Weighted analytic regularity in polyhedra

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Outline

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Question of regularity

Consider a (elliptic) boundary value problem, written in compact form as

$$\mathbb{P}\boldsymbol{u} = \boldsymbol{q}$$

where $\boldsymbol{q}$ may include interior, boundary, or interface data.
For a possible numerical approximation, answering (a priori) the question of regularity for $\boldsymbol{u}$ is of fundamental importance.

Any regularity statement takes the form

$$\boxed{\boldsymbol{u} \in \mathbb{U}_{\mathrm{base}} \quad \text{and} \quad \boldsymbol{q} \in \mathbb{Q}_{\mathrm{data}} \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{U}_{\mathrm{sol}}}$$

Ideally

- $\mathbb{U}_{\mathrm{base}}$ is a space where existence of solutions is known (e.g. variational space)
- $\mathbb{U}_{\mathrm{sol}}$ is optimal in the sense that $\mathbb{P}$ is bounded $\mathbb{U}_{\mathrm{sol}} \to \mathbb{Q}_{\mathrm{data}}$.
- If $\mathbb{Q}_{\mathrm{data}}$ is a space of piecewise analytic data, $\mathbb{U}_{\mathrm{sol}}$ is a space of piecewise analytic solutions.

Three types of possible theorems

Type C: Existence of solutions in a space $\mathbb{V}$

Coercivity or Fredholm alternative.

Type B: Basic regularity

$$\boxed{\boldsymbol{u} \in \mathbb{U}_{\mathsf{base}} \quad \text{and} \quad \boldsymbol{q} \in \mathbb{Q}_{\mathsf{data}} \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{U}_{\mathsf{sol}}} \quad \text{with}$$

1. $\mathbb{U}_{\mathsf{base}} = \mathbb{V}$
2. $\mathbb{U}_{\mathsf{sol}} = \mathbb{U}^{\mathsf{B}}_{\mathsf{sol}}$ involving estimates on a finite number of derivatives (e.g. space of strong solutions) — for suitable $\mathbb{Q}_{\mathsf{data}}$.

Type A: Analytic regularity

$$\boxed{\boldsymbol{u} \in \mathbb{U}_{\mathsf{base}} \quad \text{and} \quad \boldsymbol{q} \in \mathbb{Q}_{\mathsf{data}} \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{U}_{\mathsf{sol}}} \quad \text{with}$$

1. $\mathbb{U}_{\mathsf{base}} = \mathbb{V}$
2. $\mathbb{U}_{\mathsf{sol}} = \mathbb{U}^{\mathsf{A}}_{\mathsf{sol}}$ involving Cauchy-type estimates on all derivatives — for suitable $\mathbb{Q}_{\mathsf{data}}$.

A fourth type of statement and a strategy

Type S: Regularity Shift

$$\boxed{u \in \mathbb{U}_{\mathsf{base}} \quad \text{and} \quad q \in \mathbb{Q}_{\mathsf{data}} \quad \Longrightarrow \quad u \in \mathbb{U}_{\mathsf{sol}}} \quad \text{with}$$

1. $\mathbb{U}_{\mathsf{base}} = \mathbb{U}^{\mathsf{B}}_{\mathsf{sol}}$ improved basic regularity
2. $\mathbb{U}_{\mathsf{sol}} = \mathbb{U}^{\mathsf{A}}_{\mathsf{sol}}$ involving Cauchy-type estimates on all derivatives — for suitable $\mathbb{Q}_{\mathsf{data}}$.

Strategy

Type B + Type S → Type A

Find suitable “pairs” $(\mathbb{U}^{\mathsf{B}}_{\mathsf{sol}}, \mathbb{U}^{\mathsf{A}}_{\mathsf{sol}})$ so that

1. Type B is known
2. Type S is true (our job to prove it)

Families of semi-norms

The objects $\mathbb{U}^{\mathsf{B}}_{\mathsf{sol}}$ and $\mathbb{U}^{\mathsf{A}}_{\mathsf{sol}}$ are realized by countable sets of semi-norms

$$|\cdot|_{\mathbb{X}^m}, \quad m \in \mathbb{N}$$

Typically, the semi-norm $\mathbb{X}^m$ is a norm on derivatives ∂^α of length $|\alpha| = m$.

Several spaces are associated in a natural way:

1. $\mathbb{X}^k = \{\boldsymbol{u} : \ |\boldsymbol{u}|_{\mathbb{X}^m} < \infty, \ 0 \le m \le k\}$ and $\|\boldsymbol{u}\|_{\mathbb{X}^k} = \sup_{m=0}^{k} |\boldsymbol{u}|_{\mathbb{X}^m}$
2. $\mathbb{X}^\infty = \{\boldsymbol{u} : \ |\boldsymbol{u}|_{\mathbb{X}^m} < \infty, \ \forall m \in \mathbb{N}\}$
3. $\mathbb{X}^\varpi = \left\{\boldsymbol{u} \in \mathbb{X}^\infty : \ \sup_{m\in\mathbb{N}} \left(\frac{1}{m!} |\boldsymbol{u}|_{\mathbb{X}^m}\right)^{1/m} < \infty\right\}$ — analytic class
4. $\left\{\boldsymbol{u} \in \mathbb{X}^\infty : \ \sup_{m\in\mathbb{N}} \left(\frac{1}{(m!)^s} |\boldsymbol{u}|_{\mathbb{X}^m}\right)^{1/m} < \infty\right\}$ — Gevrey class

Similar definitions for right hand sides $\boldsymbol{q}$. Denote the semi-norms by $|\cdot|_{\mathbb{Y}^m}$.

Types B and A associated with families of semi-norms

Type B: Basic regularity

Exists $m \in \mathbb{N}$ such that

$$\boxed{\boldsymbol{u} \in \mathbb{V} \quad \text{and} \quad \boldsymbol{q} \in \mathbb{Y}^m \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^m}$$

with estimates

$$\|\boldsymbol{u}\|_{\mathbb{X}^m} \leq C\big(\|\mathbb{P}\boldsymbol{u}\|_{\mathbb{Y}^m} + \|\boldsymbol{u}\|_{\mathbb{V}}\big)$$

Type A: Analytic regularity

$$\boxed{\boldsymbol{u} \in \mathbb{V} \quad \text{and} \quad \boldsymbol{q} \in \mathbb{Y}^\varpi \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^\varpi}$$

Types S associated with families of semi-norms

Type S standard

Exists $m \in \mathbb{N}$ such that for all $k > m$

$$\boxed{\boldsymbol{u} \in \mathbb{X}^m \quad \text{and} \quad \boldsymbol{q} \in \mathbb{Y}^k \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^k}$$

with estimates

$$\|\boldsymbol{u}\|_{\mathbb{X}^k} \le C\big(\|\mathbb{P}\boldsymbol{u}\|_{\mathbb{Y}^k} + \|\boldsymbol{u}\|_{\mathbb{X}^m}\big)$$

Type S with Cauchy estimates

Exists $m \in \mathbb{N}$ such that for all $k > m$

$$\boxed{\boldsymbol{u} \in \mathbb{X}^m \quad \text{and} \quad \boldsymbol{q} \in \mathbb{Y}^k \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^k}$$

with estimates (constant A independent from k)

(S-Cauchy) $$\frac{1}{k!}\,|\boldsymbol{u}|_{\mathbb{X}^k} \le A^{k+1}\Big(\sum_{\ell=0}^{k}\frac{1}{\ell!}\,|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell} + \|\boldsymbol{u}\|_{\mathbb{X}^m}\Big)$$

A true theorem, at last

Theorem

If there exists $m \in \mathbb{N}$ such that

1. Type B is true for m
2. Type S Cauchy is true for m

then Type A is true: $\boxed{\boldsymbol{u} \in \mathbb{V} \quad \text{and} \quad \boldsymbol{q} \in \mathbb{Y}^{\varpi} \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^{\varpi}}$

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Proof. (S-Cauchy) implies, after enlarging A:

$$\frac{1}{k!}\,|\boldsymbol{u}|_{\mathbb{X}^k} \le A^{k+1}\Big(\max_{\ell=0}^{k} \frac{1}{\ell!}\,|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell} + \|\boldsymbol{u}\|_{\mathbb{X}^m}\Big)$$

Take power $1/k$

$$\Big(\frac{1}{k!}\,|\boldsymbol{u}|_{\mathbb{X}^k}\Big)^{1/k} \le A'\Big(\max_{\ell=0}^{k} \frac{1}{\ell!}\,|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell} + \|\boldsymbol{u}\|_{\mathbb{X}^m}\Big)^{1/k}$$

$$\le A'\Big\{\max_{\ell=0}^{k} \Big(\frac{1}{\ell!}\,|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell}\Big)^{1/k} + \|\boldsymbol{u}\|_{\mathbb{X}^m}^{1/k}\Big\}$$

□

It suffices to realize the program

Type B + Type S

It suffices to realize the program

Type B + Type S

to obtain Type A

... in any situation we want

Outline

First things first

- Ω smooth domain with analytic boundary.
- $\partial_{\boldsymbol{s}}\Omega$ for $\boldsymbol{s} \in \mathscr{S}$, connected components of $\partial\Omega$.
- $\mathbb{P}$ elliptic 2d order boundary value problem (system).
- $\mathbb{P} = (L, T_{\boldsymbol{s}}, D_{\boldsymbol{s}})$, operators with analytic coefficients:
  - L interior operator
  - $T_{\boldsymbol{s}}$ boundary operator of order 1, $\boldsymbol{s} \in \mathscr{S}_N$
  - $D_{\boldsymbol{s}}$ boundary operator of order 0, $\boldsymbol{s} \in \mathscr{S}_D$

[1] for more than 50 years

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Theorems of Type C, B, and A known[1] in the framework of Sobolev spaces:

$$|\boldsymbol{u}|_{\mathbb{X}^m} = \sum_{|\alpha|=m} \|\partial_{\boldsymbol{x}}^{\alpha}\boldsymbol{u}\|_{L^2(\Omega)}$$

$$|\boldsymbol{q}|_{\mathbb{Y}^m} = \sum_{|\alpha|=m-2} \|\partial_{\boldsymbol{x}}^{\alpha}\boldsymbol{f}\|_{L^2(\Omega)} + \sum_{\substack{\boldsymbol{s}\in\mathscr{S}_N \\ |\alpha|=m-2}} \|\partial_{\boldsymbol{x}}^{\alpha}\boldsymbol{g}_{\boldsymbol{s}}\|_{H^{\frac12}(\partial_{\boldsymbol{s}}\Omega)} + \sum_{\substack{\boldsymbol{s}\in\mathscr{S}_D \\ |\alpha|=m-1}} \|\partial_{\boldsymbol{x}}^{\alpha}\boldsymbol{h}_{\boldsymbol{s}}\|_{H^{\frac12}(\partial_{\boldsymbol{s}}\Omega)}$$

[1] for more than 50 years

ADN & Morrey

Theorem (Type C) [ADN, 1959, 1964]

$\mathbb{P} : \mathbb{X}^m \longrightarrow \mathbb{Y}^m$ is Fredholm for any $m \geq 2$.

Theorem (Type B) [ADN, 1959, 1964]

For any $k \geq 2$, $\boxed{\boldsymbol{u} \in \mathbb{X}^2 \quad \text{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^k \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^k}$

Theorem (Type A) [Morrey-Nirenberg, 1957]

For any $k \geq 2$, $\boxed{\boldsymbol{u} \in \mathbb{X}^2 \quad \text{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^\varpi \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^\varpi}$

Anything to add?

Yes: Regularity Shift with Cauchy-type estimates

Theorem (Type S) [CoDaNi, 2010]

There exists $A > 0$ such that for any $k \geq 2$ and $\boldsymbol{u} \in \mathbb{X}^2$

(S-Cauchy) $$\frac{1}{k!}\,|\boldsymbol{u}|_{\mathbb{X}^k} \leq A^{k+1}\Big(\sum_{\ell=0}^{k}\frac{1}{\ell!}\,|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell} + \|\boldsymbol{u}\|_{\mathbb{X}^2}\Big)$$

Proof. Clean old proofs:

- Nested open sets on model problems
- Faà di Bruno formula for local maps

Coercive variational form

If $\mathbb{P}$ issues from a variational form coercive on $\mathbb{V} \subset \boldsymbol{H}^1(\Omega)$, all thms adapt:

C Existence in $\mathbb{V}$

B Basic regularity in $\mathbb{X}^2$ if $\mathbb{P}\boldsymbol{u} \in \mathbb{Y}^2$ (var. solutions are strong solutions)

A, S Estimates with $\|\boldsymbol{u}\|_{H^1(\Omega)}$ in the RHS.

Outline

3D Examples

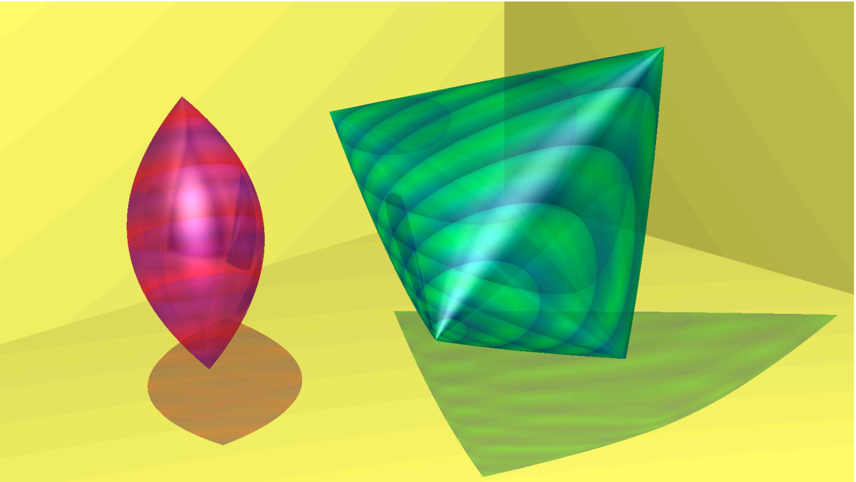

Figure: Axisymmetric domain & Cayley's tetrahedron (M. Costabel with POV-Ray)

Domains with conical points

- Ω analytic corner domain (analytic cones and maps) with corner set $\mathscr{C}$ (in 2D, piecewise analytic in 2D — polygonal domains).
- $\mathbb{P} = (L, T_{\boldsymbol{s}}, D_{\boldsymbol{s}})$ elliptic 2d order with analytic coefficients.
- To simplify: coercive problems with zero boundary data ($\boldsymbol{q} \equiv \boldsymbol{f}$).

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Theorems of Type C based on Lax-Milgram, no regularity required.

Theorems of Type B and "S standard" known, starting with [Kondratev '67].
Use of weighted Sobolev spaces:

$$|\boldsymbol{u}|_{\mathbb{X}^m} = \sum_{|\alpha|=m} \|w_m\,\partial_{\boldsymbol{x}}^{\alpha}\boldsymbol{u}\|_{L^2(\Omega)} \quad \text{and} \quad |\boldsymbol{f}|_{\mathbb{Y}^m} = \sum_{|\alpha|=m-2} \|w_m\,\partial_{\boldsymbol{x}}^{\alpha}\boldsymbol{f}\|_{L^2(\Omega)}$$

where $w_0(\boldsymbol{x})$, $w_1(\boldsymbol{x})$,..., $w_m(\boldsymbol{x})$, ... family of weights of general type

$$w_m(\boldsymbol{x}) = r(\boldsymbol{x})^{m+\beta}, \quad r(\boldsymbol{x}) = dist(\boldsymbol{x}, \mathscr{C}), \quad \beta \in \mathbb{R}.$$

An old friend: The Dirichlet Laplacian on a polygon

Let $\omega = \omega_c$ be the largest opening angle of the polygon Ω (corner c).

Theorem (Type B) [Kondratev '67]

- Let $u \in H^1_0(\Omega)$ solution of $\Delta u = f$.
- Let $m \geq 2$.
- Let β such that $0 < -\beta - 1 < \frac{\pi}{\omega}$ and $w_\ell = r^{\ell+\beta}$, $0 \leq \ell \leq m$.

Then $\boxed{f \in \mathbb{Y}^m \quad \Longrightarrow \quad u \in \mathbb{X}^m}$

1. Why $0 < -\beta - 1$, i.e. $\beta < -1$? $\Longrightarrow$ w_1 unbounded.
   Because this condition implies $\mathbb{X}^2$ compactly embedded in $H^1(\Omega)$.
2. Why $-\beta - 1 < \frac{\pi}{\omega}$? Because under this condition the strongest singularity

$$x \longmapsto r_c^{\pi/\omega} \sin \frac{\pi\theta_c}{\omega_c}$$

belongs to $\mathbb{X}^m$ for all m.

What about the Neumann Laplacian on a polygon?

- The previous functional setting is unpleasant for the Neumann Δ.
- Independent pointwise values arise at each corner.
- The constant function $1 \notin \mathbb{X}^2$ if $0 < -\beta - 1$ because $w_0 = r^\beta$. But no problem for derivatives...

Remedy: modify the first weights. Take

$$w_\ell = r^{\max\{0,\ell+\beta\}} \simeq \min\{1, r^{\ell+\beta}\}, \quad \ell \in \mathbb{N}$$

Example: If $\beta = -\frac{3}{2}$, $w_0 = w_1 = 1$, and $w_\ell = r^{\ell+\beta}$ as before if $\ell \geq 2$.

Theorem (Type B) [Mazya-Plamenevskii, 1984]

- Let $\boldsymbol{u} \in H^1(\Omega)$ solution of $\Delta \boldsymbol{u} = \boldsymbol{f}$ with $\partial_n \boldsymbol{u} = 0$.
- Let $m \geq 2$.
- Let β such that $0 < -\beta - 1 < \frac{\pi}{\omega}$ and $w_\ell = r^{\max\{0,\ell+\beta\}}$, $0 \leq \ell \leq m$.

Then $\boxed{\boldsymbol{f} \in \mathbb{Y}^m \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^m}$

General case, Type B in corner domains

- $\Omega \subset \mathbb{R}^d$. Dimension $d \geq 2$.
- Coercive variational formulation in $\mathbb{V} \subset \boldsymbol{H}^1(\Omega)$
- Smooth coefficients

Theorem (Type B)

Exists an optimal number $b^*(\Omega, \mathbb{P}) > 1 - \frac{d}{2}$ such that the following holds.

- Let $m \geq 2$.
- Let $\beta < -1$ such that $-\beta - \frac{d}{2} < b^*(\Omega, \mathbb{P})$
- Choose the weights $w_\ell = r^{\max\{0,\ell+\beta\}}$ (non-homogeneous norms).

Then $\boxed{\boldsymbol{u} \in \mathbb{V} \quad \text{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^m \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^m}$

Remark.

Weights $w_\ell = r^{\ell+\beta}$ (homogeneous norms) suitable if $\boldsymbol{u} \in \mathbb{V} \Rightarrow \frac{\boldsymbol{u}}{r} \in L^2(\Omega)$.
There holds a similar statement involving another positive number $b(\Omega, \mathbb{P})$ determined by Mellin corner spectra $\sigma(\mathfrak{A}_{\boldsymbol{c}})$, $\boldsymbol{c} \in \mathscr{C}$.

Type A for Δ, Lamé in polygonal domains

Weighted analytic regularity has been invented by Babuška and Guo, and proved for model operators in polygonal domains.

Theorem (Type A) [Babuška-Guo 1988, 1989, 1993]

There exists $\beta \in (-2, -1)$ such that with the weights $w_\ell = r^{\max\{0,\ell+\beta\}}$:

$$\boxed{\boldsymbol{u} \in \mathbb{V} \quad \text{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^{\varpi} \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^{\varpi}}$$

Exponential convergence [Babuška-Guo 1988, 1989, 1993]

This weighted analytic regularity allows to prove the exponential convergence of the *h*-*p* method of finite elements.

This is not the end of the story for corner domains

Theorem (Type S standard) [Kondratev 1967]

With homogeneous weights $w_\ell = r^{\ell+\beta}$:

For all $k \geq 2$ and $\boxed{\text{all } \beta \in \mathbb{R}}$

$$\boxed{\boldsymbol{u} \in \mathbb{X}^2 \quad \text{and} \quad \boldsymbol{q} \in \mathbb{Y}^k \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^k}$$

with estimates (C depends on β and k)

$$\|\boldsymbol{u}\|_{\mathbb{X}^k} \leq C\big(\|\mathbb{P}\boldsymbol{u}\|_{\mathbb{Y}^k} + \|\boldsymbol{u}\|_{\mathbb{X}^2}\big)$$

In other words :

For any β, if $r^{|\alpha|+\beta}\boldsymbol{u} \in L^2(\Omega)$ for $|\alpha| \leq 1$, then $r^{|\alpha|+\beta}\boldsymbol{u} \in L^2(\Omega)$ for $|\alpha| \leq k$ if the rhs has the corresponding regularity.

This is an unconditional elliptic regularity shift for corner domains.

Regularity Shift with Cauchy-type estimates

Theorem (Type S) [CoDaNi, 2010]

1. With homogeneous weights $w_\ell = r^{\ell+\beta}$:
For all $\beta \in \mathbb{R}$ exists $A > 0$ such that for any $k \geq 2$ and $\boldsymbol{u} \in \mathbb{X}^2$
$$\frac{1}{k!}\,|\boldsymbol{u}|_{\mathbb{X}^k} \leq A^{k+1}\Big(\sum_{\ell=0}^{k}\frac{1}{\ell!}\,|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell} + \|\boldsymbol{u}\|_{\mathbb{X}^1}\Big)$$

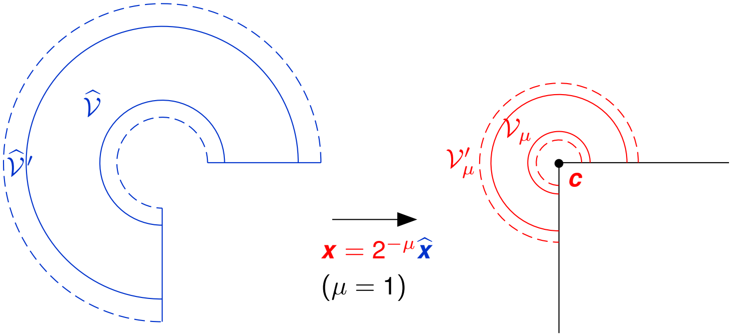
2. With non-homogeneous weights $w_\ell = r^{\max\{0,\ell+\beta\}}$:
For all $\beta \in \mathbb{R}$ and $m \geq \max\{-\beta, 2\}$ exists $A > 0$ such that for any $k \geq m$ and $\boldsymbol{u} \in \mathbb{X}^m$
$$\frac{1}{k!}\,|\boldsymbol{u}|_{\mathbb{X}^k} \leq A^{k+1}\Big(\sum_{\ell=m+1}^{k}\frac{1}{\ell!}\,|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell} + |\boldsymbol{u}|_{\mathbb{X}^m}\Big)$$

Proof

- Unweighted estimates (S-Cauchy) in fixed annulus far from corner $\boldsymbol{c}$
- Scale estimates to closer annuli. Weight appears.
- Sum over a dyadic partition of a neighborhood of $\boldsymbol{c}$

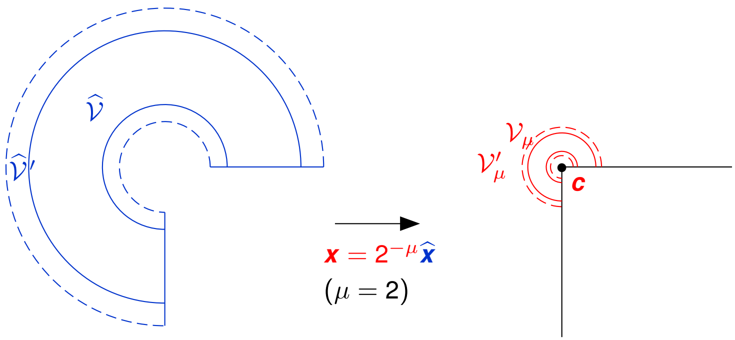
Proof of weighted analytic estimates

Scale on $\mathcal{V}_\mu = 2^{-\mu}\mathcal{V}$ and $\mathcal{V}'_\mu = 2^{-\mu}\mathcal{V}'$, for $\mu = 1$



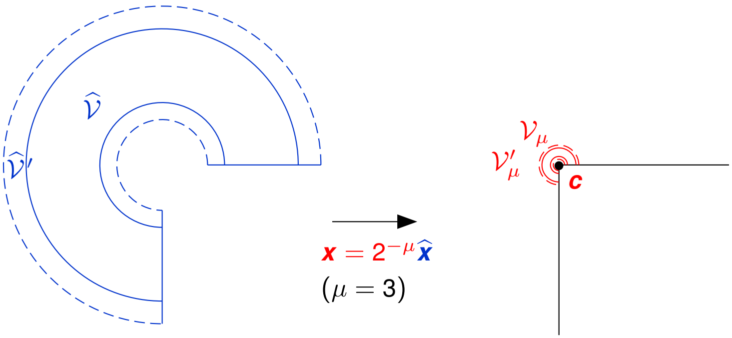
Proof of weighted analytic estimates

Scale on $\mathcal{V}_\mu = 2^{-\mu}\mathcal{V}$ and $\mathcal{V}'_\mu = 2^{-\mu}\mathcal{V}'$, for $\mu = 2$



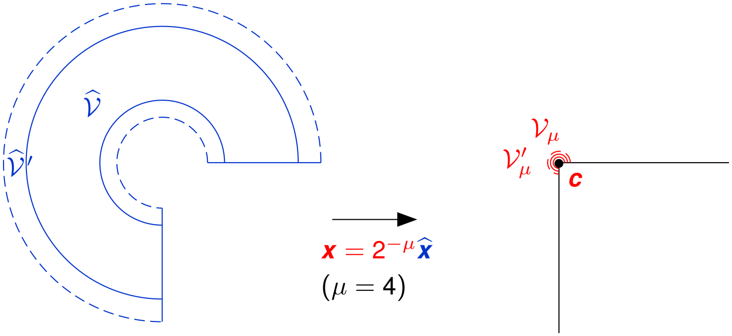
Proof of weighted analytic estimates

Scale on $\mathcal{V}_\mu = 2^{-\mu}\mathcal{V}$ and $\mathcal{V}'_\mu = 2^{-\mu}\mathcal{V}'$, for $\mu = 3$



Proof of weighted analytic estimates

Scale on $\mathcal{V}_\mu = 2^{-\mu}\mathcal{V}$ and $\mathcal{V}'_\mu = 2^{-\mu}\mathcal{V}'$, for $\mu = 4, \dots$



As a corollary of B & S: Type A in corner domains

General case *sine dolore*

- $\Omega \subset \mathbb{R}^d$. Dimension $d \geq 2$.
- Coercive variational formulation in $\mathbb{V} \subset \boldsymbol{H}^1(\Omega)$
- Analytic coefficients

Theorem (Type A) [CoDaNi, 2010]

With the same optimal number $b^*(\Omega, \mathbb{P})$ as in Theorem B, there holds.

- Let $\beta < -1$ such that $-\beta - \frac{d}{2} < b^*(\Omega, \mathbb{P})$
- Choose the weights $w_\ell = r^{\max\{0, \ell+\beta\}}$, $\ell \in \mathbb{N}$.

Then $\boxed{\boldsymbol{u} \in \mathbb{V} \quad \text{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^\varpi \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^\varpi}$

Remark.
Homogeneous weights $w_\ell = r^{\ell+\beta}$ can be used instead if

$$\boldsymbol{u} \in \mathbb{V} \Longrightarrow \frac{\boldsymbol{u}}{r} \in L^2(\Omega).$$

If $\beta < -1$ & $-\beta - \frac{d}{2} < b(\Omega, \mathbb{P})$, $\boxed{\boldsymbol{u} \in \mathbb{V} \quad \text{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^\varpi \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^\varpi}$

Outline

Corners, edges, distance functions and weights

- Ω polyhedral domain in $\mathbb{R}^3$. Distance to singular points: $\boldsymbol{x} \mapsto r(\boldsymbol{x})$
- Corners $\boldsymbol{c}$, set of corners $\mathscr{C}$, distance functions: $r_{\boldsymbol{c}}$ to $\boldsymbol{c}$, $r_{\mathscr{C}}$ to $\mathscr{C}$,
- Edges $\boldsymbol{e}$, set of edges $\mathscr{E}$, distance functions: $r_{\boldsymbol{e}}$ to $\boldsymbol{e}$.

Two ways of generating weights

1. A simple way: choose $\beta \in \mathbb{R}$ and use powers of r

$$\boxed{w_\ell = r^{\ell+\beta}} \quad \text{or} \quad \boxed{w_\ell = r^{\max\{0,\ell+\beta\}}}$$

2. A finer tool: choose a multi-β, i.e. $\underline{\beta} = (\beta_{\boldsymbol{c}}, \beta_{\boldsymbol{e}})$

$$\boxed{w_\ell = \prod_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}}^{\ell+\beta_{\boldsymbol{c}}} \times \prod_{\boldsymbol{e}\in\mathscr{E}} \Big(\frac{r_{\boldsymbol{e}}}{r_{\mathscr{C}}}\Big)^{\ell+\beta_{\boldsymbol{e}}}} \quad \text{or} \quad \boxed{w_\ell = \prod_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}}^{\max\{0,\ell+\beta_{\boldsymbol{c}}\}} \times \prod_{\boldsymbol{e}\in\mathscr{E}} \Big(\frac{r_{\boldsymbol{e}}}{r_{\mathscr{C}}}\Big)^{\max\{0,\ell+\beta_{\boldsymbol{e}}\}}}$$

Note: If $\beta_{\boldsymbol{c}} \equiv \beta$, then $\prod_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}}^{\ell+\beta_{\boldsymbol{c}}} \simeq r_{\mathscr{C}}^{\ell+\beta}$

If $\beta_{\boldsymbol{c}} \equiv \beta_{\boldsymbol{e}} \equiv \beta$, then $\prod_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}}^{\ell+\beta_{\boldsymbol{c}}} \times \prod_{\boldsymbol{e}\in\mathscr{E}} \big(\frac{r_{\boldsymbol{e}}}{r_{\mathscr{C}}}\big)^{\ell+\beta_{\boldsymbol{e}}} \simeq r^{\ell+\beta}$.

Type B in polyhedral domains

- Coercive variational formulation in $\mathbb{V} \subset \boldsymbol{H}^1(\Omega)$
- Smooth coefficients

Theorem (Type B) [Mazya-Rossmann 2003] [CoDaNi, 2012]

For optimal numbers $b_{\boldsymbol{c}}^*(\Omega, \mathbb{P}) > -\frac{1}{2}$ and $b_{\boldsymbol{e}}(\Omega, \mathbb{P}) > 0$ depending on Mellin corner and edge spectra $\sigma(\mathfrak{A}_{\boldsymbol{c}})$ and $\sigma(\mathfrak{A}_{\boldsymbol{e}})$, the following holds.

- Let $m \geq 2$.
- Let $\underline{\beta} < -1$ such that $-\beta_{\boldsymbol{c}} - \frac{3}{2} < b_{\boldsymbol{c}}^*(\Omega, \mathbb{P})$ and $-\beta_{\boldsymbol{e}} - 1 < b_{\boldsymbol{e}}(\Omega, \mathbb{P})$
- Choose the weights $w_\ell = \prod_{\boldsymbol{c} \in \mathscr{C}} r_{\boldsymbol{c}}^{\max\{0, \ell+\beta_{\boldsymbol{c}}\}} \times \prod_{\boldsymbol{e} \in \mathscr{E}} \left(\frac{r_{\boldsymbol{e}}}{r_{\mathscr{C}}}\right)^{\max\{0, \ell+\beta_{\boldsymbol{e}}\}}$.

Then $\boxed{\boldsymbol{u} \in \mathbb{V} \quad \text{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^m \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^m}$

Example. For Dirichlet Laplacian,

$$b_{\boldsymbol{e}}(\Omega, \mathbb{P}) = \frac{\pi}{\omega_{\boldsymbol{e}}}, \quad b_{\boldsymbol{c}}(\Omega, \mathbb{P}) = -\frac{1}{2} + \sqrt{\mu_{\boldsymbol{c},1}^{\mathrm{Dir}} + \frac{1}{4}}, \quad b_{\boldsymbol{c}}^*(\Omega, \mathbb{P}) = \min\{2, b_{\boldsymbol{c}}\}$$

The edge issue (without the corners)

Theorems of type S and, hence, of type A, can be proved without difficulty. But in connection with *h-p* version of finite elements, this would not help.

The edge issue (without the corners)

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Choose tubular coordinates $\boldsymbol{x}_{\boldsymbol{e}} = (x_{\boldsymbol{e}}^\perp, x_{\boldsymbol{e}}^\parallel)$ and corresponding multi-indices $\alpha_{\boldsymbol{e}} = (\alpha_{\boldsymbol{e}}^\perp, \alpha_{\boldsymbol{e}}^\parallel)$, — perpendicular and parallel to $\boldsymbol{e}$. Typically

$$\boxed{w_{\boldsymbol{e},\alpha} = r_{\boldsymbol{e}}^{\beta_{\boldsymbol{e}}+|\alpha_{\boldsymbol{e}}^\perp|}}$$

independent of derivatives $\partial_{\boldsymbol{x}}^{\alpha_{\boldsymbol{e}}^\parallel}$ along $\boldsymbol{e}$.

Anisotropic weights (edges & corners)

To simplify, assume that all edges are parallel to coordinate axes.
The non-homogeneous version of anisotropic weights is

$$w_\alpha = \prod_{c\in\mathscr{C}} r_c^{\max\{0,\beta_c+|\alpha|\}} \times \prod_{e\in\mathscr{E}} \Big(\frac{r_e}{r_{\mathscr{C}}}\Big)^{\max\{0,\beta_e+|\alpha_e^\perp|\}}$$

Theorem (Type S) [CoDaNi, 2010]

- Let $\beta = (\beta_c, \beta_e)$ such that
  $$\forall e \in \mathscr{E}, \quad 0 < -\beta_e - 1 \text{ and } -\beta_e - 1 \notin \operatorname{Re}\sigma(\mathfrak{A}_e)$$
- Let $m \geq 1$ and $\geq \max\{-\beta_e, -\beta_c\}$

Then for any $k \geq m$ and $u \in \mathbb{X}^m$

$$\frac{1}{k!}\,|u|_{\mathbb{X}^k} \leq A^{k+1}\Big(\sum_{\ell=0}^{k}\frac{1}{\ell!}\,|\mathbb{P}u|_{\mathbb{Y}^\ell} + |u|_{\mathbb{X}^m}\Big)$$

Example. For Dirichlet Laplacian, $\sigma(\mathfrak{A}_e) = \{\frac{k\pi}{\omega_e} : k \in \mathbb{Z}^*\}$.

Corollary of B & S: Type AA in polyhedral domains

- $\Omega \subset \mathbb{R}^3$ polyhedron
- Coercive variational formulation in $\mathbb{V} \subset \boldsymbol{H}^1(\Omega)$
- Homogeneous constant coefficients (Analytic coefficients possible)

Theorem (Type AA) [CoDaNi, 2010]

With the same numbers $b^*_{\boldsymbol{c}}(\Omega,\mathbb{P})$ and $b_{\boldsymbol{e}}(\Omega,\mathbb{P})$ as in Theorem B:

- Let $\beta < -1$ such that $-\beta_{\boldsymbol{c}} - \frac{3}{2} < b^*_{\boldsymbol{c}}(\Omega,\mathbb{P})$ and $-\beta_{\boldsymbol{e}} - 1 < b_{\boldsymbol{e}}(\Omega,\mathbb{P})$
- Choose the weights $w_\ell = \prod_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}}^{\max\{0,\ell+\beta_{\boldsymbol{c}}\}} \times \prod_{\boldsymbol{e}\in\mathscr{E}} \left(\frac{r_{\boldsymbol{e}}}{r_{\mathscr{C}}}\right)^{\max\{0,\ell+\beta_{\boldsymbol{e}}\}}$

Then $\boxed{\boldsymbol{u} \in \mathbb{V} \quad \text{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^{\varpi} \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^{\varpi}}$

Remark.
Homogeneous weights $w_\ell = \prod_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}}^{\ell+\beta_{\boldsymbol{c}}} \times \prod_{\boldsymbol{e}\in\mathscr{E}} \left(\frac{r_{\boldsymbol{e}}}{r_{\mathscr{C}}}\right)^{\ell+\beta_{\boldsymbol{e}}}$ can be used if

$$\boldsymbol{u} \in \mathbb{V} \Longrightarrow \frac{\boldsymbol{u}}{r} \in L^2(\Omega)$$

Numbers b for Δ on examples

Domain Ω	$b_e(\Omega)$	$b_c(\Omega)$	$b_c^*(\Omega)$
Cube, Dirichlet	2	3	2
Cube, Neumann	2	0	2
Thick L, Dirichlet	0.66666	1.66666	1.66666
Thick L, Neumann	0.66666	0	1.66666
Fichera corner, Dirichlet	0.66666	0.45418	0.45418
Fichera corner, Neumann	0.66666	0	0.84001

Thick L : $\{(-1,1)^2 \setminus (0,1)^2\} \times (-1,1)$

Fichera corner : $(-1,1)^3 \setminus (0,1)^3$

Sources

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Analytic Regularity for Linear Elliptic Systems in Polygons and Polyhedra
Math. Models Methods Appl. Sci. 08(22) (2012), 59 p.
DOI: 10.1142/S0218202512500157

M. COSTABEL, M. DAUGE, S. NICAISE
Book project:
Corner Singularities and Analytic Regularity for Linear Elliptic Systems
Part I: Smooth domains. *HAL: hal-00453934* (2010), 211 pages

3D Lexicon

Type	Homogeneous norms	Non-homogeneous norms
Isotropic	$K^k_\beta(\Omega)$	$J^k_\beta(\Omega)$
Anisotropic	$M^k_\beta(\Omega)$	$N^k_\beta(\Omega)$
Anisotropic Analytic	$A^k_\beta(\Omega)$	$B^k_\beta(\Omega)$

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Thank you for your attention