Skin effect in electromagnetism

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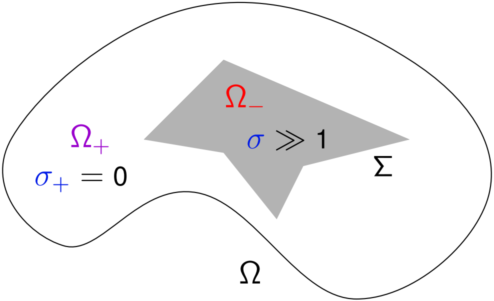
2010 ISFMA Symposium

(Fudan University Shanghai, July 26-29, 2010)

Outline

The Skin Effect : A 3-D Problem



- Ω_- Highly Conducting body $\subset\subset \Omega$: Conductivity $\sigma_- \equiv \sigma \gg 1$
- $\Sigma = \partial\Omega_-$: Interface
- Ω_+ Insulating or Dielectric body: Conductivity $\sigma_+ = 0$

The Skin Effect : rapid decay of electromagnetic fields inside the conductor.
The classical Skin Depth : $\ell(\sigma) = \sqrt{2/\omega\mu_0\sigma}$

Our references

- V. PÉRON (PhD thesis, Université Rennes 1, 2009) Link
  Modélisation mathématique de phénomènes électromagnétiques dans des matériaux à fort contraste.
- M. DAUGE, E. FAOU, V. PÉRON (Note CRAS, 2010) Link
  Comportement asymptotique à haute conductivité de l'épaisseur de peau en électromagnétisme
- G. CALOZ, M. DAUGE, V. PÉRON (Article JMAA, 2010) Link
  Uniform estimates for transmission problems with high contrast in heat conduction and electromagnetism
- G. CALOZ, M. DAUGE, E. FAOU, V. PÉRON (Preprint, 2010) Link
  On the influence of the geometry on skin effect in electromagnetism
- M. DAUGE, V. PÉRON, C. POIGNARD (In preparation, 2010)
  Asymptotic expansion for the solution of a stiff transmission problem in electromagnetism with a singular interface

- Aim : Understanding the influence of the geometry on the skin effect.

Outline

Maxwell Problem

Maxwell equations with perfectly insulating exterior b.c.

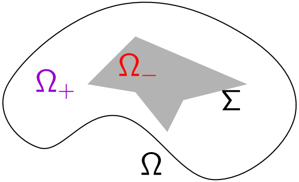
$$(\mathbf{P}_{\underline{\sigma}}) \quad \begin{cases} \operatorname{curl}\mathbf{E} - i\omega\mu_0\mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl}\mathbf{H} + (i\omega\varepsilon_0 - \underline{\sigma})\mathbf{E} = \mathbf{J} \\ \mathbf{E}\cdot\mathbf{n} = 0 \quad \text{and} \quad \mathbf{H}\times\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

with the piecewise constant conductivity

$$\underline{\sigma} = (\sigma_+, \sigma_-) = (0, \sigma \gg 1)$$

and the rhs

$$\mathbf{J} \in H_0(\operatorname{div}, \Omega) = \{\mathbf{u} \in L^2(\Omega)^3 \mid \operatorname{div}\mathbf{u} \in L^2(\Omega),\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$$



Existence of solutions

Hypothesis (*SH*)

The angular frequency ω *is not an eigenfrequency of the problem*

$$\begin{cases} \operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0 \quad \textit{and} \quad \operatorname{curl} \mathbf{H} + i\omega\varepsilon_0 \mathbf{E} = 0 & \textit{in} \quad \Omega_+ \\ \mathbf{E} \times \mathbf{n} = 0 \quad \textit{and} \quad \mathbf{H} \cdot \mathbf{n} = 0 & \textit{on} \quad \Sigma \\ \mathbf{E} \cdot \mathbf{n} = 0 \quad \textit{and} \quad \mathbf{H} \times \mathbf{n} = 0 & \textit{on} \quad \partial\Omega \end{cases}$$

Theorem (Caloz, Dauge, Péron, 2009)

If the surface Σ *is Lipschitz, under Hypothesis* (SH)*, there exist* σ_0 *and* $C > 0$*, such that for all* $\sigma \geqslant \sigma_0$*,* $(\mathbf{P}_{\underline{\sigma}})$ *with B.C. and* $\mathbf{J} \in H_0(\operatorname{div}, \Omega)$ *has a unique solution* $(\mathbf{E}, \mathbf{H})$ *in* $L^2(\Omega)^6$*, and*

$$\|\mathbf{E}\|_{0,\Omega} + \|\mathbf{H}\|_{0,\Omega} + \sqrt{\sigma}\, \|\mathbf{E}\|_{0,\Omega_-} \leqslant C\, \|\mathbf{J}\|_{H(\operatorname{div},\Omega)}$$

Application: Convergence of asymptotic expansion for large conductivity.

Outline

Earlier related references for asymptotics when $\sigma \longrightarrow \infty$

S. M. RYTOV.
Calcul du skin effect par la méthodes des perturbations.
Journal of Physics (1940)

E. STEPHAN.
Solution procedures for interface problems in [...] electromagnetics.
CISM Courses and Lectures, **277**, 291–348 (1983).

R. C. MACCAMY, E. STEPHAN.
Solution procedures for three-dimensional eddy current problems.
J. Math. Anal. Appl. **101**(2) (1984) 348–379.

R. C. MACCAMY, E. STEPHAN.
A skin effect approximation for eddy current problems.
Arch. Rational Mech. Anal. **90**(1) (1985) 87–98.

H. HADDAR, P. JOLY, H.-M. NGUYEN.
Generalized impedance [...] for strongly absorbing obstacles [...]
Math. Models Methods Appl. Sci. **18**(10) (2008) 1787–1827.

Asymptotic Expansion

Hypothesis

1. Σ *is a smooth surface, with* (y_β, y_3) *"normal coordinates" to* Σ
2. ω *satisfies the Spectral Hypothesis (SH)*
3. $\mathbf{J}$ *is smooth and* $\mathbf{J} = 0$ *in* Ω_-

Small parameter

$$\delta := \sqrt{\omega\varepsilon_0/\sigma} \longrightarrow 0 \quad \text{as} \quad \sigma \to \infty$$

Pb $(\mathbf{P}_{\sigma})$ has a unique sol. $\mathbf{H}_{(\delta)}$ for δ small enough. Expansion:

$$\mathbf{H}^+_{(\delta)}(\mathbf{x}) = \mathbf{H}^+_0(\mathbf{x}) + \delta\mathbf{H}^+_1(\mathbf{x}) + \delta^2\mathbf{H}^+_2(\mathbf{x}) + \cdots + \mathcal{O}(\delta^N) \ \text{ in } \ \Omega_+$$

$$\mathbf{H}^-_{(\delta)}(\mathbf{x}) = \mathfrak{H}_0(y_\beta, \frac{y_3}{\delta}) + \delta\mathfrak{H}_1(y_\beta, \frac{y_3}{\delta}) + \delta^2\mathfrak{H}_2(y_\beta, \frac{y_3}{\delta}) + \cdots + \mathcal{O}(\delta^N) \ \text{ in } \ \Omega_-$$

The fields $\mathfrak{H}_j \in H(\mathrm{curl}, \Sigma \times \mathbb{R}_+)$ are *exponentially decreasing profiles*

$\cdots / \cdots$

Profiles of the Magnetic Field

Exponential decrease rate λ in coordinate Y_3 with $Y_3 = \frac{y_3}{\delta}$

$$\lambda = \omega\sqrt{\varepsilon_0\mu_0}\,\mathrm{e}^{-i\pi/4}$$

1. Denote $\mathbf{h}_0(y_\beta) := (\mathbf{n}\times\mathbf{H}_0^+)\times\mathbf{n}(y_\beta,0)$. Profile $\mathfrak{H}_0$ is *tangential*:

$$\boxed{\mathfrak{H}_0(y_\beta, Y_3) = \mathbf{h}_0(y_\beta)\,\mathrm{e}^{-\lambda Y_3}}$$

2. Denote by $\mathfrak{H}_1^\alpha$ and $\mathfrak{H}_1^3$ the *tangential and normal components* of $\mathfrak{H}_1$.

$$\mathfrak{H}_1^\alpha(y_\beta, Y_3) = \Big[\mathrm{h}_1^\alpha + Y_3\Big(\mathcal{H}\,\mathrm{h}_0^\alpha + b_\sigma^\alpha\,\mathrm{h}_0^\sigma\Big)\Big](y_\beta)\,\mathrm{e}^{-\lambda Y_3}$$

$$\mathfrak{H}_1^3(y_\beta, Y_3) = \lambda^{-1}\,D_\alpha\,\mathrm{h}_0^\alpha(y_\beta)\,\mathrm{e}^{-\lambda Y_3}$$

Here, b_σ^α is the symmetric curvature tensor of Σ, and $\mathcal{H} = \frac{1}{2}b_\alpha^\alpha$ its mean curvature, and D_α is the covariant derivative. Finally,

$$\mathrm{h}_j^\alpha(y_\beta) := (\mathbf{H}_j^+)^\alpha(y_\beta, 0) \quad \text{(tangential traces).}$$

A new definition of the skin depth (smooth interface Σ)

Denote $\mathfrak{H}_{(\delta)}(y_\alpha, y_3) := \mathbf{H}^-_{(\delta)}(\mathbf{x})$, for $y_\alpha \in \Sigma$ and $0 \le y_3$ small enough.

Recall the relation $\delta = \sqrt{\omega\varepsilon_0/\sigma}$.

Definition

Let $y_\alpha \in \Sigma$ and $\sigma \ge \sigma_0$. Assume $\mathfrak{H}_{(\delta)}(y_\alpha, 0) \neq 0$.

The *skin depth* $\mathcal{L}(\sigma, y_\alpha)$ is the smallest length s.t.

$$\|\mathfrak{H}_{(\delta)}\big(y_\alpha, \mathcal{L}(\sigma, y_\alpha)\big)\| = \|\mathfrak{H}_{(\delta)}(y_\alpha, 0)\| \,\mathrm{e}^{-1}$$

Theorem (Dauge, Faou, Péron, 2010)

Recall: $\mathcal{H}$ *mean curvature and* $\ell(\sigma) = \sqrt{2/\omega\mu_0\sigma}$ *the classical skin depth.*
Assume $\mathbf{h}_0(y_\alpha) \neq 0$.

$$\mathcal{L}(\sigma, y_\alpha) = \ell(\sigma)\Big(1 + \mathcal{H}(y_\alpha)\,\ell(\sigma) + \mathcal{O}(\sigma^{-1})\Big), \quad \sigma \to \infty$$

Outline

Axisymmetric domains

The meridian domain

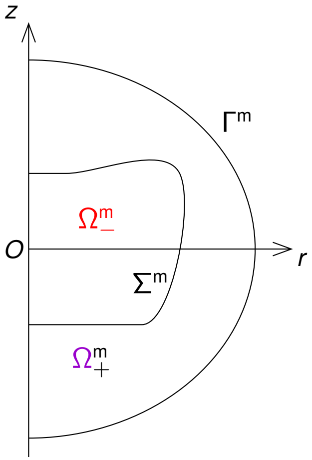


Figure: The meridian domain $\Omega^{\rm m} = \Omega^{\rm m}_{-} \cup \Omega^{\rm m}_{+} \cup \Sigma^{\rm m}$

Case of orthoradial data: a scalar problem

The curl in cylindrical coordinates:

$$\begin{cases} (\mathrm{curl}\,\mathbf{H})_r = \frac{1}{r}\partial_\theta H_z - \partial_z H_\theta \;, \\ (\mathrm{curl}\,\mathbf{H})_\theta = \partial_z H_r - \partial_r H_z \;, \\ (\mathrm{curl}\,\mathbf{H})_z = \frac{1}{r}\big(\partial_r(rH_\theta) - \partial_\theta H_r\big) \;. \end{cases}$$

The Maxwell problem is axisymmetric.

H is axisymmetric iff $\breve{\mathbf{H}} := \big(H_r, H_\theta, H_z\big)$ does not depend on θ.

H is orthoradial iff $\breve{\mathbf{H}} = \big(0, H_\theta, 0\big)$.

Assume that the right-hand side is axisymmetric and orthoradial

Then, $\mathbf{H}_{(\delta)}$ is axisymmetric and orthoradial

$$\breve{\mathbf{H}}_{(\delta)}(r,\theta,z) = (0, \mathrm{h}_{\theta\,(\delta)}(r,z), 0).$$

Configurations chosen for computations

Configuration A (Cylinder)

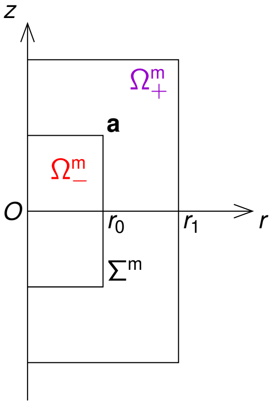


Figure: The meridian domain Ω^{m} in configuration A

Meshes
Configuration A

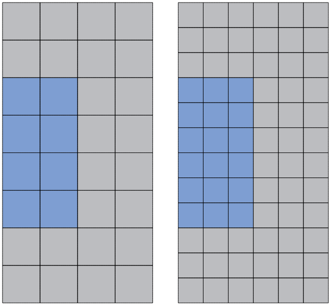


Figure: Meshes $\mathfrak{M}_2$, and $\mathfrak{M}_3$ in configuration A

Configurations chosen for computations

Configuration B (Spheroid)

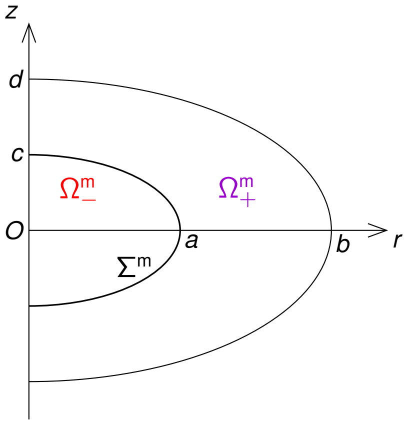


Figure: The meridian domain Ω^m in configuration B

Meshes

Configuration B

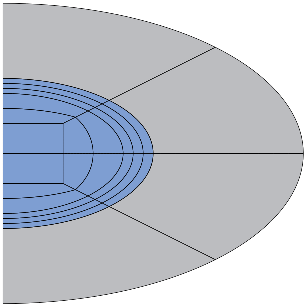


Figure: The meshes $\mathfrak{M}_3$ and $\mathfrak{M}_6$

Outline

Finite Element Method

In FEM computations, we use

1. the angular frequency $\omega = 3.10^7$.
2. the rhs $g = r$ (trace on Γ^m). It is real.
3. the high order quadrangular elements available in the finite element library MÉLINA

We compute $h_{\theta(\delta)}$. Denote the discrete solution by

$$\boxed{\tilde{h}_{\theta(\delta)} =: \tilde{h}_{\theta,\sigma}} \quad \text{with} \quad \delta = \sqrt{\omega\varepsilon_0/\sigma}.$$

We note that

1. The first term $h^{+}_{\theta,0}$ of the asymptotics of $\mathrm{h}_{\theta(\delta)}$ is real.
2. Hence, the imaginary part Im $h_{\theta(\delta)}$ is $\mathcal{O}(\delta)$ in the dielectric Ω^{m}_{+}.
3. Therefore the imaginary part of the computed field is expected to be larger in the conductor and to show the skin effect.

Skin effect in configuration B

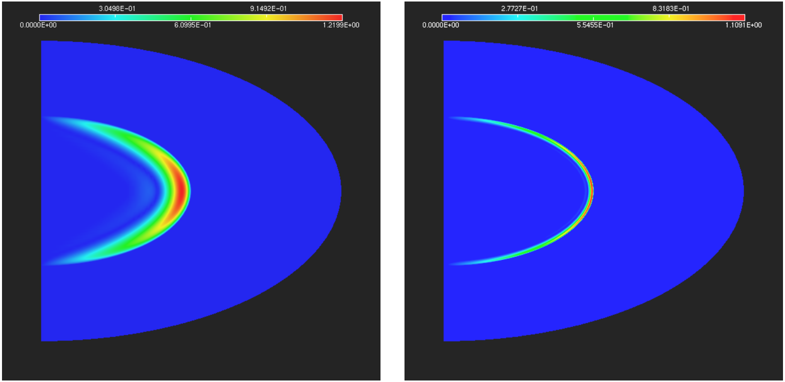
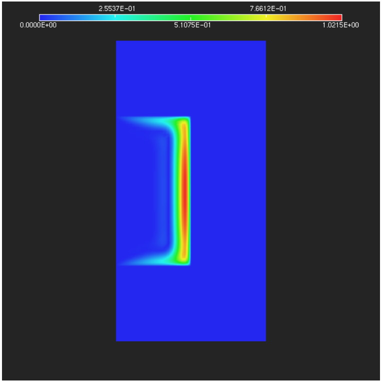


Figure: Configuration B. $\left|\operatorname{Im}\tilde{h}_{\theta,\sigma}\right|$ when $\sigma = 5$ and $\sigma = 80$

Skin effect in configuration A



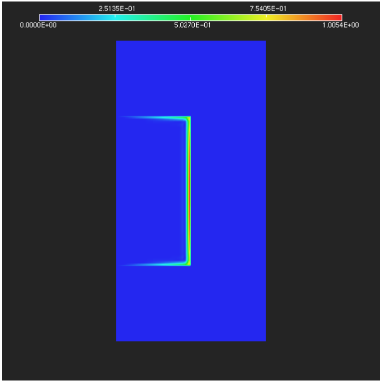


Figure: Configuration A. $|\operatorname{Im} \tilde{h}_{\theta,\sigma}|$ when $\sigma = 5$ and $\sigma = 80$

Influence of the geometry on the skin effect

Configuration B and swaped configuration B

$\mathcal{H} > 0$ on the left, and $\mathcal{H} < 0$ on the right

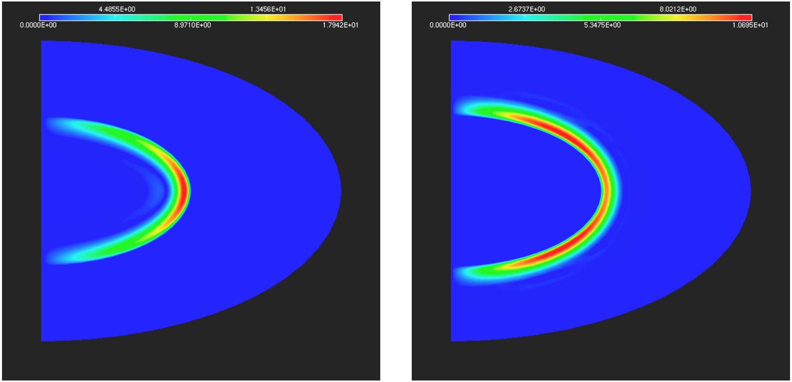
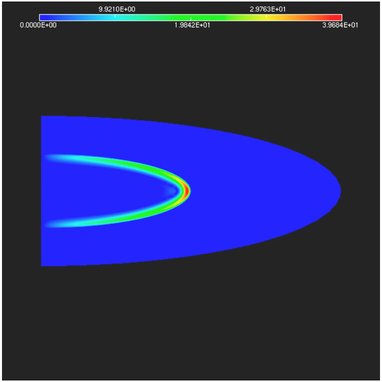


Figure: $|\,\mathrm{Im}\,\tilde{h}_{\theta,\sigma}|$, $\sigma = 5$

Influence of the geometry on the skin effect

Configuration B2 and swaped configuration B2

$\mathcal{H} > 0$ on the left, and $\mathcal{H} < 0$ on the right, and more prolate ellipsoids



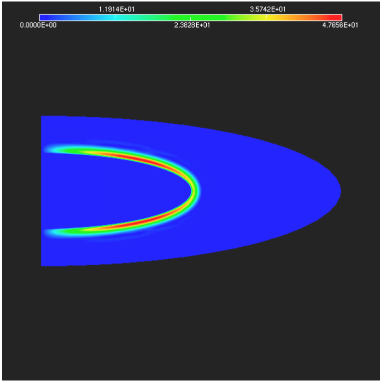


Figure: $\left|\operatorname{Im}\tilde{h}_{\theta,\sigma}\right|$, $\sigma = 5$

Influence of the geometry on the skin effect

Configuration B2 and swaped configuration B2

Zoom of the previous figures.

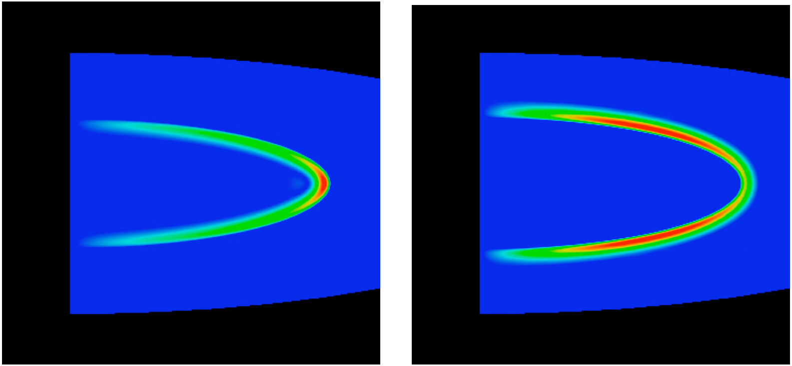


Figure: $\left|\,\mathrm{Im}\,\tilde{h}_{\theta,\sigma}\right|$, $\sigma = 5$

Outline

Quantitative information

- The qualitative behavior of skin effect is visible on previous figs.
- We want now to *measure* the exponential rate, if relevant.

We extract point values of computed solution $|\tilde{h}_{\theta,\sigma}|$ in Ω^{m}_{-} along a line D crossing Σ and calculate *slope* $\tilde{s}(\sigma)$ of the line close to

$$D \ni d \mapsto \log_{10} \left|\tilde{h}_{\theta,\sigma}\big(r(d), z(d)\big)\right|.$$

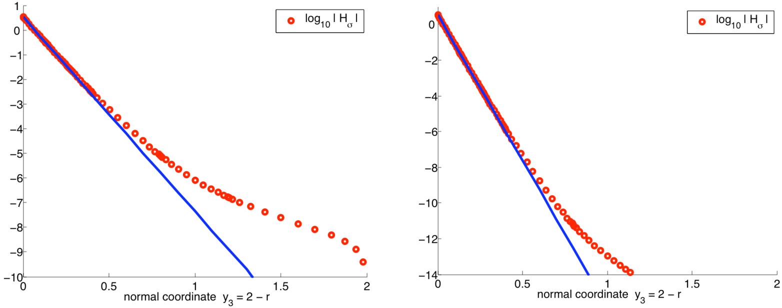


Figure: Config. B, line $z = 0$ (with $d = 2 - r$). Left, $\sigma = 20$. Right, $\sigma = 80$.

Theoretical value of the slope

1. Recall the *actual* skin depth $\mathcal{L}(\sigma, y_\alpha)$.
   The theoretical slope $s(\sigma, y_\alpha)$ is such that

$$s(\sigma, y_\alpha) = \frac{1}{\log 10} \frac{1}{\mathcal{L}(\sigma, y_\alpha)}$$

2. Recall the asymptotics

$$\mathcal{L}(\sigma, y_\alpha) = \ell(\sigma)\Big(1 + \mathcal{H}(y_\alpha)\,\ell(\sigma) + \mathcal{O}(\sigma^{-1})\Big), \quad \sigma \to \infty$$

   with $\mathcal{H}$ mean curvature and $\ell(\sigma) = \sqrt{2/\omega\mu_0\sigma}$ classical skin depth.

3. Therefore theoretical slope $s(\sigma, y_\alpha)$ satisfies

$$s(\sigma, y_\alpha) = \frac{1}{\log 10}\Big(\frac{1}{\ell(\sigma)} - \mathcal{H}(y_\alpha)\Big) + \mathcal{O}(\sigma^{-1/2})$$

Numerical results

Config. B, line $z = 0$. $\mathcal{H} = \frac{5}{4}$.

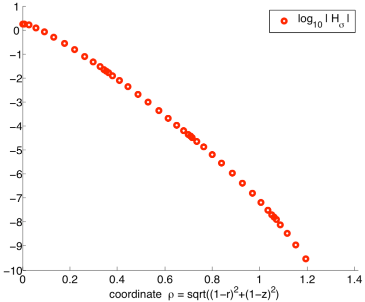
$$\text{curv\_ratio}(\sigma) := \frac{\frac{5}{4}}{\frac{1}{\ell(\sigma)} - \frac{5}{4}}$$

$$\text{err}(\sigma) := \left| \frac{s(\sigma) - \tilde{s}(\sigma)}{s(\sigma)} \right|$$

σ	5	20	80
$\ell(\sigma)$	0.103	0.0515	0.0258
$s(\sigma)$	3.67332	7.88951	16.32188
curv_ratio(σ)	0.148	0.069	0.033
$\tilde{s}(\sigma)$	3.64686	7.87347	16.308279
err(σ)	0.0072	0.002	0.0008

Configuration A

We extract values of $\log_{10}\left|\tilde{h}_{\theta,\sigma}\right|$ in Ω^{m}_{-} along the diagonal axis $r = z$



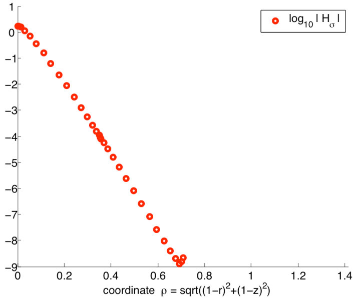
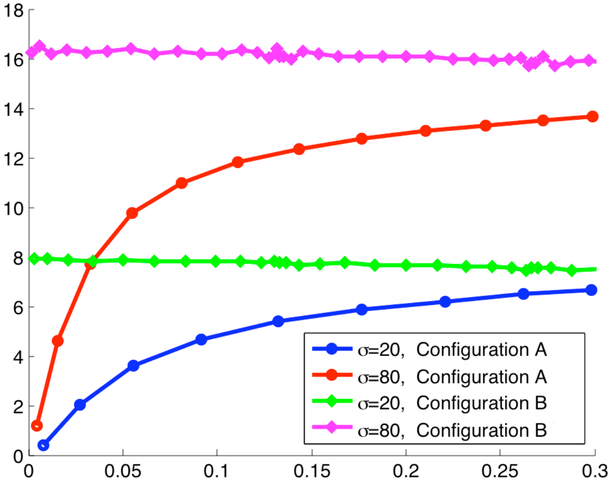


Figure: On the left $\sigma = 20$. On the right, $\sigma = 80$.

The behavior is not exactly linear...

Rates of exponential decay

We plot the *slopes* in the 4 previous figures.



Conclusion

- In config. B, slopes tend to positive limits as $y_3 \longrightarrow 0$ (exponential decay).
- The values of the slopes are very close to theoretical ones.
- In config. A, slopes tend to 0 as $\rho \longrightarrow 0$ (no exponential decay at corner **a**).
- But exponential decay is restored further away from **a**.
- The principal asymptotic contribution inside the conductor is a profile v_0 globally defined on a sector $\mathcal{S}$ solving the model Dirichlet pb

$$\begin{cases} (\partial_X^2 + \partial_Y^2)\mathrm{v}_0 - \lambda^2 \mathrm{v}_0 &= 0 \quad \text{in} \quad \mathcal{S}\,, \\ \mathrm{v}_0 &= \mathrm{h}_0^+(\mathbf{a}) \quad \text{on} \quad \partial\mathcal{S}\,, \end{cases}$$

instead the 1D problem in configuration B

$$\begin{cases} \partial_Y^2 \mathrm{v}_0 - \lambda^2 \mathrm{v}_0 &= 0 \quad \text{for} \quad 0 < Y < +\infty\,, \\ \mathrm{v}_0 &= \mathrm{h}_0^+ \quad \text{for} \quad Y = 0\,. \end{cases}$$


Thank you.
谢谢

Outline

Magnetic global formulation

$$
\begin{cases}
\operatorname{curl}\operatorname{curl}\mathbf{H}^{+}_{(\delta)} - \kappa^2\mathbf{H}^{+}_{(\delta)} = \operatorname{curl}\mathbf{J} & \text{in} \quad \Omega_+ \\
\operatorname{curl}\operatorname{curl}\mathbf{H}^{-}_{(\delta)} - \kappa^2\big(1+\frac{i}{\delta^2}\big)\mathbf{H}^{-}_{(\delta)} = 0 & \text{in} \quad \Omega_- \\
\operatorname{curl}\mathbf{H}^{+}_{(\delta)} \times \mathbf{n} = \big(1+\frac{i}{\delta^2}\big)^{-1}\operatorname{curl}\mathbf{H}^{-}_{(\delta)} \times \mathbf{n} & \text{on} \quad \Sigma \\
\mathbf{H}^{+}_{(\delta)} \times \mathbf{n} = \mathbf{H}^{-}_{(\delta)} \times \mathbf{n} & \text{on} \quad \Sigma \\
\mathbf{H}^{+}_{(\delta)} \cdot \mathbf{n} = \mathbf{H}^{-}_{(\delta)} \cdot \mathbf{n} & \text{on} \quad \Sigma \\
\mathbf{H}^{+}_{(\delta)} \times \mathbf{n} = 0 & \text{on} \quad \Gamma.
\end{cases}
$$

Magnetic power series identification

System of equations, for all $m \geq 0$

$$\begin{cases}
(1) & -\lambda^2 \mathfrak{H}_{m,3} = \sum_{j=0}^{m-1} L_3^{m-j}(\mathfrak{H}_j) & \text{in} \quad \Sigma \times I \\
(3a) & \partial_3^2 \mathfrak{H}_{m,\alpha} - \lambda^2 \mathfrak{H}_{m,\alpha} = \sum_{j=0}^{m-1} L_\alpha^{m-j}(\mathfrak{H}_j) & \text{in} \quad \Sigma \times I \\
(3b) & \mathfrak{H}_{m,\alpha} = \mathbf{H}^+_{m,\alpha} & \text{on} \quad \Sigma \\
(2a) & \operatorname{curl} \operatorname{curl} \mathbf{H}^+_m - \kappa^2 \mathbf{H}^+_m = (\text{if } m = 0) \cdot \operatorname{curl} \mathbf{J} & \text{in} \quad \Omega_+ \\
(2b) & \mathbf{H}^+_m \cdot \mathbf{n} = \mathfrak{H}_{m,3} & \text{on} \quad \Sigma \\
(2c) & \operatorname{curl} \mathbf{H}^+_m \times \mathbf{n} = \sum_{j=0}^{m-1} \mathrm{T}^{m-j} \mathfrak{H}_j & \text{on} \quad \Sigma \\
(2d) & \mathbf{H}^+_m \times \mathbf{n} = 0 & \text{on} \quad \Gamma
\end{cases}$$

Electric global formulation

$$
\begin{cases}
\operatorname{curl}\operatorname{curl}\mathbf{E}^{+}_{(\delta)} - \kappa^2\mathbf{E}^{+}_{(\delta)} = i\omega\mu_0\mathbf{J} & \text{in} \quad \Omega_+ \\
\operatorname{curl}\operatorname{curl}\mathbf{E}^{-}_{(\delta)} - \kappa^2(1+\frac{i}{\delta^2})\mathbf{E}^{-}_{(\delta)} = 0 & \text{in} \quad \Omega_- \\
\operatorname{curl}\mathbf{E}^{+}_{(\delta)} \times \mathbf{n} = \operatorname{curl}\mathbf{E}^{-}_{(\delta)} \times \mathbf{n} & \text{on} \quad \Sigma \\
\mathbf{E}^{+}_{(\delta)} \times \mathbf{n} = \mathbf{E}^{-}_{(\delta)} \times \mathbf{n} & \text{on} \quad \Sigma \\
\mathbf{E}^{+}_{(\delta)} \cdot \mathbf{n} = 0 \quad \text{and} \quad \operatorname{curl}\mathbf{E}^{+}_{(\delta)} \times \mathbf{n} = 0 & \text{on} \quad \Gamma.
\end{cases}
$$

Electric power series identification

System of equations, for all $m \geq 0$

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(1) & -\lambda^2 \mathfrak{E}_{m,3} = \sum_{j=0}^{m-1} L_3^{m-j}(\mathfrak{E}_j) & \text{in} \quad \Sigma \times I \\
(2a) & \partial_3^2 \mathfrak{E}_{m,\alpha} - \lambda^2 \mathfrak{E}_{m,\alpha} = \sum_{j=0}^{m-1} L_\alpha^{m-j}(\mathfrak{E}_j) & \text{in} \quad \Sigma \times I \\
(2b) & \partial_3 \mathfrak{E}_{m,\alpha} = D_\alpha \mathfrak{E}_{m-1,3} + \left(\operatorname{curl} \mathbf{E}_{m-1}^+ \times \mathbf{n}\right)_\alpha & \text{on} \quad \Sigma \\
(3a) & \operatorname{curl} \operatorname{curl} \mathbf{E}_m^+ - \kappa^2 \mathbf{E}_m^+ = (\text{if } m = 0) \cdot i\omega\mu_0 \mathbf{J} & \text{in} \quad \Omega_+ \\
(3b) & \mathbf{E}_m^+ \times \mathbf{n} = \mathfrak{E}_m \times \mathbf{n} & \text{on} \quad \Sigma \\
(3c) & \mathbf{E}_m^+ \cdot \mathbf{n} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{E}_m^+ \times \mathbf{n} = 0 & \text{on} \quad \Gamma.
\end{cases}$$