Regularity and singularities in polyhedral domains

Complement: The case of Helmholtz equations and impedance bc

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Corner expansion in a polygon (Dirichlet)

Consider the Helmholtz equation with Dirichlet bc

$$\Delta u + \kappa^2 u = f \quad \text{in} \quad \Omega, \quad u \in H^1_0(\Omega)$$

For diffraction, we may think of Ω as the complementary set of a bounded polygonal or polyhedral domain in a large sphere.
Let $\boldsymbol{c} \in \mathscr{C}$ and $\chi_{\boldsymbol{c}}$ be a smooth cut-off, $\chi_{\boldsymbol{c}} \equiv 1$ near corner $\boldsymbol{c}$. Expansion of solutions

$$u = u^{\text{reg}} + \sum_{\boldsymbol{c}\in\mathscr{C}} \chi_{\boldsymbol{c}}\, u^{\text{sing}}_{\boldsymbol{c}} \circ \phi_{\boldsymbol{c}}\,, \quad u^{\text{reg}} \in H^s(\Omega)$$

where we can write $u^{\text{sing}}_{\boldsymbol{c}}$ as

$$u^{\text{sing}}_{\boldsymbol{c}} = \sum_{\substack{\lambda=k\pi/\omega_{\boldsymbol{c}}\\ 0<\lambda<s-1}} d^{\lambda}_{\boldsymbol{c}} \Bigg(\sum_{\substack{p\in\mathbb{N}_0\\ \lambda+2p\le s-1}} \kappa^{2p}\Phi^{\lambda,2p}_{\boldsymbol{c}} \Bigg)$$

with $\Phi^{\lambda,0}_{\boldsymbol{c}} = r^{\lambda}\sin\lambda\theta$ and $\Phi^{\lambda,2p}_{\boldsymbol{c}} \in S^{\lambda+2p}_0(K_{\boldsymbol{c}})$ solution *(always exist!)* of

$$\Delta\Phi^{\lambda,2p}_{\boldsymbol{c}} + \Phi^{\lambda,2p-2}_{\boldsymbol{c}} = 0, \quad p = 1, 2, \dots$$

Alternative expression of singularities when $n = 2$

By separation of variables, for non-integer λ we can calculate a function Ψ_c^λ such that

$$\Delta\Psi_c^\lambda + \kappa^2\Psi_c^\lambda = 0 \text{ in } K_c \quad \text{and} \quad \Psi_c^\lambda \to \Phi_c^{\lambda,0} \text{ as } r \to 0$$

Recall that $\Phi_c^{\lambda,0} = r^\lambda \sin\lambda\theta$. We find

Bessel type singularites

$$\Psi_c^\lambda = 2^\lambda\,\Gamma(\lambda+1)\,J_\lambda\left(\frac{r}{\kappa}\right)\sin\lambda\theta$$

with the Bessel function of the first kind J_λ.

In fact

$$\Psi_c^\lambda = \sum_{p=0}^{\infty} \kappa^{2p}\Phi_c^{\lambda,2p}.$$

Alternative expression of singularities when $n = 3$

Dirichlet Eigenpairs provide singularities

$$\Delta^{\mathrm{dir}}_{G_c}\psi^\lambda_c = \mu\psi \quad \text{yields} \quad \Phi^{\lambda,0}_c = r^\lambda \psi^\lambda_c(\theta) \quad \text{with} \quad \lambda^2 + \lambda = \mu$$

By separation of variables, we can still calculate a function Ψ^λ_c such that

$$\Delta\Psi^\lambda_c + \kappa^2\Psi^\lambda_c = 0 \text{ in } K_c \quad \text{and} \quad \Psi^\lambda_c \to \Phi^{\lambda,0}_c \text{ as } r \to 0$$

We find

$$\Psi^\lambda_c = \gamma_\lambda\, j_\lambda\left(\frac{r}{\kappa}\right)\psi^\lambda_c(\theta)$$

with the spherical Bessel function j_λ. There holds

$$j_\lambda(\rho) = \sqrt{\frac{\pi}{2\rho}}\, J_{\lambda+1/2}(\rho)$$

Bessel type singularites

$$\Psi^\lambda_c = \frac{2^{\lambda+1}}{\sqrt{\pi}}\,\Gamma(\lambda + \frac{3}{2})\, j_\lambda\left(\frac{r}{\kappa}\right)\psi^\lambda_c(\theta)$$

Impedance boundary condition for Δ

With $\alpha \in \mathbb{C}$, the problem is

$$\Delta u = f \ \text{ in } \ \Omega \quad \text{and} \quad \partial_n u + \alpha u = 0 \ \text{ on } \ \partial\Omega$$

From the corner point of view, is a perturbation of the Neumann bc.
Singularities have the general form

$$u_{\boldsymbol{c}}^{\text{sing}} = \sum_{\substack{\lambda = k\pi/\omega_{\boldsymbol{c}} \\ 0<\lambda<s-1}} d_{\boldsymbol{c}}^{\lambda} \Big(\sum_{\substack{p\in\mathbb{N}_0 \\ \lambda+p\le s-1}} \Phi_{\boldsymbol{c}}^{\lambda,p} \Big) \ \text{ with } \ \begin{cases} \Phi_{\boldsymbol{c}}^{\lambda,0} = \Phi_{\boldsymbol{c}}^{\lambda} \\ \Phi_{\boldsymbol{c}}^{\lambda,p} \in S^{\lambda+p}(K_{\boldsymbol{c}}), \ p = 1, 2, \ldots \end{cases}$$

Recall

$$S^{\lambda}(K_{\boldsymbol{c}}) = \Big\{ \Phi = \sum_{q\ge 0,\ \text{finite}} r^{\lambda} \log^{q} r \, \varphi_q(\theta), \ \varphi_q \in H^1(G_{\boldsymbol{c}}) \Big\}$$

The *shadows* of singularities $\Phi_{\boldsymbol{c}}^{\lambda,p}$ are solution in $S^{\lambda+p}(K_{\boldsymbol{c}})$ of the problems

$$\Delta\Phi_{\boldsymbol{c}}^{\lambda,p} = 0 \text{ in } K_{\boldsymbol{c}} \quad \text{and} \quad \partial_n \Phi_{\boldsymbol{c}}^{\lambda,p} + \alpha \Phi_{\boldsymbol{c}}^{\lambda,p-1} = 0 \text{ on } \partial K_{\boldsymbol{c}}, \quad p = 1, 2 \ldots$$

Helmholtz

With $\alpha \in \mathbb{C}$ and $\kappa > 0$, the problem is

$$\Delta u = f \text{ in } \Omega \quad \text{and} \quad \partial_n u + \alpha u = 0 \text{ on } \partial\Omega$$

The *shadows* of singularities $\Phi_{\mathbf{c}}^{\lambda,p}$ are solution of

$$\Delta \Phi_{\mathbf{c}}^{\lambda,1} = 0 \text{ in } K_{\mathbf{c}} \quad \text{and} \quad \partial_n \Phi_{\mathbf{c}}^{\lambda,1} + \alpha \Phi_{\mathbf{c}}^{\lambda,0} = 0 \text{ on } \partial K_{\mathbf{c}}$$

and for $p = 2, 3 \ldots$

$$\Delta \Phi_{\mathbf{c}}^{\lambda,p} + \kappa^2 \Phi_{\mathbf{c}}^{\lambda,p-2} = 0 \text{ in } K_{\mathbf{c}} \quad \text{and} \quad \partial_n \Phi_{\mathbf{c}}^{\lambda,p} + \alpha \Phi_{\mathbf{c}}^{\lambda,p-1} = 0 \text{ on } \partial K_{\mathbf{c}}$$

If $\alpha = i\kappa$, one can define shadows independent of κ

$$\Delta \Phi_{\mathbf{c}}^{\lambda,1} = 0 \text{ in } K_{\mathbf{c}} \quad \text{and} \quad \partial_n \Phi_{\mathbf{c}}^{\lambda,1} + i \Phi_{\mathbf{c}}^{\lambda,0} = 0 \text{ on } \partial K_{\mathbf{c}}$$

and for $p = 2, 3 \ldots$

$$\Delta \Phi_{\mathbf{c}}^{\lambda,p} + \Phi_{\mathbf{c}}^{\lambda,p-2} = 0 \text{ in } K_{\mathbf{c}} \quad \text{and} \quad \partial_n \Phi_{\mathbf{c}}^{\lambda,p} + i \Phi_{\mathbf{c}}^{\lambda,p-1} = 0 \text{ on } \partial K_{\mathbf{c}}$$

Then

$$u_{\mathbf{c}}^{\text{sing}} = \sum_{0<\lambda<s-1} d_{\mathbf{c}}^{\lambda} \Big(\sum_{\lambda+p\leq s-1} \kappa^p \Phi_{\mathbf{c}}^{\lambda,p} \Big)$$