Harmonic Maxwell equations and their Finite Element discretization
Les équations de Maxwell harmoniques et leurs discrétisations par éléments finis

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Outline / *Plan* / Planned contributions by MD and MC

Part I. Introduction to Maxwell equations [MD]

1. **Notations**
2. **Maxwell equations**
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Part II. Traps in Finite element discretization / *Quelques pièges* [MD]

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Part I

Introduction to Maxwell equations

Outline

Before stating equations, agree on notations and conventions

Slides written in English / *Transparents en anglais, avec quelques traductions*

Colors

- Direction that we will follow
- Direction that we will leave
- Important expressions
- Emphasize or Danger

General notation

- $t \in \mathbb{R}$, time variable
- $\partial_t := \frac{\partial}{\partial t}$, time derivative
- $\boldsymbol{x}$, space variable
  - In 3 dimensions $\boldsymbol{x} = (x_1, x_2, x_3)$
  - In 2 dimensions $\boldsymbol{x} = (x_1, x_2)$
- For $j \in \{1, 2, 3\}$, $\partial_j := \frac{\partial}{\partial x_j}$ partial space derivative

Operators of order 1 and 2 in 3 dimensions of space

∇ is the gradient operator. For scalar distribution φ

$$\nabla\varphi = \begin{pmatrix} \partial_1\varphi \\ \partial_2\varphi \\ \partial_3\varphi \end{pmatrix}$$

div is the divergence operator: For vector distributions $\boldsymbol{u} = (u_1, u_2, u_3)$

$$\operatorname{div} \boldsymbol{u} = \nabla \cdot \boldsymbol{u} = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3$$

curl is the curl operator / *rotationnel*: For vector distributions $\boldsymbol{u} = (u_1, u_2, u_3)$

$$\mathbf{curl}\, \boldsymbol{u} = \nabla \times \boldsymbol{u} = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix}$$

Δ is the Laplace operator (aka Laplacian). For scalar distribution φ

$$\Delta\varphi = \partial_1^2\varphi + \partial_2^2\varphi + \partial_3^2\varphi$$

Important relations

$$\operatorname{div} \nabla\varphi = \Delta\varphi$$

$$\operatorname{div} \mathbf{curl}\, \boldsymbol{u} = 0$$

$$\mathbf{curl}\, \nabla\varphi = 0$$

$$\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{u} - \nabla \operatorname{div} \boldsymbol{u} = -\boldsymbol{\Delta u}$$

where the vector Laplacian is

$$\boldsymbol{\Delta u} = \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{pmatrix}$$

Outline

Time dependent Maxwell equations

Unknowns are 4 vector functions (fields / *champ*) with 3 components each

- $\mathscr{E}$ electric field
- $\mathscr{H}$ magnetic field
- $\mathscr{D}$ electric displacement
- $\mathscr{B}$ magnetic induction

Maxwell equations consist of the 4 relations

$$\partial_t \mathscr{B} + \mathbf{curl}\, \mathscr{E} = 0 \tag{1a}$$

$$\operatorname{div} \mathscr{D} = \rho \tag{1b}$$

$$\partial_t \mathscr{D} - \mathbf{curl}\, \mathscr{H} = -\mathscr{J} \tag{1c}$$

$$\operatorname{div} \mathscr{B} = 0 \tag{1d}$$

- (1a) Faraday's law
- (1b) Gauss's law with ρ the scalar charge density
- (1c) Ampère's circuital law, modified by Maxwell, with current density $\mathscr{J}$
- (1d) tells that $\mathscr{B}$ is solenoidal

Time harmonic Maxwell equations

By partial in time Fourier transformation, or because the data $\mathscr{J}$ and ρ are time harmonic, we assume that $\mathscr{E}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{B}$ are time harmonic, i.e. that there exists $\omega \in \mathbb{R}$ such that

$$\mathscr{E}(t,\boldsymbol{x}) = e^{-i\omega t}\boldsymbol{E}(\boldsymbol{x}), \qquad \mathscr{H}(t,\boldsymbol{x}) = e^{-i\omega t}\boldsymbol{H}(\boldsymbol{x}),$$
$$\mathscr{B}(t,\boldsymbol{x}) = e^{-i\omega t}\boldsymbol{B}(\boldsymbol{x}), \qquad \mathscr{D}(t,\boldsymbol{x}) = e^{-i\omega t}\boldsymbol{D}(\boldsymbol{x})$$

Then the 4-equation system becomes

$$\mathbf{curl}\,\boldsymbol{E} - i\omega\boldsymbol{B} = 0 \tag{2a}$$
$$\operatorname{div}\boldsymbol{D} = \rho \tag{2b}$$
$$\mathbf{curl}\,\boldsymbol{H} + i\omega\boldsymbol{D} = \boldsymbol{J} \tag{2c}$$
$$\operatorname{div}\boldsymbol{B} = 0 \tag{2d}$$

Divergence constraints

- Apply div to (2a) $\Longrightarrow i\omega \operatorname{div}\boldsymbol{B} = 0$. Hence (2d) implied if $\omega \neq 0$
- Apply div to (2c) $\Longrightarrow i\omega \operatorname{div}\boldsymbol{D} = \operatorname{div}\boldsymbol{J}$. Hence the relation $i\omega\rho = \operatorname{div}\boldsymbol{J}$

The 4-equation system is not closed.

Constitutive equations for linear media

Then $\boldsymbol{D}$ is proportional to $\boldsymbol{E}$ and $\boldsymbol{B}$ is proportional to $\boldsymbol{H}$

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} \quad \text{and} \quad \boldsymbol{B} = \mu \boldsymbol{H}$$

with coefficients $\varepsilon = \varepsilon(\boldsymbol{x})$ (electric permittivity) and $\mu = \mu(\boldsymbol{x})$ (magnetic permeability) depending on the material property at $\boldsymbol{x}$.
Material coefficients ε and μ can be matrix valued (anisotropic materials).
We consider here isotropic materials for which ε and μ are scalar.
Particular materials

- Vacuum (or free space): $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ [1]
- Dielectric material: ε and μ real, $\varepsilon \geq \varepsilon_0$ and $\mu \geq \mu_0$ for classical materials, possibly negative for metamaterials.
- Conducting material: $\mu \geq \mu_0$ real and ε complex valued, with $\operatorname{Im} \varepsilon = \sigma\omega^{-1}$ where σ is the conductivity.

Globally in $\mathbb{R}^3$, ε and μ are piecewise constant depending on which material occupies the space at each point.

[1] $\varepsilon_0 = 8.854 \times 10^{-12}\ \mathrm{Fm}^{-1}$ and $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{Hm}^{-1}$. Speed of light $c = (\varepsilon_0\mu_0)^{-1/2}$.

Time harmonic Maxwell equations with constitutive laws

Putting all together we obtain

$$\mathbf{curl}\,\boldsymbol{E} - i\omega\mu\boldsymbol{H} = 0 \tag{3a}$$
$$\operatorname{div}\varepsilon\boldsymbol{E} = \rho \tag{3b}$$
$$\mathbf{curl}\,\boldsymbol{H} + i\omega\varepsilon\boldsymbol{E} = \boldsymbol{J} \tag{3c}$$
$$\operatorname{div}\mu\boldsymbol{H} = 0 \tag{3d}$$

Leaving aside the source problem we take $\rho = 0$ and $\boldsymbol{J} = 0$:

$$\mathbf{curl}\,\boldsymbol{E} - i\omega\mu\boldsymbol{H} = 0 \tag{4a}$$
$$\operatorname{div}\varepsilon\boldsymbol{E} = 0 \tag{4b}$$
$$\mathbf{curl}\,\boldsymbol{H} + i\omega\varepsilon\boldsymbol{E} = 0 \tag{4c}$$
$$\operatorname{div}\mu\boldsymbol{H} = 0 \tag{4d}$$

The problem is to find triples $(\omega, \boldsymbol{E}, \boldsymbol{H})$ with $\omega \in \mathbb{C}$, and $(\boldsymbol{E}, \boldsymbol{H}) \neq (0, 0)$ in admissible function spaces

- In $\mathbb{R}^3$, this is the problem of finding scattering resonances.Suitable radiation conditions at infinity have to be imposed. In general $\operatorname{Im}\omega < 0$.
- In bounded domains, combined with suitable boundary conditions, this is the problem of finding cavity resonances. In general $\omega \in \mathbb{R}$.

The cavity problem

An electromagnetic cavity Ω is a bounded region of $\mathbb{R}^3$ that is isolated from an electromagnetic point of view from the outside region $\mathbb{R}^3 \setminus \Omega$.

This is an idealization of a Faraday cage for which we consider that Ω is surrounded by a layer of infinite conductivity σ. Then the electric field $\boldsymbol{E}$ is zero outside Ω and this causes the boundary condition

$$\boldsymbol{E} \times \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega \quad \text{(the tangential component of } \boldsymbol{E} \text{ is 0} \tag{5}$$

Here $\boldsymbol{n}$ is the unitary outward normal field to $\partial\Omega$.

This can be rigorously proved by setting Maxwell equation in a region containing Ω and its surrounding conductive medium and let σ tend to infinity. Going to this limit exhibits the skin effect / *effet de peau* / in conductive media.

Outline

Elimination of magnetic field

Recall equations

$$\mathbf{curl}\,\boldsymbol{E} - i\omega\mu\boldsymbol{H} = 0 \quad \text{in} \quad \Omega \tag{6a}$$
$$\operatorname{div}\varepsilon\boldsymbol{E} = 0 \quad \text{in} \quad \Omega \tag{6b}$$
$$\mathbf{curl}\,\boldsymbol{H} + i\omega\varepsilon\boldsymbol{E} = 0 \quad \text{in} \quad \Omega \tag{6c}$$
$$\operatorname{div}\mu\boldsymbol{H} = 0 \quad \text{in} \quad \Omega \tag{6d}$$
$$\boldsymbol{E} \times \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega \tag{6e}$$

Using (6a) it is tempting to eliminate $\boldsymbol{H}$ by writing: $i\omega\boldsymbol{H} = \frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{E}$ which yields, formally with (6c)

$$\mathbf{curl}\,\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{E} - \omega^2\varepsilon\boldsymbol{E} = 0 \tag{7}$$

Most frequently, one finds (7) in the literature, followed by an integration by parts to find a variational formulation.

We will rather start from the system (6) to find directly the variational formulation, which allows to find variational spaces without doubt.

The space $H(\mathbf{curl};\Omega)$

Assume that $\boldsymbol{E} \in L^2(\Omega)^3$ and $\boldsymbol{H} \in L^2(\Omega)^3$. Then (6c) and (6a) yields

$$\mathbf{curl}\,\boldsymbol{E} \in L^2(\Omega)^3 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{H} \in L^2(\Omega)^3$$

This leads to introduce the space

$$H(\mathbf{curl};\Omega) = \{\boldsymbol{U} \in L^2(\Omega)^3, \quad \mathbf{curl}\,\boldsymbol{U} \in L^2(\Omega)^3\}$$

Lemma [Girault-Raviart, 86]

Let Ω be a bounded Lipschitz domain[a]. Then $\mathscr{C}^\infty(\overline{\Omega})^3$ is dense in $H(\mathbf{curl};\Omega)$.

[a] A Lipschitz domain is a domain that is (after possible rotations) the epigraph of a Lipschitz function in the neighborhood of each of its boundary points.

Consequence: if $\boldsymbol{U} \in H(\mathbf{curl};\Omega)$, the tangential trace $\boldsymbol{U} \times \boldsymbol{n}$ makes sense in $H^{-1/2}(\partial\Omega)^3$ thanks to the identity, valid for any $\Phi \in H^1(\Omega)^3$:

$$\langle \boldsymbol{U} \times \boldsymbol{n}, \Phi \rangle_{H^{-1/2}(\partial\Omega)^3 \mid H^{1/2}(\partial\Omega)^3} = \int_\Omega \boldsymbol{U} \cdot \mathbf{curl}\,\Phi \,\mathrm{d}\boldsymbol{x} - \int_\Omega \mathbf{curl}\,\boldsymbol{U} \cdot \Phi \,\mathrm{d}\boldsymbol{x}$$

The space $H_0(\mathbf{curl};\Omega)$

Then we can introduce the H-curl space with zero tangential traces

$$H_0(\mathbf{curl};\Omega) = \{\boldsymbol{U} \in H(\mathbf{curl};\Omega), \quad \boldsymbol{u} \times \boldsymbol{n}\big|_{\partial\Omega} = 0\}$$

Then

Lemma [Girault-Raviart, 86]

Let Ω be a bounded Lipschitz domain. Then $\mathscr{C}_0^\infty(\Omega)^3$ is dense in $H_0(\mathbf{curl};\Omega)$.

And an important consequence

Lemma

Let Ω be a bounded Lipschitz domain. Then

$$\int_\Omega \boldsymbol{U} \cdot \mathbf{curl}\, \boldsymbol{V}\, \mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}\, \boldsymbol{U} \cdot \boldsymbol{V}\, \mathrm{d}\boldsymbol{x} \quad \forall \boldsymbol{U} \in H_0(\mathbf{curl};\Omega), \ \forall \boldsymbol{V} \in H(\mathbf{curl};\Omega).$$

Towards variational formulation of cavity problem

Recall

$$\mathbf{curl}\,\boldsymbol{E} - i\omega\mu\boldsymbol{H} = 0 \quad \text{in} \quad \Omega \tag{6a}$$

$$\mathbf{curl}\,\boldsymbol{H} + i\omega\varepsilon\boldsymbol{E} = 0 \quad \text{in} \quad \Omega \tag{6c}$$

$$\boldsymbol{E} \times \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega \tag{6e}$$

If $\boldsymbol{E} \in L^2(\Omega)^3$ and $\boldsymbol{H} \in L^2(\Omega)^3$, then $\boldsymbol{E} \in H_0(\mathbf{curl};\Omega)$ and $\boldsymbol{H} \in H(\mathbf{curl};\Omega)$.

Pick a test function $\boldsymbol{E}' \in H_0(\mathbf{curl};\Omega)$. Multiply (6a) by μ^{-1} on the left, take the $\cdot$ product with $\mathbf{curl}\,\boldsymbol{E}'$ on the right, integrate over Ω

$$\int_\Omega \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{E}' - i\omega\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{E}'\right)\mathrm{d}\boldsymbol{x} = 0 \tag{6a'}$$

Multiply (6c) by $i\omega$, take the $\cdot$ product with $\boldsymbol{E}'$ on the right, integrate over Ω

$$\int_\Omega \left(i\omega\,\mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{E}' - \omega^2\varepsilon\,\boldsymbol{E}\cdot\boldsymbol{E}'\right)\mathrm{d}\boldsymbol{x} = 0 \tag{6c'}$$

Add (6a') and (6c'), use the Lemma on previous slide and obtain

$$\boxed{\int_\Omega \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{E}' - \omega^2\varepsilon\,\boldsymbol{E}\cdot\boldsymbol{E}'\right)\mathrm{d}\boldsymbol{x} = 0}$$

Electric spectral problem

Definition

Let Ω be a bounded Lipschitz domain. The electric spectral problem is to find pairs $(\omega, \boldsymbol{E})$ with non-zero $\boldsymbol{E} \in H_0(\mathbf{curl}; \Omega)$, such that

$$\int_\Omega \left(\mu^{-1} \,\mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{E}' - \omega^2 \varepsilon\, \boldsymbol{E} \cdot \boldsymbol{E}'\right) \mathrm{d}\boldsymbol{x} = 0 \quad \forall \boldsymbol{E}' \in H_0(\mathbf{curl}; \Omega) \qquad (8)$$

Many questions arise

1. Can we find solutions?
2. Do solutions correspond to solutions of the cavity problem?
3. Can we discretize (8) by Finite Element Method (Galerkin projection)

We address these questions on a simplifed two-dimensional problem which

1. Encounters the same difficulties as the original 3D problem
2. Has solutions that can be alternatively deduced by solving a scalar equation.

Part II

Traps in Finite element discretization

Outline

4 Toy problem – Bench test

5 Numerical test / *Rien ne va plus*

From 3 to 2 dimensions

- Take ε and μ constant equal to 1.
- Take as domain Ω a 2-dim. polygon (straight sides).

To find the Maxwell cavity problem in Ω in its TE (Transverse Electric) formulation we go back to the 3-dim. formulation, considered in $\Omega \times \mathbb{R}$:

$$\begin{aligned}
\textbf{curl}\, \boldsymbol{E} - i\omega \boldsymbol{H} &= 0 \quad \text{in} \quad \Omega \times \mathbb{R} && \text{(6a)}\\
\operatorname{div} \boldsymbol{E} &= 0 \quad \text{in} \quad \Omega \times \mathbb{R} && \text{(6b)}\\
\textbf{curl}\, \boldsymbol{H} + i\omega \boldsymbol{E} &= 0 \quad \text{in} \quad \Omega \times \mathbb{R} && \text{(6c)}\\
\operatorname{div} \boldsymbol{H} &= 0 \quad \text{in} \quad \Omega \times \mathbb{R} && \text{(6d)}\\
\boldsymbol{E} \times \boldsymbol{n} &= 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R} && \text{(6e)}
\end{aligned}$$

and assume that

- $\boldsymbol{E}$ and $\boldsymbol{H}$ are function of (x_1, x_2) only (no dependence in x_3)
- $E_3 = 0$, $H_1 = H_2 = 0$, i.e.

$$\boldsymbol{E} = \begin{pmatrix} E_1 \\ E_2 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{H} = \begin{pmatrix} 0 \\ 0 \\ H_3 \end{pmatrix}$$

Note that (6d) is already satisfied. We obtain $\cdots / \cdots$

The TE cavity problem

$$\partial_1 E_2 - \partial_2 E_1 - i\omega H_3 = 0 \quad \text{in} \quad \Omega \tag{9a}$$
$$\partial_1 E_1 + \partial_2 E_2 = 0 \quad \text{in} \quad \Omega \tag{9b}$$
$$\partial_2 H_3 + i\omega E_1 = 0 \quad \text{and} \quad -\partial_1 H_3 + i\omega E_2 = 0 \quad \text{in} \quad \Omega \tag{9c}$$
$$E_1 n_2 - E_2 n_1 = 0 \quad \text{on} \quad \partial\Omega \tag{9d}$$

Define the scalar curl (denoted rot) in 2 dimensions as

$$\operatorname{rot} \boldsymbol{U} = \partial_1 U_2 - \partial_2 U_1 \quad \text{for} \quad \boldsymbol{U} = (U_1, U_2)$$

and the spaces $H(\operatorname{rot};\Omega)$ and $H_0(\operatorname{rot};\Omega)$ accordingly.

By the same method as in 3-dim. we find that $\boldsymbol{U} = (E_1, E_2)$ is solution of the

Electric Maxwell spectral problem in 2-dim.

Find pairs $(\omega, \boldsymbol{U})$ with non-zero $\boldsymbol{U} \in H_0(\operatorname{rot};\Omega)$, such that

$$\int_\Omega \Big(\operatorname{rot} \boldsymbol{U} \ \operatorname{rot} \boldsymbol{U}' - \omega^2\, \boldsymbol{U} \cdot \boldsymbol{U}'\Big)\, \mathrm{d}\boldsymbol{x} = 0 \quad \forall \boldsymbol{U}' \in H_0(\operatorname{rot};\Omega) \tag{10}$$

Observe that (9c) implies $\partial_1 H_3$ and $\partial_2 H_3$ are in $L^2(\Omega)$. Hence $H_3 \in H^1(\Omega)$. We find:

Neumann spectral problem

Find pairs (ω, H_3) with non-zero $H_3 \in H^1(\Omega)$, such that

$$\int_\Omega \Big(\nabla H_3 \cdot \nabla H' - \omega^2\, H_3\, H'\Big)\, \mathrm{d}\boldsymbol{x} = 0 \quad \forall H' \in H^1(\Omega) \tag{11}$$

The electric Maxwell spectral problem (rot-rot eigenmodes)

Proposition 1

Let Ω be a 2-dim. simply connected Lipschitz domain.
Let $(\omega, \boldsymbol{U}) \in \mathbb{C} \times H_0(\mathrm{rot};\Omega)$ be a solution of

$$(*) \qquad \int_\Omega \Big(\mathrm{rot}\,\boldsymbol{U}\ \mathrm{rot}\,\boldsymbol{U}' - \omega^2\,\boldsymbol{U}\cdot\boldsymbol{U}'\Big)\,\mathrm{d}\boldsymbol{x} = 0 \quad \forall \boldsymbol{U}' \in H_0(\mathrm{rot};\Omega)$$

1. If $\omega = 0$, then exists a scalar potential φ such that
$$\varphi \in H^1_0(\Omega) \quad \text{and} \quad \nabla\varphi = \boldsymbol{U}$$
Conversely, if $\varphi \in H^1_0(\Omega)$, then $(0, \nabla\varphi)$ solves $(*)$.
2. If $\omega \neq 0$, then div $\boldsymbol{U} = 0$ and exists a scalar potential[a] $\psi \in H^1(\Omega)$ s. t.
$$\psi \in H^1(\Omega) \quad \text{and} \quad \overrightarrow{\mathbf{rot}}\,\psi = \boldsymbol{U}$$
and (ω^2, ψ) is an eigenpair of the Neumann problem
$$(**) \qquad \int_\Omega \Big(\nabla\psi\cdot\nabla\psi' - \omega^2\,\psi\,\psi'\Big)\,\mathrm{d}\boldsymbol{x} = 0 \quad \forall \psi' \in H^1(\Omega)$$
Conversely, if (ω^2, ψ) is an eigenpair $(**)$, then $(\omega, \overrightarrow{\mathbf{rot}}\,\psi)$ solves $(*)$.

[a] $\overrightarrow{\mathbf{rot}}\,\psi$ is the vector curl in 2-dim. : $\overrightarrow{\mathbf{rot}}\,\psi = (\partial_2\psi, -\partial_1\psi)^\perp$

Proof

1. If $\omega = 0$, then rot $\boldsymbol{U} = 0$.
   - As Ω is simply connected, there exists a potential φ such that $\nabla\varphi = \boldsymbol{U}$.
   - Since $\boldsymbol{U} \times \boldsymbol{n} = 0$ on $\partial\Omega$, then φ is constant on $\partial\Omega$.
   - The simple connectedness implies that $\partial\Omega$ has one component, so φ can be chosen in $H^1_0(\Omega)$.
2. If $\omega \neq 0$, choose as test function $\boldsymbol{U}' = \nabla\varphi'$, with $\varphi' \in H^1_0(\Omega)$. Then $(*) \Rightarrow$

$$\int_\Omega \boldsymbol{U} \cdot \nabla\varphi' \, \mathrm{d}\boldsymbol{x} = 0 \quad \forall \varphi' \in H^1_0(\Omega)$$

Therefore, in the sense of duality

$$\left\langle \operatorname{div} \boldsymbol{U} \,,\, \varphi' \right\rangle_{H^{-1}(\Omega)\,|\,H^1_0(\Omega)} = 0 \quad \forall \varphi' \in H^1_0(\Omega)$$

Hence div $\boldsymbol{U} = 0$. This implies the existence of a scalar potential ψ s.t. $\overrightarrow{\textbf{rot}}\,\psi = \boldsymbol{U}$. As

$$\operatorname{rot} \overrightarrow{\textbf{rot}}\,\psi = -\Delta\psi \quad \text{and} \quad \overrightarrow{\textbf{rot}}\,\psi \cdot \overrightarrow{\textbf{rot}}\,\psi' = \nabla\psi \cdot \nabla\psi'$$

$\boldsymbol{U} \in H_0(\mathrm{rot};\Omega) \Longleftrightarrow \psi \in \mathrm{D}(\Delta^{\mathrm{Neu}};\Omega) := \left\{ v \in H^1(\Omega),\ \Delta v \in L^2(\Omega)\ \&\ \partial_n v|_{\partial\Omega} = 0 \right\}$

With the test functions $\boldsymbol{U}' = \overrightarrow{\textbf{rot}}\,\psi'$ for any $\psi' \in \mathrm{D}(\Delta^{\mathrm{Neu}};\Omega)$, $(*)$ implies that ψ satifies

$$(***) \qquad \int_\Omega \Big(\Delta\psi \, \Delta\psi' - \omega^2 \, \nabla\psi \cdot \nabla\psi' \Big) \, \mathrm{d}\boldsymbol{x} = 0 \quad \forall \psi' \in \mathrm{D}(\Delta^{\mathrm{Neu}};\Omega)$$

End of proof

Integrating by parts $(***)$ implies

$$\int_\Omega \left(\Delta\psi\,\Delta\psi' + \omega^2\,\psi\,\Delta\psi'\right) \mathrm{d}\boldsymbol{x} = 0 \quad \forall\psi' \in \mathrm{D}(\Delta^{\mathrm{Neu}};\Omega)$$

i.e.

$$\int_\Omega \left(\Delta\psi + \omega^2\,\psi\right) \Delta\psi' \mathrm{d}\boldsymbol{x} = 0 \quad \forall\psi' \in \mathrm{D}(\Delta^{\mathrm{Neu}};\Omega)$$

Denote by $L^2_\circ(\Omega)$ the space of functions $L^2(\Omega)$ orthogonal to constants on Ω

$$L^2_\circ(\Omega) = \left\{ v \in L^2(\Omega), \quad \int_\Omega v \, \mathrm{d}\boldsymbol{x} = 0 \right\}$$

Now, we can choose $\psi \in L^2_\circ(\Omega)$, and still have $\overrightarrow{\mathbf{rot}}\,\psi = \boldsymbol{U}$. The operator Δ^{Neu}

$$\Delta^{\mathrm{Neu}} : \mathrm{D}(\Delta^{\mathrm{Neu}};\Omega) \longrightarrow L^2_\circ(\Omega) \quad \text{is onto / } \textit{surjectif}$$

Hence

$$\int_\Omega \left(\Delta\psi + \omega^2\,\psi\right) v \mathrm{d}\boldsymbol{x} = 0 \quad \forall v \in L^2_\circ(\Omega)$$

and, since $\Delta\psi + \omega^2\,\psi$ belongs to $L^2_\circ(\Omega)$

$$\Delta\psi + \omega^2\,\psi = 0$$

Finishing the proof is now easy.

The TE cavity problem versus the rot-rot spectral problem

Corollary

Let Ω be a 2-dim. simply connected Lipschitz domain.
The solutions $\big(\omega, (E_1, E_2, H_3)\big)$ of the TE cavity problem (9) are

1. $\omega = 0$ with $E_1 = E_2 = 0$ and H_3 non-zero constant.
2. $\omega \neq 0$ such that ω^2 is an eigenvalue of Δ^{Neu}, the positive Laplace operator with Neumann conditions: $\Delta^{\text{Neu}} = -\Delta$ with operator domain $\mathrm{D}(\Delta^{\text{Neu}}; \Omega)$. Then
$$(E_1, E_2, H_3) = (\overrightarrow{\textbf{rot}}\,\psi, -i\omega\psi)$$
with ψ eigenvector of Δ^{Neu} associated with ω^2.

Remarks on 3-dim. domains

If Ω is a 3-dim. simply connected Lipschitz domain, the solutions of

$$(*) \qquad \int_\Omega \Big(\textbf{curl}\, \boldsymbol{U} \cdot \textbf{curl}\, \boldsymbol{U}' - \omega^2\, \boldsymbol{U} \cdot \boldsymbol{U}'\Big)\, \mathrm{d}\boldsymbol{x} = 0 \quad \forall \boldsymbol{U}' \in H_0(\textbf{curl}; \Omega)$$

are related to the cavity problem in a similar way:

$$\omega = 0 \Longrightarrow \operatorname{div} \boldsymbol{U} \neq 0 \quad \text{and} \quad \omega \neq 0 \Longrightarrow \operatorname{div} \boldsymbol{U} = 0$$

and the solutions of the cavity problem can be deduced from those of $(*)$ when $\omega \neq 0$.
But, in 3-dim. there is no scalar potential in general.
The 2-dim. serves as a bench test / *banc d'essai* / for 3-dim.

Outline

The square

Consider $\Omega = (0, \pi) \times (0, \pi)$.
By separation of variables, we find that the eigenpairs of Δ^{Neu} are

$$\begin{cases} \omega^2 = j_1^2 + j_2^2 \\ \psi(x_1, x_2) = \cos(j_1 x_1)\cos(j_2 x_2) \end{cases} \qquad \text{for any integers} \quad j_1,\ j_2 \in \{0, 1, 2, \dots\}$$

Using Proposition 1, this implies that the solutions of the electric Maxwell spectral problem

$$(*) \qquad \int_\Omega \operatorname{rot} \boldsymbol{U}\ \operatorname{rot} \boldsymbol{U}'\ \mathrm{d}\boldsymbol{x} = \omega^2 \int_\Omega \boldsymbol{U} \cdot \boldsymbol{U}'\ \mathrm{d}\boldsymbol{x} \quad \forall \boldsymbol{U}' \in H_0(\operatorname{rot}; \Omega)$$

correspond to eigenvalues ω^2 equal to

1. 0 (with infinite multiplicity)
2. 1, 1, 2, 4, 4, 5, 5, 8, 9, 9, 10, 10, 13, 13, ...
   (with repetition according to multiplicity)

Finite element method

a Let a be bilinear (or sesquilinear) form well defined on a product space $V \times V$

$$a(\boldsymbol{u}, \boldsymbol{v}) = \sum_i \sum_j \sum_{|\alpha| \leq 1} \sum_{|\beta| \leq 1} \int_\Omega \Big(a_{\alpha\beta}\, \partial^\alpha u_i\, \partial^\beta v_j \Big)\, \mathrm{d}\boldsymbol{x}$$

Spectral problem associated with a: Find pairs $(\lambda, \boldsymbol{u})$, with $0 \neq \boldsymbol{u} \in V$ s. t.

$$(\dagger) \qquad a(\boldsymbol{u}, \boldsymbol{v}) = \lambda \langle \boldsymbol{u}, \boldsymbol{v} \rangle_{L^2(\Omega)|L^2(\Omega)} \quad \forall \boldsymbol{v} \in V$$

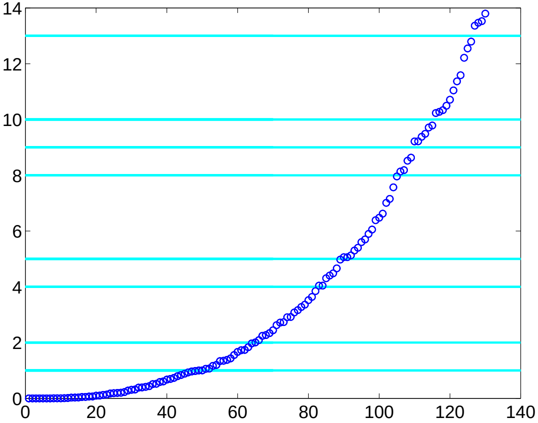
b Let $\widetilde{V}$ be a finite dimensional subspace of V.
Galerkin projection of problem $(\dagger)$: Find pairs $(\tilde{\lambda}, \widetilde{\boldsymbol{u}})$, with $0 \neq \widetilde{\boldsymbol{u}} \in \widetilde{V}$ s. t.

$$(\ddagger) \qquad a(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{v}}) = \tilde{\lambda} \langle \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{v}} \rangle_{L^2(\Omega)|L^2(\Omega)} \quad \forall \widetilde{\boldsymbol{v}} \in \widetilde{V}$$

c The Finite Element Method [FEM] consists in constructing and implementing suitable spaces $\widetilde{V}$. In general, they are based on a mesh of Ω (subdivision into triangular or quadrilateral elements in 2-dim.) and piecewise (mapped-)polynomials in each element of the mesh.

Analysis of FEM: proving (or disproving) convergence when $\dim \widetilde{V} \to \infty$.

Let's go / *On y va*



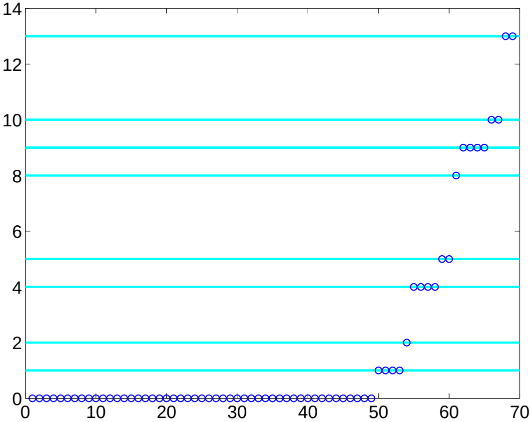
Triangular mesh
$\sim$ 450 Lagrange elements of degree 1

Sort computed eigenvalues by increasing order

$$\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots$$

- Abscissa: rank of computed eigenvalue $1 \leq n \leq 140$
- Ordinates: value of $\tilde{\lambda}_n$
- Horizontal lines = exact values for λ_j

Another try / *Un autre essai*



Mesh with one square element of degree 8 :

$$\widetilde{\boldsymbol{u}} = (\tilde{u}_1, \tilde{u}_2)$$

with

$$\widetilde{u}_1,\ \widetilde{u}_2 \in \mathbb{Q}_8 = \mathbb{P}_8 \otimes \mathbb{P}_8$$

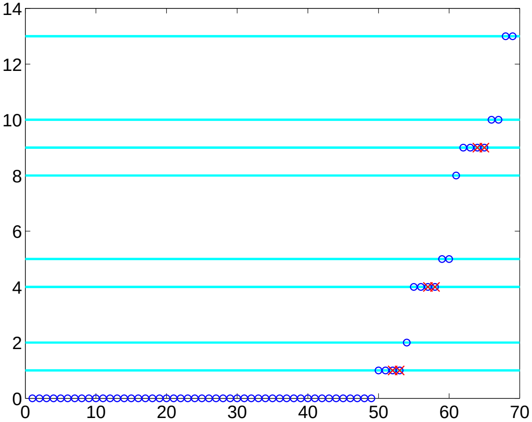
Sort computed eigenvalues by increasing order

$$\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots$$

Il y a encore un problème

- Abscissa: rank of computed eigenvalue $1 \leq n \leq 70$
- Ordinates: value of $\tilde{\lambda}_n$
- Horizontal lines = exact values for λ_j

Another try / *Un autre essai*



Mesh with one square element of degree 8 :

$$\widetilde{\boldsymbol{u}} = (\tilde{u}_1, \tilde{u}_2)$$

with

$$\widetilde{u}_1,\ \widetilde{u}_2 \in \mathbb{Q}_8 = \mathbb{P}_8 \otimes \mathbb{P}_8$$

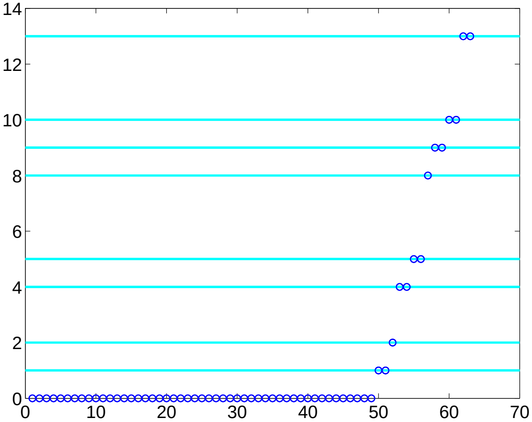
Sort computed eigenvalues by increasing order

$$\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots$$

Il y a encore un problème

- Abscissa: rank of computed eigenvalue $1 \leq n \leq 70$
- Ordinates: value of $\tilde{\lambda}_n$
- Horizontal lines = exact values for λ_j

Try something else (breaking identity between components)



Mesh with one square element of mixed degrees 7&8 :

$$\widetilde{\boldsymbol{u}} = (\tilde{u}_1, \tilde{u}_2)$$

with

$$\widetilde{u}_1 \in \mathbb{P}_7 \otimes \mathbb{P}_8 \quad \widetilde{u}_2 \in \mathbb{P}_8 \otimes \mathbb{P}_7$$

Sort computed eigenvalues by increasing order

$$\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots$$

Il n'y a plus de problème

This is an "edge element" cf lecture by Martin.

- Abscissa: rank of computed eigenvalue $1 \leq n \leq 70$
- Ordinates: value of $\tilde{\lambda}_n$
- Horizontal lines = exact values for λ_j

Part III

Elliptic regularization: bad and good methods

Outline

6 **Standard regularization**

7 Non-convex corners

8 Weighted regularization

Rappels on Dirichlet and Neumann scalar Laplace operators

Let Ω be a bounded Lipschitz domain. Denote by a_∇ the bilinear form

$$a_\nabla(u,v) := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}\boldsymbol{x}, \quad \text{for} \quad u,\ v \in H^1(\Omega).$$

ⓐ The positive Dirichlet Laplacian Δ^{Dir} is defined from $H^1_0(\Omega)$ into its dual space $H^{-1}(\Omega)$ by

$$\Delta^{\mathrm{Dir}}(u) = F \quad \text{with} \quad \langle F, v\rangle_{H^{-1}(\Omega)\,|\,H^1_0(\Omega)} := a_\nabla(u,v)$$

NB: Since $H^{-1}(\Omega)$ is a space of distributions in Ω, we have $F = -\Delta u$.

Since a_∇ is coercive on $H^1_0(\Omega)$, Δ^{Dir} is invertible with compact inverse. The domain (in the sense of domain of unbounded operators) is

$$\mathrm{D}(\Delta^{\mathrm{Dir}};\Omega) = \{v \in H^1_0(\Omega), \quad F \in L^2(\Omega)\}$$

The operator Δ^{Dir} defines an isomorphism from $\mathrm{D}(\Delta^{\mathrm{Dir}};\Omega)$ onto $L^2(\Omega)$.

The spectrum of Δ^{Dir} is discrete and formed by a sequence of positive eigenvalues λ_n^{Dir} that tends to infinity as $n \to +\infty$.

Rappels on Dirichlet and Neumann scalar Laplace operators

Let Ω be a bounded Lipschitz domain. Denote by a_∇ the bilinear form

$$a_\nabla(u,v) := \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x}, \quad \text{for} \quad u,\ v \in H^1(\Omega).$$

The non-negative Neumann Laplacian Δ^{Neu} is defined from $H^1(\Omega)$ into its dual space $H^1(\Omega)'$ by

$$\Delta^{\mathrm{Neu}}(u) = F \quad \text{with} \quad \langle F, v \rangle_{H^1(\Omega)' \,|\, H^1(\Omega)} := a_\nabla(u,v)$$

NB: Since $H^1(\Omega)'$ is not a space of distributions in Ω, it may happen that $F \neq -\Delta u$

Since $a_\nabla + \mathrm{Id}$ is coercive on $H^1(\Omega)$, $\Delta^{\mathrm{Neu}} + \mathrm{Id}$ is invertible with compact inverse.

$$\mathrm{D}(\Delta^{\mathrm{Neu}};\Omega) = \{v \in H^1(\Omega), \quad F \in L^2(\Omega)\}$$

$F \in L^2(\Omega)$ means that there exists a function $f \in L^2(\Omega)$ such that $\langle F, v \rangle = \int_\Omega f\, v \, \mathrm{d}\boldsymbol{x}$.
We deduce that

$$\mathrm{D}(\Delta^{\mathrm{Neu}};\Omega) = \{v \in H^1(\Omega), \quad \Delta v \in L^2(\Omega) \ \text{ and } \ \partial_n v\big|_{\partial\Omega} = 0\}$$

The spectrum of Δ^{Neu} is discrete and formed by a sequence of non-negative eigenvalues / *valeurs propres positives ou nulles* / $\lambda_n^{\mathrm{Neu}} \to +\infty$ as $n \to +\infty$.

Blowing up the kernel / *Exploser le noyau* /of the rot-rot operator

Recall that we want to compute FEM approximations of the eigenpairs $(\lambda, \boldsymbol{U})$ with $\lambda = \omega^2$ and non-zero $\boldsymbol{U} \in H_0(\mathrm{rot}; \Omega)$, solution of

$$(*) \qquad \int_\Omega \mathrm{rot}\, \boldsymbol{U} \;\mathrm{rot}\, \boldsymbol{U}' \;\mathrm{d}\boldsymbol{x} = \lambda \int_\Omega \boldsymbol{U} \cdot \boldsymbol{U}' \;\mathrm{d}\boldsymbol{x} \qquad \forall \boldsymbol{U}' \in H_0(\mathrm{rot}; \Omega)$$

The “standard” approximation theory [Osborn, 75] [Babuška-Osborn, 91] applies if there is a compact embedding of the space V corresponding to the left hand side of $(*)$ into the space H corresponding to its right hand side. But in our case

$$V = H_0(\mathrm{rot}; \Omega) \quad \text{and} \quad H = L^2(\Omega)^2$$

The embedding $H_0(\mathrm{rot}; \Omega) \longrightarrow L^2(\Omega)^2$ is not compact. The symptom is the infinite dimensional kernel.

Since we are interested by the divergence-free solutions of $(*)$, a natural idea is to regularize the rot-rot bilinear form by the div-div form.

Notation

For any chosen $s > 0$ / *pour tout s fixé*, / set

$$a[s](\boldsymbol{U}, \boldsymbol{U}') = \int_\Omega \big(\mathrm{rot}\, \boldsymbol{U} \;\mathrm{rot}\, \boldsymbol{U}' + s\, \mathrm{div}\, \boldsymbol{U} \;\mathrm{div}\, \boldsymbol{U}'\big) \;\mathrm{d}\boldsymbol{x}$$

well defined of the new space

$$X_N(\Omega) = \{\boldsymbol{V} \in H_0(\mathrm{rot}; \Omega), \quad \mathrm{div}\, \boldsymbol{V} \in L^2(\Omega)\}$$

The divergence

Lemma

Let Ω be a 2-dim. Lipschitz domain. Choose $s > 0$. Let $(\lambda, \boldsymbol{U}) \in \mathbb{R} \times X_N(\Omega)$ be an eigenpair of $a[s]$

$$(*) \qquad a[s](\boldsymbol{U}, \boldsymbol{U}') = \lambda \int_\Omega \boldsymbol{U} \cdot \boldsymbol{U}' \, \mathrm{d}\boldsymbol{x} \quad \forall \boldsymbol{U}' \in X_N(\Omega)$$

Then $\operatorname{div} \boldsymbol{U} \in H^1_0(\Omega)$ and [❶ or ❷ holds]

❶ $\operatorname{div} \boldsymbol{U} =: \Phi$ is an eigenvector of $s\,\Delta^{\mathrm{Dir}}$ with eigenvalue λ:

$$\Phi \in H^1_0(\Omega) \quad \text{solves} \quad s \int_\Omega \nabla\Phi \cdot \nabla\Phi' \, \mathrm{d}\boldsymbol{x} = \lambda \int_\Omega \Phi\, \Phi' \, \mathrm{d}\boldsymbol{x} \quad \forall \Phi' \in H^1_0(\Omega)$$

❷ $\operatorname{div} \boldsymbol{U} = 0$.

Proof

Set $\Phi := \operatorname{div} \boldsymbol{U}$. Choose as test function $\boldsymbol{U}' = \nabla\Phi'$ with

$$\Phi' \in \mathrm{D}(\Delta^{\mathrm{Dir}}; \Omega) = \left\{ v \in H^1_0(\Omega), \quad \Delta v \in L^2(\Omega) \right\}.$$

Then $\boldsymbol{U}' = \nabla\Phi'$ belongs to $X_N(\Omega)$ since:

- $\Phi' \in H^1(\Omega) \Longrightarrow \boldsymbol{U}' \in L^2(\Omega)$
- $\Phi'\big|_{\partial\Omega} = 0 \Longrightarrow \boldsymbol{U}' \times \boldsymbol{n}\big|_{\partial\Omega} = 0$
- $\Delta\Phi' \in L^2(\Omega) \Longrightarrow \operatorname{div} \boldsymbol{U}' \in L^2(\Omega)$

The divergence: Proof of Lemma

Set $\Phi := \operatorname{div} \boldsymbol{U}$. Choose as test function $\boldsymbol{U}' = \nabla\Phi'$ with $\Phi' \in \mathrm{D}(\Delta^{\mathrm{Dir}};\Omega)$. Then $(*) \Rightarrow$

$$s \int_\Omega \Phi \ \operatorname{div} \nabla\Phi' \,\mathrm{d}\boldsymbol{x} = \lambda \int_\Omega \boldsymbol{U} \cdot \nabla\Phi' \,\mathrm{d}\boldsymbol{x} \qquad \forall\Phi' \in \mathrm{D}(\Delta^{\mathrm{Dir}};\Omega)$$

Observe that

- $\operatorname{div} \nabla\Phi' = -\Delta^{\mathrm{Dir}}\Phi'$
- $\displaystyle\int_\Omega \boldsymbol{U} \cdot \nabla\Phi' \,\mathrm{d}\boldsymbol{x} = -\langle \operatorname{div} \boldsymbol{U}, \Phi' \rangle_{H^{-1}(\Omega)\,|\,H^1_0(\Omega)} = -\lambda \int_\Omega \Phi\ \Phi' \,\mathrm{d}\boldsymbol{x}$

Therefore, we have the orthogonality condition

$$\int_\Omega \Phi \left(s\,\Delta^{\mathrm{Dir}}\Phi' - \lambda\Phi' \right) \mathrm{d}\boldsymbol{x} = 0 \quad \forall\Phi' \in \mathrm{D}(\Delta^{\mathrm{Dir}};\Omega)$$

In other words $\operatorname{div} \boldsymbol{U} = \Phi$ belongs to the orthogonal of the range of the self-adjoint operator $s\,\Delta^{\mathrm{Dir}} - \lambda\,\mathrm{Id}$:

$$\Phi \in \left(\mathrm{range}(s\,\Delta^{\mathrm{Dir}} - \lambda\,\mathrm{Id}) \right)^{\perp}$$

Then 1 or 2 holds

1. Φ is a non-zero element in the kernel of $s\,\Delta^{\mathrm{Dir}} - \lambda\,\mathrm{Id}$, i.e. is an eigenvector of $s\,\Delta^{\mathrm{Dir}}$ with eigenvalue λ.
2. $\Phi = 0$.

The scalar rot

We have a similar statement concerning the scalar rot of $\boldsymbol{U}$:

Lemma

Let Ω be a 2-dim. Lipschitz domain. Choose $s > 0$.
Let $(\lambda, \boldsymbol{U}) \in \mathbb{R} \times X_N(\Omega)$ be an eigenpair of $a[s]$

$$(*) \qquad a[s](\boldsymbol{U}, \boldsymbol{U}') = \lambda \int_\Omega \boldsymbol{U} \cdot \boldsymbol{U}' \,\mathrm{d}\boldsymbol{x} \quad \forall \boldsymbol{U}' \in X_N(\Omega)$$

Then $\operatorname{rot} \boldsymbol{U} \in H^1(\Omega)$ and [❶ or ❷ holds]

❶ $\operatorname{rot} \boldsymbol{U} = 0$.

❷ $\operatorname{rot} \boldsymbol{U} =: \Psi$ is an eigenvector of Δ^{Neu} with eigenvalue λ:

$$\Psi \in H^1(\Omega) \quad \text{solves} \quad \int_\Omega \nabla\Psi \cdot \nabla\Psi' \,\mathrm{d}\boldsymbol{x} = \lambda \int_\Omega \Psi\, \Psi' \,\mathrm{d}\boldsymbol{x} \quad \forall \Psi' \in H^1(\Omega)$$

Proof
Similar as before. Now the test functions are $\boldsymbol{U}' = \overrightarrow{\mathbf{rot}}\, \Psi'$ with any $\Phi' \in \mathrm{D}(\Delta^{\mathrm{Neu}}; \Omega)$.

Spectrum of the regularized form $a[s]$

Theorem

Let Ω be a 2-dim. simply connedted Lipschitz domain. Choose $s > 0$.

- Let $(\lambda_n^{\mathrm{Dir}}, \Phi_n^{\mathrm{Dir}})_{n\geq 1}$ be a complete system of eigenpairs of Δ^{Dir}
- Let $(\lambda_n^{\mathrm{Neu}}, \Psi_n^{\mathrm{Neu}})_{n\geq 0}$ be a complete system of eigenpairs of Δ^{Neu}, with $\lambda_0^{\mathrm{Neu}} = 0$ and $\Psi_0^{\mathrm{Neu}} = 1$

Then a complete system of eigenpairs for $a[s]$ is given by the union of

$$\left(s\lambda_n^{\mathrm{Dir}}, \boldsymbol{U}_n^{\mathrm{Div}}\right)_{n\geq 1} \quad \text{and} \quad \left(\lambda_n^{\mathrm{Neu}}, \boldsymbol{U}_n^{\mathrm{Max}}\right)_{n\geq 1}$$

where

1. $\operatorname{rot} \boldsymbol{U}_n^{\mathrm{Div}} = 0$ and $\operatorname{div} \boldsymbol{U}_n^{\mathrm{Div}} = \Phi_n^{\mathrm{Dir}}$
2. $\operatorname{div} \boldsymbol{U}_n^{\mathrm{Max}} = 0$ and $\operatorname{rot} \boldsymbol{U}_n^{\mathrm{Max}} = \Phi_n^{\mathrm{Neu}}$

Proof. It suffices to set

$$\boldsymbol{U}_n^{\mathrm{Div}} = -\frac{1}{\lambda_n^{\mathrm{Dir}}} \nabla \Phi_n^{\mathrm{Dir}} \quad \text{and} \quad \boldsymbol{U}_n^{\mathrm{Max}} = \frac{1}{\lambda_n^{\mathrm{Neu}}} \overrightarrow{\mathbf{rot}}\, \Phi_n^{\mathrm{Dir}}$$

Since Ω is simply connected, there is no non-zero field $\boldsymbol{U} \in X_N(\Omega)$ such that $\operatorname{div} \boldsymbol{U} = \operatorname{rot} \boldsymbol{U} = 0$.

Spectrum of $a[s]$ and Maxwell spectral problem

The spectrum of $a[s]$: $\lambda \in \mathbb{R}$, $\boldsymbol{U} \in X_N(\Omega)$

$$(*) \quad \int_\Omega \left(\operatorname{rot} \boldsymbol{U} \ \operatorname{rot} \boldsymbol{U}' + s \operatorname{div} \boldsymbol{U} \ \operatorname{div} \boldsymbol{U}'\right) \, \mathrm{d}\boldsymbol{x} = \lambda[s] \int_\Omega \boldsymbol{U} \cdot \boldsymbol{U}' \, \mathrm{d}\boldsymbol{x} \quad \forall \boldsymbol{U}' \in X_N(\Omega)$$

has clearly two well separated parts:

- A part that depends linearly of s and with curl-free eigenvectors
- A part independent of s with divergence free eigenvectors. This is the spectrum we are looking for.

How to distinguish them in numerical computations?

Two techniques:

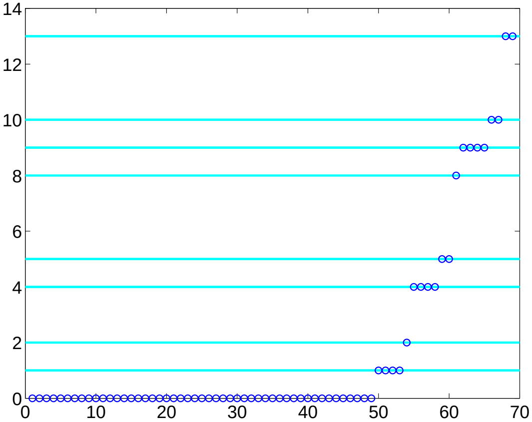
- Calculate the ratio

$$\tau(\widetilde{\boldsymbol{U}}) = \frac{\|\operatorname{rot} \widetilde{\boldsymbol{U}}\|^2}{s\|\operatorname{div} \widetilde{\boldsymbol{U}}\|^2}$$

  We expect large values for approximation of divergence free eigenvectors and small values for the others.

- Calculate eigenvalues for several different values of s

Spectrum of $a[s]$ on the square, $s = 0$



Mesh with one square element of degree 8 :

$$\widetilde{\boldsymbol{u}} = (\tilde{u}_1, \tilde{u}_2) \in (\mathbb{Q}_8)^2$$

with b.c. $\widetilde{\boldsymbol{u}} \times \boldsymbol{n}\big|_{\partial\Omega} = 0$

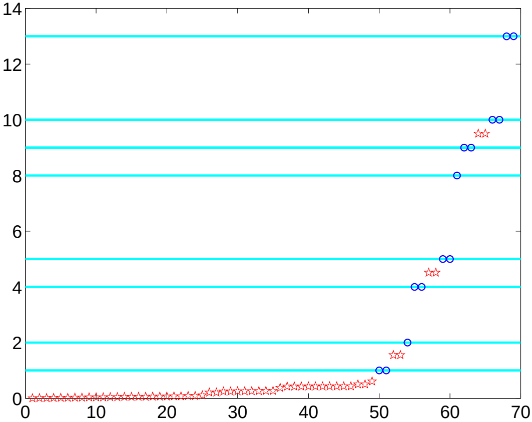
Sort computed eigenvalues by increasing order

$$\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots$$

Extra multiplicities

- Abscissa: rank of computed eigenvalue $1 \leq n \leq 70$
- Ordinates: value of $\tilde{\lambda}_n$
- Horizontal lines = exact values for λ_j

Spectrum of $a[s]$, $s = 0.002$ on the square



Mesh with one square element of degree 8 :

$$\widetilde{\boldsymbol{u}} = (\tilde{u}_1, \tilde{u}_2) \in (\mathbb{Q}_8)^2$$

with b.c. $\widetilde{\boldsymbol{u}} \times \boldsymbol{n}\big|_{\partial\Omega} = 0$

Sort computed eigenvalues by increasing order

$$\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots$$

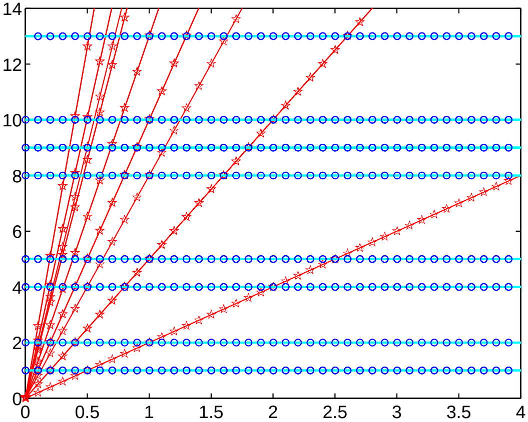
$$\tau(\widetilde{\boldsymbol{u}}) = \| \operatorname{rot} \widetilde{\boldsymbol{u}}\|^2 (s\| \operatorname{div} \widetilde{\boldsymbol{u}}\|^2)^{-1}$$

$\star$ when τ small $\circ$ when τ large

Le problème semble disparaître

- Abscissa: rank of computed eigenvalue $1 \leq n \leq 70$
- Ordinates: value of $\tilde{\lambda}_n$
- Horizontal lines = exact values for λ_j

Spectrum of $a[s]$ on the square, dependence in s



Mesh with one square element of degree 8 :

$$\widetilde{\boldsymbol{u}} = (\tilde{u}_1, \tilde{u}_2) \in (\mathbb{Q}_8)^2$$

with b.c. $\widetilde{\boldsymbol{u}} \times \boldsymbol{n}\big|_{\partial\Omega} = 0$

Sort computed eigenvalues by increasing order

$$\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots$$

$$\tau(\widetilde{\boldsymbol{u}}) = \|\operatorname{rot}\widetilde{\boldsymbol{u}}\|^2 (s\|\operatorname{div}\widetilde{\boldsymbol{u}}\|^2)^{-1}$$

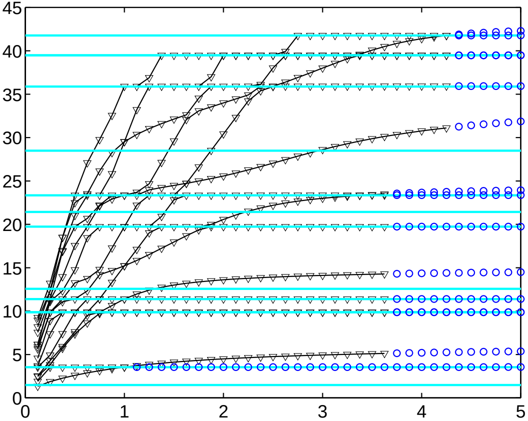
$\star$ when τ small $\circ$ when τ large

Le problème a disparu

- Abscissa: Value of s
- Ordinates: all smallest values of $\tilde{\lambda}_n \leq 14$, $n = 1, 2, 3, \ldots$
- Horizontal lines = exact values for λ_j

Outline

Spectrum of $a[s]$ on a L-shape domain, dependence in s



$$\Omega = (-1,1)^2 \setminus [0,1] \times [-1,0]$$

Mesh with 9 quadrilateral elements of degree 10 :

$$\widetilde{\boldsymbol{u}} \in (\mathbb{Q}_{10})^2$$

with b.c. $\widetilde{\boldsymbol{u}} \times \boldsymbol{n}\big|_{\partial\Omega} = 0$

Sort computed eigenvalues by increasing order

$$\tilde\lambda_1 \le \tilde\lambda_2 \le \cdots$$

$$\tau(\widetilde{\boldsymbol{u}}) = \|\operatorname{rot}\widetilde{\boldsymbol{u}}\|^2 (s\|\operatorname{div}\widetilde{\boldsymbol{u}}\|^2)^{-1}$$

$\star$ when τ small $\circ$ when τ large

Patatras !

- Abscissa: Value of s
- Ordinates: all smallest values of $\tilde\lambda_n \le 45$, $n = 1, 2, 3, \ldots$
- Horizontal lines = values close to exact for λ_j (by computing Neumann eigenvalues)

Spectrum of $a[s]$ on a L-shape domain: Interpretation

We observe

- One (large) half of eigenvalues seems to be correctly approximated
- The other (smaller) half is completely missed and replaced by something else that does not have a clear behavior in s (neither linear nor constant).
- The situation does not improve if we increase the polynomial degree or the density of the mesh (or both)

The diagnosis is that

$$\boxed{\text{We converge towards something that we don't expect}}$$

What? Why?

Spectrum of $a[s]$ on a L-shape domain: Explanation

Recall that

$$X_N(\Omega) = \{\boldsymbol{V} \in H_0(\mathrm{rot};\Omega), \quad \mathrm{div}\, \boldsymbol{V} \in L^2(\Omega)\}$$

Denote by $H_N(\Omega)$ the space

$$H_N(\Omega) = H_1(\Omega)^2 \cap X_N(\Omega) = \{\boldsymbol{V} \in H_1(\Omega)^2, \quad \boldsymbol{V} \times \boldsymbol{n}\big|_{\partial\Omega} = 0\}$$

The explanation is the conjunction of three facts:

1. In L-shape domain Ω, $H_N(\Omega)$ is strictly smaller that $X_N(\Omega)$. Moreover, a large part of eigenvectors $\boldsymbol{U}_n^{\mathrm{Div}}$ and $\boldsymbol{U}_n^{\mathrm{Max}}$ do not belong to $H_N(\Omega)$
2. The discrete Finite Element spaces are contained in $H_N(\Omega)$
3. $H_N(\Omega)$ is closed for the topology of $X_N(\Omega)$

Conclusion: A large part of the eigenvectors of $a[s]$ cannot be approximated by a plain Finire Element discretization in $X_N(\Omega)$.

Let us explain each point in moe detail.

Spectrum of $a[s]$ on a L-shape domain. Point 1

There exists one corner singularity $\boldsymbol{S} = \nabla \Phi_{\text{sing}}$ s. t. [Birman-Solomyak 87]

$$X_N(\Omega) = H_N(\Omega) \oplus \langle \boldsymbol{S} \rangle$$

where $\Phi_{\text{sing}} \in \mathrm{D}(\Delta^{\text{Dir}}; \Omega)$ but $\Phi_{\text{sing}} \notin H^2(\Omega)$. In polar coordinates (r, θ)

$$\Phi_{\text{sing}}(\boldsymbol{x}) = \chi(r)\, r^{2/3} \sin(\frac{2\theta}{3})$$

where χ is a smooth function equal to 1 if $r < \frac{1}{4}$ and to 0 if $r > \frac{1}{2}$. We have

$$\mathrm{D}(\Delta^{\text{Dir}}; \Omega) = (H^2 \cap H^1_0)(\Omega) \oplus \langle \Phi_{\text{sing}} \rangle$$

In fact, since we are in 2-dim.

$$\boldsymbol{S} = \nabla \Phi_{\text{sing}} = \overrightarrow{\textbf{rot}}\, \Psi_{\text{sing}}$$

where $\Psi_{\text{sing}} \in \mathrm{D}(\Delta^{\text{Neu}}; \Omega)$ but $\Psi_{\text{sing}} \notin H^2(\Omega)$. Note $\Psi_{\text{sing}} = \chi(r)\, r^{2/3} \cos(\frac{2\theta}{3})$

Almost all eigenvectors Ψ_n^{Neu} of Δ^{Neu} that are even with respect to the diagonal $x_1 + x_2 = 0$ “contain” this singularity, i.e.

$$\Psi_n^{\text{Neu}} - c_n \Psi_{\text{sing}} \in H^2(\Omega) \text{ for some coefficient } c_n \neq 0$$

Spectrum of $a[s]$ on a L-shape domain. Points 2 and 3

2 The FEM space are made of functions $\widetilde{\boldsymbol{u}}$ that are

- a piecewise polynomials
- b in the space $X_N(\Omega)$.

We observe

- rot $\widetilde{\boldsymbol{u}}$ is in $L^2(\Omega) \Longrightarrow$ no tangential jump for $\widetilde{\boldsymbol{u}}$ between two elements.
- div $\widetilde{\boldsymbol{u}}$ is in $L^2(\Omega) \Longrightarrow$ no normal jump for $\widetilde{\boldsymbol{u}}$ between two elements.
- Finally, both components of $\widetilde{\boldsymbol{u}}$ are continuous over Ω.
- Therefore $\widetilde{\boldsymbol{u}}$ belongs to H_N

3 A sequence $\boldsymbol{u}_m \in H_N(\Omega)$, $m \geq 1$, that is converging for the topology of $X_N(\Omega)$ will never converge to a limit outside $H_N(\Omega)$ by virtue of

Theorem [Costabel, 91] [Costabel-Dauge, 99]

Let Ω be a Lipschitz polygon.
The space $H_N(\Omega)$ is a closed subspace in $X_N(\Omega)$.

$\Longrightarrow$ In L-shape, instead of the Maxwell spectral problem, we are solving a Lamé system with elasticity coefficients depending on s

Outline

Introduction of a weight

- Let Ω be a polygon with one non-convex corner $\boldsymbol{c}$ of opening $\omega > \pi$
- This applies to the L-shape domain Ω with its non-convex corner at the origin.
- Let $r = |\boldsymbol{x} - \boldsymbol{c}|$ be the distance function to the non-convex corner $\boldsymbol{c}$.
- Choose a number $\gamma \in [0, 1]$. This will be the exponent of a weight function.

Notation

For any chosen $s > 0$ set

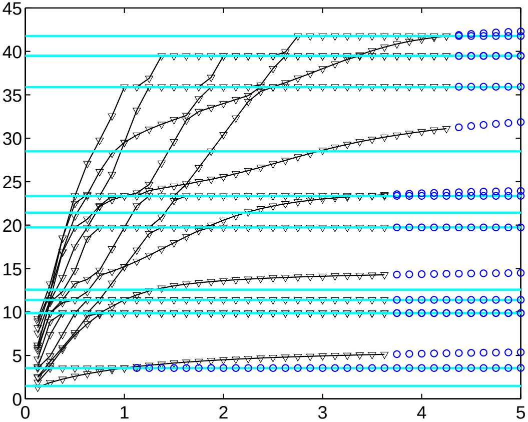
$$a_\gamma[s](\boldsymbol{U}, \boldsymbol{U}') = \int_\Omega \Big(\operatorname{rot} \boldsymbol{U} \ \operatorname{rot} \boldsymbol{U}' + s\, r^{2\gamma} \operatorname{div} \boldsymbol{U} \ \operatorname{div} \boldsymbol{U}'\Big) \ \mathrm{d}\boldsymbol{x}$$

well defined of the new space

$$X_N^\gamma(\Omega) = \{\boldsymbol{V} \in H_0(\operatorname{rot}; \Omega), \quad r^\gamma \operatorname{div} \boldsymbol{V} \in L^2(\Omega)\}$$

If $\gamma > 0$, the norm in the divergence is relaxed (the norm is smaller than without weight)

Spectrum of $a_\gamma[s]$ on a L-shape domain, $\gamma = 0$



$$\boxed{\Omega = (-1,1)^2 \setminus [0,1] \times [-1,0]}$$

Mesh with 9 quadrilateral elements of degree 10 :

$$\widetilde{\boldsymbol{u}} \in (\mathbb{Q}_{10})^2$$

with b.c. $\widetilde{\boldsymbol{u}} \times \boldsymbol{n}\big|_{\partial\Omega} = 0$

Sort computed eigenvalues by increasing order

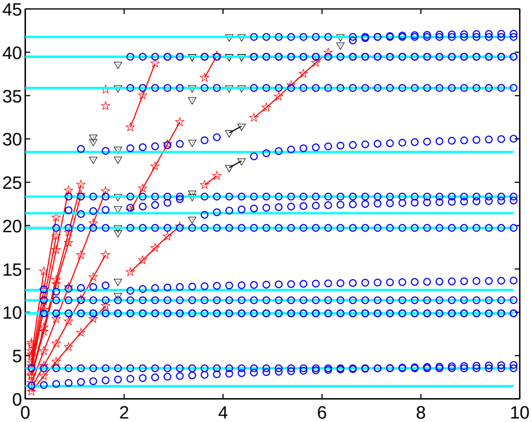
$$\tilde\lambda_1 \leq \tilde\lambda_2 \leq \cdots$$

$\tau(\widetilde{\boldsymbol{u}}) = \|\operatorname{rot} \widetilde{\boldsymbol{u}}\|^2 (s\|\operatorname{div} \widetilde{\boldsymbol{u}}\|^2)^{-1}$

$\star$ when τ small $\circ$ when τ large

- Abscissa: Value of s
- Ordinates: all smallest values of $\tilde\lambda_n \leq 14$, $n = 1, 2, 3, \ldots$
- Horizontal lines = values close to exact for λ_j (by computing Neumann eigenvalues)

Spectrum of $a_\gamma[s]$ on a L-shape domain, $\gamma = 0.35$



$$\Omega = (-1,1)^2 \setminus [0,1] \times [-1,0]$$

Mesh with 9 quadrilateral elements of degree 10 :

$$\widetilde{\boldsymbol{u}} \in (\mathbb{Q}_{10})^2$$

with b.c. $\widetilde{\boldsymbol{u}} \times \boldsymbol{n}\big|_{\partial\Omega} = 0$

Sort computed eigenvalues by increasing order

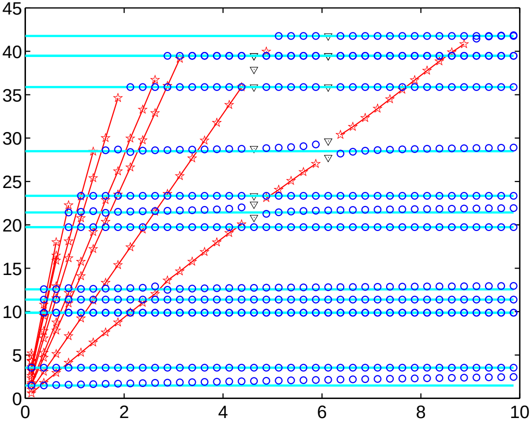
$$\tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \cdots$$

$$\tau(\widetilde{\boldsymbol{u}}) = \|\operatorname{rot}\widetilde{\boldsymbol{u}}\|^2 (s\|\operatorname{div}\widetilde{\boldsymbol{u}}\|^2)^{-1}$$

$\star$ when τ small $\circ$ when τ large

- Abscissa: Value of s
- Ordinates: all smallest values of $\tilde{\lambda}_n \le 14$, $n = 1, 2, 3, \ldots$
- Horizontal lines = values close to exact for λ_j (by computing Neumann eigenvalues)

Spectrum of $a_\gamma[s]$ on a L-shape domain, $\gamma = 0.5$



$$\boxed{\Omega = (-1,1)^2 \setminus [0,1] \times [-1,0]}$$

Mesh with 9 quadrilateral elements of degree 10 :

$$\widetilde{\boldsymbol{u}} \in (\mathbb{Q}_{10})^2$$

with b.c. $\widetilde{\boldsymbol{u}} \times \boldsymbol{n}\big|_{\partial\Omega} = 0$

Sort computed eigenvalues by increasing order

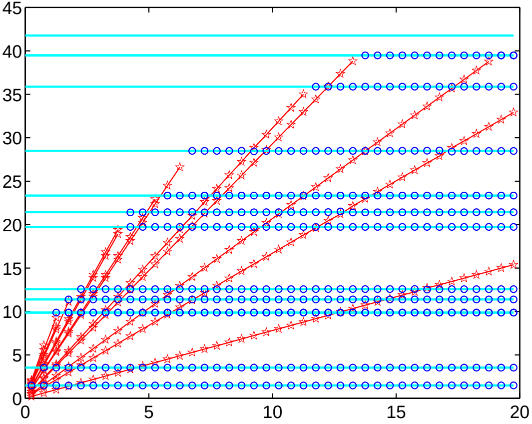
$$\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots$$

$$\tau(\widetilde{\boldsymbol{u}}) = \|\operatorname{rot}\widetilde{\boldsymbol{u}}\|^2 (s\|\operatorname{div}\widetilde{\boldsymbol{u}}\|^2)^{-1}$$

$\star$ when τ small $\circ$ when τ large

- Abscissa: Value of s
- Ordinates: all smallest values of $\tilde{\lambda}_n \leq 45$, $n = 1, 2, 3, \ldots$
- Horizontal lines = values close to exact for λ_j (by computing Neumann eigenvalues)

Spectrum of $a_\gamma[s]$ on a L-shape domain, $\gamma = 1$



$$\boxed{\Omega = (-1,1)^2 \setminus [0,1] \times [-1,0]}$$

Mesh with 9 quadrilateral elements of degree 10 :

$$\widetilde{\boldsymbol{u}} \in (\mathbb{Q}_{10})^2$$

with b.c. $\widetilde{\boldsymbol{u}} \times \boldsymbol{n}\big|_{\partial\Omega} = 0$

Sort computed eigenvalues by increasing order

$$\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots$$

$\tau(\widetilde{\boldsymbol{u}}) = \|\operatorname{rot}\widetilde{\boldsymbol{u}}\|^2 (s\|\operatorname{div}\widetilde{\boldsymbol{u}}\|^2)^{-1}$

$\star$ when τ small $\circ$ when τ large

- Abscissa: Value of s
- Ordinates: all smallest values of $\tilde{\lambda}_n \leq 45$, $n = 1, 2, 3, \ldots$
- Horizontal lines = values close to exact for λ_j (by computing Neumann eigenvalues)

Theoretical result

Theorem Costabel-Dauge, 02

Let Ω be a polygon with one non-convex corner $\boldsymbol{c}$ of opening $\omega > \pi$ and let r be the distance fonction to $\boldsymbol{c}$. Let γ be such that

$$1 - \frac{\pi}{\omega} < \gamma < 1$$

Then, for any $s > 0$

- $H_N(\Omega)$ is dense in $X_N^{\gamma}(\Omega)$
- The eigenvalues of $a_{\gamma}[s]$ are correctly approximated by Lagrange finite elements

The eigenvalues of $a_{\gamma}[s]$ are

- the $s\lambda_n^{\mathrm{Dir},\gamma}$ with the eigenvalues $\lambda_n^{\mathrm{Dir},\gamma}$ of the Dirichlet realization of $v \mapsto r^{\gamma}\Delta(r^{\gamma}v)$
- the λ^{Neu} (independent of s and γ)

This theorem extends to 3-dim. polyhedra. The weight is the distance to the union of non-convex edges.

Singularities and weighted spaces

Let Ω be a polygon with one non-convex corner $\boldsymbol{c}$ of opening $\omega > \pi$
Define for $m \in \mathbb{N}$ and $\beta \in \mathbb{R}$ the weighted Sobolev space

$$K^m_\beta(\Omega) = \{v \in L^2_{\mathrm{loc}}(\Omega), \quad r^{\beta+|\alpha|}\partial^\alpha_{\boldsymbol{x}} v \in L^2(\Omega)\ \forall \alpha \in \mathbb{N}^2, |\alpha| \le m\}$$

Observe that

- The Laplacian $\Delta = \partial_1^2 + \partial_2^2$ is continuous from $K^2_{\gamma-2}(\Omega)$ into $K^0_\gamma(\Omega)$.
- If $\boldsymbol{U} \in X^\gamma_N(\Omega)$, then div $\boldsymbol{U} \in K^0_\gamma(\Omega)$.
- The singularity Φ_{sing} has the form, in polar coordinates centered at $\boldsymbol{c}$

$$\Phi_{\mathrm{sing}}(\boldsymbol{x}) = \chi(r)\, r^{\pi/\omega} \sin\left(\frac{\pi\theta}{\omega}\right)$$

and $\Phi_{\mathrm{sing}} \in K^2_{\gamma-2}(\Omega)$ if and only if $\frac{\pi}{\omega} > 1 - \gamma$, i.e. $\boxed{\gamma > 1 - \frac{\pi}{\omega}}$

Theorem [Kondrat'ev, 67]

Let $\gamma \in [0, 1]$. If $\gamma > 1 - \frac{\pi}{\omega}$,

$$\Delta \quad \text{isomorphism} \quad K^2_{\gamma-2}(\Omega) \cap H^1_0(\Omega) \longrightarrow K^0_\gamma(\Omega)$$

If $\gamma < 1 - \frac{\pi}{\omega}$,

$$\Delta \quad \text{isomorphism} \quad K^2_{\gamma-2}(\Omega) \cap H^1_0(\Omega) \oplus \langle \Phi_{\mathrm{sing}} \rangle \longrightarrow K^0_\gamma(\Omega)$$

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