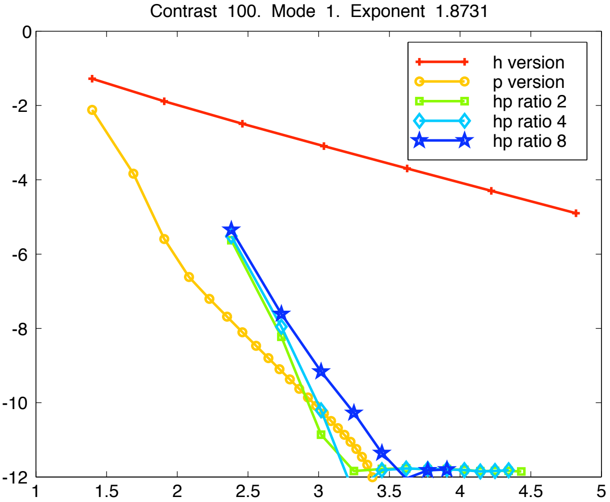
Abcissa = log 10(#DOF)   *Ordinate* = log 10(rel. error for eigenvalue)

Error plots for more and more regular eigenvectors



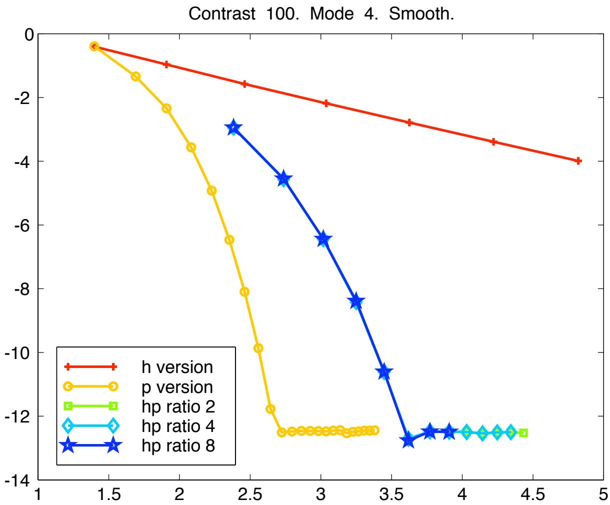
Abcissa = log 10(#DOF) *Ordinate* = log 10(rel. error for eigenvalue)

Error plots for more and more regular eigenvectors



Abcissa = log 10(#DOF) *Ordinate* = log 10(rel. error for eigenvalue)

Error plots for more and more regular eigenvectors



Abcissa $=\log 10(\#\text{DOF})$ *Ordinate* $=\log 10(\text{rel. error for eigenvalue})$

Outline

Polyhedral domains

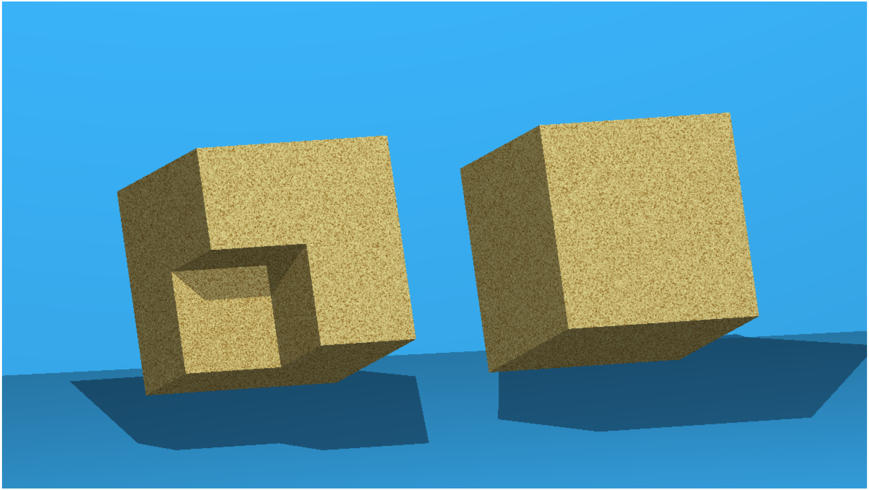


Figure: Fichera corner and cube (M. Costabel with POV-Ray)

Weighted spaces

Weight multi-exponent $\underline{\beta} = \{\beta_{\boldsymbol{e}}, \beta_{\boldsymbol{c}}\}_{\boldsymbol{e}\in\mathscr{E},\, \boldsymbol{c}\in\mathscr{C}}$ with $\mathscr{E}$ edge set, $\mathscr{C}$ corner set

$\mathrm{K}^m_{\underline{\beta}}(\Omega)$ defined as space of $v \in \mathscr{D}'(\Omega)$ such that

- In smooth region Ω_{smo} : $v \in \mathrm{H}^m(\Omega_{\mathrm{smo}})$
- In pure edge region $\Omega_{\boldsymbol{e}}$, with $r_{\boldsymbol{e}}$ distance to $\boldsymbol{e}$

$$r_{\boldsymbol{e}}^{|\boldsymbol{\alpha}|+\beta_{\boldsymbol{e}}} \partial^{\boldsymbol{\alpha}}_{\boldsymbol{x}} v \in \mathrm{L}^2(\Omega_{\boldsymbol{e}}), \ |\boldsymbol{\alpha}| \le m$$

- In pure corner region $\Omega_{\boldsymbol{c}}$

$$|\boldsymbol{x} - \boldsymbol{c}|^{|\boldsymbol{\alpha}|+\beta_{\boldsymbol{c}}} \partial^{\boldsymbol{\alpha}}_{\boldsymbol{x}} v \in \mathrm{L}^2(\Omega_{\boldsymbol{c}}), \ |\boldsymbol{\alpha}| \le m$$

- In corner-edge region $\Omega_{\boldsymbol{c},\boldsymbol{e}}$

$$|\boldsymbol{x} - \boldsymbol{c}|^{|\boldsymbol{\alpha}|+\beta_{\boldsymbol{c}}} \left(\frac{r_{\boldsymbol{e}}}{|\boldsymbol{x} - \boldsymbol{c}|}\right)^{|\boldsymbol{\alpha}|+\beta_{\boldsymbol{e}}} \partial^{\alpha}_{\boldsymbol{x}} v \in \mathrm{L}^2(\Omega_{\boldsymbol{c},\boldsymbol{e}}), \ |\boldsymbol{\alpha}| \le m$$

Anisotropic weighted spaces

Weight multi-exponent $\underline{\beta} = \{\beta_{\boldsymbol{e}}, \beta_{\boldsymbol{c}}\}_{\boldsymbol{c}\in\mathscr{C},\, \boldsymbol{e}\in\mathscr{E}}$

$\mathrm{M}^m_{\underline{\beta}}(\Omega)$ defined as space of $v \in \mathscr{D}'(\Omega)$ such that

- In smooth region Ω_{smo} : $v \in \mathrm{H}^m(\Omega_{\mathrm{smo}})$
- In pure edge region $\Omega_{\boldsymbol{e}}$ with coord. $\boldsymbol{y}$ transverse and $\boldsymbol{z}$ aligned with $\boldsymbol{e}$

$$r_{\boldsymbol{e}}^{|\boldsymbol{\alpha}_\perp|+\beta_{\boldsymbol{e}}} \partial_{\boldsymbol{y}}^{\boldsymbol{\alpha}_\perp} \partial_{\boldsymbol{z}}^{\boldsymbol{\alpha}_\parallel} v \in \mathrm{L}^2(\Omega_{\boldsymbol{e}}),\ |\boldsymbol{\alpha}_\perp| + |\boldsymbol{\alpha}_\parallel| \le m$$

- In pure corner region $\Omega_{\boldsymbol{c}}$

$$|\boldsymbol{x} - \boldsymbol{c}|^{|\boldsymbol{\alpha}|+\beta_{\boldsymbol{c}}} \partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} v \in \mathrm{L}^2(\Omega_{\boldsymbol{c}}),\ |\boldsymbol{\alpha}| \le m$$

- In corner-edge region $\Omega_{\boldsymbol{c},\boldsymbol{e}}$

$$|\boldsymbol{x} - \boldsymbol{c}|^{|\boldsymbol{\alpha}|+\beta_{\boldsymbol{c}}} \left(\frac{r_{\boldsymbol{e}}}{|\boldsymbol{x} - \boldsymbol{c}|}\right)^{|\boldsymbol{\alpha}_\perp|+\beta_{\boldsymbol{e}}} \partial_{\boldsymbol{y}}^{\boldsymbol{\alpha}_\perp} \partial_{\boldsymbol{z}}^{\boldsymbol{\alpha}_\parallel} v \in \mathrm{L}^2(\Omega_{\boldsymbol{c},\boldsymbol{e}}),\ |\boldsymbol{\alpha}_\perp| + |\boldsymbol{\alpha}_\parallel| \le m$$

Using semi-norms, define the corresponding analytic class $\mathrm{A}_{\underline{\beta}}(\Omega)$.

Anisotropy: Why? How?

NB: Could prove analytic estimates like before in $K^m_\beta(\Omega)$.

But: Exponential convergence of FEM based on such a result would require refinement towards edges in all directions. Too many elements.

Fact: For $\mathcal{C}^\infty$ data, solutions are more regular in the direction of edges

Assumption $\mathcal{A}(\boldsymbol{e},\beta)$

Along the edge $\boldsymbol{e}$, closed range estimates are valid in isotropic spaces

$$\|u\|_{\mathrm{K}^2_\beta(\Omega_{\boldsymbol{e}})} \le C_0\Big\{\|Lu\|_{\mathrm{K}^0_{\beta+2}(\Omega'_{\boldsymbol{e}})} + \gamma_u\Big\}$$

with $\gamma_u := \|u\|_{\mathrm{K}^1_{\beta+1}(\Omega'_{\boldsymbol{e}})}$

Proposition 1 : *Under assumption $\mathcal{A}(\boldsymbol{e},\beta)$, solution $u \in K^1_\beta(\Omega'_{\boldsymbol{e}})$ satisfies*

$$\frac{1}{k!}\sum_{|\alpha|=k}\|r_{\boldsymbol{e}}^{|\alpha_\perp|+\beta}\partial^\alpha_{\boldsymbol{x}} u\|_{\Omega_{\boldsymbol{e}}} \le C^{k+1}\Big\{\sum_{\ell=0}^{k}\frac{1}{\ell!}\sum_{|\alpha|=\ell}\|r_{\boldsymbol{e}}^{|\alpha_\perp|+\beta+2}\partial^\alpha_{\boldsymbol{x}} Lu\|_{\Omega'_{\boldsymbol{e}}} + \gamma_u\Big\}$$

Elements of proof

Step 1. By dyadic partition, proof of isotropic estimate

$$\frac{1}{k!}\sum_{|\alpha|=k}\|r_e^{|\alpha|+\beta}\partial_x^\alpha u\|_{\Omega_e} \le C^{k+1}\left\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\sum_{|\alpha|=\ell}\|r_e^{|\alpha|+\beta+2}\partial_x^\alpha Lu\|_{\Omega'_e} + \gamma_u\right\}$$

Step 2. By differential quotients on estimate of Assumption $\mathcal{A}(\boldsymbol{e},\beta)$ in nested open sets, proof of tangential estimates

$$\frac{1}{k!}\sum_{\substack{|\alpha|=k\\|\alpha_\perp|\le 2}}\|r_e^{|\alpha_\perp|+\beta}\partial_x^\alpha u\|_{\Omega_e} \le C^{k+1}\left\{\sum_{\ell=0}^{k}\frac{1}{\ell!}\sum_{\substack{|\alpha|=\ell\\|\alpha_\perp|=0}}\|r_e^{|\alpha_\perp|+\beta+2}\partial_x^\alpha Lu\|_{\Omega'_e} + \gamma_u\right\}$$

Step 3. Combine steps 1 and 2 to obtain

$$\frac{1}{k!}\sum_{|\alpha|=k}\|r_e^{|\alpha_\perp|+\beta}\partial_x^\alpha u\|_{\Omega_e} \le C^{k+1}\left\{\sum_{\ell=0}^{k}\frac{1}{\ell!}\sum_{|\alpha|=\ell}\|r_e^{|\alpha_\perp|+\beta+2}\partial_x^\alpha Lu\|_{\Omega'_e} + \gamma_u\right\}$$

Anisotropic analytic regularity...

Homogeneous constant coefficient case

Theorem 1 : *Under assumption $\mathcal{A}(\boldsymbol{e}, \beta_{\boldsymbol{e}})$ for all $\boldsymbol{e} \in \mathscr{E}$*

$$u \in \mathsf{K}^1_{\underline{\beta}}(\Omega) \quad \text{and} \quad f \in \mathsf{A}_{\underline{\beta}+2}(\Omega) \quad \Longrightarrow \quad u \in \mathsf{A}_{\underline{\beta}}(\Omega)$$

Proof

1. Proposition 1 gives suitable estimates in pure edge region $\Omega_{\boldsymbol{e}}$
2. This estimate is scaled and transported in a corner dyadic partition.
   Hence suitable estimates in corner-edge region $\Omega_{\boldsymbol{c},\boldsymbol{e}}$
3. The estimate in smooth domains is scaled and transported in a corner dyadic partition of $\Omega_{\boldsymbol{c}}$.
   Hence suitable estimates in pure corner region $\Omega_{\boldsymbol{c}}$

... Non-homogeneous version

Assumption $\mathcal{B}(\boldsymbol{e}, \beta)$

Along the edge $\boldsymbol{e}$, closed range estimates are valid in isotropic spaces

$$\|u\|_{\mathsf{J}^2_\beta(\Omega_{\boldsymbol{e}})} \leq C_0 \Big\{ \|Lu\|_{\mathsf{J}^0_{\beta+2}(\Omega'_{\boldsymbol{e}})} + \|u\|_{\mathsf{J}^1_{\beta+1}(\Omega'_{\boldsymbol{e}})} \Big\}$$

Theorem 2 : *Under assumption $\mathcal{B}(\boldsymbol{e}, \beta_{\boldsymbol{e}})$ for all $\boldsymbol{e} \in \mathscr{E}$*

$$u \in \mathsf{J}^1_{\underline{\beta}}(\Omega) \quad \text{and} \quad f \in \mathsf{B}_{\underline{\beta}+2}(\Omega) \quad \Longrightarrow \quad u \in \mathsf{B}_{\underline{\beta}}(\Omega)$$

References

M. Costabel, M. Dauge, and S. Nicaise.
Analytic Regularity for Linear Elliptic Systems in Polygons & Polyhedra
Preprint Rennes 2010.

M. Costabel, M. Dauge, and S. Nicaise.
Corner Singularities and Analytic Regularity for Linear Elliptic Systems
In preparation.

Conclusion

Combine Theorem 1 or Theorem 2 with regularity and a priori estimates in $\mathrm{K}^2_{\underline{\beta}}(\Omega)$ or $\mathrm{J}^2_{\underline{\beta}}(\Omega)$ proved by [MAZ'YA-ROSSMANN 2003].

Hence Anisotropic Analytic Regularity holds for variational solutions with sufficiently smooth RHS.

Thank you for your attention!