Regularity for Corner Problems in Anisotropic Weighted Spaces

Theory and Applications

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Outline

Outline

Elliptic boundary value problems in smooth domains

Ω: smooth domain in $\mathbb{R}^n$ ($n \geq 2$): bounded and regular boundary.
Example: Ball, Ellipsoid.

L: second order elliptic operator or system with smooth coefficients.
Example: $L = \Delta$ (Laplacian), $L =$ Lamé system (elasticity)

B: operator of order $k = 0$ or 1 with smooth coeff. which “covers” L on $\partial\Omega$
Example: $B = Id$ (Dirichlet, $k = 0$),
$B =$ conormal derivative associated with L (Neumann, $k = 1$)

Problem :

Given f and g, find u

$$(BVP) \qquad \begin{cases} Lu = f & \text{in } \Omega \\ Bu = g & \text{on } \partial\Omega. \end{cases}$$

Sobolev Regularity

Sobolev spaces

$$\mathrm{H}^m(\Omega) = \{v \in \mathscr{D}'(\Omega) : \ \partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} v \in L^2(\Omega), \ |\boldsymbol{\alpha}| \le m\}$$

Theorem: [AGMON-DOUGLIS-NIRENBERG 1959, 1964]

Let $m \ge 2$. If $u \in \mathrm{H}^2(\Omega)$ solves (BVP) with

$$f \in \mathrm{H}^{m-2}(\Omega) \quad \text{and} \quad g \in \mathrm{H}^{m-k-1/2}(\partial\Omega)$$

then $u \in \mathrm{H}^m(\Omega)$ with estimates

$$\|u\|_{\mathrm{H}^m(\Omega)} \le C \left\{ \|f\|_{\mathrm{H}^{m-2}(\Omega)} + \|g\|_{\mathrm{H}^{m-k-1/2}(\partial\Omega)} + \|u\|_{\mathrm{H}^2(\Omega)} \right\}.$$

Remark

If (BVP) has a coercive variational formulation in H^1, the above statement holds for $u \in \mathrm{H}^1(\Omega)$ with estimates (if the solution is unique)

$$\|u\|_{\mathrm{H}^m(\Omega)} \le C \left\{ \|f\|_{\mathrm{H}^{m-2}(\Omega)} + \|g\|_{\mathrm{H}^{m-k-1/2}(\partial\Omega)} \right\}.$$

p-version of Finite Element Method

In the coercive variational framework.

- $\widehat{K}$: reference element (triangle, tetrahedron, simplexe,... square, cube, hypercube...)
- K: mesh element, — mapped from a reference element
- $p \in \mathbb{N}$: degree of approximation
- $\mathfrak{M}$: mesh, — partition of Ω by a finite number of elements K
- $\mathfrak{V}_p$: discrete space of piecewise mapped polynomials of degree $\leq p$ on each K
- p-version (or p-extension) Family $\left(\mathfrak{V}_p\right)_{p\in\mathbb{N}}$ of discrete spaces
- u_p: solution of Galerkin projection on space $\mathfrak{V}_p$

Theorem

If $u \in \mathrm{H}^m(\Omega)$, the error $u - u_p$ satisfies the estimate

$$\left\| u - u_p \right\|_{\mathrm{H}^1(\Omega)} \leq C\, p^{-m+1} \left\| u \right\|_{\mathrm{H}^m(\Omega)} .$$

Analytic regularity and exponential convergence

Theorem: [Morrey-Nirenberg 1957] and [Schwab 1998]

Assume

- $\partial\Omega$ is analytic,
- the coefficients of L and B are analytic,
- the data f and g are analytic,

then u is analytic and there is a $\delta > 0$ independent of u and p such that

$$\|u - u_p\|_{H^1(\Omega)} \leq C\, e^{-\delta p}.$$

But :

The number of degrees of freedom is a $\mathcal{O}(p^n)$.

This is the *curse of dimensionality*.

Outline

Corner domains (definition)

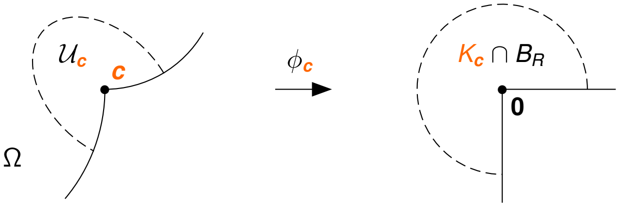


Figure: Corner domain: Local map (made with Fig4TeX)

Ω has a finite set $\mathscr{C}$ of corners $\boldsymbol{c}$:

- *All corners are points*
- All corners $\boldsymbol{c}$ are in the boundary $\partial\Omega$ of Ω
- Around each boundary point $\boldsymbol{x}_0 \notin \mathscr{C}$, Ω is smooth
- Around each corner point $\boldsymbol{c} \in \mathscr{C}$, Ω is *diffeomorphic to a cone $K_{\boldsymbol{c}}$*

Corner domains (3D)

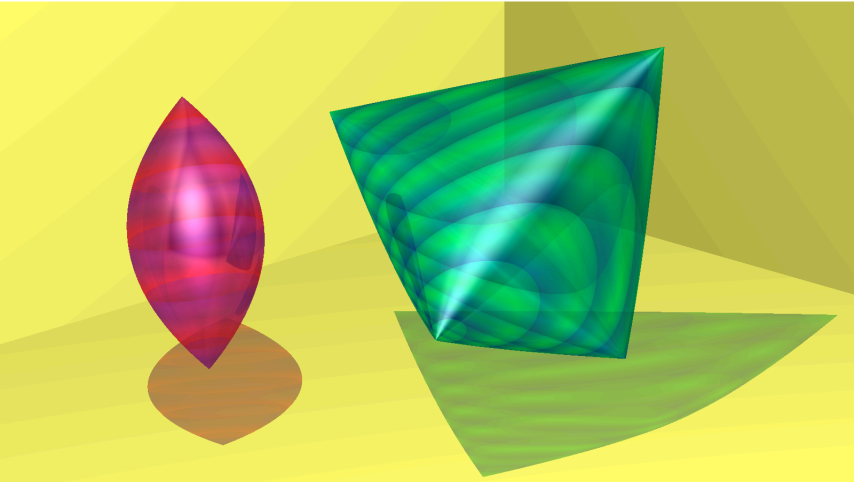


Figure: Axisymmetric domain & Cayley's tetrahedron (M. Costabel with POV-Ray)

Corner expansion

Take smooth right hand side $f \in \mathcal{C}^\infty(\overline{\Omega})$.
At any corner $\boldsymbol{c}$, f has a Taylor expansion at any order N

$$f(\boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}|\leq N} \frac{\partial^{\boldsymbol{\alpha}} f(\boldsymbol{c})}{\boldsymbol{\alpha}!} (\boldsymbol{x}-\boldsymbol{c})^{\boldsymbol{\alpha}} + \mathcal{O}(|\boldsymbol{x}-\boldsymbol{c}|^{N+1})$$

Instead, u has a corner expansion with polynomial and singular parts

$$u(\boldsymbol{x}) = \underbrace{\sum_{|\boldsymbol{\alpha}|\leq N} d_{\boldsymbol{\alpha}} (\boldsymbol{x}-\boldsymbol{c})^{\boldsymbol{\alpha}}}_{\text{polynomial part}} + \underbrace{\sum_{\mathfrak{Re}\,\lambda_k\leq N} d_k\, |\boldsymbol{x}-\boldsymbol{c}|^{\lambda_k}\, \varphi_k(\theta_{\boldsymbol{c}})}_{\text{singular part}} + o(|\boldsymbol{x}-\boldsymbol{c}|^N)$$

Note:

- The exponents $\lambda_k \in \mathbb{C}$ depend on Ω, L and B
- The sum is finite, and $1-\frac{n}{2} < \mathfrak{Re}\,\lambda_k$.
- The singular part may also contain log terms

Weighted Sobolev spaces

- Weight: powers of $r(\boldsymbol{x}) = \min_{\boldsymbol{c}\in\mathscr{C}} |\boldsymbol{x} - \boldsymbol{c}|$
- Exponent: $\gamma \in \mathbb{R}$.
- **Homogeneous weighted Sobolev spaces**
  KONDRAT'EV, MAZ'YA-PLAMENEVSKII, NAZAROV

$$\mathrm{K}^m_\gamma(\Omega) = \{v \in \mathscr{D}'(\Omega) : \quad \underbrace{r(\boldsymbol{x})^{|\boldsymbol{\alpha}|+\gamma}}_{\text{Depending on } \boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}}_{\boldsymbol{x}} v \in L^2(\Omega),\ |\boldsymbol{\alpha}| \le m\}$$

  Remainder (+ *singularities*) well described in scale $\mathrm{K}^m_\gamma(\Omega)$.
- **Non-homogeneous weighted Sobolev spaces**
  BABUŠKA-GUO, MAZ'YA-PLAMENEVSKII, COSTABEL-DAUGE-NICAISE

$$\mathrm{J}^m_\gamma(\Omega) = \{v \in \mathscr{D}'(\Omega) : \quad \underbrace{r(\boldsymbol{x})^{m+\gamma}}_{\text{Not depending on } \boldsymbol{\alpha}} \quad \partial^{\boldsymbol{\alpha}}_{\boldsymbol{x}} v \in L^2(\Omega),\ |\boldsymbol{\alpha}| \le m\}$$

  Polynomials (+ *remainder*, *singularities*) well described in scale $\mathrm{J}^m_\gamma(\Omega)$.
  Scale $\mathrm{J}^m_\gamma(\Omega)$ more versatile to describe the global regularity of u.

Weighted Sobolev Regularity

Assume: (BVP) has a coercive variational form. in H^1 and u variational sol.

Theorem: [Kondrat'ev 1967]

- In the Dirichlet case
- or if $n \geq 3$

there exists $\gamma_{\Omega,L,B} < -1$ such that the following regularity holds.
For any $m \geq 2$ and any γ, $\boxed{\gamma_{\Omega,L,B} < \gamma < -1}$

$$f \in \mathrm{K}^{m-2}_{\gamma+2}(\Omega) \quad \text{and} \quad g \in \operatorname{trace} \mathrm{K}^{m-k}_{\gamma+k}(\Omega) \quad \Longrightarrow \quad u \in \mathrm{K}^{m}_{\gamma}(\Omega)$$

Theorem: [Maz'ya-Plamenevskii 1984] [Costabel-Dauge-Nicaise]

There exists $\gamma^*_{\Omega,L,B} < -1$ such that the following regularity holds.
For any $m \geq 2$ and any γ, $\boxed{\gamma^*_{\Omega,L,B} < \gamma < -1}$

$$f \in \mathrm{J}^{m-2}_{\gamma+2}(\Omega) \quad \text{and} \quad g \in \operatorname{trace} \mathrm{J}^{m-k}_{\gamma+k}(\Omega) \quad \Longrightarrow \quad u \in \mathrm{J}^{m}_{\gamma}(\Omega)$$

Three questions and three answers

1. Why consider $\gamma < -1$?
   Because of compact embeddings: For $m \geq 2$,

$$\mathrm{K}^m_\gamma(\Omega) \overset{\text{comp}}{\hookrightarrow} \mathrm{H}^1(\Omega) \quad \Longleftrightarrow \quad \gamma < -1$$

   and the same for $\mathrm{J}^m_\gamma(\Omega)$.

2. Why are $\mathrm{J}^m_\gamma(\Omega)$ better than $\mathrm{K}^m_\gamma(\Omega)$?
   Because for any $\gamma < -1$, if m is large enough ($m > -\gamma - \frac{n}{2}$), $\mathrm{J}^m_\gamma(\Omega)$ contains all polynomials, which is not the case for $\mathrm{K}^m_\gamma(\Omega)$.

3. Why not consider standard spaces H^m instead? (note: $\mathrm{H}^m = \mathrm{J}^m_{-m}$)
   Because of associated spaces of analytic functions

$$\mathrm{B}_\gamma(\Omega) = \Big\{ v \in \bigcap_{m > -\gamma - \frac{n}{2}} \mathrm{J}^m_\gamma(\Omega) : \sum_{|\boldsymbol{\alpha}| = m} \| r^{m+\gamma} \partial^{\boldsymbol{\alpha}}_{\boldsymbol{x}} v \|_{L^2(\Omega)} \leq C^{m+1} m! \Big\}$$

   here C is independent from m, for all $m > -\gamma - \frac{n}{2}$.

Analytic regularity

Spaces B_γ coincide with countably normed spaces B^ℓ_β introduced by BABUŠKA-GUO according to

$$B^\ell_\beta(\Omega) = \mathsf{B}_{\beta-\ell}, \quad \ell = 0, 1, 2, \ldots, \ \beta \in (0, 1)$$

Theorem: [BABUŠKA-GUO, 1988-] and [COSTABEL-DAUGE-NICAISE]

Assume

- Ω is an analytic corner domain,
- the coefficients of L and B are analytic.

Then with the same $\gamma^*_{\Omega,L,B} < -1$ as above: For any γ, $\gamma^*_{\Omega,L,B} < \gamma < -1$

$$f \in \mathsf{B}_{\gamma+2}(\Omega) \quad \text{and} \quad g \in \operatorname{trace} \mathsf{B}_{\gamma+k}(\Omega) \quad \Longrightarrow \quad u \in \mathsf{B}_\gamma(\Omega)$$

$\Longrightarrow$ Exponential convergence of hp-version of Finite Element Method. [BABUŠKA-GUO]

The mesh is geometrically refined near corners. Contains $\mathcal{O}(p)$ elements. Discrete spaces $\mathfrak{V}_p$ contain $\mathcal{O}(p) \cdot \mathcal{O}(p^n) = \mathcal{O}(p^{n+1})$ degrees of freedom.

hp-version on a L-shape domain (full refinement)

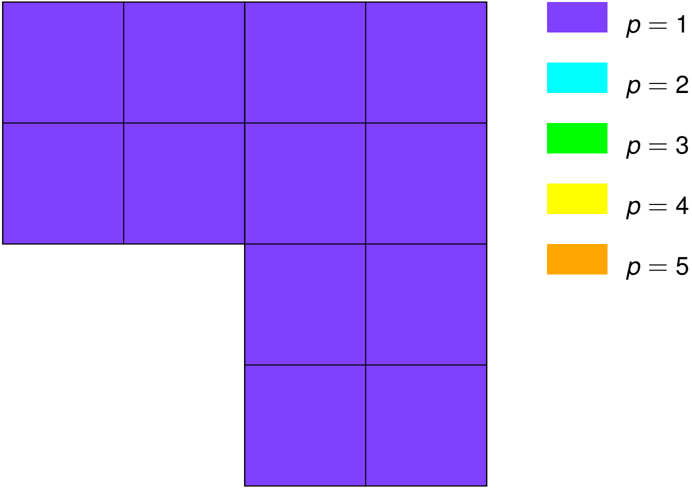


Figure: Level 1

hp-version on a L-shape domain (full refinement)

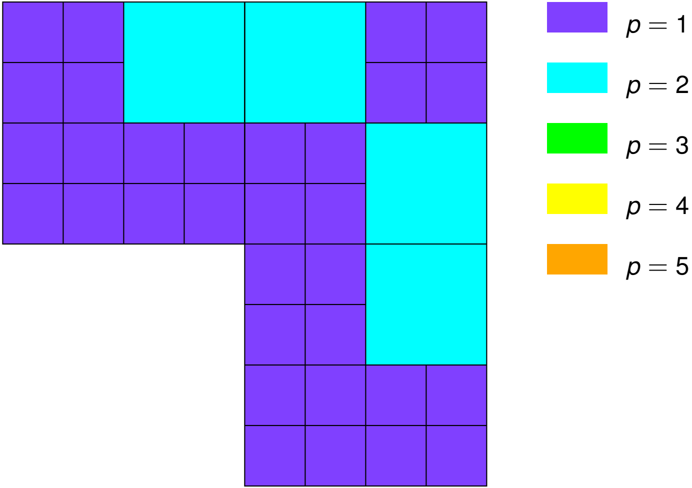


Figure: Level 2

hp-version on a L-shape domain (full refinement)

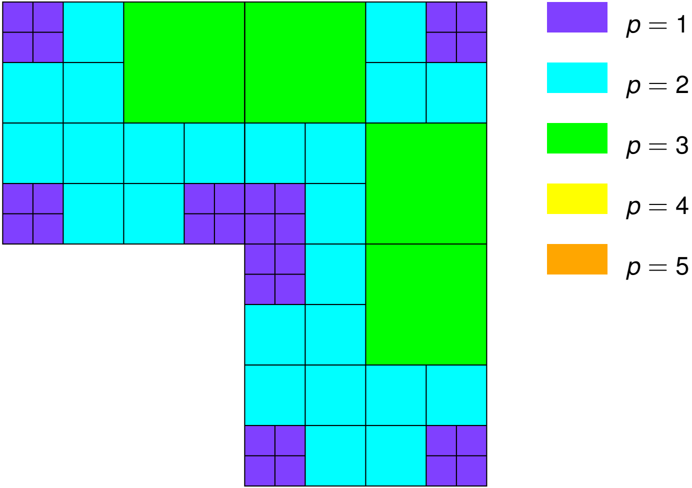


Figure: Level 3

hp-version on a L-shape domain (full refinement)

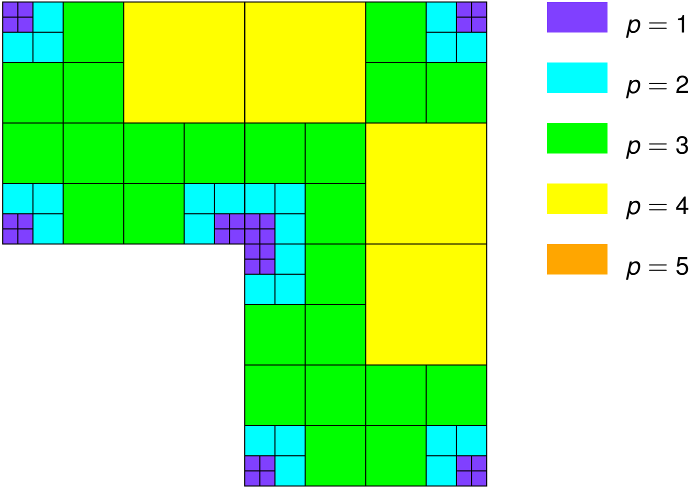


Figure: Level 4

hp-version on a L-shape domain (full refinement)

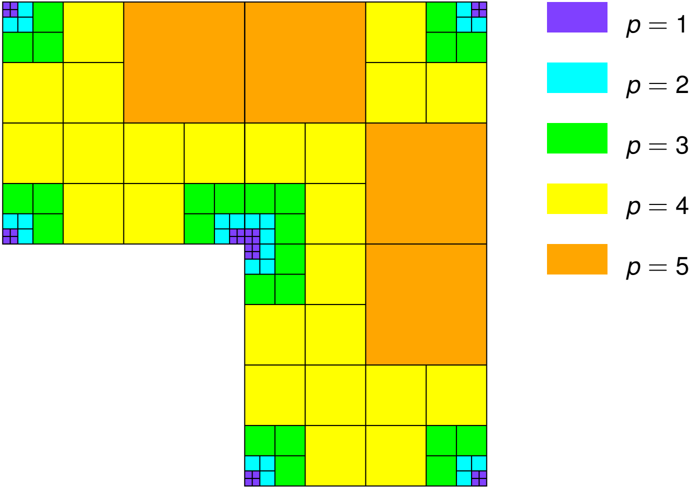


Figure: Level 5

hp-version on a L-shape domain (partial refinement)

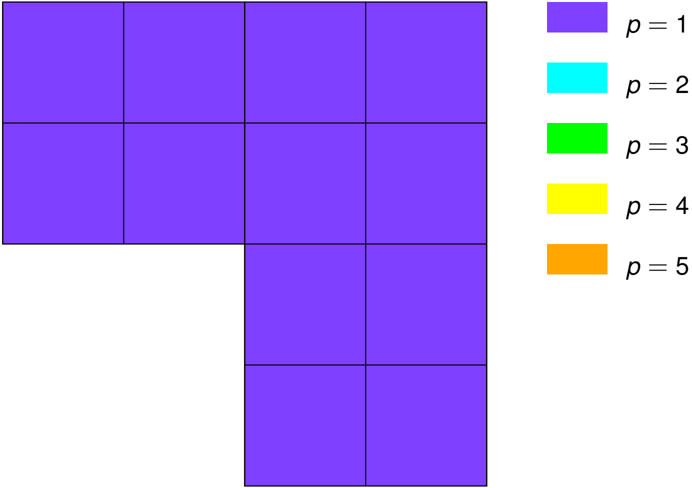
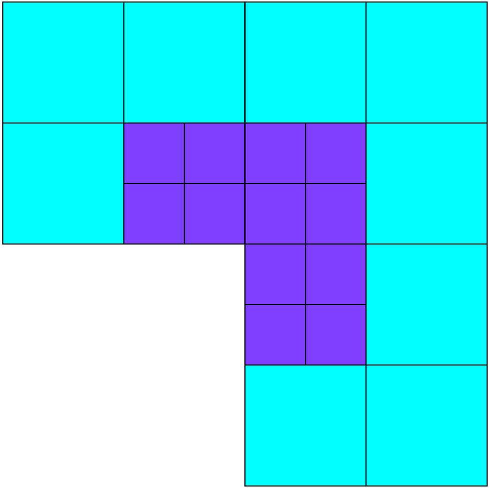


Figure: Level 1

hp-version on a L-shape domain (partial refinement)



$p = 1$

$p = 2$

$p = 3$

$p = 4$

$p = 5$

Figure: Level 2

hp-version on a L-shape domain (partial refinement)

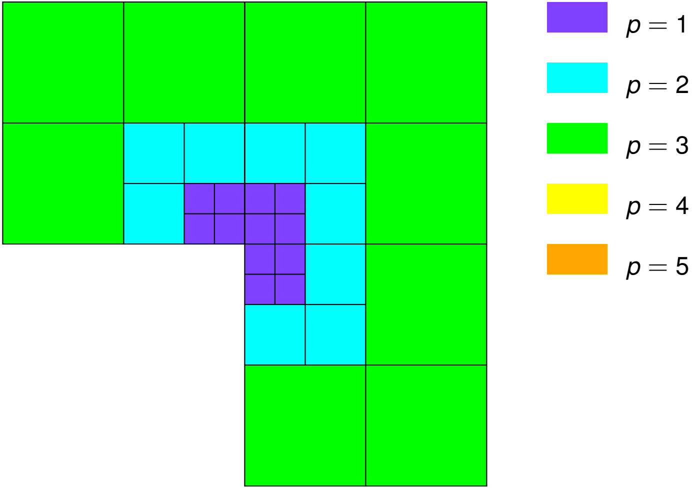


Figure: Level 3

hp-version on a L-shape domain (partial refinement)

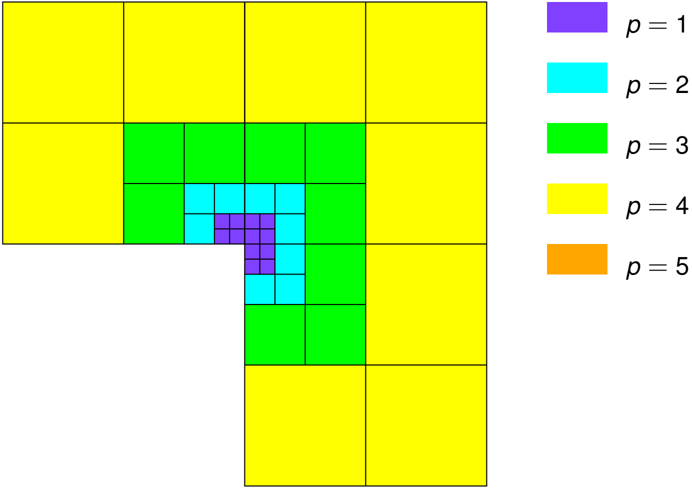


Figure: Level 4

hp-version on a L-shape domain (partial refinement)

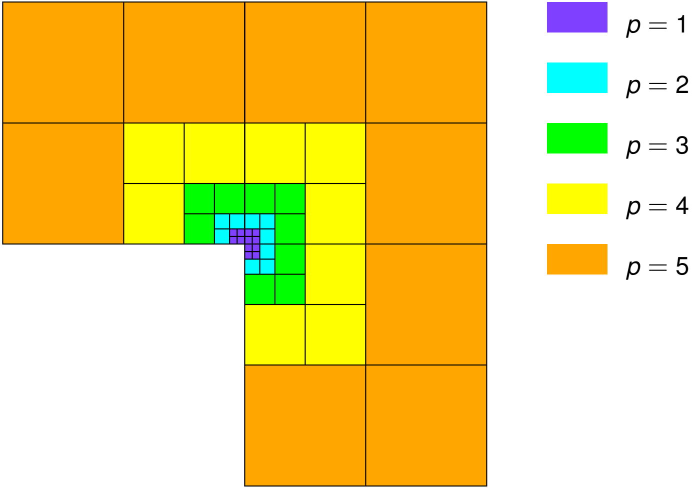


Figure: Level 5

Outline

Edge domains

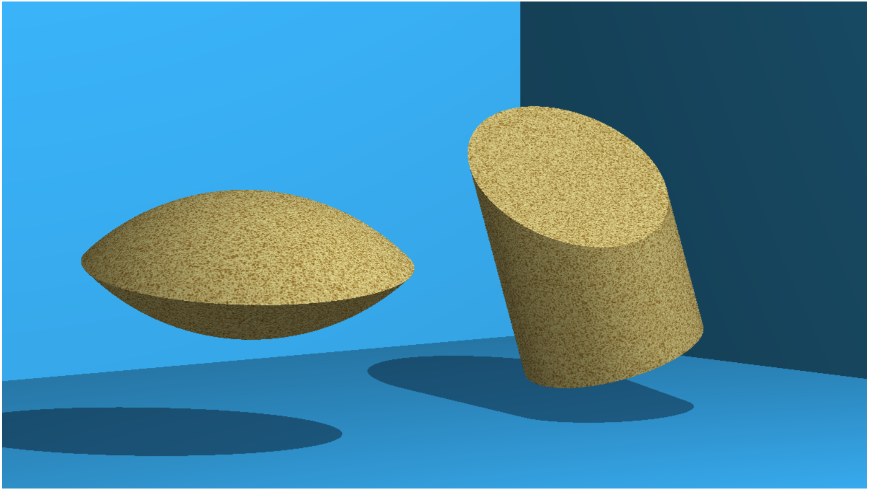


Figure: Flying saucer and skew cylinder (M. Costabel with POV-Ray)

Edge domains: Definition

$\Omega \subset \mathbb{R}^n$, $n \geq 3$.
Ω has a finite set $\mathscr{E}$ of edges $\boldsymbol{e}$:

- *All edges are closed $n-d$ manifolds, $d \geq 2$, – curves if $n = 3$*
- All edges $\boldsymbol{e}$ are subsets of $\partial\Omega$
- Around each boundary point $\boldsymbol{x}_0 \notin \cup_{\boldsymbol{e}\in\mathscr{E}}\boldsymbol{e}$, Ω is smooth
- Around each edge point $\boldsymbol{z} \in \boldsymbol{e}$, Ω is *diffeomorphic to a wedge* $\Gamma_{\boldsymbol{z}} \times \mathbb{R}^{n-d}$ for a cone $\Gamma_{\boldsymbol{z}} \subset \mathbb{R}^d$.

Edge expansions

Fix an edge $\boldsymbol{e}$ and a system of coordinates $\boldsymbol{x} = (\boldsymbol{y}, \boldsymbol{z})$ with

- $\boldsymbol{y}$ normal to $\boldsymbol{e}$, and $\boldsymbol{y} = 0$ on the edge
- $\boldsymbol{z}$ tangent to $\boldsymbol{e}$ (the variable along the edge)

Polynomial part of Edge Expansion for solution with regular rhs

$$(\boldsymbol{y}, \boldsymbol{z}) \mapsto \sum_{|\boldsymbol{\alpha}_\perp| \le N} d_{\boldsymbol{\alpha}_\perp}(\boldsymbol{z})\, \boldsymbol{y}^{\boldsymbol{\alpha}_\perp} \quad \text{with regular coefficients } \boldsymbol{z} \mapsto d_{\boldsymbol{\alpha}_\perp}(\boldsymbol{z})$$

Simplified Singular part of Edge Expansion for solution with regular rhs

$$(\boldsymbol{y}, \boldsymbol{z}) \mapsto \sum_{\Re\mathrm{e}\, \lambda_k \le N} d_k(\boldsymbol{z})\, |\boldsymbol{y}|^{\lambda_k} \varphi_k(\theta_{\boldsymbol{e}}) \quad \text{with regular coefficients } \boldsymbol{z} \mapsto d_k(\boldsymbol{z})$$

But

- Terms in $\log^q |\boldsymbol{y}|$ may appear, and in case of finite regularity of data, coefficients d_k have finite Sobolev regularity depending on $\Re\mathrm{e}\, \lambda_k$.
- In case of curved edge, varying opening or variable coefficients : Exponents $\lambda_k = \lambda_k(\boldsymbol{z})$ interact with each other or with polynomials $\Longrightarrow$ Crossing and Branching phenomena [COSTABEL-DAUGE], [MAZ'YA-ROSSMANN]

K and J, a right choice for function spaces?

1. Defining

$$r(\boldsymbol{x}) = \min_{\boldsymbol{e}\in\mathscr{E}} \min_{\boldsymbol{z}\in\boldsymbol{e}} |\boldsymbol{x}-\boldsymbol{z}| \simeq \min_{\boldsymbol{e}\in\mathscr{E}} |\boldsymbol{y}_{\boldsymbol{e}}|$$

$\mathrm{K}^m_\gamma(\Omega)$ and $\mathrm{J}^m_\gamma(\Omega)$ can be copied from corner case.
In the coercive case, **Fredholm and Regularity** results can be proved for a certain range of weight exponents γ around -1.

2. When applied to the design of hp-version, these results are useless: The spaces $\mathrm{K}^m_\gamma(\Omega)$ and $\mathrm{J}^m_\gamma(\Omega)$ are isotropic $\Longrightarrow$ The corresponding mesh-refinement produces (very) small elements in all directions near edges $\Longrightarrow$ **Exponential Blow-Up** of number of degrees of freedom.

Not exactly

Anisotropic weighted spaces

The fundamental fact is:
The regularity of edge coefficients follows exactly the regularity of data, without loss. The tangential regularity in edge variables $\boldsymbol{z}$ is not limited.
Assume one edge $\boldsymbol{e}$ for simplicity, with $(\boldsymbol{y}, \boldsymbol{z})$ normal-tangential coord. to $\boldsymbol{e}$.

- **Homogeneous anisotropic weighted Sobolev spaces**
  BUFFA-COSTABEL-DAUGE

$$\mathrm{M}^m_\gamma(\Omega) = \{ v \in \mathscr{D}'(\Omega) : \quad \underbrace{|\boldsymbol{y}|^{|\boldsymbol{\alpha}_\perp|+\gamma}}_{\text{Depending on } \boldsymbol{\alpha}_\perp} \ \partial_{\boldsymbol{y}}^{\boldsymbol{\alpha}_\perp} \partial_{\boldsymbol{z}}^{\boldsymbol{\alpha}_\parallel} v \in L^2(\Omega), \ \underbrace{|\boldsymbol{\alpha}_\perp| + |\boldsymbol{\alpha}_\parallel|}_{=|\boldsymbol{\alpha}|} \le m \}$$

- **Non-homogeneous weighted Sobolev spaces**
  Something like:

$$\mathrm{N}^m_\gamma(\Omega) = \{ v \in \mathscr{D}'(\Omega) : \ |\boldsymbol{y}|^{\max\{|\boldsymbol{\alpha}_\perp|+\gamma, 0\}} \partial_{\boldsymbol{y}}^{\boldsymbol{\alpha}_\perp} \partial_{\boldsymbol{z}}^{\boldsymbol{\alpha}_\parallel} v \in L^2(\Omega), \ |\boldsymbol{\alpha}| \le m \}$$

No Fredholm theorems in these spaces.
Valuable for their $\mathcal{C}^\infty$ and analytic limits. Compatible with GUO's definitions.

Tensor product structure

Locally near the edge $\boldsymbol{e}$, $\Omega \simeq S \times \mathbb{T}^{n-d}$ where S is a bounded cone in $\mathbb{R}^d$ and $\mathbb{T}^{n-d}$ is the $n-d$ dimensional torus.

$$\boldsymbol{y} \in S, \quad \boldsymbol{z} \in \mathbb{T}^{n-d}$$

The corner of S is the origin $\boldsymbol{y} = \boldsymbol{0}$. Recall

$$\mathrm{K}^m_\gamma(S) = \{v \in \mathscr{D}'(S) : \quad |\boldsymbol{y}|^{|\boldsymbol{\alpha}_\perp|+\gamma}\partial^{\boldsymbol{\alpha}_\perp}_{\boldsymbol{y}} v \in L^2(S),\ |\boldsymbol{\alpha}_\perp| \leq m\}$$

Then

$$\mathrm{M}^{2m}_\gamma(S \times \mathbb{T}^{n-d}) \subset \mathrm{K}^m_\gamma(S) \otimes \mathrm{H}^m(\mathbb{T}^{n-d}) \subset \mathrm{M}^m_\gamma(S \times \mathbb{T}^{n-d})$$

Similarly, for m large enough

$$\mathrm{N}^{2m}_\gamma(S \times \mathbb{T}^{n-d}) \hookrightarrow \mathrm{J}^m_\gamma(S) \otimes \mathrm{H}^m(\mathbb{T}^{n-d}) \hookrightarrow \mathrm{N}^m_\gamma(S \times \mathbb{T}^{n-d})$$

Hence, for analytic classes:

$$\mathrm{B}_\gamma(S \times \mathbb{T}^{n-d}) = \mathrm{B}_\gamma(S) \otimes \mathrm{A}(\mathbb{T}^{n-d})$$

Tensor product hp-version

If the solution $u \in \mathrm{B}_\gamma(S) \otimes \mathrm{A}(\mathbb{T}^{n-d})$ with suitable $\gamma < -1$, we expect exponential convergence for a tensor mesh:

Geometrically refined in S and finite in $\mathbb{T}^{n-d}$.

Number of degrees of freedom

$$\mathcal{O}(p^{d+1}) \cdot \mathcal{O}(p^{n-d}) = \mathcal{O}(p^{n+1}).$$

Outline

Polyhedral domains

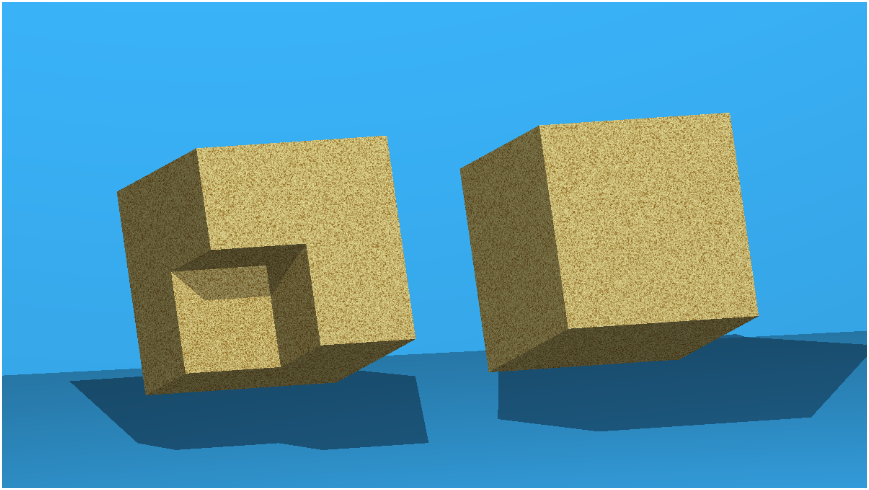


Figure: Fichera corner and cube (M. Costabel with POV-Ray)

A local example

Polyhedral domains: Definition

$\Omega \subset \mathbb{R}^3$.

Ω has a finite set $\mathscr{E}$ of edges $\boldsymbol{e}$ and a finite set $\mathscr{C}$ of corners $\boldsymbol{c}$:

- *All edges are segments*
- *All edge tips $\boldsymbol{c} \in \overline{\boldsymbol{e}} \setminus \boldsymbol{e}$ are corners*
- All edges $\boldsymbol{e}$ and corners $\boldsymbol{c}$ are subsets of $\partial\Omega$
- Around each...
  - Boundary point $\boldsymbol{x}_0 \notin \cup_{\overline{\boldsymbol{e}} \in \mathscr{E}} \boldsymbol{e}$, Ω is affine diffeomorphic to $\mathbb{R}_+ \times \mathbb{R}^2$
  - Edge point $\boldsymbol{z} \in \boldsymbol{e}$, Ω is *affine diffeomorphic to a wedge $\Gamma_{\boldsymbol{e}} \times \mathbb{R}$*
  - Corner point $\boldsymbol{c} \in \mathscr{C}$, Ω is *affine diffeomorphic to a polyhedral cone $K_{\boldsymbol{c}}$*

As a consequence, the regular part of the boundary

$$\partial\Omega \setminus \Big\{ \bigcup_{\boldsymbol{e} \in \mathscr{E}} \overline{\boldsymbol{e}} \Big\}$$

is a finite union of plane faces which are polygonal.

Anisotropic (homogeneous) weighted spaces

Weight multi-exponent $\gamma = \{\gamma_{\boldsymbol{e}}, \gamma_{\boldsymbol{c}}\}_{\boldsymbol{c}\in\mathscr{C},\, \boldsymbol{e}\in\mathscr{E}}$
$\mathrm{M}^m_\gamma(\Omega)$ defined as set of $v \in \mathscr{D}'(\Omega)$ such that

- In smooth region Ω_{smo} : $v \in H^m(\Omega_{\mathrm{smo}})$
- In pure edge region $\Omega_{\boldsymbol{e}}$

$$|\boldsymbol{y_e}|^{|\boldsymbol{\alpha}_\perp|+\gamma_e}\partial_{\boldsymbol{y}}^{\boldsymbol{\alpha}_\perp}\partial_{\boldsymbol{z}}^{\boldsymbol{\alpha}_\parallel} v \in L^2(\Omega_{\boldsymbol{e}}),\ |\boldsymbol{\alpha}_\perp| + |\boldsymbol{\alpha}_\parallel| \le m$$

- In pure corner region $\Omega_{\boldsymbol{c}}$

$$|\boldsymbol{x}-\boldsymbol{c}|^{|\boldsymbol{\alpha}|+\gamma_c}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} v \in L^2(\Omega_{\boldsymbol{c}}),\ |\boldsymbol{\alpha}| \le m$$

- In corner-edge region $\Omega_{\boldsymbol{c},\boldsymbol{e}}$

$$|\boldsymbol{x}-\boldsymbol{c}|^{|\boldsymbol{\alpha}|+\gamma_c}\left(\frac{|\boldsymbol{y_e}|}{|\boldsymbol{x}-\boldsymbol{c}|}\right)^{|\boldsymbol{\alpha}_\perp|+\gamma_e}\partial_{\boldsymbol{y}}^{\boldsymbol{\alpha}_\perp}\partial_{\boldsymbol{z}}^{\boldsymbol{\alpha}_\parallel} v \in L^2(\Omega_{\boldsymbol{c},\boldsymbol{e}}),\ |\boldsymbol{\alpha}_\perp| + |\boldsymbol{\alpha}_\parallel| \le m$$

[GUO,1995]'s definitions amount to the non-homogeneous version of this.

Anisotropic weighted spaces in a cube

Unit cube $\Omega = \mathrm{I}^3$ with $\mathrm{I} = (0,1)$. Coordinates x_1, x_2, x_3.
Isolate corner $\boldsymbol{c} = \boldsymbol{0}$ by considering $\Omega^*_{\boldsymbol{0}} = (0, \frac{1}{2})^3$.
Split $\Omega^*_{\boldsymbol{0}}$ in 6 parts $\Omega^j_{\boldsymbol{0}}$, $j = 1, \ldots, 6$, by ordering coordinates: E.g.

$$\Omega^1_{\boldsymbol{0}} := \{\boldsymbol{x} = (x_1, x_2, x_3) : \quad x_1 < x_2 < x_3\}$$

The only edge such that $\boldsymbol{e} \cap \overline{\Omega}^1_{\boldsymbol{0}} \neq \emptyset$ is $x_1 = x_2 = 0$, and

$$\boldsymbol{y}_{\boldsymbol{e}} = (x_1, x_2), \quad \boldsymbol{z} = x_3, \quad \text{hence} \quad |\boldsymbol{y}| \simeq x_2, \quad |\boldsymbol{x} - \boldsymbol{c}| = |\boldsymbol{x}| \simeq x_3.$$

Then

$$\begin{aligned} \mathrm{M}^m_\gamma(\Omega^1_{\boldsymbol{0}}) &= \{v : \; x_3^{\alpha_1+\alpha_2+\alpha_3+\gamma_{\boldsymbol{c}}} \left(\frac{x_2}{x_3}\right)^{\alpha_1+\alpha_2+\gamma_{\boldsymbol{e}}} \partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2} \partial^{\alpha_3}_{x_3} v \in L^2(\Omega^1_{\boldsymbol{0}}), \; |\boldsymbol{\alpha}| \leq s\} \\ &= \{v : \; x_3^{\alpha_3+\gamma_{\boldsymbol{c}}-\gamma_{\boldsymbol{e}}} x_2^{\alpha_1+\alpha_2+\gamma_{\boldsymbol{e}}} \partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2} \partial^{\alpha_3}_{x_3} v \in L^2(\Omega^1_{\boldsymbol{0}}), \; |\boldsymbol{\alpha}| \leq s\} \end{aligned}$$

Note

$$\boldsymbol{x} \in \Omega^1_{\boldsymbol{0}} \cap \Omega_{\boldsymbol{c}} \iff x_2 \simeq x_3 \quad \text{and} \quad \boldsymbol{x} \in \Omega^1_{\boldsymbol{0}} \cap \Omega_{\boldsymbol{e}} \iff x_3 > c > 0$$

Tensor weighted spaces in a cube

Recall: On interval I:

$$\mathrm{M}^m_\omega(\mathrm{I})\big|_{(0,\frac12)} = \mathrm{K}^m_\omega(\mathrm{I})\big|_{(0,\frac12)} = \{v:\ x^{\alpha+\omega}\partial_x^\alpha v \in L^2((0,\frac12)),\ \alpha \le m\}$$

Weight multi-exponent $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$. Define new M space so that

$$\overset{\otimes}{M}{}^m_{\boldsymbol{\omega}}(\mathrm{I}^3)\big|_{\Omega_{\mathbf{0}}^*} = \{v:\ x_1^{\alpha_1+\omega_1} x_2^{\alpha_2+\omega_2} x_3^{\alpha_3+\omega_3}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} v \in L^2(\Omega^*\mathbf{0}),\ |\boldsymbol{\alpha}| \le m\}$$

Two remarks

1. Relation with tensor product spaces

$$\overset{\otimes}{M}{}^{3m}_{\boldsymbol{\omega}}(\mathrm{I}^3) \hookrightarrow \mathrm{M}^m_{\omega_1}(\mathrm{I}) \otimes \mathrm{M}^m_{\omega_2}(\mathrm{I}) \otimes \mathrm{M}^m_{\omega_3}(\mathrm{I}) \hookrightarrow \overset{\otimes}{M}{}^m_{\boldsymbol{\omega}}(\mathrm{I}^3)$$

2. Relation with corner-edge weighted spaces (N-version): if

$$\gamma_{\boldsymbol{c}} \le \omega_1 + \omega_2 + \omega_3 \quad \text{and} \quad \gamma_{\boldsymbol{e}} \le \omega_i + \omega_j \text{ (with } \boldsymbol{e} \parallel x_i = x_j = 0)$$

$$\Longrightarrow \mathrm{M}^m_\gamma(\mathrm{I}^3) \hookrightarrow \mathrm{N}^m_\gamma(\mathrm{I}^3) \hookrightarrow \overset{\otimes}{N}{}^m_{\boldsymbol{\omega}}(\mathrm{I}^3)$$

Anisotropic weighted regularity in hyper-cubes

$\Omega = \mathrm{I}^n$ unit cube in $\mathbb{R}^n$, $n \geq 2$.
Dirichlet problem for Laplace operator

$$(DLP) \qquad \begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem: [Dauge-Stevenson, 2009]

Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ such that

$$-\tfrac{3}{2} < \omega_i < 0 \ (i = 1, \ldots, n) \quad \text{and} \quad \omega_1 + \cdots + \omega_n > -2.$$

Let $m \geq 2$. Then u solution of (DLP) satisfies

$$f \in \overset{\otimes}{M}{}^{m+2n-4}_{\mathbf{0}}(\mathrm{I}^n) \quad \Longrightarrow \quad u \in \overset{\otimes}{M}{}^{m}_{\boldsymbol{\omega}}(\mathrm{I}^n)$$

In particular, if $f \in \mathcal{C}^\infty(\bar{\mathrm{I}}^n)$, then

$$u \in \mathrm{M}^\infty_{\omega_1}(\mathrm{I}) \otimes \cdots \otimes \mathrm{M}^\infty_{\omega_n}(\mathrm{I}).$$

Toward analytic estimates

$\Omega = \mathrm{I}^n$ unit cube in $\mathbb{R}^n$, $n \geq 2$, and Dirichlet problem for Laplace operator

$$(DLP) \qquad \begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The proof of [DAUGE-STEVENSON, 2009] seems to adapt to analytic anisotropic weighted spaces – by a combination of nested a priori estimates with the $n+1$-level two-step recurrence of [DS09].

Almost-Theorem:

Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ such that

$$-\tfrac{3}{2} < \omega_i < 0 \ (i = 1, \ldots, n) \quad \text{and} \quad \omega_1 + \cdots + \omega_n > -2.$$

Then u solution of (DLP) satisfies

$$f \in \mathrm{A}_0(\mathrm{I}) \otimes \cdots \otimes \mathrm{A}_0(\mathrm{I}) \quad \Longrightarrow \quad u \in \mathrm{A}_{\omega_1}(\mathrm{I}) \otimes \cdots \otimes \mathrm{A}_{\omega_n}(\mathrm{I}).$$

Here $\mathrm{A}_\gamma(\mathrm{I}) = \{v : \|(1-x^2)^{\alpha+\gamma}\partial_x^\alpha v\|_{L^2(\mathrm{I})} \leq C^{\alpha+1}\alpha!,\ \alpha \in \mathbb{N}\}$

Conclusions: Approximation properties

1. Finite degree approximation (h-version type of degree d). The sparse tensor wavelet approximation of [DS09] yields the following error estimate between u solution of (DLP) and u_L the Galerkin approximation of level L

$$\left\| u - u_L \right\|_{H^1(\mathrm{I}^n)} \leq C_n(u)\, 2^{-L(d-1)} \quad \text{where} \quad \#(DOF) =: N = \mathcal{O}(2^L)$$

Note that the standard approximation in h-version would yield $\mathcal{O}(N^{-(d-1)/n})$ instead of $\boxed{\mathcal{O}(N^{-(d-1)})}$

2. If Almost-Theorem is true, tensor hp-version will yield exponential convergence: For $n = 2$, similar approximation properties are proved by [MAISCHAK-STEPHAN] in relation with a Boundary Integral Method.

3. For $n = 3$, if the regularity in analytic spaces with edge-corner weights is true, then exponential convergence in edge-corner hp-version will hold (approximation properties proved by GUO).

The end

感谢您关注

Thank you for your attention

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