On the inf-sup constant of the divergence

alias LBB constant

Monique DAUGE et Martin COSTABEL

with contributions of

Christine BERNARDI, Vivette GIRAULT and Frédéric HECHT

Conference on Computational Electromagnetism and Acoustics
Oberwolfach, January 20–26, 2013

Plan

The inf-sup constant, alias LBB constant

- Ω bounded connected open set of $\mathbb{R}^d$ $(d \geq 1)$
- $L^2(\Omega)$ space of square integrable functions q on Ω. Norm $\|q\|_{0,\Omega}$
- $H^1(\Omega)$ Sobolev space of $v \in L^2(\Omega)$ with gradient $\nabla v \in L^2(\Omega)^d$
- $L^2_\circ(\Omega)$ sub-space of $q \in L^2(\Omega)$ with $\int_\Omega q = 0$.
- $H^1_0(\Omega)$ sub-space of $u \in H^1(\Omega)$ with null trace on $\partial\Omega$. Poincaré inequ.
  $\rightarrow$ Semi-norm $|u|_{1,\Omega} := \|\nabla u\|_{0,\Omega}$ equivalent to norm $\|u\|_{1,\Omega}$

The inf-sup constant, alias LBB constant [a]

[a] After Ladyzhenskaya, Babuška and Brezzi

[B1]
$$\beta(\Omega) = \inf_{q \in L^2_\circ(\Omega)} \sup_{\boldsymbol{v} \in H^1_0(\Omega)^d} \frac{\displaystyle\int_\Omega \operatorname{div} \boldsymbol{v}\, q}{|\boldsymbol{v}|_{1,\Omega}\, \|q\|_{0,\Omega}}$$

Elementary properties

$$\beta(\Omega) = \inf_{q \in L^2_\circ(\Omega)} \sup_{\boldsymbol{v} \in H^1_0(\Omega)^d} \frac{\int_\Omega \operatorname{div} \boldsymbol{v}\, q}{|\boldsymbol{v}|_{1,\Omega} \, \|q\|_{0,\Omega}}$$

- If $d = 1$, $\beta(\Omega) = 1$:
  Ω is an interval (a, b). For $q \in L^2_\circ(\Omega)$, take $v(x) = \int_a^x q(t)\, dt$.
- In any dimension $\beta(\Omega) \geq 0$
- In any dimension $\beta(\Omega) \leq 1$, because of the identity

$$\forall \boldsymbol{v} \in H^1_0(\Omega)^d, \quad |\boldsymbol{v}|^2_{1,\Omega} = \|\nabla \boldsymbol{v}\|^2_{0,\Omega} = \|\overrightarrow{\operatorname{curl}}\, \boldsymbol{v}\|^2_{0,\Omega} + \|\operatorname{div} \boldsymbol{v}\|^2_{0,\Omega}$$

- $\beta(\Omega)$ is invariant by dilations, translations, symmetries and rotations.
  Depends only on the *shape* of Ω
- Less elementary: [Michlin 1973] GO [Cosserat-Cosserat 1898] GO

$\forall d \geq 2, \ \beta(\Omega)^2 \leq \frac{1}{2}$ *and* if $d = 2, \ \beta(\Omega)^2 = \frac{1}{2}$ for discs

Positiveness: $\beta(\Omega) > 0$ for Lipschitz domains

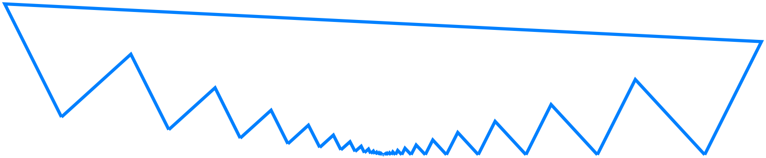


Figure: A Lipschitz domain with infinitely many corners

The boundary $\partial\Omega$ is locally a Lipschitz epigraph.
[Nečas 1964]

Positiveness: $\beta(\Omega) > 0$ for finite $\cup$ of Lipschitz dom.

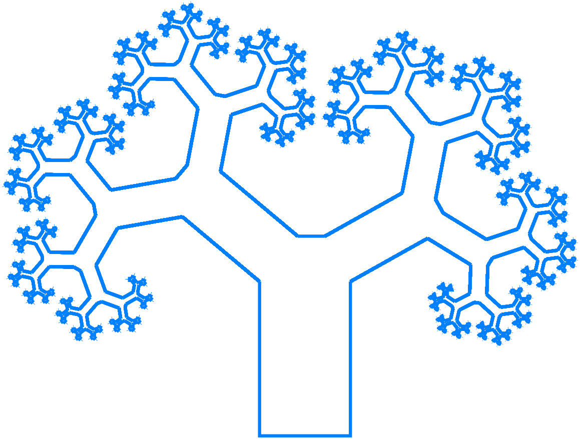
Picture by Martin at San Juan de la Peña, near Jaca (Spain), Sept 18, 2012.

Positiveness: $\beta(\Omega) > 0$ for weakly Lipschitz domains

Figure: A weakly Lipschitz domain: the self-similar zigzag

The boundary $\partial\Omega$ is locally described by bi-Lipschitz maps.

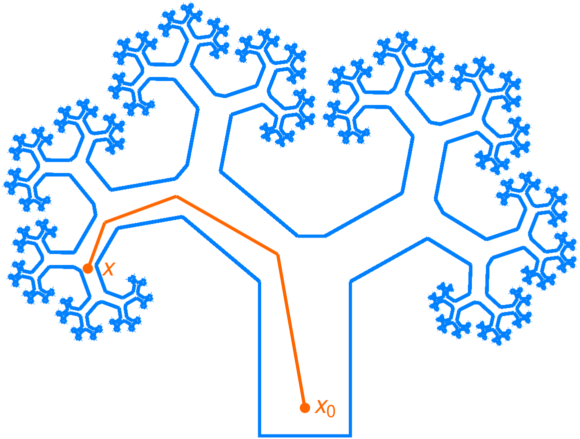
Positiveness: $\beta(\Omega) > 0$ for John domains



$\exists x_0 \in \Omega$, $\exists \rho > 0$, s.t. any point $x \in \Omega$ is joined to x_0 by a rectifiable curve γ parametrized by arclength $t \in [0, \ell = \ell(x)]$, $\forall t \in [0, \ell], \quad \rho\, t \leq d(\partial\Omega, \gamma(t))$

[Acosta-Durán-Muschietti, 2006]

Positiveness: $\beta(\Omega) > 0$ for John domains



$\exists x_0 \in \Omega$, $\exists \rho > 0$, s.t. any point $x \in \Omega$ is joined to x_0 by a rectifiable curve γ parametrized by arclength $t \in [0, \ell = \ell(x)]$, $\forall t \in [0, \ell], \quad \rho\, t \leq d(\partial\Omega, \gamma(t))$

[Acosta-Durán-Muschietti, 2006]

Non-positiveness: $\beta(\Omega) = 0$ for outward cusps

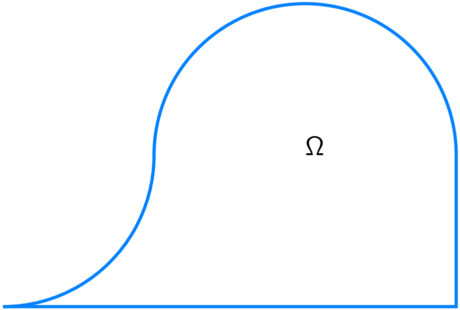


Figure: A domain with an external cusp

[Tartar 2006]

Relation with the Schur complement

Denote

- D the positive vector Laplacian $D = (-\Delta)\,\mathrm{I}_d : H_0^1(\Omega)^d \to H^{-1}(\Omega)^d$
- S the Schur complement [of the Stokes operator]

$$S: \quad \begin{array}{ccc} L^2_\circ(\Omega) & \longrightarrow & L^2_\circ(\Omega) \\ q & \longmapsto & -\operatorname{div} D^{-1}\nabla q \end{array}$$

Bounded, self-adj, non-negative. Not compact, no compact resolvent.

- $\sigma(\Omega)$ the bottom of spectrum : $\sigma(\Omega) = \inf_{q\in L^2_\circ(\Omega)} \dfrac{\langle Sq, q\rangle_\Omega}{\langle q, q\rangle_\Omega}$

Lemma

[B3] $$\sigma(\Omega) = \beta(\Omega)^2$$

Proof ingredients:

1. LBB constant as duality estimate $\beta(\Omega) = \inf_{q\in L^2_\circ(\Omega)} \dfrac{|\nabla q|_{-1,\Omega}}{\|q\|_{0,\Omega}}$
2. Conversion by D the Riesz isometry between $H_0^1(\Omega)^d$ and $H^{-1}(\Omega)^d$

Relation with the Cosserat problem

Introduce the family of operators $\sigma \mapsto L_\sigma$

$$L: \quad H_0^1(\Omega)^d \longrightarrow H^{-1}(\Omega)^d$$
$$\boldsymbol{v} \longmapsto \sigma\Delta\boldsymbol{v} - \nabla \operatorname{div} \boldsymbol{v}$$

Spectrum

- Cosserat spectrum, $\mathfrak{S}(L)$: set of σ such that L_σ is not invertible,
- Essential spectrum, $\mathfrak{S}_{ess}(L)$: set of σ such that L_σ is not Fredholm.

Relation of Cosserat spectrum with Schur complement S:

$$\mathfrak{S}(L) = \mathfrak{S}(S) \cup \{0\} \quad \text{and} \quad \mathfrak{S}_{ess}(L) = \mathfrak{S}_{ess}(S) \cup \{0\}$$

[Michlin 1973]:

- $0, 1 \in \mathfrak{S}_{ess}(L)$ because L_σ is not elliptic inside the domain
- $\frac{1}{2} \in \mathfrak{S}_{ess}(L)$ because Dirichlet conditions do not cover L_σ

▸ RETURN

Plan

Friedrichs constant

- $\Omega \subset \mathbb{R}^2 \sim \mathbb{C}$
- $\mathfrak{F}(\Omega)$ space of complex valued $L^2(\Omega)$ holomorphic functions.
- $\mathfrak{F}_\circ(\Omega)$ the subspace of $\mathfrak{F}(\Omega)$ with mean value 0.

Definition

Let $h, g \in L^2(\Omega)$ with real values.

- h is *harmonic conjugate* to g if the function $(x_1 + ix_2) \mapsto h(x_1, x_2) + ig(x_1, x_2)$ is holomorphic. Equivalently:

$$\Delta h = 0, \quad \Delta g = 0, \quad \text{and} \quad \nabla h = \overrightarrow{\text{curl}}\, g \quad \text{in} \quad \Omega.$$

- The *Friedrichs constant* $\Gamma(\Omega) \in \mathbb{R} \cup \{\infty\}$ is the smallest constant Γ such that for all $h + ig \in \mathfrak{F}_\circ(\Omega)$

$$\|h\|^2_{L^2(\Omega)} \leq \Gamma \, \|g\|^2_{L^2(\Omega)}$$

Friedrichs constant and LBB constant

Theorem [Horgan-Payne 1983] [Costabel-Dauge 2013]

1. If $\beta(\Omega) > 0$, then $\Gamma(\Omega)$ is finite and
$$\Gamma(\Omega) + 1 \leq \frac{1}{\beta(\Omega)^2}$$
2. If $\Gamma(\Omega)$ is finite, then $\beta(\Omega) > 0$ and
$$\frac{1}{\beta(\Omega)^2} \leq \Gamma(\Omega) + 1$$

1. Let $h + ig \in \mathfrak{F}_\circ(\Omega)$. Define $\boldsymbol{u} \in H^1_0(\Omega)^2$ such that
$$\operatorname{div} \boldsymbol{u} = h \quad \text{and} \quad |\boldsymbol{u}|^2_{1,\Omega} \leq \frac{1}{\beta(\Omega)^2} \|h\|^2_{0,\Omega}$$
2. Let $p \in L^2_\circ(\Omega)$. Define $\boldsymbol{u} = D^{-1}\nabla p$, $q = \operatorname{div} \boldsymbol{u}$ and $g = \operatorname{curl} \boldsymbol{u}$. Then g and $q - p$ are conjugate harmonic functions.

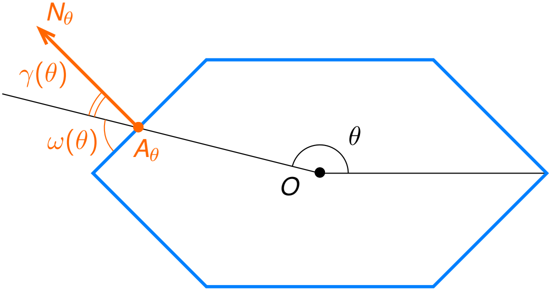
Plan

Horgan-Payne ingredients

- Domain Ω star-shaped with respect to a ball $\mathcal{B}$ (Ω *strictly star-shaped*).
- Center $O \in \mathcal{B}$
- Polar coordinates (r, θ) centered at O
- Polar parametrization $f \in W^{1,\infty}(\mathbb{T})$ of the boundary:

$$
\begin{array}{rlcl}
f: & \mathbb{R}/2\pi\mathbb{Z} =: \mathbb{T} & \longrightarrow & \partial\Omega \\
 & \theta & \longmapsto & A_\theta = \big(f(\theta)\cos\theta, f(\theta)\sin\theta\big)
\end{array}
$$

- Normal N_θ to $\partial\Omega$ at A_θ
- Angle $\gamma(\theta)$ between line $[O, A_\theta]$ and N_θ, and $\omega(\theta) = \frac{\pi}{2} - \gamma(\theta)$.



Horgan-Payne bounds (original and updated)

Bounds for the "Friedrichs constant" $\Gamma(\Omega) \overset{Theorem}{=} \dfrac{1}{\beta(\Omega)^2} - 1$

- ☹ m_{HP} original bound, nicer, but flawed proof
- ☺ M_{HP} updated bound, uglier, but correct proof (by us)

Defined by means of function P (assuming without restriction $\max_{\theta\in\mathbb{T}} f(\theta) = 1$)

$$(0,1)\times\mathbb{T} \ni (\alpha,\theta) \longmapsto P(\alpha,\theta) = \frac{1}{\alpha f(\theta)^2}\left(1 + \frac{\tan^2\gamma(\theta)}{1-\alpha f(\theta)^2}\right)$$

α is a parameter introduced to optimize the sum of two terms in the proof

Original HP-bound

$$\Gamma(\Omega) \le m_{HP} = \sup_{\theta\in\mathbb{T}} \left\{ \inf_{\alpha\in\left(0,\frac{1}{f(\theta)^2}\right)} P(\alpha,\theta) \right\}$$

Updated HP-like bound (the correct one)

$$\Gamma(\Omega) \le M_{HP} := \inf_{\alpha\in(0,1)} \left\{ \sup_{\theta\in\mathbb{T}} P(\alpha,\theta) \right\}$$

An example where $M_{HP} \neq m_{HP}$

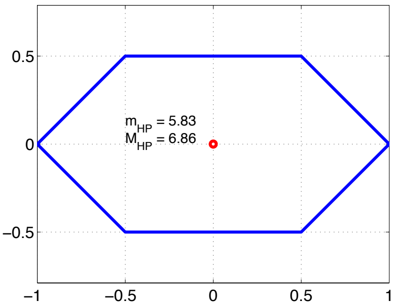
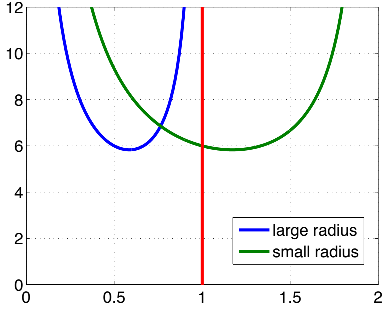


Figure: Ω with its center O



Plot of $\alpha \mapsto P(\alpha, \theta_j),\ j = 1, 2$

$\theta_1 = 0, \quad \theta_2 = \frac{\pi}{4}$

Horgan-Payne angles (original and updated)

Original HP-angle (introduced by Stoyan)

$$\omega_{HP} := \frac{\pi}{2} - \sup_{\theta\in\mathbb{T}} \gamma(\theta) = \inf_{\theta\in\mathbb{T}} \omega(\theta) = \arccos\left(\frac{m_{HP}-1}{m_{HP}+1}\right)$$

ω_{HP} is the minimal angle between radius and tangent at boundary pts A_θ.

Updated HP-like angle (the correct one)

$$\mu_{HP} = \arccos\left(\frac{M_{HP}-1}{M_{HP}+1}\right)$$

"Theorem" Original by Horgan & Payne 1983 and Stoyan 2001 ϟ

$$\Gamma(\Omega) \le m_{HP} \quad \text{i.e.} \quad \beta(\Omega) \ge \sin\frac{\omega_{HP}}{2}$$

Theorem (the proved one) [Co-Da 2013] ☼

$$\Gamma(\Omega) \le M_{HP} \quad \text{i.e.} \quad \beta(\Omega) \ge \sin\frac{\mu_{HP}}{2}$$

Cases of agreement ☺

Examples where $m_{HP} = M_{HP}$ and $\omega_{HP} = \mu_{HP}$

- Disk and ellipses: equation (with $a \geq b$)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Then

$$m_{HP} = M_{HP} = \Gamma(\Omega) = \frac{a^2}{b^2}, \quad \text{i.e.} \quad \beta(\Omega) = \frac{b}{\sqrt{a^2 + b^2}}$$

▸ RETURN

- Polygons with concentric corners c_j, i.e. $r(c_j)$ are equal, $j = 1, \ldots, J$
  Example 1: Regular polygons
  Example 2: Rectangles
- Polygons with edges e_j such that $dist(O, e_j)$ are equal, $j = 1, \ldots, J$
  Example: Triangles

Plan

Cases of disagreement ☹

[Ho-Pa 1983] + [Stoyan 2001]

$$\Gamma(\Omega) \le m_{HP} \quad \text{i.e.} \quad \beta(\Omega) \ge \sin \frac{\omega_{HP}}{2}$$

Theorem [Co-Da 2013]

There exists a strictly star-shaped domain $\Omega \subset \mathbb{R}^2$ such that

$$\Gamma(\Omega) > m_{HP} \quad \text{i.e.} \quad \beta(\Omega) < \sin \frac{\omega_{HP}}{2}$$

Our counter-example is based on a family of domains with a *narrow pass* for which we can prove

an upper bound for $\beta(\Omega)$, i.e., a lower bound for $\Gamma(\Omega)$

Domains with a narrow pass

Let us define for the (connected) domain $\Omega \subset \mathbb{R}^2$

$$\begin{aligned} \Omega_+ &= \{(x_1, x_2) \in \Omega, \ \ x_1 > 0\} \\ \Omega_- &= \{(x_1, x_2) \in \Omega, \ \ x_1 < 0\} \end{aligned} \quad \text{and} \quad \Pi = \{(x_1, x_2) \in \Omega, \ \ x_1 = 0\}.$$

We assume that Ω_+, Ω_- and Π are connected and non-empty.
Let $|\Omega_\pm|$ denote the area of $\Omega_\pm$ and $\ell(\Pi)$ the length of Π

Theorem [Co-Da 2013]

We have the upper bound for $\sigma(\Omega) = \beta(\Omega)^2$

$$\beta(\Omega)^2 \leq \frac{8}{3} \, \ell(\Pi)^2 \, \frac{|\Omega_+| + |\Omega_-|}{|\Omega_+| \, |\Omega_-|}$$

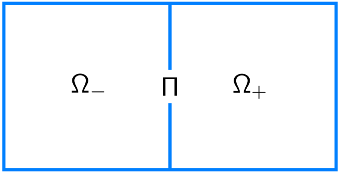
Particular case

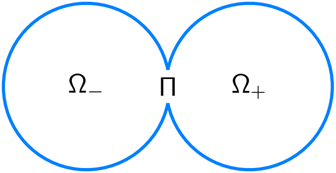
If $|\Omega_+| = |\Omega_-|$
$$\beta(\Omega)^2 \leq \frac{16}{3} \, \frac{\ell(\Pi)^2}{|\Omega_+|}$$

Example of domains with a narrow pass

The proof of the Theorem is based on property [B2]

$$\beta(\Omega) = \inf_{q \in L^2_\circ(\Omega) \text{ with } \|q\|_{0,\Omega}=1} |\nabla q|_{-1,\Omega} \quad \text{choosing} \quad q = \frac{\chi_{\Omega_+}}{|\Omega_+|} - \frac{\chi_{\Omega_-}}{|\Omega_-|}$$





$$\ell(\Pi) = 2\varepsilon \text{ with } \varepsilon = 0.2$$

$$|\Omega_+| = 4$$

$$\beta(\Omega)^2 \leq \frac{16}{3}\,\varepsilon^2 \approx 0.2133$$

Ω is not strictly star-shaped

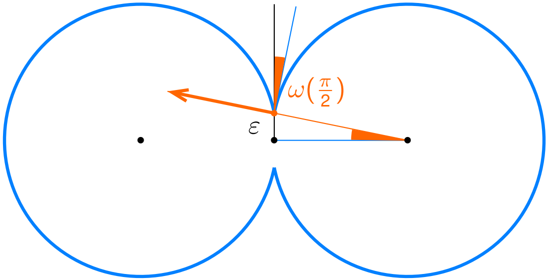
$$\ell(\Pi) = 2\varepsilon \text{ with } \varepsilon = 0.2$$

$$|\Omega_+| = \pi - \arcsin\varepsilon + \varepsilon\sqrt{1-\varepsilon^2}$$

$$\beta(\Omega)^2 \leq \frac{64}{3\pi}\,\varepsilon^2 + \mathcal{O}(\varepsilon^5) \approx 0.2720$$

Ω is strictly star-shaped

More on the two circles example



We take $\ell(\Pi) = 2\varepsilon$ with $\varepsilon = 0.2$. Hence $|\Omega_+| = \pi - \arcsin\varepsilon + \varepsilon\sqrt{1-\varepsilon^2}$.
Hence the upper bound $\beta(\Omega)^2 \leq \frac{64}{3\pi}\,\varepsilon^2 + \mathcal{O}(\varepsilon^5) \approx 0.2760$

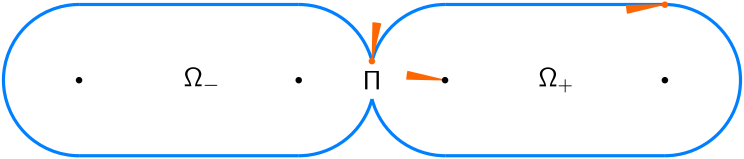
$$\omega_{HP}(\Omega) = \omega\left(\frac{\pi}{2}\right) = \arcsin\varepsilon$$

HP bound:

$$\beta(\Omega)^2 \geq \sin^2\frac{\omega_{HP}}{2} = \frac{\varepsilon^2}{4} + \mathcal{O}(\varepsilon^4)$$

Not yet a counter-example (no contradiction).

The two stadia (counter) example



Set $\rho = \sqrt{1-\varepsilon^2}$. Centers in Ω_+ have abscissa ρ and ρ/ε. Half-circles have radius 1.

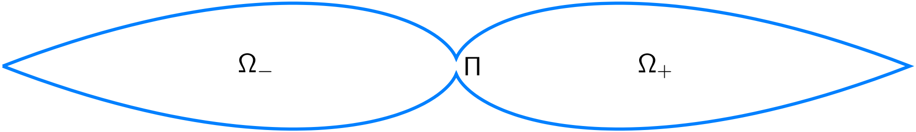
$$\ell(\Pi) = 2\varepsilon \quad \text{with} \quad \varepsilon = 0.25 \text{ (above) and } \varepsilon = 0.0234 \text{ (below)}$$

$$|\Omega_+| = 2\rho\Big(\frac{1}{\varepsilon} - 1\Big) + \pi + \mathcal{O}(\varepsilon^3) = \frac{2}{\varepsilon} - 2 + \pi + \mathcal{O}(\varepsilon^2)$$

$$\beta(\Omega)^2 \leq \frac{32}{3}\varepsilon^3 + \mathcal{O}(\varepsilon^4)$$

$$\omega_{HP}(\Omega) = \arcsin\varepsilon \quad \overset{HP\ bound}{\Longrightarrow} \quad \beta(\Omega)^2 \geq \sin^2\frac{\omega_{HP}}{2} = \frac{\varepsilon^2}{4} + \mathcal{O}_{\varepsilon\to 0}(\varepsilon^4) \quad ϟ$$

The four logarithmic spirals counter-example



In the first quadrant, the curve is defined by $f(\theta) = e^{-c\theta}$

$$\ell(\Pi) = 2e^{-c\pi/2} \quad \text{with} \quad c = 2.58$$

$$|\Omega_+| = 2\int_0^{\pi/2}\int_0^{f(\theta)} r\,dr\,d\theta = [-\frac{1}{2c}e^{-2c\theta}]_0^{\pi/2} = \frac{1}{2c}(1 - e^{-c\pi})$$

$$\beta(\Omega)^2 \leq \frac{128}{3}\,\frac{c\,e^{-c\pi}}{1 - e^{-c\pi}} \approx 0.0333$$

But

$$\forall\theta \in \mathbb{T}, \quad \tan\gamma(\theta) \overset{general}{=} \frac{f'(\theta)}{f(\theta)} \overset{specific}{=} c, \quad \text{i.e.} \quad \tan\omega(\theta) = \cot\gamma(\theta) = \frac{1}{c}$$

$$\omega_{HP}(\Omega) = \arctan\frac{1}{c} \overset{HP\ bound}{\Longrightarrow} \beta(\Omega)^2 \geq \sin^2\frac{\omega_{HP}}{2} = \frac{1}{4c^2} + \mathcal{O}(\frac{1}{c^4}) \approx 0.0337 \quad ↯$$

Plan

Inner and outer radii of strict. star-shaped domains

- $\Omega \subset \mathbb{R}^2$ star-shaped with respect to the ball $\mathcal{B}$
- O center of $\mathcal{B}$ and ρ radius of $\mathcal{B}$ *(inner radius)*
- R (smallest number) such that $\Omega \subset \mathcal{B}(O, R)$ *(outer radius)*

Lemma [Co-Da 2013]

$\Omega \subset \mathbb{R}^2$ bounded domain.

1. Ω is strictly star-shaped $\Longleftrightarrow$ $\partial\Omega$ has a polar parametrization $r = f(\theta)$ with a periodic Lipschitz continuous function f satisfying

$$\forall \theta \in \mathbb{T} : \quad f(\theta) > 0.$$

2. Optimal values for ρ and R with respect to the center O of the parametrization are given by

$$R = \max_{\theta \in \mathbb{T}} f(\theta) \quad \text{and} \quad \rho = \inf_{\theta \in \mathbb{T}} \frac{f(\theta)^2}{\sqrt{f(\theta)^2 + f'(\theta)^2}}$$

Universal lower bound for strict. star-shaped domains

Theorem [Co-Da 2013]

The inf-sup constant $\beta(\Omega)$ admits the lower bound

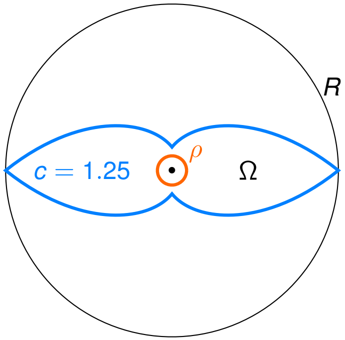
$$\beta(\Omega)^2 \geq \frac{\rho^2}{2R^2}\left(1+\sqrt{1-\frac{\rho^2}{R^2}}\right)^{-1}$$

Corollary

1. $$\beta(\Omega) \geq \frac{\rho}{2R}$$
2. $$\rho \to R \quad \Longrightarrow \quad \beta(\Omega) \to \frac{1}{\sqrt{2}}$$

Improvement of [Duran 2012].

The four logarithmic spirals: upper & lower bounds



For $f(\theta) = \mathrm{e}^{-c\theta}$

$$R = 1 \quad \text{and} \quad \rho = \frac{\mathrm{e}^{-c\pi/2}}{\sqrt{1+c^2}}$$

Hence the lower bound

$$\beta(\Omega)^2 \geq \frac{\mathrm{e}^{-c\pi}}{4(1+c^2)}$$

Hence

$$\frac{\mathrm{e}^{-c\pi}}{4(1+c^2)} \leq \beta(\Omega)^2 \leq \frac{64}{3}\,\frac{c\,\mathrm{e}^{-c\pi}}{1-\mathrm{e}^{-c\pi}}$$

For the counter-example ($c = 2.25$)

$$0.00014 \leq \beta(\Omega)^2 \leq 0.0409$$

Plan

References

- G. Acosta, R.G. Durán, M.A. Muschietti, *Solutions of the divergence operator on John domains*, Adv. Math. **206** (2006), 373–401.
- M.E. Bogovskiĭ, *Solution of the first boundary value problem for the equation of continuity of an incompressible medium*, Soviet Math. Dokl. **20** (1979), 1094–1098.
- E. Cosserat, F. Cosserat, *Sur la déformation infiniment petite d'un ellipsoide élastique*, Note aux C.R.A.S., Paris **127** (1898), 315–318.
- M. Costabel, M. Dauge, *On the inequalities of Babuška–Aziz, Friedrichs and Horgan–Payne*, In preparation (2013).
- M. Crouzeix, *On an operator related to the convergence of Uzawa's algorithm for the Stokes equation*, in Computational Science for the 21 century, M.-O. Bristeau and al. eds, Wiley (1997), 242–249.
- R.G. Durán, *An elementary proof of the continuity from $L^2_0(\Omega)$ to $H^1_0(\Omega)^n$ of Bogovskii's right inverse of the divergence*, Revista de la Unión Matemática Argentina **53(2)** (2012), 59–78.

References

- K.O. Friedrichs, *On certain inequalities and characteristic value problems for analytic functions and for functions of two variables*, Trans. Amer. Math. Soc. **41** (1937), 321–364.
- C.O. Horgan, L.E. Payne, *On inequalities of Korn, Friedrichs and Babuška–Aziz*, Arch. Rat. Mech. Anal. **82** (1983), 165–179.
- S. G. Mihlin, *The spectrum of the pencil of operators of elasticity theory*. (Russian) Uspehi Mat. Nauk **28** (1973), no. 3(171), 43–82
- J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson et Cie (1967).
- G. Stoyan, *Iterative Stokes solvers in the harmonic Velte subspace*, Computing **67** (2001), 12–33.
- L. Tartar, *An Introduction to Navier-Stokes Equation and Oceanography*, Lecture Notes of the Unione Matematica Italiane **1**, Springer (2006).

A couple of irritating questions

1. The *question of continuity* of $\beta(\Omega)$ wrt Ω. Positive answer if $\mathscr{C}^2$ variable changes are involved. No answer if less regular, so no answer for rounded corners or piecewise linear approximation. Though flawed, the approach by Horgan-Payne is remarkable because only first derivatives of f are involved.
2. The precise value of $\beta(\square)$ (the square). The Schur complement S has *essential spectrum*

$$
\begin{aligned}
\mathfrak{S}_{\text{ess}}(S) &= \left[\frac{1}{2} - \frac{|\sin\alpha|}{2\alpha}, \frac{1}{2} + \frac{|\sin\alpha|}{2\alpha}\right] \cup \{1\} \\
&= \left[\frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} + \frac{1}{\pi}\right] \cup \{1\} \\
&\simeq \left[0.1817, 0.8183\right] \cup \{1\}.
\end{aligned}
$$

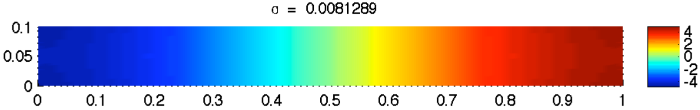
   Difficult to say whether S has discrete spectrum below 0.1817.
3. Less irritating: The characterization of domains such that $\beta(\Omega) > 0$. Almost optimal characterization: the *John domains*.

Plan

Appendices: Computations of eigenpairs

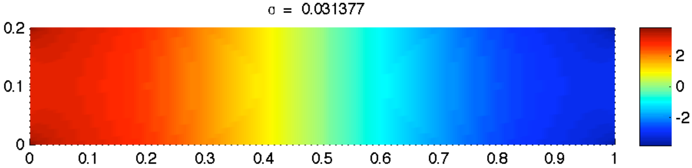
1. Computation in rectangles by Monique with an *ad hoc* spectral method (Matlab)
2. Computation in finite fractal-like domains by Frédéric Hecht with mixed $\mathbb{P}_2$-$\mathbb{P}_1$ finite elements (FreeFEM++)

First eigenpair of S in $(0,1) \times (0,\rho)$ with kmax = 100



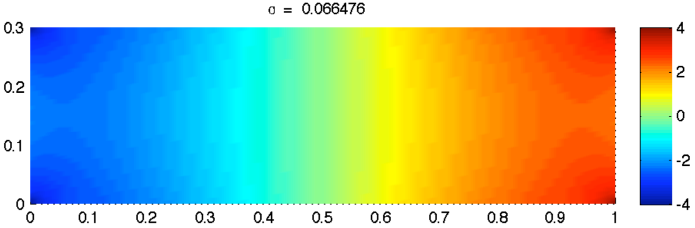
$$\rho = 0.1 \qquad \sigma_{\text{app}} = 0.0081$$

First eigenpair of S in $(0,1) \times (0,\rho)$ with kmax = 100



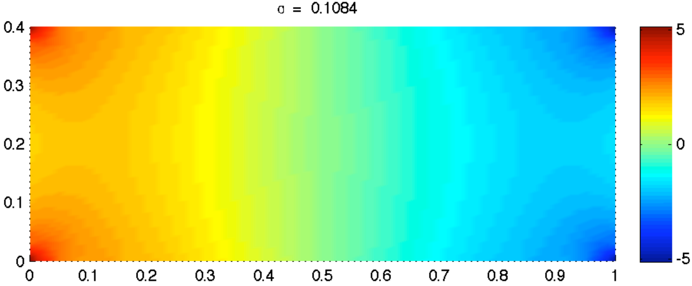
$\rho = 0.2 \qquad \sigma_{\text{app}} = 0.0314$

First eigenpair of S in $(0, 1) \times (0, \rho)$ with kmax = 100



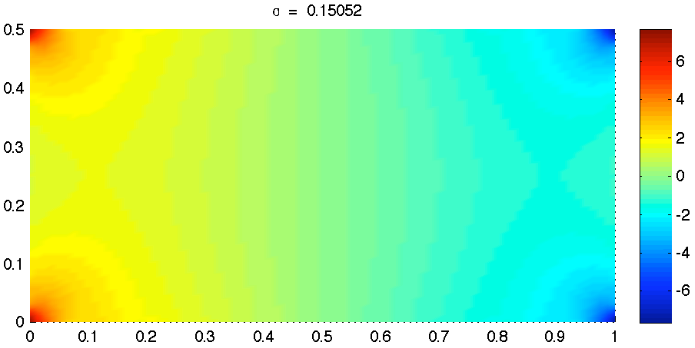
$$\rho = 0.3 \qquad \sigma_{\text{app}} = 0.0665$$

First eigenpair of S in $(0,1) \times (0,\rho)$ with kmax = 100



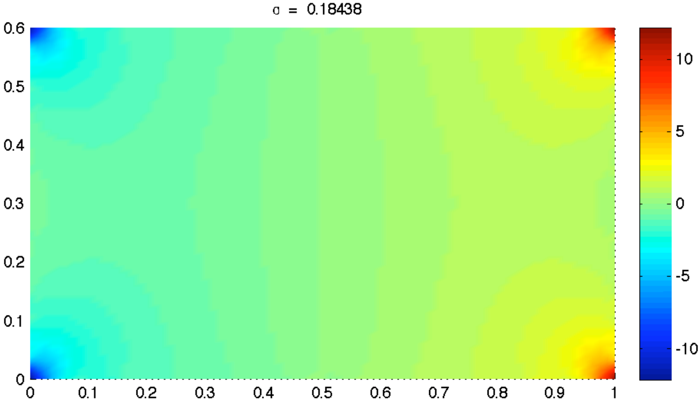
$$\rho = 0.4 \qquad \sigma_{\text{app}} = 0.1084$$

First eigenpair of S in $(0,1) \times (0,\rho)$ with kmax = 100



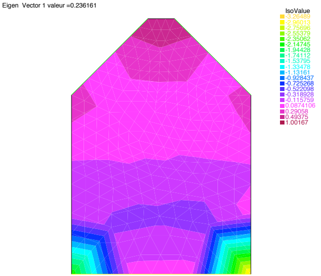
$\rho = 0.5 \qquad \sigma_{\text{app}} = 0.1505$

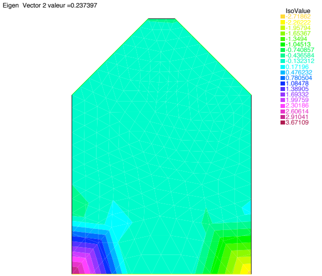
First eigenpair of S in $(0,1)\times(0,\rho)$ with kmax = 100

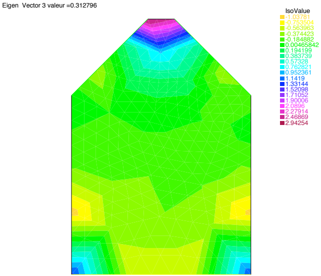


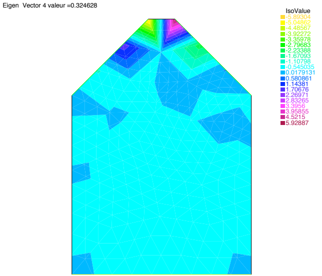
$\rho = 0.6 \quad \sigma_{\text{app}} = 0.1844.$ Note: $\inf \mathfrak{S}_{\text{ess}}(S) \simeq 0.1817.$

First 4 eigenpairs on “fractal” structure, level 0

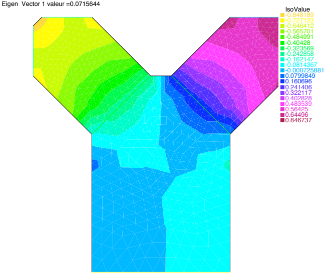


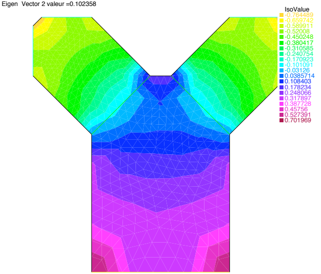


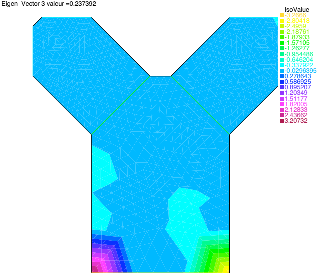


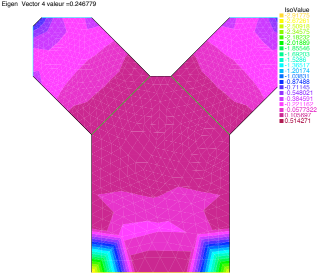


First 4 eigenpairs on “fractal” structure, level 1

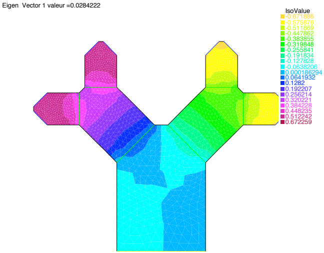


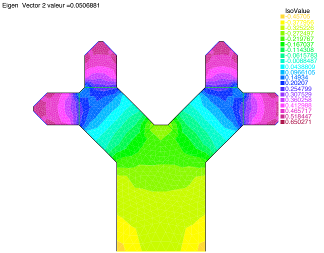


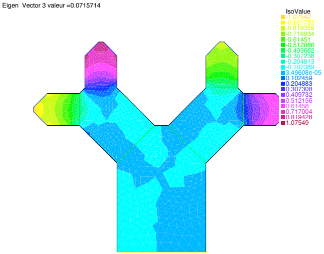


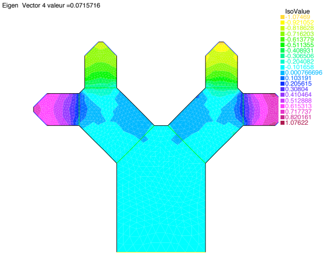


First 4 eigenpairs on “fractal” structure, level 2

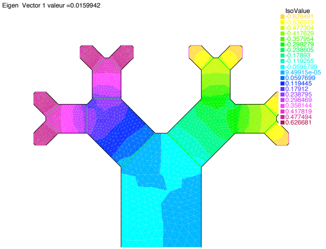


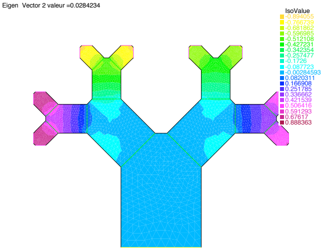


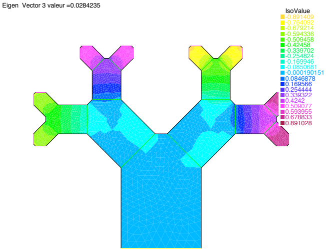


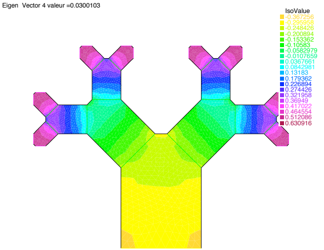


First 4 eigenpairs on “fractal” structure, level 3

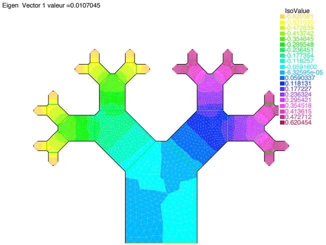


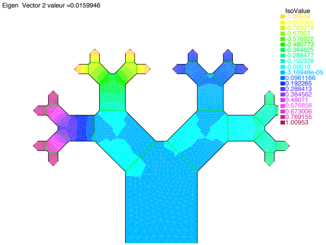


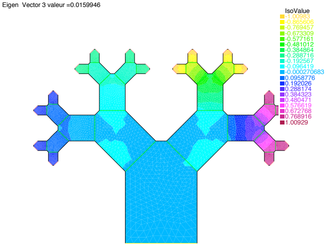


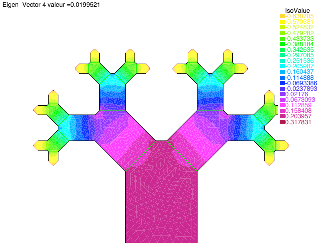


First 4 eigenpairs on “fractal” structure, level 4









First 4 eigenpairs on “fractal” structure, level 5