Regularity and singularities in polyhedral domains

The case of Laplace and Maxwell equations

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Outline

Outline

Corner domains (3D)

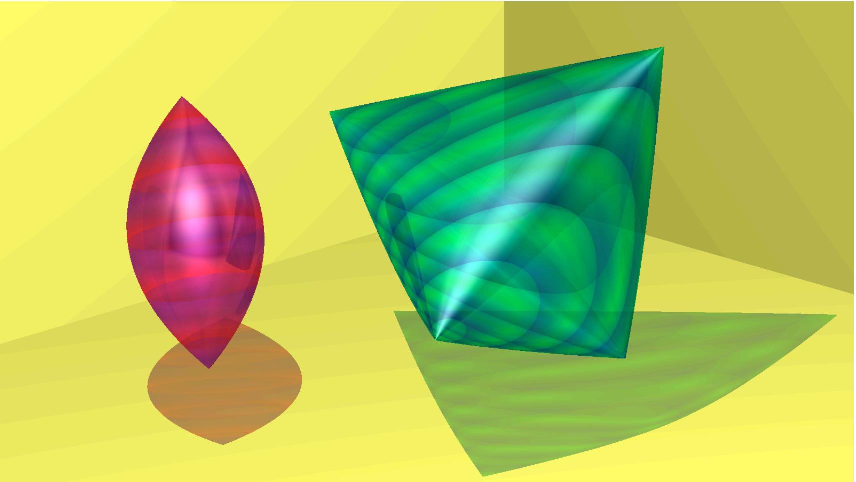

Figure: Axisymmetric domain & Cayley's tetrahedron (M. Costabel with POV-Ray)

Corner domains (definition)

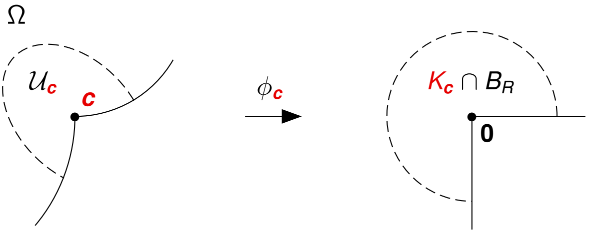

Figure: Corner domain: Local map (made with Fig4TeX)

Ω has a finite set $\mathscr{C}$ of corners c:

- *All corners are points*
- All corners c are in the boundary $\partial\Omega$ of Ω
- Around each boundary point $x_0 \notin \mathscr{C}$, Ω is smooth
- Around each corner point $c \in \mathscr{C}$, Ω is *diffeomorphic to a cone K_c*
- A *polygonal domain* is a plane corner domain whose boundary is a union of segments

Edge domains

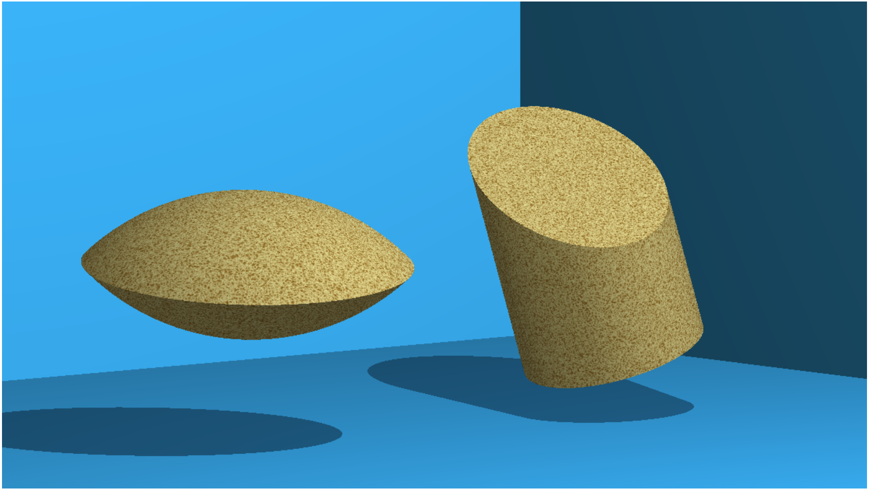

Figure: Flying saucer and skew cylinder (M. Costabel with POV-Ray)

Edge domains: Definition

$\Omega \subset \mathbb{R}^3$.

Ω has a finite set $\mathscr{E}$ of edges $\boldsymbol{e}$:

- *All edges are closed curves*
- All edges $\boldsymbol{e}$ are subsets of $\partial\Omega$
- Around each boundary point $\boldsymbol{x}_0 \notin \cup_{\boldsymbol{e}\in\mathscr{E}}\boldsymbol{e}$, Ω is smooth
- Around each edge point $\boldsymbol{z} \in \boldsymbol{e}$, Ω is *diffeomorphic to a wedge $K_{\boldsymbol{z}} \times \mathbb{R}$*

Edge opening

- The diffeomorphism $\phi_{\boldsymbol{z}}$ is tangent to a rotation: $\nabla\phi_{\boldsymbol{z}}(\boldsymbol{z}) \in \mathbb{O}_3$
- $K_{\boldsymbol{z}}$ is a plane sector: Let $\omega_{\boldsymbol{z}}$ be its opening.
- $\boldsymbol{z} \mapsto \omega_{\boldsymbol{z}}$ can be variable or constant
- If $\boldsymbol{z} \mapsto \omega_{\boldsymbol{z}}$ is constant, it defines $\omega_{\boldsymbol{e}}$

Polyhedral domains

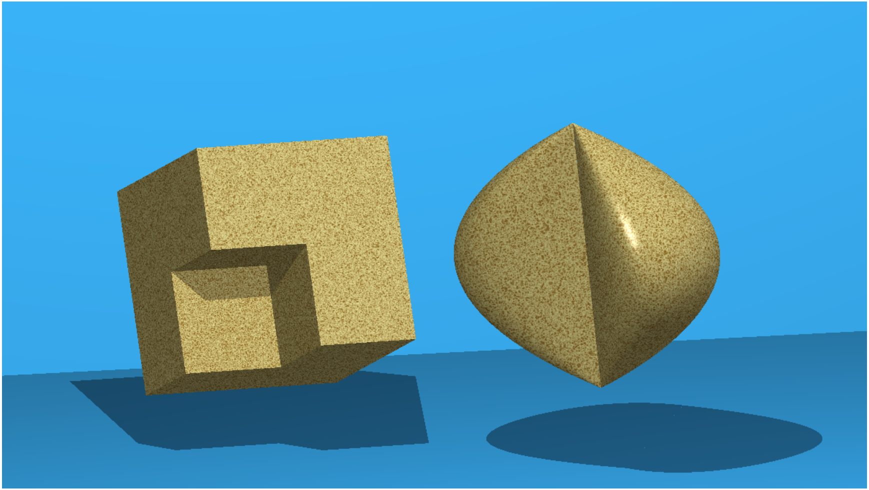

Figure: Fichera corner and seed (M. Costabel with POV-Ray)

Curvilinear Polyhedral domains: Definition

$\Omega \subset \mathbb{R}^3$.

Ω has a finite set $\mathscr{E}$ of edges $\boldsymbol{e}$ and a finite set $\mathscr{C}$ of corners $\boldsymbol{c}$:

- *All edges are smooth open arcs of curve*
- *All edge boundary points $\boldsymbol{c} \in \overline{\mathbf{e}} \setminus \boldsymbol{e}$ are corners*
- All edges $\boldsymbol{e}$ and corners $\boldsymbol{c}$ are subsets of $\partial\Omega$
- Around each boundary point $\boldsymbol{x}_0 \notin \mathscr{C} \cup (\cup_{\boldsymbol{e} \in \mathscr{E}} \boldsymbol{e})$, Ω is smooth
- Around each edge point $\boldsymbol{z} \in \boldsymbol{e}$, Ω is *diffeomorphic to a wedge* $K_{\boldsymbol{z}} \times \mathbb{R}$
- Around each corner point $\boldsymbol{c} \in \mathscr{C}$, Ω is *diffeomorphic to a cone* $K_{\boldsymbol{c}}$
- Let $G_{\boldsymbol{c}}$ be the *solid angle* of $K_{\boldsymbol{c}}$, i.e. $G_{\boldsymbol{c}} = K_{\boldsymbol{c}} \cap \mathbb{S}^2$

Polyhedral cone

- If $\boldsymbol{c}$ does not belong to a $\overline{\mathbf{e}}$, $K_{\boldsymbol{c}}$ is a regular cone, i.e. $G_{\boldsymbol{c}}$ is smooth
- If $\boldsymbol{c} \in \overline{\mathbf{e}}$, $K_{\boldsymbol{c}}$ is a polyhedral cone, i.e. $G_{\boldsymbol{c}}$ is a 2D corner domain

Outline

Dirichlet and Neumann problems for Δ

Two typical problems:

Dirichlet

For $f \in L^2(\Omega)$,
find $u \in H^1(\Omega)$:

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Existence and uniqueness.
If Ω is smooth, regularity shift:

$$f \in H^{s-2}(\Omega) \Longrightarrow u \in H^s(\Omega),$$

with estimates

$$\|u\|_{s;\Omega} \leq C\|f\|_{s-2;\Omega}\,.$$

Neumann

For $f \in L^2(\Omega)$, $g \in H^{-1/2}(\partial\Omega)$,
find $u \in H^1(\Omega)$:

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \partial_n u = g & \text{on } \partial\Omega. \end{cases}$$

Existence if compatible data.
If Ω is smooth, regularity shift with estimates

$$\|u\|_{s;\Omega} \leq C\Big(\|f\|_{s-2;\Omega} + \|g\|_{s-3/2;\partial\Omega} + \|u\|_{1;\Omega}\Big).$$

Regularity on plane corner domains

Corner $\boldsymbol{c} \in \mathscr{C} \to$ Opening $\omega_{\boldsymbol{c}} \in (0, 2\pi]$ of the tangent sector $K_{\boldsymbol{c}}$.

Theorem 1 for Dirichlet and Neumann problems

Let $s > \frac{3}{2}$ real. If

$$\forall \boldsymbol{c} \in \mathscr{C}, \quad s - 1 < \frac{\pi}{\omega_{\boldsymbol{c}}}$$

- Dirichlet: $f \in H^{s-2}(\Omega) \implies u \in H^s(\Omega)$
- Neumann: $f \in H^{s-2}(\Omega)$ and $g \in PH^{s-3/2}(\partial\Omega) \implies u \in H^s(\Omega)$

Here $PH^\sigma(\partial\Omega) = \prod_j H^\sigma(\partial_j\Omega)$ with $\partial_j\Omega$ the sides of Ω.

Mixed Dirichlet-Neumann problems

The regularity condition is then (even if $\omega_{\boldsymbol{c}} = \pi$)

$$\forall \boldsymbol{c} \in \mathscr{C}, \quad s - 1 < \frac{\pi}{2\,\omega_{\boldsymbol{c}}}$$

Regularity on corner three-dimensional domains

Notations

- Corner $\boldsymbol{c} \in \mathscr{C} \to$ Solid angle $G_{\boldsymbol{c}}$ of the tangent cone $K_{\boldsymbol{c}}$.
- $\mu_{\boldsymbol{c}}^{\text{dir}} = 1^{\text{rst}}$ eigenvalue of Laplace-Beltrami with Dirichlet bc on $G_{\boldsymbol{c}}$
- $\mu_{\boldsymbol{c}}^{\text{neu}} = 2^{\text{nd}}$ eigenvalue of Laplace-Beltrami with Neumann bc on $G_{\boldsymbol{c}}$

Theorem 2 for Dirichlet and Neumann problems

Let $s > \frac{3}{2}$ real.

- Dirichlet: $\lambda_{\boldsymbol{c}}^{\text{dir}}$ positive root of $\lambda^2 + \lambda = \mu_{\boldsymbol{c}}^{\text{dir}}$. If

$$\forall \boldsymbol{c} \in \mathscr{C}, \quad s - \tfrac{3}{2} < \min\{\lambda_{\boldsymbol{c}}^{\text{dir}}, 2\}$$

  Then: $\quad f \in H^{s-2}(\Omega) \implies u \in H^s(\Omega)$

- Neumann: $\lambda_{\boldsymbol{c}}^{\text{neu}}$ positive root of $\lambda^2 + \lambda = \mu_{\boldsymbol{c}}^{\text{neu}}$. If

$$\forall \boldsymbol{c} \in \mathscr{C}, \quad s - \tfrac{3}{2} < \min\{\lambda_{\boldsymbol{c}}^{\text{neu}}, 1\}$$

  Then: $\quad f \in H^{s-2}(\Omega)$ and $g \in PH^{s-3/2}(\partial\Omega) \implies u \in H^s(\Omega)$

Corner Regularity: The secret of generality (Dirichlet)

Laplace-Beltrami first eigenvalue

$$\mu_{\boldsymbol{c}}^{\mathrm{dir}} = 1^{\mathrm{rst}} \text{ eigenvalue of Laplace-Beltrami with Dirichlet bc on } G_{\boldsymbol{c}}$$

In dimension $n = 2$, $G_{\boldsymbol{c}} = (0, \omega_{\boldsymbol{c}})$ and $\mu_{\boldsymbol{c}}^{\mathrm{dir}} = (\frac{\pi}{\omega_{\boldsymbol{c}}})^2$.

Regularity conditions

- $n = 2$: $s - 1 < \lambda_{\boldsymbol{c}}^{\mathrm{dir}}$ with $(\lambda_{\boldsymbol{c}}^{\mathrm{dir}})^2 = \mu_{\boldsymbol{c}}^{\mathrm{dir}}$.
- $n = 3$: $s - \frac{3}{2} < \lambda_{\boldsymbol{c}}^{\mathrm{dir}}$ with $\lambda_{\boldsymbol{c}}^{\mathrm{dir}}(\lambda_{\boldsymbol{c}}^{\mathrm{dir}} + 1) = \mu_{\boldsymbol{c}}^{\mathrm{dir}}$

Δ in polar coordinates

$$\Delta = r^{-2}\big((r\partial_r)^2 + (n-2)r\partial_r - \Delta_{\mathbb{S}^{n-1}}\big)$$

General regularity condition (Lapace-Dirichlet)

$$\forall \boldsymbol{c} \in \mathscr{C}, \quad s - \frac{n}{2} < \lambda_{\boldsymbol{c}}^{\mathrm{dir}}$$

with n = dimension and $\lambda_{\boldsymbol{c}}^{\mathrm{dir}}$ positive root of $\lambda(\lambda + n - 2) = \mu_{\boldsymbol{c}}^{\mathrm{dir}}$.

3D Corner Regularity: Values for λ^{dir}

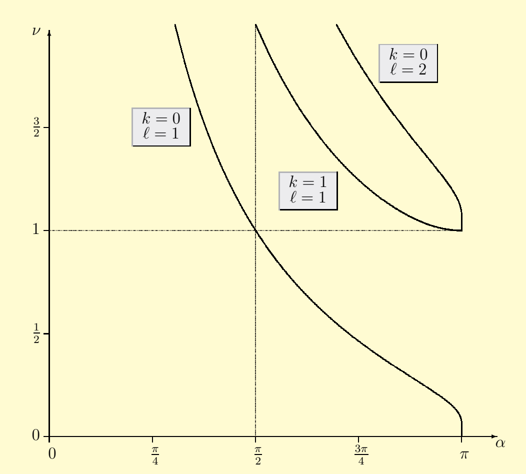

Figure: λ_c^{dir} for axisymmetric cones (corresponds to $k = 0$, $\ell = 1$)

3D Corner Regularity: Comparison for λ^{dir} and λ^{neu}

Dirichlet

If $G_c \subset G$, then $\lambda_c^{\mathrm{dir}} \geq \lambda^{\mathrm{dir}}(G)$

Known values for $\lambda^{\mathrm{dir}}(G)$

- Half-sphere: $\lambda^{\mathrm{dir}}(G) = 1$
- Dihedron of opening ω: $\lambda^{\mathrm{dir}}(G) = \frac{\pi}{\omega}$
- Half-dihedron of opening ω: $\lambda^{\mathrm{dir}}(G) = \frac{\pi}{\omega} + 1$
- Axisymmetric cone (spherical cup)

Neumann

- No comparison principle for Neumann
- But, if K_c is convex, $\mu_c^{\mathrm{neu}} \geq 1$, hence $\lambda_c^{\mathrm{neu}} \geq \frac{\sqrt{5}-1}{2}$

Regularity on edge domains

Edge $\boldsymbol{e} \in \mathscr{E}$, $\boldsymbol{z} \in \boldsymbol{e}$ $\rightarrow$ Opening $\omega_{\boldsymbol{z}}$ of the tangent wedge $K_{\boldsymbol{z}} \times \mathbb{R}$.

Theorem 3 for Dirichlet and Neumann problems

Let $s > \frac{3}{2}$ real. If

$$\forall \boldsymbol{e} \in \mathscr{E},\ \forall \boldsymbol{z} \in \boldsymbol{e}, \quad s - 1 < \frac{\pi}{\omega_{\boldsymbol{z}}}$$

- Dirichlet: $f \in H^{s-2}(\Omega) \implies u \in H^s(\Omega)$
- Neumann: $f \in H^{s-2}(\Omega)$ and $g \in PH^{s-3/2}(\partial\Omega) \implies u \in H^s(\Omega)$

Regularity on curvilinear polyhedral domains

Notations

- Edge $\boldsymbol{e} \in \mathscr{E}$, $\boldsymbol{z} \in \boldsymbol{e} \rightarrow$ Opening $\omega_{\boldsymbol{z}}$ of tangent wedge $K_{\boldsymbol{z}} \times \mathbb{R}$.
- Corner $\boldsymbol{c} \in \mathscr{C} \rightarrow$ Solid angle $G_{\boldsymbol{c}}$ of tangent cone $K_{\boldsymbol{c}}$, $\rightarrow \mu_{\boldsymbol{c}}^{\text{dir}}$ & $\mu_{\boldsymbol{c}}^{\text{neu}}$.

Theorem 4 for Dirichlet and Neumann problems

Let $s > \frac{3}{2}$ real.

- Dirichlet: $\lambda_{\boldsymbol{c}}^{\text{dir}}$ positive root of $\lambda^2 + \lambda = \mu_{\boldsymbol{c}}^{\text{dir}}$. If

$$\begin{cases} \forall \boldsymbol{c} \in \mathscr{C}, & s - \frac{3}{2} < \min\{\lambda_{\boldsymbol{c}}^{\text{dir}}, 2\} \\ \forall \boldsymbol{e} \in \mathscr{E},\ \forall \boldsymbol{z} \in \boldsymbol{e}, & s - 1 < \pi/\omega_{\boldsymbol{z}} \end{cases}$$

  Then: $f \in H^{s-2}(\Omega) \implies u \in H^s(\Omega)$

- Neumann: $\lambda_{\boldsymbol{c}}^{\text{neu}}$ positive root of $\lambda^2 + \lambda = \mu_{\boldsymbol{c}}^{\text{neu}}$. If

$$\begin{cases} \forall \boldsymbol{c} \in \mathscr{C}, & s - \frac{3}{2} < \min\{\lambda_{\boldsymbol{c}}^{\text{neu}}, 1\} \\ \forall \boldsymbol{e} \in \mathscr{E},\ \forall \boldsymbol{z} \in \boldsymbol{e}, & s - 1 < \pi/\omega_{\boldsymbol{z}} \end{cases}$$

  Then: $f \in H^{s-2}(\Omega)$ and $g \in PH^{s-3/2}(\partial\Omega) \implies u \in H^s(\Omega)$

Outline

Definition of corner singularities (Dirichlet)

Let $\boldsymbol{c} \in \mathscr{C}$. **The real condition:**

Corner regularity at *c* is ensured by the general condition:

$$\forall\lambda \in \mathbb{C},\ \Re\mathrm{e}\,\lambda \in (1-\tfrac{n}{2}, s-\tfrac{n}{2}],$$
$$\Phi \in S_0^\lambda(K_{\boldsymbol{c}}) \text{ and } \Delta\Phi \in P^{\lambda-2} \implies \Phi \in P^\lambda$$

where P^λ is the space of polynomials, homogeneous of degree λ, and

$$S_0^\lambda(K_{\boldsymbol{c}}) = \Big\{\Phi = \sum_{q\geq 0,\ \text{finite}} r^\lambda \log^q r\, \varphi_q(\theta),\ \varphi_q \in H_0^1(G_{\boldsymbol{c}})\Big\}$$

Singularities are the default:

$\Phi \in S_0^\lambda(K_{\boldsymbol{c}})$ with $\Re\mathrm{e}\,\lambda \in (1-\tfrac{n}{2}, s-\tfrac{n}{2}]$, such that

$$\Phi \notin P^\lambda \quad \text{and} \quad \Delta\Phi \in P^{\lambda-2}$$

Principles for finding corner singularities (Dirichlet)

Δ in polar coordinates

$$\Delta = r^{-2}\big((r\partial_r)^2 + (n-2)r\partial_r - \Delta_{\mathbb{S}^{n-1}}\big)$$

Hence

$$\Delta\big(r^\lambda\varphi(\theta)\big) = r^{\lambda-2}\big(\lambda^2 + (n-2)\lambda - \Delta_{\mathbb{S}^{n-1}}\big)\varphi(\theta)$$

- If $\lambda \notin \mathbb{N}$, singularities are the non-zero $\Phi \in S_0^\lambda(K_{\boldsymbol{c}})$ s.t. $\Delta\Phi = 0$.

Dirichlet Eigenpairs provide us with singularities

$\Delta^{\mathrm{dir}}_{G_{\boldsymbol{c}}}\psi = \mu\psi$ yields $\Phi = r^\lambda\psi(\theta)$ with $\lambda^2 + (n-2)\lambda = \mu$

Conversely, all singularities have this form (feature of Δ).

- If $\lambda \in \mathbb{N}$, interactions with polynomials have to be considered.

Corner singularities versus polynomials (Dirichlet)

- $\lambda \in \mathbb{N}$. Space of polynomials: $P_0^\lambda(K_c) = \{\Phi \in P^\lambda \mid \Phi = 0 \text{ on } \partial K_c\}$.
- $\mathfrak{S}(\Delta^{\text{dir}}_{G_c})$ the spectrum of $\Delta_{\mathbb{S}^{n-1}}$ with Dirichlet bc on ∂G_c.

Four possibilities:

1. $\dim P_0^\lambda(K_c) = \dim P^{\lambda-2}$ and $\lambda^2 + (n-2)\lambda \notin \mathfrak{S}(\Delta^{\text{dir}}_{G_c})$. No singularity.
2. $\dim P_0^\lambda(K_c) = \dim P^{\lambda-2}$ and $\lambda^2 + (n-2)\lambda \in \mathfrak{S}(\Delta^{\text{dir}}_{G_c})$. Singularities.
   $n = 2$, $\omega_c \neq 2\pi$ or $n \geq 3$, K_c quadratic cone.
3. $\dim P_0^\lambda(K_c) < \dim P^{\lambda-2}$. Singularities.
   $n \geq 3$, K_c general non-quadratic cone.
4. $\dim P_0^\lambda(K_c) > \dim P^{\lambda-2} \Longrightarrow \lambda^2 + (n-2)\lambda \in \mathfrak{S}(\Delta^{\text{dir}}_{G_c})$. Singularities?
   Let m be the multiplicity of eigenvalue $\lambda^2 + (n-2)\lambda$.
   - $m = \dim P_0^\lambda(K_c) - \dim P^{\lambda-2}$: No singularity. $n = 2$, $\omega_c = 2\pi$
   - $m > \dim P_0^\lambda(K_c) - \dim P^{\lambda-2}$: Singularities.
   - $m < \dim P_0^\lambda(K_c) - \dim P^{\lambda-2}$? *Impossible*.

Expansions in polygonal domains

Notations:

- Smooth Cut-off $\chi_{\boldsymbol{c}} \equiv 1$ near corner $\boldsymbol{c}$ (partition of unity).

Theorem 5 for Dirichlet and Neumann problems

Let $s > \frac{3}{2}$ real. Let $f \in H^{s-2}(\Omega)$. If

$$\forall \boldsymbol{c} \in \mathscr{C},\ \forall k \in \mathbb{N}\ (k \text{ odd if } \omega_{\boldsymbol{c}} = 2\pi), \quad s - 1 \neq \frac{k\pi}{\omega_{\boldsymbol{c}}},$$

then any solution $u \in H^1(\Omega)$ has an expansion

$$u = u^{\mathrm{reg}} + \sum_{\boldsymbol{c} \in \mathscr{C}} \chi_{\boldsymbol{c}}\, u_{\boldsymbol{c}}^{\mathrm{sing}}, \quad u^{\mathrm{reg}} \in H^s(\Omega),$$

Structure of singular terms

$$u_{\boldsymbol{c}}^{\mathrm{sing}} = \sum_{\substack{\lambda = k\pi/\omega_{\boldsymbol{c}} \\ k \in \mathbb{N},\ k \text{ odd if } \omega_{\boldsymbol{c}} = 2\pi \\ 0 < \lambda < s-1}} d_{\boldsymbol{c}}^{\lambda}\, \Phi_{\boldsymbol{c}}^{\lambda}, \qquad \text{with} \quad d_{\boldsymbol{c}}^{\lambda} \in \mathbb{R} \ \&\ \Phi_{\boldsymbol{c}}^{\lambda} \in S^{\lambda}(K_{\boldsymbol{c}})$$

Singularities in a plane sector

Notations:

- ω opening of the sector K
- (r,θ) polar coordinates so that $K = \{x \in \mathbb{R}^2 \mid \theta \in (0,\omega)\}$

Formulas for Φ^λ (Dirichlet)

k positive integer and $\lambda = \frac{k\pi}{\omega}$

$$\Phi^\lambda = \begin{cases} r^\lambda \sin\lambda\theta & \text{if } \lambda \notin \mathbb{N} \\ r^\lambda \big(\log r \sin\lambda\theta + \theta\cos\lambda\theta\big) & \text{if } \lambda \in \mathbb{N} \end{cases}$$

Formulas for Φ^λ (Neumann)

k positive integer and $\lambda = \frac{k\pi}{\omega}$

$$\Phi^\lambda = \begin{cases} r^\lambda \cos\lambda\theta & \text{if } \lambda \notin \mathbb{N} \\ r^\lambda \big(\log r \cos\lambda\theta - \theta\sin\lambda\theta\big) & \text{if } \lambda \in \mathbb{N} \end{cases}$$

Expansions in plane corner domains

Notations:

- Cut-off $\chi_{\boldsymbol{c}} \equiv 1$ near corner $\boldsymbol{c}$.
- $\phi_{\boldsymbol{c}}$ local map: neighborhood of $\boldsymbol{c}$ in Ω to neighborhood of 0 in $K_{\boldsymbol{c}}$.

Theorem 6 for Dirichlet and Neumann problems

Same assumptions as in Theorem 5: Any solution $u \in H^1(\Omega)$ has an expansion

$$u = u^{\mathrm{reg}} + \sum_{\boldsymbol{c}\in\mathscr{C}} \chi_{\boldsymbol{c}}\, u_{\boldsymbol{c}}^{\mathrm{sing}} \circ \phi_{\boldsymbol{c}}\,, \quad u^{\mathrm{reg}} \in H^s(\Omega)$$

The corner terms have a new level of complexity:

$$u_{\boldsymbol{c}}^{\mathrm{sing}} = \sum_{\substack{\lambda=k\pi/\omega_{\boldsymbol{c}}\\ 0<\lambda<s-1}} d_{\boldsymbol{c}}^{\lambda} \Bigg(\sum_{\substack{p\in\mathbb{N}_0\\ \lambda+p<s-1}} \Phi_{\boldsymbol{c}}^{\lambda,p} \Bigg), \quad \text{with} \quad \Phi_{\boldsymbol{c}}^{\lambda,p} \in S^{\lambda+p}(K_{\boldsymbol{c}})$$

Leading terms

$$\Phi_{\boldsymbol{c}}^{\lambda,0} = \Phi_{\boldsymbol{c}}^{\lambda}$$

Outline

Expansions in edge domains with constant openings

Notations:

- Assume that $\forall \boldsymbol{z} \in \boldsymbol{e}$, $\omega_{\boldsymbol{z}} = \omega_{\boldsymbol{e}}$ (constant opening).
- $(r_{\boldsymbol{e}}, \theta_{\boldsymbol{e}}, \boldsymbol{z})$ cylindrical coordinates with axis along $\boldsymbol{e}$.
- Cut-off $\chi_{\boldsymbol{e}} \equiv 1$ near edge $\boldsymbol{e}$.
- $\mathfrak{K}_{\boldsymbol{e}}$ lifting (smoothing) operator from $\boldsymbol{e}$ to Ω.

Theorem 7 for Dirichlet and Neumann problems

Let $s > \frac{3}{2}$ real. Let $f \in H^{s-2}(\Omega)$.
If $\forall \boldsymbol{e} \in \mathscr{E}$, $\forall k \in \mathbb{N}$ (k odd if $\omega_{\boldsymbol{e}} = 2\pi$), $s - 1 \neq \frac{k\pi}{\omega_{\boldsymbol{e}}}$, then u expands as

$$u = u^{\text{reg}} + \sum_{\boldsymbol{e} \in \mathscr{E}} \chi_{\boldsymbol{e}}\, u_{\boldsymbol{e}}^{\text{sing}}, \quad u^{\text{reg}} \in H^s(\Omega),$$

Structure of the edge singular terms when $2\pi/\omega_{\boldsymbol{e}} > s - 1$

$$u_{\boldsymbol{e}}^{\text{sing}} = \sum_{\omega_{\boldsymbol{e}} > \pi} \mathfrak{K}_{\boldsymbol{e}}[d_{\boldsymbol{e}}]\, r_{\boldsymbol{e}}^{\pi/\omega_{\boldsymbol{e}}} \sin \frac{\pi\theta_{\boldsymbol{e}}}{\omega_{\boldsymbol{e}}}, \quad \text{for} \quad d_{\boldsymbol{e}} \in H^{s-1-\pi/\omega_{\boldsymbol{e}}}(\boldsymbol{e})$$

Edge Expansions: Simplified, but more general

Drop the assumption $2\pi/\omega_{\boldsymbol{e}} > s-1$.

The simplified expression below is valid if:

- The edge $\boldsymbol{e}$ is a straight line.
- There is no resonnance between exponents: The sets

$$\mathbb{N} \cap (0, s-1), \quad \{\lambda + 2p \,|\, p \in \mathbb{N}_0\} \cap (0, s-1), \quad \lambda = \frac{k\pi}{\omega_{\boldsymbol{e}}}$$

are pairwise disjoint.

Simple Structure of the edge singular terms

$$u_{\boldsymbol{e}}^{\text{sing}} = \sum_{\substack{\lambda = k\pi/\omega_{\boldsymbol{e}} \\ 0<\lambda<s-1}} \Bigg(\sum_{\substack{p \in \mathbb{N}_0 \\ \lambda+2p<s-1}} \mathfrak{K}_{\boldsymbol{e}}[\partial_{\boldsymbol{z}}^{2p} d_{\boldsymbol{e}}^{\lambda}]\, \Phi_{\boldsymbol{e}}^{\lambda,2p} \Bigg), \quad \text{with} \quad \begin{cases} d_{\boldsymbol{e}}^{\lambda} \in H^{s-1-\lambda}(\boldsymbol{e}) \\ \Phi_{\boldsymbol{e}}^{\lambda,p} \in S^{\lambda+2p}(K_{\boldsymbol{e}}) \end{cases}$$

Note: $\Phi_{\boldsymbol{e}}^{\lambda,0} = r_{\boldsymbol{e}}^{\lambda} \sin \lambda\theta_{\boldsymbol{e}}$

Edge Expansions: The real thing

Now we hide each singular block into one packet.

Most general form of singular terms

$$u_{e}^{\text{sing}} = \sum_{\substack{\lambda=k\pi/\omega_{e} \\ 0<\lambda<s-1}} \mathfrak{U}_{e,s-1}^{\lambda}[d_{e}^{\lambda}], \quad \text{with} \quad d_{e}^{\lambda} \in H^{s-1-\lambda}(e)$$

Edge packet

$$\mathfrak{U}_{e,s-1}^{\lambda}[d_{e}^{\lambda}] = \mathfrak{K}_{e}[d_{e}^{\lambda}]\, r_{e}^{\lambda} \sin \lambda\theta_{e} + \text{h.o.t.} \quad \text{as} \quad r_{e} \to 0$$

Here h.o.t. is a finite sum of terms of the form

$$\mathfrak{K}_{e}[d_{e}^{\lambda,p,q}]\, \Phi_{e}^{\lambda,p,q} \quad \text{with} \quad \Phi_{e}^{\lambda,p,q} \in S^{\lambda+p}(K_{e})$$

with

- $p \geq 1$ such that $\lambda + p \leq s - 1$
- $d_{e}^{\lambda,p,q}$ functions of $z \in e$ obtained from d_{e}^{λ} by a Ψdo-differential operator with polynomial-logarithmic symbol of “degree” p.

Expansions in polyhedral domains: Preliminaries

Notations:

- $\Lambda_{\boldsymbol{e}}^{\Delta}$ = set of exponents λ associated with $\boldsymbol{e}$, i.e.

$$\Lambda_{\boldsymbol{e}}^{\Delta} = \left\{ \frac{k\pi}{\omega_{\boldsymbol{e}}} \mid \ k \in \mathbb{N}_1, \ \text{with odd} \ \ k \text{ if } \omega_{\boldsymbol{e}} = 2\pi \right\}$$

- $\Lambda_{\boldsymbol{c}}^{\Delta}$ = set of exponents λ associated with $\boldsymbol{c}$
  - Dirichlet: $\exists$ non-polynomial $\Phi \in S_0^{\lambda}(K_{\boldsymbol{c}})$ with polynomial $\Delta\Phi$.

$$\Lambda_{\boldsymbol{c}}^{\Delta} \subset \{\lambda \mid \ \lambda^2 + \lambda \in \mathfrak{S}(\Delta_{G_{\boldsymbol{c}}}^{\text{dir}})\} \cup \mathbb{N}_2$$

  - Neumann: $\exists$ non-polynomial $\Phi \in S^{\lambda}(K_{\boldsymbol{c}})$ with polynomial $(\Delta\Phi, \partial_n\Phi)$.

$$\Lambda_{\boldsymbol{c}}^{\Delta} \subset \{\lambda \mid \ \lambda^2 + \lambda \in \mathfrak{S}(\Delta_{G_{\boldsymbol{c}}}^{\text{neu}})\} \cup \mathbb{N}_1$$

Assumptions of Theorem 8

Let $s > \frac{3}{2}$ real. $f \in H^{s-2}(\Omega)$.

$$\begin{cases} \forall \boldsymbol{c} \in \mathscr{C}, & s - \frac{3}{2} \notin \Lambda_{\boldsymbol{c}}^{\Delta} \\ \forall \boldsymbol{e} \in \mathscr{E}, & s - 1 \notin \Lambda_{\boldsymbol{e}}^{\Delta} \end{cases}$$

Expansions in polyhedral domains: Statement

Notations:

- Cut-off $\chi_{\boldsymbol{c}} \equiv 1$ near corner $\boldsymbol{c}$.
- Cut-off $\chi_{\boldsymbol{e}} \equiv 1$ near edge $\boldsymbol{e}$ (conical near corners).

Conclusion of Theorem 8: Corner-Edge expansion

Then u expands as

$$u = u^{\text{reg}} + \sum_{\boldsymbol{c}\in\mathscr{C}} \chi_{\boldsymbol{c}}\, u^{\text{sing}}_{\boldsymbol{c}} + \sum_{\boldsymbol{e}\in\mathscr{E}} \chi_{\boldsymbol{e}}\, u^{\text{sing}}_{\boldsymbol{e}}, \quad u^{\text{reg}} \in H^s(\Omega),$$

Structure of singular terms

$$u^{\text{sing}}_{\boldsymbol{c}} = \sum_{\substack{\lambda\in\Lambda^{\Delta}_{\boldsymbol{c}} \\ -\frac12<\lambda<s-\frac32}} \ \sum_{q,\ \text{finite}} d^{\lambda,q}_{\boldsymbol{c}}\, \Phi^{\lambda,q}_{\boldsymbol{c}}, \quad \text{with} \quad d^{\lambda,q}_{\boldsymbol{c}} \in \mathbb{R} \ \ \& \ \ \Phi^{\lambda,q}_{\boldsymbol{c}} \in S^{\lambda}(K_{\boldsymbol{c}})$$

$$u^{\text{sing}}_{\boldsymbol{e}} = \sum_{\substack{\lambda\in\Lambda^{\Delta}_{\boldsymbol{e}} \\ 0<\lambda<s-1}} \mathfrak{U}^{\lambda}_{\boldsymbol{e},s-1}[d^{\lambda}_{\boldsymbol{e}}], \quad \text{with} \quad d^{\lambda}_{\boldsymbol{e}} \in V^{s-1-\lambda}(\boldsymbol{e})$$

Expansions in polyhedral domains: What is hidden?

1. The space $V^{s-1-\lambda}(\boldsymbol{e})$: Weighted space, defined for real $\sigma \geq 0$

$$V^{\sigma}(\boldsymbol{e}) = \{d \in H^{\sigma}(\boldsymbol{e}) \mid \ \tau_{\boldsymbol{e}}^{k-\sigma}\partial_{\boldsymbol{z}}^{k} d \in L^2(\boldsymbol{e}), \ \ \forall k, \ 0 \leq k \leq \sigma\}$$

with the distance function $\boldsymbol{z} \to \tau_{\boldsymbol{e}}(\boldsymbol{z})$ to the two ends of the edge $\boldsymbol{e}$.

Edge coefficients

$V^{s-1-\lambda}(\boldsymbol{e})$ is the subspace of $H^{s-1-\lambda}(\boldsymbol{e})$ of functions *flat* at the corners

2. The generating term in the packet $\mathfrak{U}^{\lambda}_{\boldsymbol{e},s-1}[d^{\lambda}_{\boldsymbol{e}}]$:

Edge packet

$$\mathfrak{U}^{\lambda}_{\boldsymbol{e},s-1}[d^{\lambda}_{\boldsymbol{e}}] \ = \ \mathfrak{K}_{\boldsymbol{e}}[d^{\lambda}_{\boldsymbol{e}}]\, r_{\boldsymbol{e}}^{\lambda} \sin\lambda\theta_{\boldsymbol{e}} + \text{h.o.t.} \quad \text{as} \quad r_{\boldsymbol{e}} \to 0$$

Expansions in polyhedral domains: Comments

1. Theorem 8 states a corner-edge expansion: The converse (edge-corner) approach is less natural.
2. The edge expansion alone provides different edge coefficients $D_{\boldsymbol{e}}^{\lambda}$:

The Generalized Stress Intensity Functions

$$D_{\boldsymbol{e}}^{\lambda} = \sum_{\substack{\mu \in \Lambda_{\boldsymbol{c}}^{\Delta} \\ -\frac{1}{2} < \mu < s - \frac{3}{2}}} \sum_{q,\ \text{finite}} a^{\mu,q} \boxed{r_{\boldsymbol{c}}^{\mu-\lambda}} \log^q r_{\boldsymbol{c}} + d_{\boldsymbol{e}}^{\lambda}$$

Outline

Harmonic Maxwell equations with PEC conditions

Ω three-dimensional domain. $\kappa \in \mathbb{R}$. $\boldsymbol{J}$ such that $\operatorname{div} \boldsymbol{J} = 0$.

Maxwell equations with Perfectly Electric Conducting conditions

$$\begin{cases} \textbf{curl}\, \boldsymbol{E} - i\kappa \boldsymbol{H} = 0 \quad \text{and} \quad \textbf{curl}\, \boldsymbol{H} + i\kappa \boldsymbol{E} = \boldsymbol{J} \quad \text{in } \Omega \\ \boldsymbol{E} \times \boldsymbol{n} = 0 \quad \text{and} \quad \boldsymbol{H} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

If $\kappa = 0$, add the jauge conditions (if $\kappa \neq 0$, they are implied).

Jauge conditions

$$\operatorname{div} \boldsymbol{E} = 0 \quad \text{and} \quad \operatorname{div} \boldsymbol{H} = 0 \quad \text{in } \Omega.$$

Look for $\boldsymbol{E}$ and $\boldsymbol{H}$ in $L^2(\Omega)^3$. Then

$$\begin{cases} \boldsymbol{E} \in X_N := \{\boldsymbol{u} \in H(\textbf{curl};\Omega) \cap H(\operatorname{div};\Omega) \mid \boldsymbol{u} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\} \\ \boldsymbol{H} \in X_T := \{\boldsymbol{u} \in H(\textbf{curl};\Omega) \cap H(\operatorname{div};\Omega) \mid \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\} \end{cases}$$

Regularized formulations

- *Elimination* of $\boldsymbol{H}$
- Variational formulation for $\boldsymbol{E}$ in $H_0(\mathbf{curl};\Omega)$
- *Regularization* by a $\operatorname{div}\boldsymbol{E}\operatorname{div}\boldsymbol{E}'$ term
- Variational formulation in X_N: *Find* $\boldsymbol{E}\in X_N$, $\forall \boldsymbol{E}'\in X_N$:

$$\int_\Omega \mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{E}' + \operatorname{div}\boldsymbol{E}\operatorname{div}\boldsymbol{E}' - \kappa^2\boldsymbol{E}\cdot\boldsymbol{E}' = i\kappa\int_\Omega \boldsymbol{J}\cdot\boldsymbol{E}'$$

- Consider the *Principal Part* (i.e. $\kappa=0$): Same regularity properties. Same singularites, at least up to $H^2(\Omega)$.

Theorem: Regularity of the divergence

For $\boldsymbol{f}\in L^2(\Omega)^3$, let $\boldsymbol{u}\in X_N$, $\forall \boldsymbol{u}'\in X_N$:

$$\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{u}' + \operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{u}' = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}'$$

Then $\operatorname{div}\boldsymbol{u}\in H^1_0(\Omega)$, and is solution of $\Delta(\operatorname{div}\boldsymbol{u}) = \operatorname{div}\boldsymbol{f}$.

The curse of the variational space X_N

Ω polyhedral domain (i.e. with planar faces).

- The potentials φ such that **grad** $\varphi \in X_N$ are all elements of

$$D(\Delta^{\text{dir}}) := \{\psi \in H_0^1(\Omega) \mid \quad \Delta\psi \in L^2(\Omega)\}$$

- If $D(\Delta^{\text{dir}}) \not\subset H^2(\Omega)$, then $X_N \not\subset H^1(\Omega)$
- Application of first part: *When is $D(\Delta^{\text{dir}})$ a subset of $H^2(\Omega)$*?

The curse of the variational space X_N

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- The potentials φ such that $\mathbf{grad}\,\varphi \in X_N$ are all elements of

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- If $D(\Delta^{\text{dir}}) \not\subset H^2(\Omega)$, then $X_N \not\subset H^1(\Omega)$
- Application of first part: *When is $D(\Delta^{\text{dir}})$ a subset of $H^2(\Omega)$*?
- If and only if all edge openings $\omega_{\boldsymbol{e}} < \pi$
- If and only if Ω is convex

Theorem [BiSo'87]

Let $C(\Delta^{\text{dir}})$ be a closed complement of $H^2 \cap H^1_0(\Omega)$ in $D(\Delta^{\text{dir}})$. Then

$$X_N = \mathbf{grad}\, C(\Delta^{\text{dir}}) \oplus H_N\,, \quad \text{with} \quad H_N = X_N \cap H^1(\Omega)^3$$

A bad news for FEM approximations: Theorem [Co'91]

H_N is closed in X_N for X_N's norm

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Formulation as boundary value problem

Maxwell regularized problem (principal part)

Let $\boldsymbol{f} \in L^2(\Omega)^3$.

$$\begin{cases} \Delta \boldsymbol{u} & = & \boldsymbol{f} & \text{in } \Omega \\ \boldsymbol{u} \times \boldsymbol{n} & = & 0 & \text{on } \partial\Omega \\ \operatorname{div} \boldsymbol{u} & = & 0 & \text{on } \partial\Omega \end{cases}$$

with **curl** $\boldsymbol{u} \in L^2(\Omega)^3$ and div $\boldsymbol{u} \in H^1(\Omega)$.

Written in abstract form as

$$\begin{cases} L\boldsymbol{u} & = & \boldsymbol{f} & \text{in } \Omega \\ D\boldsymbol{u} & = & 0 & \text{on } \partial\Omega \\ T\boldsymbol{u} & = & 0 & \text{on } \partial\Omega \end{cases}$$

Outline

Edge analysis

Edge $\boldsymbol{e}$.

- Local coordinates $\boldsymbol{x} = (\boldsymbol{y}, \boldsymbol{z})$, with $\boldsymbol{y} = (y_1, y_2) \in K_{\boldsymbol{e}}$ and $\boldsymbol{z} \in \boldsymbol{e}$.
- $\boldsymbol{u} = (\boldsymbol{v}, w)$ with $\boldsymbol{v} = (v_1, v_2)$ normal to $\boldsymbol{e}$ and w tangent to $\boldsymbol{e}$.
- Write L as $L(\partial_{\boldsymbol{y}}, \partial_{\boldsymbol{z}})$ and set $L_{\boldsymbol{e}} = L(\partial_{\boldsymbol{y}}, 0)$.
- Same with boundary operators: $T_{\boldsymbol{e}} = T(\partial_{\boldsymbol{y}}, 0)$, and $D_{\boldsymbol{e}} = D$.

$$T_{\boldsymbol{e}}(\boldsymbol{v}, w) = \operatorname{div}_{\boldsymbol{y}} \boldsymbol{v} \quad \text{and} \quad D\boldsymbol{u} = (\boldsymbol{v} \times \boldsymbol{n}, w).$$

- Model problem

$$\begin{cases} L_{\boldsymbol{e}}\boldsymbol{u} = 0 & \text{in } K_{\boldsymbol{e}} \\ D_{\boldsymbol{e}}\boldsymbol{u} = 0 & \text{on } \partial K_{\boldsymbol{e}} \\ T_{\boldsymbol{e}}\boldsymbol{u} = 0 & \text{on } \partial K_{\boldsymbol{e}} \end{cases}$$

Model edge problem

$$\begin{cases} \Delta_{\boldsymbol{y}}\boldsymbol{v} = 0 & \text{in } K_{\boldsymbol{e}} \\ \boldsymbol{v} \times \boldsymbol{n} = 0 & \text{on } \partial K_{\boldsymbol{e}} \\ \operatorname{div}_{\boldsymbol{y}} \boldsymbol{v} = 0 & \text{on } \partial K_{\boldsymbol{e}} \end{cases} \quad \text{and} \quad \begin{cases} \Delta_{\boldsymbol{y}} w = 0 & \text{in } K_{\boldsymbol{e}} \\ w = 0 & \text{on } \partial K_{\boldsymbol{e}} \end{cases}$$

Model analysis

Model edge problem

$$\begin{cases} \Delta_y \boldsymbol{v} &= 0 \quad \text{in } K_e \\ \boldsymbol{v}\times\boldsymbol{n} &= 0 \quad \text{on } \partial K_e \\ \operatorname{div}_y \boldsymbol{v} &= 0 \quad \text{on } \partial K_e \end{cases} \quad \text{and} \quad \begin{cases} \Delta_y w &= 0 \quad \text{in } K_e \\ w &= 0 \quad \text{on } \partial K_e \end{cases}$$

Model corner problem

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } K_c \\ \boldsymbol{u}\times\boldsymbol{n} &= 0 \quad \text{on } \partial K_c \\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{on } \partial K_c \end{cases}$$

Look for $\boldsymbol{u}$, $\boldsymbol{v}$ and w in the homogeneous form $r^\lambda\varphi(\theta)$, with

- $\Re\mathrm{e}\,\lambda > -\frac{n}{2}$ (L^2 fields) and $\Re\mathrm{e}\,\lambda \le 2-\frac{n}{2}$ (L^2 RHS)
- $\mathbf{curl}\,\boldsymbol{u} = 0$, $\operatorname{rot}\boldsymbol{v} = 0$, or $\Re\mathrm{e}\,\lambda > 1-\frac{n}{2}$ (L^2 curls)
- $\operatorname{div}\boldsymbol{u} = 0$, $\operatorname{div}\boldsymbol{v} = 0$, or $\Re\mathrm{e}\,\lambda > 2-\frac{n}{2}$ (H^1 divergence)
- For w, $\Re\mathrm{e}\,\lambda > 1-\frac{n}{2} = 0$ (L^2 vector curl)

Model singularities

w = Laplace-Dirichlet singularity on the sector K_e

$$w = r^{\pi/\omega_e} \sin\left(\frac{\pi}{\omega_e}\right), \quad \text{if} \quad \omega_e > \pi$$

$\boldsymbol{v}$ = gradient of Laplace-Dirichlet singularity on the sector K_e

$$\boldsymbol{v} = \mathbf{grad}_{\boldsymbol{y}} \left(r^{k\pi/\omega_e} \sin\left(\frac{k\pi}{\omega_e}\right)\right), \quad 0 < \frac{k\pi}{\omega_e} \leq 2.$$

Smallest exponent $\lambda_e = \frac{\pi}{\omega_e} - 1$. Set of exponents $=: \Lambda_e$

$\boldsymbol{u}$ can have two types (for simply connected G_c)

1. $\boldsymbol{u}$ = gradient of Laplace-Dirichlet singularity on the cone K_c

$$\boldsymbol{u} = \mathbf{grad}\left(r^\lambda \varphi(\theta)\right) \quad \lambda \leq \tfrac{3}{2}, \quad \lambda(\lambda+1) \in \mathfrak{S}(\Delta^{\text{dir}}_{G_c}).$$

2. $\mathbf{curl}\,\boldsymbol{u} = \mathbf{grad}\,\Phi$ with Φ Laplace-Neumann singularity on K_c, $\lambda \leq \frac{1}{2}$.

Smallest exponent $\lambda_c = \lambda_c^{\text{dir}} - 1$. Set of exponents $=: \Lambda_c$

Regularity for PEC Maxwell in a polyhedron

Theorem [CoDa'00]

Let $\boldsymbol{f} \in L^2(\Omega)^3$. Let $s \in (0,2]$. If

$$\begin{cases} \forall \boldsymbol{c} \in \mathscr{C}, & s - \frac{3}{2} < \lambda^{\mathrm{dir}}_{\boldsymbol{c}} - 1 \\ \forall \boldsymbol{e} \in \mathscr{E},\ \forall \boldsymbol{z} \in \boldsymbol{e}, & s - 1 < \pi/\omega_{\boldsymbol{z}} - 1 \end{cases}$$

Then: $\boldsymbol{u} \in H^s(\Omega)$

Remark

"Regularity Electric Maxwell" = "Regularity Dirichlet Δ" -1

Singularities for PEC Maxwell in a polyhedron

Theorem [CoDa'00]

$\boldsymbol{f} \in L^2(\Omega)^3$. Assumption: $\forall \boldsymbol{c} \in \mathscr{C}$, $\frac{1}{2} \notin \Lambda_{\boldsymbol{c}}$. Then

$$\boldsymbol{u} = \boldsymbol{u}^{\mathrm{reg}} + \sum_{\boldsymbol{c}\in\mathscr{C}} \chi_{\boldsymbol{c}}\, \boldsymbol{u}_{\boldsymbol{c}}^{\mathrm{sing}} + \sum_{\boldsymbol{e}\in\mathscr{E}} \chi_{\boldsymbol{e}}\, \boldsymbol{u}_{\boldsymbol{e}}^{\mathrm{sing}}, \qquad \boldsymbol{u}^{\mathrm{reg}} \in H^2(\Omega)^3,$$

Structure of singular terms

$$\boldsymbol{u}_{\boldsymbol{c}}^{\mathrm{sing}} = \sum_{\substack{\lambda\in\Lambda_{\boldsymbol{c}} \\ -\frac{3}{2}<\lambda<\frac{1}{2}}} D_{\boldsymbol{c}}^{\lambda}\, \boldsymbol{\Phi}_{\boldsymbol{c}}^{\lambda}, \quad \text{with} \quad D_{\boldsymbol{c}}^{\lambda} \in \mathbb{R} \text{ and } \boldsymbol{\Phi}_{\boldsymbol{c}}^{\lambda} \in S^{\lambda}(K_{\boldsymbol{c}})^3$$

$$u_{\boldsymbol{e}}^{\mathrm{sing}} = \sum_{\substack{\lambda\in\Lambda_{\boldsymbol{e}} \\ -1<\lambda<1}} \mathfrak{U}_{\boldsymbol{e}}^{\lambda}[D_{\boldsymbol{e}}^{\lambda}], \quad \text{with} \quad D_{\boldsymbol{e}}^{\lambda} \in V^{1-\lambda}(\boldsymbol{e})$$

Edge packet

$$\mathfrak{U}_{\boldsymbol{e}}^{\lambda}[D_{\boldsymbol{e}}^{\lambda}] = \mathfrak{K}_{\boldsymbol{e}}[D_{\boldsymbol{e}}^{\lambda}]\boldsymbol{\Phi}_{\boldsymbol{e}}^{\lambda}(r_{\boldsymbol{e}}, \theta_{\boldsymbol{e}}) + \text{h.o.t.} \quad \text{as} \quad r_{\boldsymbol{e}} \to 0$$

Improving regularity (removing a gradient)

Let $\boldsymbol{f} \in L^2(\Omega)^3$. The main singularities can be gathered into a gradient.

Theorem [CoDa'00]

Let $s \leq 2$ such that $\begin{cases} \forall \boldsymbol{c} \in \mathscr{C}, & s - \frac{3}{2} < \lambda^{\text{dir}}_{\boldsymbol{c}} \\ \forall \boldsymbol{e} \in \mathscr{E},\ \forall \boldsymbol{z} \in \boldsymbol{e}, & s - 1 < \pi/\omega_{\boldsymbol{z}} \end{cases}$

Then $\quad \boldsymbol{u} = \boldsymbol{u}^{\text{reg}} + \mathbf{grad}\,\psi, \quad \boldsymbol{u}^{\text{reg}} \in H^s(\Omega)^3 \quad$ and $\quad \psi \in H^s(\Omega)$,

Structure of potential term: $\psi = \sum_{\boldsymbol{c}} \psi_{\boldsymbol{c}} + \sum_{\boldsymbol{e}} \psi_{\boldsymbol{e}}$ with:

$$\psi_{\boldsymbol{c}} = \sum_{\lambda \in \Lambda^{\text{dir}}_{\boldsymbol{c}},\ \lambda < \frac{1}{2}} d^\lambda_{\boldsymbol{c}}\, \Phi^\lambda_{\boldsymbol{c}} \qquad d^\lambda_{\boldsymbol{c}} \in \mathbb{R} \qquad \Phi^\lambda_{\boldsymbol{c}} = r^\lambda_{\boldsymbol{c}}\varphi(\theta_{\boldsymbol{c}})$$

$$\psi_{\boldsymbol{e}} = \sum_{\lambda = \frac{\pi}{\omega_{\boldsymbol{e}}},\ \lambda < 1} \mathfrak{K}_{\boldsymbol{e}}[d_{\boldsymbol{e}}]\, \Phi^\lambda_{\boldsymbol{e}} \qquad d_{\boldsymbol{e}} \in V^{2-\lambda}(\boldsymbol{e}) \qquad \Phi^\lambda_{\boldsymbol{e}} = r^\lambda_{\boldsymbol{e}} \sin\lambda\theta_{\boldsymbol{e}}$$

Comparison

$$\nabla\Phi^\lambda_{\boldsymbol{c}} = \boldsymbol{\Phi}^{\lambda-1}_{\boldsymbol{c}} \implies d^\lambda_{\boldsymbol{c}} = D^{\lambda-1}_{\boldsymbol{c}}$$

Comparison

$$\nabla\Phi^\lambda_{\boldsymbol{e}} = \boldsymbol{\Phi}^{\lambda-1}_{\boldsymbol{e}} \implies d^\lambda_{\boldsymbol{e}} = D^{\lambda-1}_{\boldsymbol{e}}$$

Outline

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