Théories des Coques

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“Shell theory attempts the impossible: to provide a two-dimensional representation of an intrinsically three-dimensional phenomenon.”

KOITER & SIMMONDS, 1972.

Outline

- Three-dimensional linear elasticity in a shell and the three related problematics:
  1. "Shell theory", 2. Asymptotic analysis, 3. Multi-scale expansions.
- Three categories of shells:
  a. Plates, b. Shallow shells (a. & b. = plate-like), c. Real shells.
- Two-dimensional models for the three categories of shells and their relations between each other.
- Limits as the thickness tends to zero. Comparison between scaled and unscaled theories allows unification of results.
- Convergence in energy : reconstruction operators from the mean surface into the shell and "estimates" by Koiter, compared to optimal estimates provided by the multi-scale expansions (for plate-like domains and elliptic shells).

What is a shell ?

Less and less inaccurately:

1. A three-dimensional body with one dimension small (h).
2. A bounded domain $\hat{\Omega} \in \mathbb{R}^3$ with one dimension small (h) and its two other characteristic lengths $>> h$.
3. A bounded domain $\hat{\Omega} \in \mathbb{R}^3$ associated with a smooth surface S and a smooth diffeomorphism $\Phi : S \times (-h, h) \longrightarrow \hat{\Omega} = \Phi\big(S \times (-h, h)\big)$. The pointwise values of Φ and $\nabla\Phi$ are bounded independently of h.
4. A bounded domain $\hat{\Omega} \in \mathbb{R}^3$ associated with its smooth mean surface S and with its constant thickness $(2h)$. Let a unit normal field $n : x \longrightarrow n(x)$ be fixed on S. The following mapping is a smooth diffeomorphism:

$$\begin{array}{rccc} \Phi : & S \times (-h, h) & \longrightarrow & \hat{\Omega} = \Phi\big(S \times (-h, h)\big) \\ & (x, t) & \longmapsto & x + tn(x) \end{array}$$

Three-dimensional linear elasticity for an isotropic body

Simply in cartesian coordinates

- Displacement **tensor :** $u = (u_i)$
- Linearized strain **tensor :** $e = (e_{ij})$ **:** $e_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$.
- Stress **tensor :** $\sigma = (\sigma_{ij})$ **:** $\boxed{\sigma = A\,e}$ (Constitutive Equations)
- Elasticity **tensor of the constitutive material** $A = (A^{ijkl})$ **:**

$$A^{ijkl} = \lambda\,\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$$ **with Lamé coeff.** λ **and** μ

- Volume force **field :** $f = (f^j)$

$$\boxed{\operatorname{div}\sigma = f}$$ (Equations of Equilibrium)

- **Boundary conditions on the lower and upper surfaces : traction-free (or imposed).**
- **Boundary conditions on the lateral surface : here, clamped, in general.**

Problematics

Given a shell $\hat{\Omega}$ with mean surface S and (half-)thickness h, and a volume load $\hat{f}$, let $\hat{u}$ be the solution displacement. Problem $(\mathfrak{E})$.

1. **Shell Theory**

   Find a problem $(\mathfrak{P})$ on S (the two-dim. model) whose solution z provides, via a reconstruction operator $\hat{V}$ a good approximation $\hat{V}z$ of $\hat{u}$ in energy norm on $\hat{\Omega}$.

2. **Asymptotic Analysis**

   Define a family of 3-dim problems $\varepsilon \mapsto (\mathfrak{E}_\varepsilon)$ s. t. $(\mathfrak{E}_h) = (\mathfrak{E})$. Let $\hat{u}_\varepsilon$ be the sol. of $(\mathfrak{E}_\varepsilon)$. Find $(\mathfrak{L}) := \lim_{\varepsilon\longrightarrow 0}(\mathfrak{E}_\varepsilon)$. The solution ζ of $(\mathfrak{L})$ will be $\lim_{\varepsilon\longrightarrow 0} \hat{u}_\varepsilon$.

3. **Multi-scale Expansions**

   Define a family of solutions $\hat{u}_\varepsilon$ as in 2. and expand $\varepsilon \mapsto \hat{u}_\varepsilon$ with respect to ε $\boxed{\hat{u}_\varepsilon \sim \sum_p \varepsilon^p u^p(y)}$ with profiles u^p and arguments $y = (x, X(\varepsilon))$. The partial (finite) sums for $p \le N$ are improved approximations of $\hat{u}_\varepsilon$ as $N \nearrow$.

Classification

The body $\hat{\Omega}$ with mean surface S and half-thickness h is given once for all.

Let R be the minimal principal curvature radius of S and D its geodesic diameter.

$R \equiv \infty$ **Plate :** S is a domain $\omega \subset \mathbb{R}^2$.

$R \sim 1$ **Shell :** S is a manifold embedded in $\mathbb{R}^3$.

$R \sim 1/h$ **Shallow Shell :**

If $R > 2D$, then S can represented by only one chart Ξ above a flat surface ω immersed in $\mathbb{R}^2$. And Ξ satisfies the estimates

$$|\Xi| + |\nabla\Xi| \leq \frac{C}{R}, \quad \text{with} \quad C = C(D).$$

[ANDREOIU-DA.-FAOU, 2000]

Factorize by h if $1/h > 2D$:

$$\Xi = h\xi.$$

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This definition of shallow shells is distinct from that used
- in engineering and also
- in the framework of a particular mathematical analysis.

This other definition starts from a similar inequality $D << R$ but considers R as fixed and D small. This corresponds to a spatial localization on a manifold S, concentrating around one point.

Associated families of domains

Plate: $\hat{\Omega}_\varepsilon = \omega \times (-\varepsilon, \varepsilon)$. We have $S = \omega$.

Shell: $\hat{\Omega}_\varepsilon = \{x + tn(x), \quad x \in S, \ t \in (-\varepsilon, \varepsilon)\}$.

Sh.Sh.: $\hat{\Omega}_\varepsilon = \{x + tn(x), \quad x \in S_\varepsilon, \ t \in (-\varepsilon, \varepsilon)\}$
with $S_\varepsilon = \{(x_*, \varepsilon\xi(x_*)), \quad x_* \in \omega\}$.

Normal coordinates (x_α, t) on $\hat{\Omega}_\varepsilon$, with x_α local surface coordinates in S (S_ε in the case of Sh. Sh.) and $t \in (-\varepsilon, \varepsilon)$.

Extend the volume force $\hat{f}$ in $\varepsilon \mapsto \hat{f}_\varepsilon$

- either by restriction $\hat{f}_\varepsilon = \hat{f}\,\big|_{\hat{\Omega}^\varepsilon}$
- or by the formula $\hat{f}_\varepsilon(x_\alpha, t) = \hat{f}(x_\alpha, \frac{th}{\varepsilon})$ for $|t| \leq \varepsilon$.

The problem $(\mathfrak{E}_\varepsilon)$ is posed on $\hat{\Omega}_\varepsilon$ with the same Lamé constants λ and μ and the volume force $\hat{f}_\varepsilon$.

Plate theory: the Kirchhoff-Love models

Use the scaling $\tilde{u} = (u_\alpha, \varepsilon u_3)$ *and* $\tilde{f} = (f^\alpha, \varepsilon^{-1} f^3)$ *in an essential step.*

The two-dimensional models are written in variational form on the displacements. The elasticity tensor $M^{\alpha\beta\sigma\delta}$ is that of the "plain stress model"

$$M^{\alpha\beta\sigma\delta} = \tilde{\lambda}\,\delta^{\alpha\beta}\delta^{\sigma\delta} + \mu(\delta^{\alpha\sigma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\sigma}) \quad \text{with} \quad \tilde{\lambda} = \frac{2\lambda\mu}{\lambda + 2\mu}\,.$$

Plate :

$$(\zeta, \zeta') \longmapsto \int_\omega M^{\alpha\beta\sigma\delta}\Big(e_{\alpha\beta}(\zeta_*)\, e_{\sigma\delta}(\zeta'_*) + \frac{1}{3}\partial_{\alpha\beta}\zeta_3\, \partial_{\sigma\delta}\zeta'_3\Big)\mathrm{d}x_*$$

Shallow Shell (in curvilinear components) :

$$(\zeta, \zeta') \longmapsto \int_\omega M^{\alpha\beta\sigma\delta}\Big(\tilde{e}_{\alpha\beta}(\zeta)\, \tilde{e}_{\sigma\delta}(\zeta') + \frac{1}{3}\partial_{\alpha\beta}\zeta_3\, \partial_{\sigma\delta}\zeta'_3\Big)\mathrm{d}x_*$$

with $\tilde{e}_{\alpha\beta}(\zeta) = \frac{1}{2}(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha) - \partial_{\alpha\beta}\xi\,\zeta_3\,.$

Shell Theory: the Koiter model

"It is shown that Love's so-called first approximation for the strain energy as the sum of stretching or extensional energy and bending or flexural energy, is a consistent first approximation, and that no refinement of this first approximation is justified, in general, if the basic Love-Kirchhoff assumptions are retained." **[KOITER, 1959]**

$$(z, z') \longmapsto \int_S M^{\alpha\beta\sigma\delta} \Big(\gamma_{\alpha\beta}(z)\, \gamma_{\sigma\delta}(z') + \frac{\varepsilon^2}{3} \rho_{\alpha\beta}(z)\, \rho_{\sigma\delta}(z') \Big) \mathrm{d}S$$

with the elasticity tensor (where $a_{\alpha\beta}$ is the 1^{st} fundamental form on S)

$$M^{\alpha\beta\sigma\delta} = \tilde{\lambda}\, a^{\alpha\beta} a^{\sigma\delta} + \mu (a^{\alpha\sigma} a^{\beta\delta} + a^{\alpha\delta} a^{\beta\sigma}) \quad \text{with} \quad \tilde{\lambda} = \frac{2\lambda\mu}{\lambda + 2\mu} .$$

with the *membrane* or *extensional* strain tensor associated with the change of metrics

$$\gamma_{\alpha\beta}(z) = \frac{1}{2} (\mathrm{D}_\alpha z_\beta + \mathrm{D}_\beta z_\alpha) - b_{\alpha\beta} z_3$$

where D_α is the covariant derivative on S, **and the *bending* strain tensor.../...**

Shell Theory: the Koiter model and its intrinsic variants

and the *bending* or *flexural* strain tensor associated with the change of curvature

$$\rho_{\alpha\beta}(z) = \mathrm{D}_\alpha \mathrm{D}_\beta z_3 - c_{\alpha\beta} z_3 + b_\alpha^\sigma \mathrm{D}_\beta z_\sigma + \mathrm{D}_\alpha b_\beta^\sigma z_\sigma$$

where b_β^α is the 2^{d} fundamental form and $c_{\alpha\beta} = b_\alpha^\sigma b_{\sigma\beta}$ the 3^{d} fundamental form.

In the variants of the Koiter model the membrane strain is still $\gamma_{\alpha\beta}$.

The bending strain is modified

$$\chi_{\alpha\beta}(z) = \tfrac{1}{2}(\mathrm{D}_\alpha \theta_\beta + \mathrm{D}_\beta \theta_\alpha) + \tfrac{1}{2}(b_\alpha^\sigma \omega_{\beta\sigma} + b_\beta^\sigma \omega_{\alpha\sigma}) \quad \text{[BUDIANSKI-SANDERS, 1967]}$$

$$\tau_{\alpha\beta}(z) = \mathrm{D}_\alpha \mathrm{D}_\beta z_3 \quad \text{(minimal model)},$$

with the surface vorticity $\omega_{\alpha\beta}$ and the transverse vorticity θ_α :

$$\omega_{\alpha\beta}(z) = \tfrac{1}{2}(\mathrm{D}_\alpha z_\beta - \mathrm{D}_\beta z_\alpha) \quad \text{and} \quad \theta_\alpha(z) = \mathrm{D}_\alpha z_3 + b_\alpha^\beta z_\beta.$$

There holds

$$\chi_{\alpha\beta} = \rho_{\alpha\beta} - \tfrac{1}{2}(b_\alpha^\sigma \gamma_{\beta\sigma} + b_\beta^\sigma \gamma_{\alpha\sigma}).$$

Quote non-intrinsic variants by [LOVE, 1888] [REISSNER, 1941] [NOVOZHILOV, 1959].

Shell theory...

... gives back the plate theory in plate-like domains if the scalings are eliminated.

Recall A_{Plate} **the bilinear form on plates**

$$(\zeta, \zeta') \longmapsto \int_\omega M^{\alpha\beta\sigma\delta} \Big(e_{\alpha\beta}(\zeta_*)\, e_{\sigma\delta}(\zeta'_*) + \frac{1}{3} \partial_{\alpha\beta}\zeta_3\, \partial_{\sigma\delta}\zeta'_3 \Big) \mathrm{d}x_*$$

Set $z = (z_\alpha, z_3) := (\zeta_\alpha, \varepsilon^{-1}\zeta_3)$

Written in ***unscaled*** **unknowns, the bilinear form become :**

$$(z, z') \longmapsto \int_\omega M^{\alpha\beta\sigma\delta} \Big(e_{\alpha\beta}(z_*)\, e_{\sigma\delta}(z'_*) + \frac{\varepsilon^2}{3} \partial_{\alpha\beta} z_3\, \partial_{\sigma\delta} z'_3 \Big) \mathrm{d}x_*.$$

We have $e_{\alpha\beta}(z_*) = \gamma_{\alpha\beta}(z)$ **and** $\partial_{\alpha\beta} z_3 = \rho_{\alpha\beta}(z) = \chi_{\alpha\beta}(z) = \tau_{\alpha\beta}(z)$ **on** S**.**

Works similarly for shallow shells.

Limits in scaled variables, references

For plate-like domains, find the limit as $\varepsilon \longrightarrow 0$ of the *scaled* displacements $\tilde{u}(\varepsilon)$ written in scaled variables. Convergence in $\mathrm{H}^1(\Omega)^3$ with $\Omega = \omega \times (-1, 1)$.

[CIARLET-DESTUYNDER, 1979] for Plates.

[BUSSE-CIARLET-MIARA, 1997] for Sh.Sh.

For shells, find the limit as $\varepsilon \longrightarrow 0$ of $\varepsilon^2 u(\varepsilon)$ with $u(\varepsilon)$ the *unscaled* displacement written in scaled variables. Convergence in $\mathrm{H}^1(\Omega)^3$ with $\Omega = S \times (-1, 1)$.

[CIARLET-LODS-MIARA, 1996].

Similar result for the solution $z(\varepsilon)$ of Koiter model. [CIARLET-LODS, 1996].

If moreover, S is elliptic, find the limit as $\varepsilon \longrightarrow 0$ of $u(\varepsilon)$ with $u(\varepsilon)$ the *unscaled* displacement written in scaled variables. Convergence in $\mathrm{H}^1(\Omega)^2 \times \mathrm{L}^2(\Omega)$.

[CIARLET-LODS, 1996].

Similar result for the solution $z(\varepsilon)$ of Koiter model. [CIARLET-LODS, 1996].

Limits in scaled variables

For plate-like domains, define the scaled Kirchoff-Love displacement $\tilde{V}^{\mathrm{KL}}\zeta$ by

$$\tilde{V}^{\mathrm{KL}}\zeta = (\zeta_\alpha - X_3\partial_\alpha\zeta_3, \zeta_3) \quad \text{on} \quad \Omega := \omega \times (-1,1) \quad \text{and} \quad X_3 = t/\varepsilon.$$

If $f^3 = \mathcal{O}(\varepsilon)$, $\boxed{\tilde{u}(\varepsilon) \xrightarrow{\mathrm{H}^1(\Omega)^3} \tilde{V}^{\mathrm{KL}}\tilde{\zeta}^0}$ with $\tilde{\zeta}^0$ sol. of Dirichlet pb for P and rhs $\tilde{g}^0$.

Shells. Inextensional disp. $\mathcal{V}_{\mathrm{F}} := \{\zeta \in \mathrm{H}^1_0 \times \mathrm{H}^1_0 \times \mathrm{H}^2_0(S)\ ;\ \gamma_{\alpha\beta}(\zeta) = 0\}$.

If $f = \mathcal{O}(1)$, $\boxed{\varepsilon^2 u(\varepsilon) \xrightarrow{\mathrm{H}^1(\Omega)^3} \zeta^{-2}}$ with $\zeta^{-2} \in \mathcal{V}_{\mathrm{F}}$ the solution of

$$\forall \zeta' \in \mathcal{V}_{\mathrm{F}}, \quad \frac{1}{3}\int_S M^{\alpha\beta\sigma\delta}\rho_{\alpha\beta}(\zeta)\,\rho_{\sigma\delta}(\zeta')\,\mathrm{d}S = \int_S \zeta\, g_0\,\mathrm{d}S.$$

If S is elliptic, $\mathcal{V}_{\mathrm{F}} = \{0\}$ and $\boxed{u(\varepsilon) \xrightarrow{\mathrm{H}^1\times\mathrm{H}^1\times\mathrm{L}^2} \zeta^0}$ with $\zeta^0 \in \mathrm{H}^1_0 \times \mathrm{H}^1_0 \times \mathrm{L}^2$ sol

$$\forall \zeta' \in \mathrm{H}^1_0 \times \mathrm{H}^1_0 \times \mathrm{L}^2(S), \quad \int_S M^{\alpha\beta\sigma\delta}\gamma_{\alpha\beta}(\zeta)\,\gamma_{\sigma\delta}(\zeta')\,\mathrm{d}S = \int_S \zeta\, g_0\,\mathrm{d}S.$$

Here g_0 is the first momentum of forces across the thick shell $S \times (-1, 1)$

$$g_0(x_\alpha) = \int_{-1}^{1} f(x_\alpha, X_3)\,\mathrm{d}X_3.$$

Plates seen as shells

Consider $f = \mathcal{O}(1)$ on plates. Split f in $\varepsilon^{-1}(0, \varepsilon f^3) + (f^\alpha, 0)$. Plate limit yields that $\tilde{u}(\varepsilon) \simeq \tilde{V}^{\mathrm{KL}}(\varepsilon^{-1}\tilde{\zeta}^{-1} + \tilde{\zeta}^0)$ with $\tilde{\zeta}^{-1} \in \mathrm{H}^1_0 \times \mathrm{H}^1_0 \times \mathrm{H}^2_0(\omega)$ solution of

$$\forall \zeta' \in \mathrm{H}^1_0 \times \mathrm{H}^1_0 \times \mathrm{H}^2_0(\omega), \quad A_{\mathrm{Plate}}(\tilde{\zeta}^{-1}, \zeta') = \frac{1}{2}\int_{-1}^{1}\int_\omega f^3\, \zeta'_3\, \mathrm{d}x_*\, \mathrm{d}X_3.$$

$\Longrightarrow \tilde{\zeta}^{-1} = (0, \tilde{\zeta}_3^{-1})$ and the *unscaled* displacement $u(\varepsilon)$ then satisfies

$$u(\varepsilon) \simeq \varepsilon^{-2}(0, \tilde{\zeta}_3^{-1}) + \varepsilon^{-1}(0, \tilde{\zeta}_3^0) + \mathcal{O}(1).$$

On a plate $\mathcal{V}_{\mathrm{F}} = \{(0, \zeta_3) \;;\; \zeta_3 \in \mathrm{H}^2_0(\omega)\}$. Shell limit is $u(\varepsilon) \simeq \varepsilon^{-2}\zeta^{-2}$ with $\zeta^{-2} \in \mathcal{V}_{\mathrm{F}}$ the solution of

$$\forall \zeta' \in \mathcal{V}_{\mathrm{F}}, \quad \frac{1}{3}\int_S M^{\alpha\beta\sigma\delta} \rho_{\alpha\beta}(\zeta)\, \rho_{\sigma\delta}(\zeta')\, \mathrm{d}S = \frac{1}{2}\int_{-1}^{1}\int_S f^j\, \zeta'_j\, \mathrm{d}S\, \mathrm{d}X_3$$

$\Longrightarrow \zeta^{-2} = (0, \zeta_3^{-2})$.
We see that $\tilde{\zeta}_3^{-1}$ and ζ_3^{-2} solve the same bending equation, thus coincide.

Reconstruction operators

Canonical injections $\hat{\mathcal{I}}: \; S \longrightarrow \hat{\Omega}_\varepsilon\,, \quad (\hat{\mathcal{I}} z)(x_\alpha, t) = z(x_\alpha)\,.$

Note that the scaled-variable version $\mathcal{I}$ of $\hat{\mathcal{I}}$ was understood in S.13.

Kirchhoff-Love $\hat{V}^{\mathrm{KL}}$ for shells (gives back standard KL in the case of plates)

$$\hat{V}^{\mathrm{KL}}(z) := \begin{cases} z_\sigma - t\big(\mathrm{D}_\sigma z_3 + 2b_\sigma^\alpha z_\alpha\big)\,, \\ z_3\,. \end{cases}$$

Modified Kirchhoff $\hat{V}^{\mathrm{modKL}}$ for shells [Koiter, 1970]

$$\hat{V}^{\mathrm{modKL}}(z) := \begin{cases} z_\sigma - t\big(\mathrm{D}_\sigma z_3 + 2b_\sigma^\alpha z_\alpha\big)\,, \\ z_3 - \frac{\lambda}{\lambda+2\mu}\big(t\gamma_\alpha^\alpha(z) - \frac{1}{2}t^2\rho_\alpha^\alpha(z)\big)\,. \end{cases}$$

Formal series of reconstruction operators $S \longrightarrow S \times (-1,1)$ in scaled var. $\frac{t}{\varepsilon}$

$V[\varepsilon] = V^0 + \varepsilon V^1 + \varepsilon^2 V^2 + \cdots$ [Faou, 2000].

Convergence in energy

Let $E(\hat{\Omega})$ be the three-dimensional strain energy norm

$$\|\hat{u}\|_{E(\hat{\Omega})} = \Big(\int_{\hat{\Omega}} |e_{ij}(\hat{u})|^2 \,\mathrm{d}x \Big)^{1/2}.$$

Wanted : a reconstruction operator $\hat{V}$ and a two-dimensional model such that, if z is the solution of the 2D model there holds an estimate

$$\|\hat{u} - \hat{V}(z)\|^2_{E(\hat{\Omega})} \le \delta \|\hat{u}\|^2_{E(\hat{\Omega})}, \quad \text{with “small” } \delta.$$

Hope for “universal” constants δ tending to 0 as $h \to 0$ involving only

h the thickness of the shell $\hat{\Omega}$,

R the minimal principal radius of curvature of S

L the wave length associated with z, i.e. the largest constant such that the following pointwise estimates hold everywhere in S, for $\ell = 1, 2$

$$|D^\ell \gamma_{\alpha\beta}(z)| \le L^{-\ell} \sum_{\sigma\delta} |\gamma_{\sigma\delta}(z)| \quad \text{and} \quad |D^\ell \rho_{\alpha\beta}(z)| \le L^{-\ell} \sum_{\sigma\delta} |\rho_{\sigma\delta}(z)|.$$

Convergence in energy: Koiter's estimates

Let z the solution of the Koiter model for thickness h and L its wave length.

The paper [KOITER, 1970] yields the formula

$$\delta = \frac{h^2}{L^2} + \frac{h}{R} \, .$$

But, in the paper [KOITER-SIMMONDS, 1972] we read : *"The somewhat depressing conclusion for most shell problems is, similar to the earlier conclusions of* GOL'DENWEIZER, *that no better accuracy of the solutions can be expected than of order*

$$\delta^* = \frac{h}{L} + \frac{h}{R},$$

even if the equations of first-approximation shell theory would permit, in principle, an accuracy of order δ *."*

The obstruction in Koiter's estimates

The obstruction comes from the three-dimensional boundary layers, which cannot be described by any two-dimensional model. In the '60, a correct multi-scale analysis was seemingly out of reach.

"Concentrating on the interior we sidestep all kinds of delicate questions, with an attendant gain in certainty and generality. The information about the interior behavior can be obtained much more cheaply (in the mathematical sense) than that required for the discussion of boundary value problems, which form a more "transcendental" stage."

[JOHN, 1965]

We are going to revisit Koiter's estimates in the light of recent results involving multi-scale expansions.

Multi-scale expansions: references

t **normal coordinate,** r **distance to the edge,** s **tangential coordinate along** ∂S **.**

Clamped plates, 2-scale expansion : [NAZAROV-ZORIN, 1989] and [DA.-GRUAIS, 1996].

Simply supported and free plates, 2-scale expansion : [DA.-GRUAIS-RÖSSLE, 2000].

Clamped and free shallow shells, 2-scale expansion : [ANDREOIU-DA.-FAOU, 2000].

$$\varepsilon^k, k \in \mathbb{N}, \quad \text{scales} \quad \left(x_\alpha, \tfrac{t}{\varepsilon}\right) \quad \text{and} \quad \left(s, \tfrac{r}{\varepsilon}, \tfrac{t}{\varepsilon}\right).$$

Clamped elliptic surfaces, 2-scale expansion for the Koiter model : [FAOU, 2000].

$$\varepsilon^{k/2}, k \in \mathbb{N}, \quad \text{scales} \quad (x_\alpha) \quad \text{and} \quad \left(s, \tfrac{r}{\sqrt{\varepsilon}}\right).$$

Clamped elliptic shells, 3-scale expansion : [FAOU, 2000].

$$\varepsilon^{k/2}, k \in \mathbb{N}, \quad \text{scales} \quad \left(x_\alpha, \tfrac{t}{\varepsilon}\right), \quad \left(s, \tfrac{r}{\sqrt{\varepsilon}}, \tfrac{t}{\varepsilon}\right) \quad \text{and} \quad \left(s, \tfrac{r}{\varepsilon}, \tfrac{t}{\varepsilon}\right).$$

Strain-energy and H^1-norm of the different "objects"

Name	Type of displacement	Scale	$\|\hat{u}\|^2_{E(\hat{\Omega})}$	$\|\hat{u}\|^2_{H^1(\hat{\Omega})}$
$u_{\mathrm{KL,b}}$	bending KL on plates	(x_α, t)	$\mathcal{O}(h^3)$	$\mathcal{O}(h)$
$u_{\mathrm{KL,m}}$	membrane KL on plates	(x_α, t)	$\mathcal{O}(h)$	$\mathcal{O}(h)$
u_{KL}	general KL on shells	(x_α, t)	$\mathcal{O}(h)$	$\mathcal{O}(h)$
v	non-KL	$(x_\alpha, \frac{t}{h})$	$\mathcal{O}(h^{-1})$	$\mathcal{O}(h^{-1})$
Φ	3D boundary layer	$(s, \frac{r}{h}, \frac{t}{h})$	$\mathcal{O}(h^0)$	$\mathcal{O}(h^0)$
Z	2D boundary layer	$(s, \frac{r}{\sqrt{h}})$	$\mathcal{O}(h^{1/2})$	$\mathcal{O}(h^{1/2})$
Z_{KL}	2D boundary layer KL	$(s, \frac{r}{\sqrt{h}}, t)$	$\mathcal{O}(h^{3/2})$	$\mathcal{O}(h^{1/2})$
W	2D-3D boundary layer	$(s, \frac{r}{\sqrt{h}}, \frac{t}{h})$	$\mathcal{O}(h^{-1/2})$	$\mathcal{O}(h^{-1/2})$

The above Table gives the behavior with respect to h of the strain-energy of typical profiles (which could also be called "objects") regardless of any particular asymptotics. This means for v for example that we fix a profile V on $S \times (-1,1)$ and evaluate the energy of $v(x_\alpha, t) := V(x_\alpha, \frac{t}{h})$ on $S \times (-h, h)$ as a function of h, and similarly for the others: the profile is fixed and only h varies.

The object type Z_{KL} is specific to shells and is a KL displacement generated by a 2D boundary layer profile of the form

$$\big(0, 0, Z_3(s, \frac{r}{\sqrt{h}})\big).$$

In Slides 21 and 23 this is applied to real asymptotics with the aim of evaluating the asymptotic behavior of the relative error in energy norm with respect to h, in the "generic" case when the first terms u^{-2} and u^0 in plates and shells respectively, are not zero. In Slide.22 we do a similar procedure in squared H^1-norm.

Reappraisal of Koiter's estimates: plate-like domains

In plate-like domains:

$$\hat{u} = h^{-2}u^{-2}_{\mathrm{KL,b}} + h^{-1}u^{-1}_{\mathrm{KL}} + h^{0}(u^{0}_{\mathrm{KL}} + v^{0} + \boxed{\Phi^{0}}\,) + \cdots$$

Energy $\quad h^{-1} \qquad h^{-1} \qquad h^{1} \qquad h^{-1} \qquad \boxed{h^{0}}$

Note that $z = h^{-2}(0, \zeta_3^{-2}) + h^{-1}(\zeta_\alpha^{-1}, 0)$.

It is necessary to include $h^0 v^0$ in the reconstruction operator. We have

$$V^{\mathrm{modKL}}(z) = h^{-2}u^{-2}_{\mathrm{KL,b}} + h^{-1}u^{-1}_{\mathrm{KL,m}} + h^{0}v^{0}.$$

Hence

$$\left\|\hat{u} - \hat{V}^{\mathrm{modKL}}(z)\right\|^{2}_{E(\hat{\Omega})} \le ch\left\|\hat{u}\right\|^{2}_{E(\hat{\Omega})}.$$

The wave-length $L = \mathcal{O}(1)$, for plates $R = \infty$ and for sh.sh $R = 1/h$.

Hence $\delta \simeq h^2$ and $\delta^* \simeq h$.

Koiter's estimate holds for δ^* and not δ.

Relative squared H^1-norm estimates for clamped elliptic shells

Expansion:

$$\hat{u} = u^0_{\mathrm{KL}} + Z^0 + h^{1/2}(u^{1/2}_{\mathrm{KL}} + \boxed{Z^{1/2}} + W^{1/2}) + h(u^1_{\mathrm{KL}} + \boxed{W^1} + v^1 + \Phi^1) + \cdots$$

$$\begin{array}{ccccccccc} h & h^{1/2} & h^2 & h^{3/2} & h^{1/2} & h^3 & h^{3/2} & h & h^2 \end{array}$$

The most energetic part in H^1 -norm is the two-dimensional boundary layer Z^0 of the Koiter model.

The solution of the Koiter model expands as

$$z = \zeta^0 + Z^0 + h^{1/2}(\zeta^{1/2} + \bar{Z}^{1/2}) + \mathcal{O}(h), \quad \text{with } \bar{Z}^{1/2} \neq Z^{1/2}.$$

The reconstruction operator V^{modKL} includes $W^{1/2}$.There holds the expansion :

$$V^{\mathrm{modKL}}(z) = u^0_{\mathrm{KL}} + Z^0 + h^{1/2}(u^{1/2}_{\mathrm{KL}} + \bar{Z}^{1/2} + W^{1/2}) + hv^1 + \cdots$$

Error estimate :

$$\|\hat{u} - \hat{V}^{\mathrm{modKL}}(z)\|^2_{H^1(\hat{\Omega})} \le ch\|\hat{u}\|^2_{H^1(\hat{\Omega})}.$$

Reappraisal of Koiter's estimates: clamped elliptic shells

Expansion:

$$\hat{u} = u^0_{\mathrm{KL}} + Z^0_{\mathrm{KL}} + h^{1/2}(u^{1/2}_{\mathrm{KL}} + Z^{1/2}_{\mathrm{KL}} + Z^{1/2}_{\mathrm{surf}}) + h(W^1 + v^1 + \boxed{\Phi^1} + \cdots)$$

$$h \qquad h^{3/2} \qquad h^2 \qquad h^{5/2} \qquad h^{3/2} \qquad h^{3/2} \qquad h \qquad h^2$$

The solution of the Koiter model expands as

$$z = \zeta^0 + Z^0 + h^{1/2}(\zeta^{1/2} + \bar{Z}^{1/2}) + \mathcal{O}(h), \quad \text{with } \bar{Z}^{1/2}_\sigma = Z^{1/2}_\sigma .$$

The reconstruction operator V^{modKL} includes W^1. There holds the expansion:

$$V^{\mathrm{modKL}}(z) = u^0_{\mathrm{KL}} + Z^0_{\mathrm{KL}} + h^{1/2}(u^{1/2}_{\mathrm{KL}} + \bar{Z}^{1/2}_{\mathrm{KL}} + Z^{1/2}_{\mathrm{surf}} + h(W^1 + v^1 + \cdots)$$

Error estimate (still due to the 3D boundary layer like in plates!):

$$\|\hat{u} - \hat{V}^{\mathrm{modKL}}(z)\|^2_{E(\hat{\Omega})} \leq ch\|\hat{u}\|^2_{E(\hat{\Omega})} .$$

The wave-length $L = \mathcal{O}(h^{1/2})$ and $R = \mathcal{O}(1)$. Hence $\delta \simeq \delta^* \simeq h$.

IRMAR

For plates the same as Slide.21 holds if the forces have the membrane parity and non-vanishing first momenta across the thickness.

With free lateral edge, the same holds for plates with bending load having a non-vanishing first momentum across the thickness. But if the load has the membrane parity and still non-vanishing first momenta across the thickness, the constant is now h^2, i.e. the "nice" δ !

Thus for clamped and free plates, for clamped elliptic plates, we obtain an asymptotic Koiter-like relative estimate in energy norm if the first momenta g_0 of forces are not vanishing. Of course if it happens that $g_0 \equiv 0$, this estimate is definitely wrong.

Towards universal models, a motivation

Until now we have seen

1. Reduced operators acting between tensors on the mean surface S, in the form $P(\varepsilon) = P^0 + \varepsilon P^1 + \varepsilon^2 P^2$, namely the Koiter operator,

2. Reconstruction operators transforming tensors on the mean surface S into tensors in the scaled shell $S \times (-1, 1)$ (or in the shell $\hat{\Omega}^\varepsilon$) in the form $V(\varepsilon) = V^0 + \varepsilon V^1 + \varepsilon^2 V^2$

so that, if $z(\varepsilon)$ solves a problem of the type $P(\varepsilon)z(\varepsilon) = g$ then $V(\varepsilon)z(\varepsilon)$ is a good approximation of the three-dimensional shell solution $u(\varepsilon)$.

Here the edge boundary conditions are discarded, and the above statement is almost meaningless...

A correct statement in this direction is... given on the next slide.

Formal series approach, case without loading

Consider the equations inside the scaled shell Ω and the boundary conditions on lower and upper surfaces in the form a formal series equation

$$(\mathfrak{F}) \qquad \begin{cases} L[\varepsilon]w[\varepsilon] & = & f[\varepsilon] \\ B[\varepsilon]w[\varepsilon] & = & 0, \end{cases}$$

(and compute the coefficients L^k and B^k). There exist

1. **A formal series of reduction operators $A[\varepsilon] = \sum_k \varepsilon^k A^k$,**
2. **A formal series of reconstruction operators $V[\varepsilon] = \sum_k \varepsilon^k V^k$,**

so that if $f \equiv 0$ *(i)* For any sol of $w[\varepsilon]$ of $(\mathfrak{F})$, the f. s. $z[\varepsilon] := w[\varepsilon]\big|_S$ satisfies

$$(1) \qquad A[\varepsilon]z[\varepsilon] = 0,$$

$$(2) \qquad w[\varepsilon] = V[\varepsilon]z[\varepsilon].$$

***(ii)* For any $z[\varepsilon]$ sol of (1), the formal series $w[\varepsilon]$ defined by (2) solves $(\mathfrak{F})$.**

Formal series approach, general case

For general right hand side $f[\varepsilon]$ in $(\mathfrak{F})$, prove the existence of two extra formal series acting on $f[\varepsilon]$:

1. A reduction series $G[\varepsilon]$ from $S \times (-1,1)$ to S,
2. A solution series $Q[\varepsilon]$ from S to $S \times (-1,1)$

so that if *(i)* For any sol of $w[\varepsilon]$ of $(\mathfrak{F})$, the f. s. $z[\varepsilon] := w[\varepsilon]\big|_S$ satisfies

$$(1') \qquad A[\varepsilon]z[\varepsilon] = G[\varepsilon]f[\varepsilon],$$

$$(2') \qquad w[\varepsilon] = V[\varepsilon]z[\varepsilon] + Q[\varepsilon]f[\varepsilon].$$

(ii) For any $z[\varepsilon]$ sol of $(1')$, the formal series $w[\varepsilon]$ defined by $(2')$ solves $(\mathfrak{F})$.

A^0 classical membrane operator. $A^1 = 0$ (in relation with the isotropy).
$A^2 = F + B$ where F is Koiter's flexural operator and $B\big|_{\mathcal{V}_F} \equiv 0$.
G^0 first tranverse momentum operator. [FAOU, 2000].

Towards universal models

1. When $\partial S = \emptyset$: If eq. $A[\varepsilon]z[\varepsilon] = G[\varepsilon]f[\varepsilon]$ can be solved (such is the case for elliptic arches and elliptic shells), then the reconstruction equation gives back the complete asymptotics of $u(\varepsilon)$.

2. When S is a manifold with boundary, the reconstruction equation does not yield in general a displacement satisfying the edge boundary condition: this would require infinitely many boundary conditions on the two-dimensional series $z[\varepsilon]$. Equation (1') completed with all these boundary conditions is never solvable. But eq. $A[\varepsilon]z[\varepsilon] = G[\varepsilon]f[\varepsilon]$ is completed by the only 4 Dirichlet boundary conditions of the space $\mathrm{H}^1_0 \times \mathrm{H}^1_0 \times \mathrm{H}^2_0(S)$. These are just the right conditions so that the adjunction of plate-type 3 dimensional boundary layers yield the full clamped boundary condition on the edge of the shell.

Open problems

- Is asymptotic Koiter-like estimate valid in a more general framework (subject to g_0 is not zero)?
- Can we find a more universal error indicator between the 3D solution and a 3D displacement reconstructed from the solution of a 2D problem?
- Do the 3D solution always admits a multi-scale expansion, when the data are smooth, piecewise-smooth,... Probably not in any case.
- Special attention within these approaches to lateral boundary conditions, which may change dramatically the nature of the solutions.