Mathematische Analyse von FEM für Probleme in der Mechanik

Oberwolfach, 7-13 Februar 1999

Boundary layers in thin structures

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Plan

- A Lamé boundary value problem in thin plates.
- A two-scale membrane or bending asymptotics for the displacement in thin plates.
- A three-scale asymptotics for the displacement in thin elliptic shells.
- A numerical evidence of the boundary layer for strains in a bending plate.
- Prospects.

Connection with literature, concerning theoretical aspects

- **Philippe CIARLET and his group, concerning the limits as the thickness goes to 0.**
- **Sergei NAZAROV with co-authors and students, concerning asymptotics using refined Korn inequalities (in general requires clamped conditions).**
- **Doug ARNOLD with co-authors and students, concerning the asymptotics of the 2D models, e.g. the Reissner-Mindlin model.**

Three-Dimensional Equations in Linearly Elastic Plates

- Displacement Field $u = (u_1, u_2, u_3)$
- Linearized Strain Tensor $e = (e_{ij})_{1 \le i,j \le 3}$: $e_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$.
- Stress Tensor $\sigma = (\sigma_{ij})_{1 \le i,j \le 3}$: $\boxed{\sigma = A\, e}$ (Hooke's law)
- Matrix of Rigidity of the constitutive material $A = (A_{ijkl})$: symmetries

$$A_{ijkl} = A_{jikl} = A_{ijlk} = A_{klij}$$

Here we consider the simplest situation

Isotropic $A_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ with Lamé coeff. λ and μ

- Volumic Force Field $f = (f_1, f_2, f_3)$ (Equations of Equilibrium)

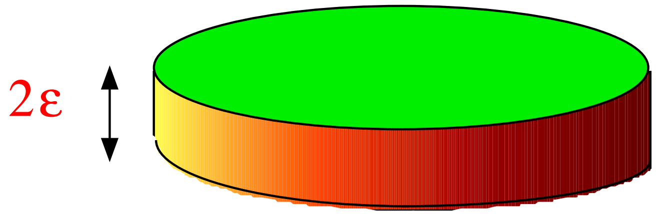
$$\boxed{\operatorname{div} \sigma = f} \quad \text{in} \quad \Omega.$$

Boundary Conditions

Plate $\Omega = \omega \times (-\varepsilon, +\varepsilon)$

with small ε

Mean surface ω = smooth plane domain.



Horizontal boundaries $\omega \times \{\pm\varepsilon\}$

- **Zero Traction** $T := \sigma(n) = 0.$

Lateral boundary $\partial\omega \times (-\varepsilon, +\varepsilon)$

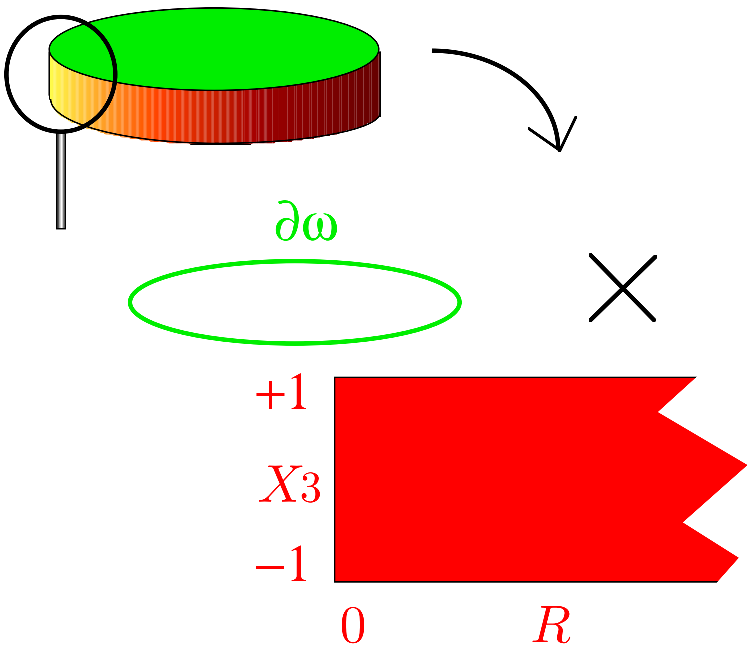
① **Hard Clamped** $u_s = 0,\ u_n = 0,\ u_3 = 0.$

② **Hard Simple Supported** $u_s = 0,\ T_n = 0,\ u_3 = 0.$

③ **Sliding Edge** $T_s = 0,\ u_n = 0,\ T_3 = 0.$

④ **Free** $T_s = 0,\ T_n = 0,\ T_3 = 0.$

Scaled variables



In Ω^ε :

variables $x = (x_1, x_2, x_3)$

In ω :

variables $x_* = (x_1, x_2)$

In ω, near $\partial\omega$:

r distance to $\partial\omega$, s arclength,

$\kappa = \kappa(s)$ curvature.

Scaled vertical variable:

$$X_3 = \frac{x_3}{\varepsilon}$$

Scaled distance to $\partial\omega$:

$$R = \frac{r}{\varepsilon}$$

Membrane answer

Membrane load: $f^\varepsilon(x_*, x_3) = f(x_*, X_3)$, i.e. such that

$$f_\alpha(X_3) = f_\alpha(-X_3) \quad \text{and} \quad f_3(X_3) = -f_3(-X_3)$$

Ingredients

Kirchhoff-Love displacements $u^k_{\mathrm{KL,m}}(x_*, x_3) = \big(\zeta^k_1(x_*), \zeta^k_2(x_*), 0\big)$

Displacements with mean values zero across each fiber

$$\int_{-1}^{+1} v^k_{\mathrm{m}}(x_*, X_3)\, dX_3 = 0, \qquad \forall x_* \in \omega$$

Boundary layer terms $\Phi^k_{\mathrm{m}}(s, R, X_3)$, Φ^k_{m} exp. decaying as $R \to \infty$

Expansion

$$u^\varepsilon \simeq u^0_{\mathrm{KL,m}} + \sum_{k\geq 1} \varepsilon^k \left(u^k_{\mathrm{KL,m}} + v^k_{\mathrm{m}} + \Phi^k_{\mathrm{m}} \right)$$

Bending answer

Bending load: $f^\varepsilon(x_*, x_3) = f(x_*, X_3)$, i.e. such that

$$f_\alpha(X_3) = -f_\alpha(-X_3) \quad \text{and} \quad f_3(X_3) = f_3(-X_3)$$

Ingredients

Kirchhoff-Love displacements

$$u^k_{\mathrm{KL,b}}(x_*, x_3) = \left(-x_3\partial_1\zeta^k_3(x_*), -x_3\partial_2\zeta^k_3(x_*), \zeta^k_3(x_*)\right)$$

Displacements with mean values zero across each fiber

$$\int_{-1}^{+1} v^k_{\mathrm{b}}(x_*, X_3)\, dX_3 = 0, \qquad \forall x_* \in \omega$$

Boundary layer terms $\Phi^k_{\mathrm{b}}(s, R, X_3)$, Φ^k_{b} exp. decaying as $R \to \infty$

Expansion

$$u^\varepsilon \simeq \varepsilon^{-2} u^{-2}_{\mathrm{KL,b}} + \varepsilon^{-1} u^{-1}_{\mathrm{KL,b}} + \sum_{k\geq 0} \varepsilon^k \left(u^k_{\mathrm{KL,b}} + v^k_{\mathrm{b}} + \Phi^k_{\mathrm{b}}\right)$$

(Clamped Elliptic Shells – Result by Erwan FAOU)

Loading $f^\varepsilon = f(x_*, X_3)$.

New scale $T = \dfrac{r}{\sqrt{\varepsilon}}$ (*cf* Novozhilov & Koiter models) and new profiles

$Z = Z(s, T, X_3)$, exp. decaying as $T \to \infty$, polynomial in X_3

Expansion

$$u^\varepsilon \simeq u^0_{\mathrm{KL}} + Z^0 + \sqrt{\varepsilon}\Big(u^{\frac{1}{2}}_{\mathrm{KL}} + Z^{\frac{1}{2}}\Big) + \sum_{k=1,\frac{3}{2},2,\ldots} \varepsilon^k \Big(u^k_{\mathrm{KL}} + v^k + Z^k + \Phi^k\Big)$$

Elastic energy	Membrane Plate	Bending Plate	Elliptic shell
Boundary layer terms	ε^2	ε^0	$\sqrt{\varepsilon}$
In-plane terms	ε	ε^{-1}	ε

Structure of first terms in plates

Displacements with mean values zero — with $p = \frac{\lambda}{6(\lambda+2\mu)}$

$$v^1_{\mathrm{m}} = p\Big(0,\ 0,\ -6X_3\,\mathrm{div}_*\,\zeta^0_*\Big) \quad \text{and} \quad v^0_{\mathrm{b}} = p\Big(0,\ 0,\ (3X_3^2-1)\,\Delta_*\zeta_3^{-2}\Big)$$

Boundary layer profiles (in tensor product form)

$$\Phi^1_{\mathrm{m}}(s,R,X_3) = \ell_{\mathrm{m}}(s)\,\bar\Phi_{\mathrm{m}}(R,X_3) \quad \text{and} \quad \Phi^0_{\mathrm{b}}(s,R,X_3) = \ell_{\mathrm{b}}(s)\,\bar\Phi_{\mathrm{b}}(R,X_3)$$

Case	ℓ_{m}	$\bar\Phi_{\mathrm{m}}$	ℓ_{b}	$\bar\Phi_{\mathrm{b}}$
①	$\mathrm{div}_*\,\zeta^0_*$	$(0,\bar\Phi_n,\bar\Phi_3)$	$\Delta_*\zeta_3^{-2}$	$(0,\bar\Phi_n,\bar\Phi_3)$
②	$\kappa\,\zeta^0_n$	$(0,\bar\Phi_n,\bar\Phi_3)$	$\kappa\,\partial_n\zeta_3^{-2}$	$(0,\bar\Phi_n,\bar\Phi_3)$
③	0	0	$\kappa\,\partial_s\zeta_3^{-2}$	$(\bar\Phi_s,\ 0,\ 0)$
④	0	0	$(\partial_n+\kappa)\partial_s\zeta_3^{-2}$	$(\bar\Phi_s,\ 0,\ 0)$

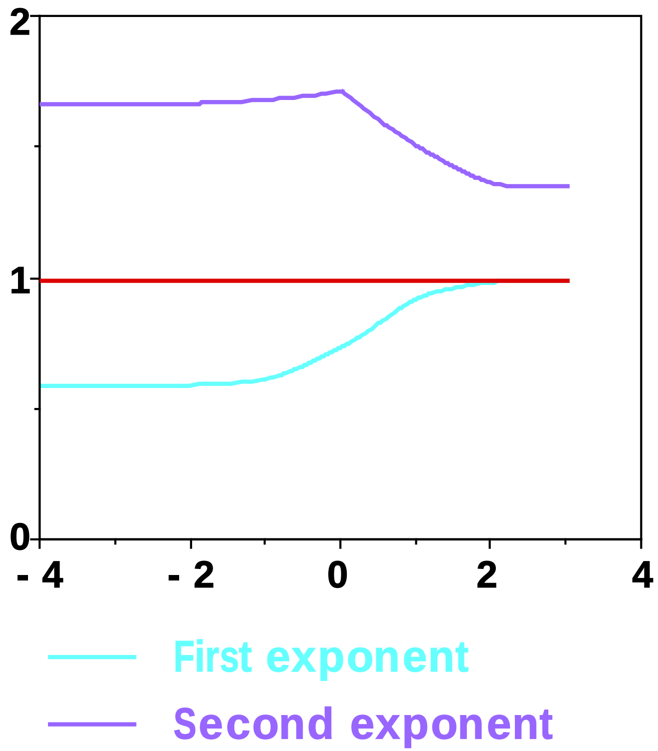
Singularities

The singularities concentrate in the boundary layer terms, via the typical profiles $\bar{\Phi}$.

The $\bar{\Phi}$ have singularities at the corners $(0, \pm 1)$ of the half-strip:

$$\bar{\Phi} = \bar{\Phi}_{\text{reg}}(R, X_3) + \sum_{+,-} \mathcal{S}^{\pm}(\varrho^{\pm}, \vartheta^{\pm})$$

① Exponents versus $\log_{10} \frac{\mu}{\lambda}$ $\longrightarrow$
Never H^2.



② Singularity in $\varrho \log \varrho$. Almost H^2.

③ Singularity in $\varrho^2 \log \varrho$. Almost H^3.

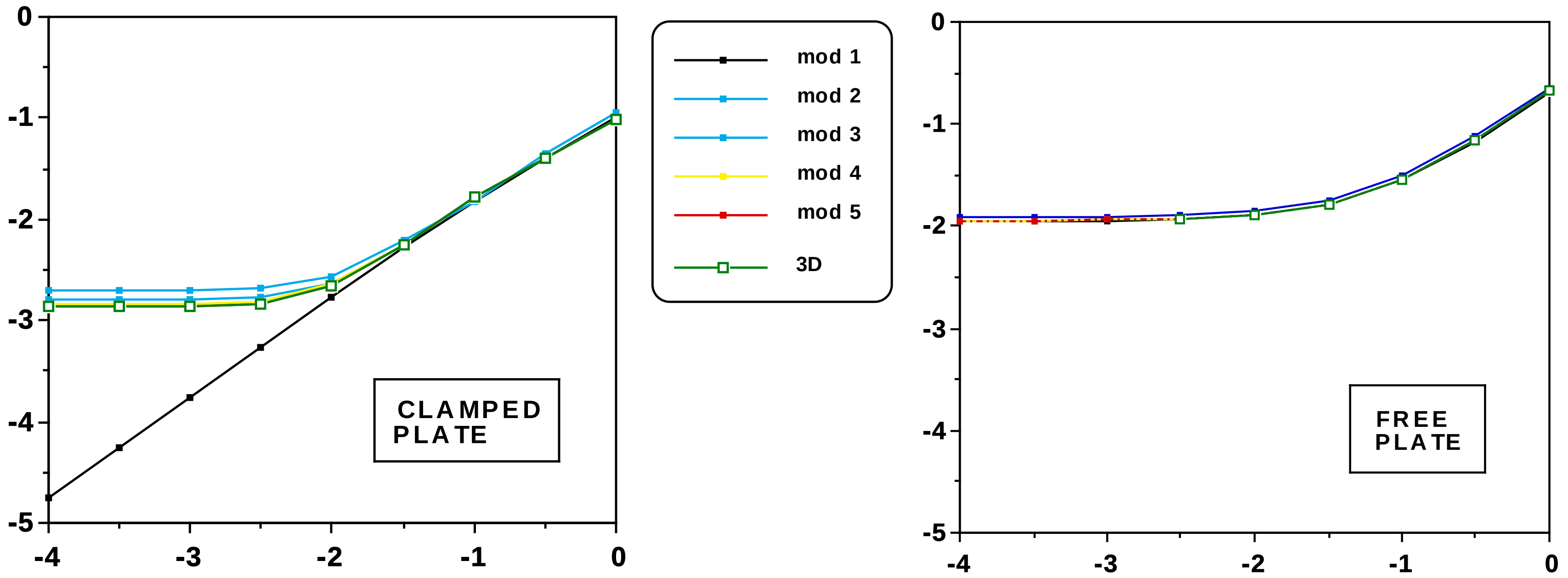
④ Idem.

Boundary layers in Strains

In order to measure boundary layer effects we introduce the quantity

$$I_{ij} = \left(\frac{1}{V} \int_{s=s_1}^{s_2} \int_{r=0}^{\varepsilon} \int_{x_3=-\varepsilon}^{\varepsilon} |e_{ij}|^2 dV \right)^{\frac{1}{2}}$$

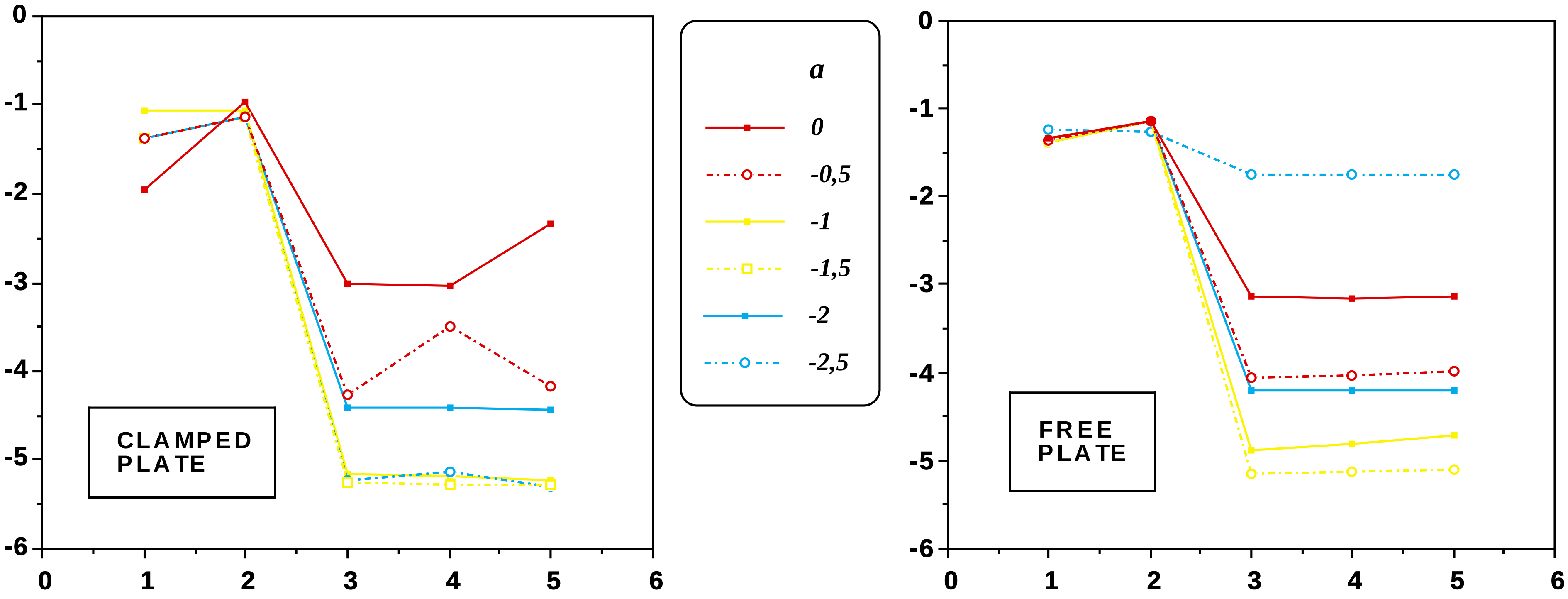
Clamped plate: $e_{r3} = e_{RX_3}[\Phi^1] + \varepsilon(\cdots)$. **Free plate:** $e_{s3} = e_{sX_3}[\Phi^1] + \varepsilon(\cdots)$.



$\log_{10} I_{r3}$ **versus** $\log_{10} \varepsilon$. **With different models.** $\log_{10} I_{s3}$ **versus** $\log_{10} \varepsilon$.

Computed by StressCheck$^{\mathrm{TM}}$. Joint work with Zohar Yosibash

Convergence of models outside the boundary layer



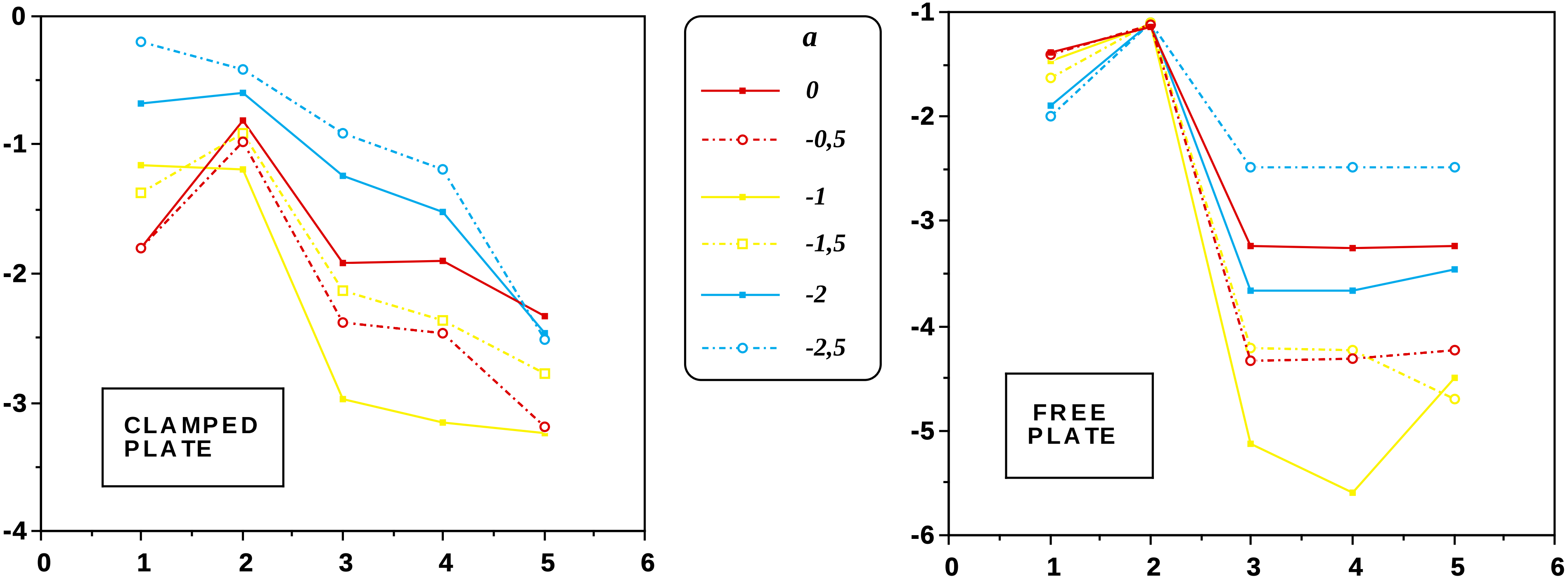
Abscissa : the degree in the hierarchy

Ordinates : $\log_{10}$ of the relative error with 3D.

Each curve : a value of $\varepsilon = 10^a$.

Degree	1	2	3	4	5
In-plane	1	1	3	3	5
Tranverse	0	2	2	4	4

Convergence of models inside the boundary layer



Abscissa : the degree in the hierarchy (in StressCheck)

Ordinates : $\log_{10}$ of the relative error with 3D. **Different ranges for clamped and free.**

Each curve : a value of $\varepsilon = 10^a$.

Prospects

- **Shallow shells**, general material law with non-necessarily clamped boundary conditions: in progress with Georgiana ANDREOIU and Erwan FAOU.

- **Modal analysis** for lowest families of eigenvalues in thin structures: in progress with Ivica DJURDJEVIC, Erwan FAOU and Andreas RÖSSLE.

- Modal analysis for **High frequencies** in thin structures: planned with Sergei NAZAROV and Andreas RÖSSLE.

- **Shells**: in progress by Erwan FAOU.

- **Plates with corners**: planned with Martin COSTABEL and Andreas RÖSSLE.

- **Numerical experiments** with Zohar YOSIBASH.