# Spectrally correct approximation for Maxwell

## Weighted Regularization and Discrete Commuting Diagrams

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# Introduction

# Cavity modes with perfectly conducting conditions

Cavity $\Omega$ bounded polygonal domain in $\mathbb{R}^2$ or polyhedral in $\mathbb{R}^3$.
Functional spaces for electric formulation

1. For plain curl-curl formulation

$$H_0(\mathbf{curl},\Omega) = \{\boldsymbol{u} \in L^2(\Omega)^3,\ \mathbf{curl}\,\boldsymbol{u} \in L^2(\Omega)^3, \quad \boldsymbol{u}\times\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$$

2. For regularized formulation

$$\boldsymbol{X}_N(\Omega) = \{\boldsymbol{u} \in H_0(\mathbf{curl},\Omega),\ \operatorname{div}\boldsymbol{u} \in L^2(\Omega)\}$$

And corresponding variational formulations

1. Find non-zero $\boldsymbol{E} \in H_0(\mathbf{curl},\Omega)$ and non-zero $\lambda$:

$$\int_\Omega \mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{E}'\,d\boldsymbol{x} = \lambda\int_\Omega \boldsymbol{E}\cdot\boldsymbol{E}'\,d\boldsymbol{x}, \quad \forall \boldsymbol{E}' \in H_0(\mathbf{curl},\Omega)$$

2. Find non-zero $\boldsymbol{E} \in \boldsymbol{X}_N(\Omega)$ and non-zero $\lambda$:

$$\int_\Omega \big(\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{E}' + s\operatorname{div}\boldsymbol{E}\operatorname{div}\boldsymbol{E}'\big)\,d\boldsymbol{x} = \lambda\int_\Omega \boldsymbol{E}\cdot\boldsymbol{E}'\,d\boldsymbol{x}, \quad \forall \boldsymbol{E}' \in \boldsymbol{X}_N(\Omega)$$

with a (free) positive parameter $s$.

# Two methods

Recall from previous talk

1. The plain curl-curl formulation provides an approximation of the infinite dimensional kernel (the gradients), and nothing else, in general.
2. The regularized formulation works in a square, on not in a L-shape.

We reverse the (natural) order and investigate

1. A modification of regularized formulation, introducing a weight.
2. Special finite elements which contain exactly the right amount of gradients.

# Weighted Regularization

# A density issue

Let

$$\boldsymbol{H}_N^1(\Omega) = H^1(\Omega)^2 \cap \boldsymbol{X}_N(\Omega).$$

FE spaces $\boldsymbol{W}_p$ which are curl and div conforming satisfy $\boldsymbol{W}_p \subset \boldsymbol{H}_N^1(\Omega)$.

1. $\boldsymbol{H}_N^1(\Omega)$ is closed in $\boldsymbol{X}_N(\Omega)$ for the norm of $\boldsymbol{X}_N(\Omega)$.
2. If $\Omega$ has non-convex corners, the embedding $\boldsymbol{H}_N^1(\Omega) \subset \boldsymbol{X}_N(\Omega)$ is strict.

For Properties (1) and (2) one can consider gradient fields $\boldsymbol{u} = \mathbf{grad}\,\varphi$ only, and obtain equivalent statements phrased in $\varphi$: Introduce

$$\mathcal{D}(\Delta^{\mathrm{dir}}(\Omega)) = \{\varphi \in H_0^1(\Omega),\ \Delta\varphi \in L^2(\Omega)\}$$

Then, we have the ***gradient correspondance principle :***

1. $H^2 \cap H_0^1(\Omega)$ is closed in $\mathcal{D}(\Delta^{\mathrm{dir}}(\Omega))$.
2. If $\Omega$ has non-convex corners, the embedding

$$H^2 \cap H_0^1(\Omega) \subset \mathcal{D}(\Delta^{\mathrm{dir}}(\Omega)) \quad \text{is strict.}$$

# Introducing a weight in the divergence term

Let $\boldsymbol{x} \mapsto \rho(\boldsymbol{x})$ be the distance function to the set of non-convex corners or edges of $\Omega$. For $s > 0$ and $\gamma \in \mathbb{R}$ we introduce the bilinear form

$$a_{\gamma,s}(\boldsymbol{E},\boldsymbol{E}') = \int_\Omega \mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\boldsymbol{E}' + s \int_\Omega \rho^\gamma \operatorname{div} \boldsymbol{E}\, \rho^\gamma \operatorname{div} \boldsymbol{E}' \, d\boldsymbol{x}$$

defined on its natural space — here $L^2_\gamma(\Omega) = \{v,\ \ \rho^\gamma v \in L^2(\Omega)\}$

$$\boldsymbol{X}_{\gamma,N}(\Omega) = \{\boldsymbol{u} \in H_0(\mathbf{curl},\Omega),\ \operatorname{div} \boldsymbol{u} \in L^2_\gamma(\Omega)\}$$

Define the Laplace-Dirichlet operator $\Delta^{\text{dir}}_\gamma$ as

$$\begin{array}{rccl} \Delta^{\text{dir}}_\gamma : & \mathcal{D}(\Delta^{\text{dir}}_\gamma(\Omega)) := \{\varphi \in H^1_0(\Omega) \mid \Delta\varphi \in L^2_\gamma(\Omega)\} & \longrightarrow & L^2_\gamma(\Omega) \\ & \varphi & \longmapsto & \Delta\varphi. \end{array}$$

**Theorem — Costabel-Dauge**

If $H^2 \cap H^1_0(\Omega)$ is dense in $\mathcal{D}(\Delta^{\text{dir}}_\gamma(\Omega))$, then $\boldsymbol{H}^1_N(\Omega)$ is dense in $\boldsymbol{X}_{\gamma,N}(\Omega)$

# Finding a suitable weight

**Theorem**

$\Omega$ polygonal or polyhedral domain. There exists $\gamma_0 = \gamma_0(\Omega) < 1$ such that

$$\forall \gamma,\ \gamma_0 < \gamma \leq 1, \quad H^2 \cap H^1_0(\Omega) \text{ is dense in } \mathcal{D}(\Delta^{\mathrm{dir}}_{\gamma}(\Omega))$$

and, therefore

$$\forall \gamma,\ \gamma_0 < \gamma \leq 1, \quad \boldsymbol{H}^1_N(\Omega) \text{ is dense in } \boldsymbol{X}_{\gamma,N}(\Omega)$$

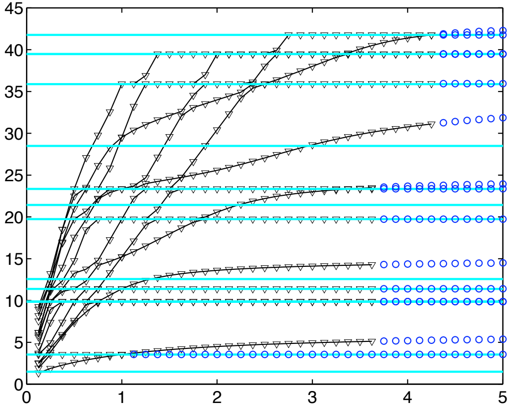
$\gamma_0$ is explicit in function of

- the largest opening $\omega$ of $\Omega$ in 2D: $\gamma_0 = 1 - \pi/\omega$
- the largest edge opening, and the corner solid angles in 3D

Example of the L-shape:

$$\gamma_0 = \frac{1}{3}$$

# Plain regularization




Sorted by ratio of curl energy / div energy

$\Omega =$ L-shape.

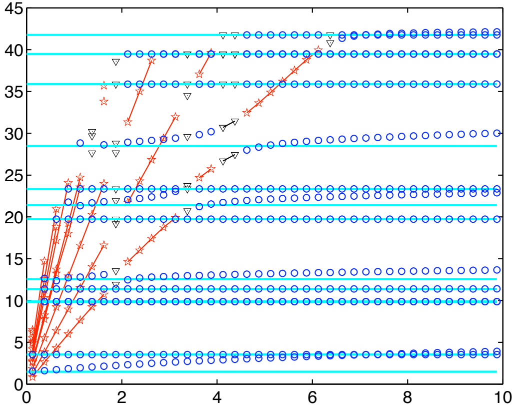
Regularizing parameter
$0 < s \leq 5$

Plot $s \to \lambda_{s,k}$.

$\mathbb{Q}_{10}$ elements on a 9-element mesh

Exact values:
**Horizontal lines**

# Weighted regularization with $\gamma = 0.35$




Sorted by ratio of curl energy / div energy

$\Omega =$ L-shape.

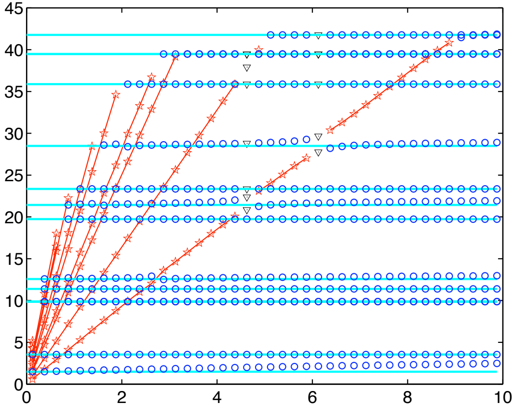
Regularizing parameter $0 < s \leq 10$

Plot $s \to \lambda_{s,k}$.

$\mathbb{Q}_{10}$ elements on a 9-element mesh

Exact values: **Horizontal lines**

# Weighted regularization with $\gamma = 0.5$




Sorted by ratio of curl energy / div energy

$\Omega =$ L-shape.

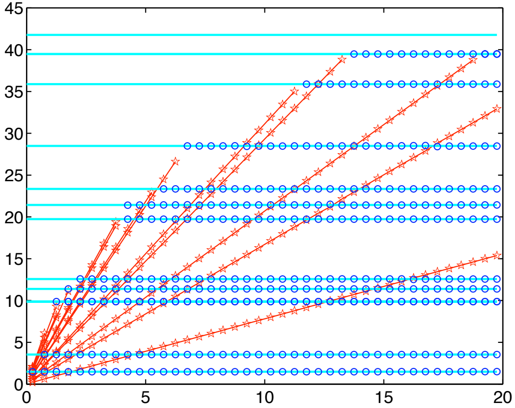
Regularizing parameter $0 < s \leq 10$

Plot $s \rightarrow \lambda_{s,k}$.

$\mathbb{Q}_{10}$ elements on a 9-element mesh

Exact values: **Horizontal lines**

# Weighted regularization with $\gamma = 1$




Sorted by ratio of curl energy / div energy

$\Omega =$ L-shape.

Regularizing parameter
$0 < s \leq 20$

Plot $s \to \lambda_{s,k}$.

$\mathbb{Q}_{10}$ elements on a 9-element mesh

Exact values:
**Horizontal lines**

# Discrete commuting diagrams

## Edge elements and discrete compactness

# Mimicking the kernel: edge elements

Sequence of discretizations

$$V_p \subset V \subset H^1_0(\Omega) \quad \text{and} \quad \boldsymbol{W}_p \subset \boldsymbol{W} \subset \boldsymbol{H}_0(\mathbf{curl}, \Omega)$$

with projection operators $\pi_p^V$ and $\pi_p^{\boldsymbol{W}}$ satisfying the *commuting diagram*:

$$\begin{array}{ccc} V & \xrightarrow{\mathbf{grad}} & \boldsymbol{W} \\ \Big\downarrow \pi_p^V & & \Big\downarrow \pi_p^{\boldsymbol{W}} \\ V_p & \xrightarrow{\mathbf{grad}} & \boldsymbol{W}_p \end{array}$$

Any $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ satisfies

$$\forall \varphi \in H^1_0(\Omega), \quad \langle \boldsymbol{u}, \mathbf{grad}\,\varphi \rangle_\Omega = 0 \quad \Longleftrightarrow \quad \operatorname{div} \boldsymbol{u} = 0$$

**Definition: $\boldsymbol{u}_p \in \boldsymbol{W}_p$ is *discrete divergence free* if**

$$\forall \varphi_p \in V_p, \quad \langle \boldsymbol{u}_p, \mathbf{grad}\,\varphi_p \rangle_\Omega = 0$$

Such a $\boldsymbol{u}_p$ is not divergence free, in general. Nevertheless...

# Spectral correctness if 3 conditions are satisfied

(CAS) ***Completeness of the Approximating Subspace***

$$\forall \boldsymbol{v} \in H_0(\mathbf{curl}, \Omega), \qquad \lim_{p} \inf_{\boldsymbol{v}_p \in \boldsymbol{W}_p} \|\boldsymbol{v} - \boldsymbol{v}_p\|_{H(\mathbf{curl},\Omega)} = 0.$$

(CDK) ***Completeness of the Discrete Kernel***

$$\forall \varphi \in H_0^1(\Omega), \qquad \lim_{p} \inf_{\varphi_p \in V_p} \|\varphi - \varphi_p\|_{H^1(\Omega)} = 0.$$

(DCP) ***Discrete Compactness Property*** (KIKUCHI)
For any sequence $\{\boldsymbol{u}_p\}$ of discrete divergence free fields bounded in $H(\mathbf{curl}, \Omega)$, there exists a subsequence $\{\boldsymbol{u}_{p'}\}$ and a limit $\boldsymbol{u} \in L^2(\Omega)^3$

$$\lim_{p'} \|\boldsymbol{u}_{p'} - \boldsymbol{u}\|_{L^2(\Omega)} = 0.$$

(CAORSI-FERNANDES-RAFFETTO)

# Discrete Compactness Property

For the *h*-version of Finite Elements:

- Proof for uniform meshes (MONK, DEMKOWICZ)
- Proof for certain anisotropically refined meshes (NICAISE, BUFFA-COSTABEL-DAUGE)

For the *p*-version of Finite Elements:

- An incomplete proof for 2D triangular meshes (BOFFI-DEMKOWICZ-COSTABEL)
- Proof for 2D rectangular elements (BOFFI-COSTABEL-DAUGE-DEMKOWICZ).
- General proof for 2D and 3D meshes ... /...

# Result for the *p*-version: The Four Assumptions

1. All spaces and operators are defined *Element-wise*, e.g.

$$\forall \boldsymbol{u} \in \boldsymbol{W}, \quad (\pi_p^{\boldsymbol{W}} \boldsymbol{u})\big|_K = \pi_{p,K}^{\boldsymbol{W}}(\boldsymbol{u}\big|_K)$$

and the basic commuting diagram is a cell of a larger exact sequence

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & V & \xrightarrow{\textbf{grad}} & \boldsymbol{W} & \xrightarrow{\textbf{curl}} & \textbf{curl}\,\boldsymbol{W} & \longrightarrow & 0 \\
 & & \Big\downarrow \pi_p^V & & \Big\downarrow \pi_p^{\boldsymbol{W}} & & \Big\downarrow \pi_p^Y & & \\
\mathbb{R} & \longrightarrow & V_p & \xrightarrow{\textbf{grad}} & \boldsymbol{W}_p & \xrightarrow{\textbf{curl}} & \textbf{curl}\,\boldsymbol{W}_p & \longrightarrow & 0
\end{array}$$

2. *Compact Embedding* $\boldsymbol{W} \Subset \boldsymbol{L}^2(\Omega)$ and *Regularity Result* :

$$\boldsymbol{u} \in H_0(\textbf{curl}, \Omega) \text{ and } \operatorname{div} \boldsymbol{u} \in L^2(\Omega) \quad \Longrightarrow \quad \boldsymbol{u} \in \boldsymbol{W}.$$

3. *Local scalar approximation property*: exists $\delta > 0$,

$$\forall \varphi \in V,\ \forall p \in \mathbb{N}, \quad \|\varphi - \pi_p^V \varphi\|_{H^1(K)} \leq C\,p^{-\delta} \|\varphi\|_{V(K)}.$$

4. $\exists$ *Poincaré operator* $\mathfrak{K}$ lifting the curl: $\forall f \in \textbf{curl}\,\boldsymbol{W}(K)$, $\textbf{curl}\,\mathfrak{K} f = f$

$$\mathfrak{K} \text{ continuous } \textbf{curl}\,\boldsymbol{W}\big|_{L^2(K)} \to \boldsymbol{W} \text{ and } \textbf{curl}\,\boldsymbol{W}_p \to \boldsymbol{W}_p$$

# Step 0 of the proof of the discrete compactness

**Theorem (Boffi-Costabel-Dauge-Demkowicz 2008)**

Under the Four Assumptions, the Discrete Compactness Property holds.

*Step 0*: (Kikuchi)

$\left(\boldsymbol{u}_p \in \boldsymbol{W}_p\right)_p$ sequ. bounded in $H_0(\mathbf{curl}, \Omega)$ and discrete divergence free.
For each $p \in \mathbb{N}$, let $\boldsymbol{u}^p$ be such that

$$\mathbf{curl}\, \boldsymbol{u}^p = \mathbf{curl}\, \boldsymbol{u}_p, \quad \boldsymbol{u}^p \times \boldsymbol{n}\big|_{\partial\Omega} = 0 \quad \text{and} \quad \operatorname{div} \boldsymbol{u}^p = 0.$$

The continuity assumption (2) yields

$$\left\|\boldsymbol{u}^p\right\|_{\boldsymbol{W}} \leq C\left\|\boldsymbol{u}_p\right\|_{H(\mathbf{curl},\Omega)},$$

and the compact embedding yields $\boldsymbol{W} \Subset \boldsymbol{L}^2(\Omega)$ that $\boldsymbol{u}^p$ converges.
It remains to estimate

$$\left\|\boldsymbol{u}^p - \boldsymbol{u}_p\right\|_{H(\mathbf{curl},\Omega)} = \left\|\boldsymbol{u}^p - \boldsymbol{u}_p\right\|_{L^2(\Omega)}$$

We are going to prove that this tends to 0.

# Step 1 of the proof of the discrete compactness

*Step 1*: Known as "Nédélec's trick"

$$\left\| \boldsymbol{u}^p - \boldsymbol{u}_p \right\|_{L^2(\Omega)} \leq \left\| \boldsymbol{u}^p - \pi_p^{\boldsymbol{W}} \boldsymbol{u}^p \right\|_{L^2(\Omega)} .$$

Essentially based on

- The discrete divergence free property of $\boldsymbol{u}_p$
- The divergence free property of $\boldsymbol{u}^p$

# Step 2 of the proof of the discrete compactness

*Step 2*: The innovative one.

*For each element $K$ of the mesh, $\exists$ potential $\psi^p \in V(K)$ satisfying*

$$\begin{cases} \|\psi^p\|_{V(K)} & \leq & C\|\boldsymbol{u}^p\|_{\boldsymbol{X}(K)} \\ \|\boldsymbol{u}^p - \pi_p^{\boldsymbol{W}}\boldsymbol{u}^p\|_{L^2(K)} & \leq & C\|\psi^p - \pi_p^V\psi^p\|_{H^1(K)} \end{cases}$$

**Proof.** Reduction to *scalar potentials* thanks to the Poincaré operator:

- $\exists$ potential $\psi^p \in H^1(K)$ such that

$$\boldsymbol{u}^p = \mathfrak{K}(\mathbf{curl}\,\boldsymbol{u}^p) + \mathbf{grad}\,\psi^p \tag{1}$$

  By continuity of $\mathfrak{K}$, we have $\mathbf{grad}\,\psi^p \in \boldsymbol{W}(K)$, hence $\psi^p \in V(K)$

- Since $\mathbf{curl}\,\boldsymbol{u}^p \in \boldsymbol{W}_p(K)$, $\exists$ potential $\psi_p \in \boldsymbol{W}_p(K)$ such that

$$\pi_p^{\boldsymbol{W}}\boldsymbol{u}^p = \mathfrak{K}(\mathbf{curl}\,\boldsymbol{u}^p) + \mathbf{grad}\,\psi_p \tag{2}$$

- (1) – (2) and commuting diagram:

$$\boldsymbol{u}^p - \pi_p^{\boldsymbol{W}}\boldsymbol{u}^p = \mathbf{grad}\,\psi^p - \mathbf{grad}\,\psi_p = \mathbf{grad}\,\psi^p - \mathbf{grad}\,\pi_p^V\psi^p$$

# Step 3 of the proof of the discrete compactness

*Step 3*: The conclusion.

We use the scalar local approximation

$$\begin{aligned}\|\psi^p - \pi_p^V \psi^p\|_{H^1(K)} &\leq C\,p^{-\delta}\|\psi^p\|_{V(K)}\\ &\leq C\,p^{-\delta}\|\boldsymbol{u}^p\|_{\boldsymbol{X}(K)}\end{aligned}$$

Hence, coming back to $\boldsymbol{u}^p - \boldsymbol{u}_p$

$$\|\boldsymbol{u}^p - \pi_p^{\boldsymbol{W}} \boldsymbol{u}^p\|_{L^2(K)} \leq C\,p^{-\delta}\|\boldsymbol{u}^p\|_{\boldsymbol{X}(K)}$$

Sum squares of estimates over mesh elements $K$, and rely on Steps 0 & 1

$$\begin{aligned}\|\boldsymbol{u}^p - \boldsymbol{u}_p\|_{L^2(\Omega)} &\leq \|\boldsymbol{u}^p - \pi_p^{\boldsymbol{W}} \boldsymbol{u}^p\|_{L^2(\Omega)}\\ &\leq C\,p^{-\delta}\|\boldsymbol{u}_p\|_{H(\mathbf{curl},\Omega)}\end{aligned}$$

Hence the convergence. **QED**

# To conclude

Two methods are proved to be spectrally correct in many configurations

- Weighted Regularization Method
- Edge Elements in the framework of exact sequences and commuting diagrams

See on the benchmark page

*http://perso.univ-rennes1.fr/monique.dauge/benchmax.html*

computations by both methods, with the codes

- Mélina (IRMAR Rennes)
- Concepts (ETH Zürich)
- Montjoie (INRIA Rocquencourt)

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