The Rotation Number for Quantum Integrable Systems

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Abstract

For a two degree of freedom quantum integrable system, a new spectral quantity is defined, the quantum rotation number. In the semiclassical limit, the quantum rotation number can be detected on a joint spectrum and is shown to converge to the well-known classical rotation number. The proof requires not only semiclassical analysis (including Bohr-Sommerfeld quantization rules) but also a detailed study on how quantum labels can be assigned to the joint spectrum in a smooth way. This leads to the definition and analysis of asymptotic lattices. The general results are applied to the semitoric case where formulas become particularly natural.

Keywords : Liouville integrable systems, rotation number, semitoric systems, quantization, pseudo-differential operators, semiclassical analysis, asymptotic lattice, good labelling, inverse problem, symplectic invariants, lattice detection.

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1 Introduction

Let M be $\mathbb{R}^4$ or, more generally, a 4-dimensional symplectic manifold. If a two-degree of freedom Hamiltonian system, given by a Hamiltonian $H \in C^\infty(M)$, is integrable in the classical Liouville sense, then it is well known that most of the dynamics takes place on invariant tori of dimension 2. On each such torus Λ, the motion is particularly simple: in suitable angle coordinates (θ_1, θ_2) on Λ, the Hamiltonian vector field $\mathcal{X}_H$ is constant:

$$(\mathcal{X}_H)\restriction_\Lambda = \alpha_1 \frac{\partial}{\partial\theta_1} + \alpha_2 \frac{\partial}{\partial\theta_2}.$$

This is the content of the classical action-angle theorem (see for instance [17] and the references therein). The direction in $\mathbb{R}P^1$ given by the frequency vector $w := (\alpha_1, \alpha_2)$ is called the rotation

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number. If w is rational (by this we mean that α_1 and α_2 are linearly dependent over $\mathbb{Q}$), then the trajectory of the Hamiltonian system on Λ is periodic. On the contrary, when w is irrational, this trajectory is dense on Λ. Thus, the knowledge of w gives important information on the nature of the dynamics. Understanding the variation of w is also crucial for the study of perturbations of H, via the various "KAM" theorems (see for instance the review article [41]).

The goal of this article is to investigate the effect of this dynamical quantity on a *quantum* system. Assume for instance that we consider a Schrödinger operator $P = -\hbar^2\Delta + V$, with a smooth potential V on a 2-dimensional Riemannian manifold; assume moreover that this operator is *quantum integrable* (see Section 3). Then, what is the manifestation of the underlying classical rotation number on the spectrum of P? A first answer was given by Hitrik and Sjöstrand in a series of papers [25, 26] where they study the case of weakly non-selfadjoint operators (*i.e.*, V has a small imaginary part), and they proved that the asymptotics of the spectrum, in the semiclassical limit $\hbar \to 0$, exhibit very different behaviors depending on the rationality of the rotation number. In the purely selfadjoint case, the construction of quasi-modes in the various situations where w is strongly irrational or not was known for a long time, see for instance [31, 8]; however, perhaps surprisingly, only the non-selfadjoint case, where the spectrum, instead of being one-dimensional, is deployed in the complex plane, gives some hope to recognize useful geometric structures from the eigenvalues themselves. For quasi-periodic dynamics, this idea was exploited to recover the Birkhoff normal form from the complex spectrum in [21]. A strong motivation for our work is the recent paper [24], which naturally leads to this intriguing question: can you detect the rationality of the rotation number from the spectrum?

It is precisely this type of inverse spectral problem that we study in this paper. Here we stick to the simpler "normal" case, which means that instead of considering the spectrum of a truly non-selfadjoint operator, we consider the joint spectrum of a pair of commuting selfadjoint operators. The fully non-selfadjoint case is still largely open. We prove that one can define a *quantum rotation number*, in a very natural and concrete way, from the joint spectrum of two commuting operators (Definition (3.27)); moreover, in a suitable sense, this quantum rotation number converges to the classical rotation number in the semiclassical limit $\hbar \to 0$ (Theorem 3.28). This result is however more delicate than one could think at first sight, because it requires to be able to detect from the joint spectrum a *good labelling*: a pair of integers (or "quantum numbers") for each joint eigenvalue, with suitable regularity properties (Definition 3.13). After going through this constructive and combinatorial issue, we finally show that one can detect the classical rotation number from the quantum spectrum. We also believe that the quantum rotation number will prove to be a useful object in the study of quantum integrable (or near-integrable) systems, and we hope to apply this idea on concrete systems in a near future. The recent article [23] on asymmetric-top molecules nicely supports this idea.

The second motivation of this paper comes from the theory of semitoric systems, see [47, 37, 38, 40]. The general conjecture for quantum semitoric systems is that one can always recover the underlying classical system from the spectrum of a quantum semitoric system [39, Conjecture 9.1]. This builds on a number of geometric and spectral invariants, like the quantum monodromy [13, 43]. Recent advances show that a general conjecture like this is not out of reach [33], although probably under suitable genericity assumptions. Thus, exhibiting a new invariant that can be recovered from the system is an additional step in this direction; and it turns out that this rotation number is particular well defined for semitoric systems, see Section 2.3. The semitoric theory should find natural applications in the study of axisymmetric Schrödinger operators, a question that we hope to investigate in a future work.

Our work contributes to the general inverse spectral theory in the semiclassical limit. Of course, in such generality, this question has a long history, especially when restricted to the particular case of Laplace-Beltrami or Schrödinger operators; see for instance the survey [14] and the references therein. The semiclassical inverse problem for more general Hamiltonians, which is our concern here, was also considered, albeit by less numerous studies, see in particular [28] for a very general treatment. In the case of Liouville integrable systems, a line of program, closely following recent development in the classification of Lagrangian fibrations, was proposed in [48, 40, 42], but prior works have been produced in the Riemannian case, see for instance [50].

The structure of the paper is as follows. In Section 2 we recall the classical definition of the rotation number, explaining under which conditions it is well defined (which seems to be an information that is difficult to locate in the literature), and its relationship with Hamiltonian monodromy (Proposition 2.10). In Section 3, after recalling known results about the joint spectrum of commuting pseudo-differential operators, we propose our quantum rotation number and prove that it determines the classical rotation number, which is the main result of the paper (Theorems 3.28, 3.34 and Corollaries 3.50 and 3.47). An informal summary would be the following:

Theorem 1.1 *Given a pair of commuting $\hbar$-pseudo-differential operators $(\hat{J}, \hat{H})$ whose principal symbols $F := (J, H)$ form a completely integrable system, let Λ be a Liouville torus and $c = F(\Lambda)$ its classical value. Then, under suitable assumptions, the following hold:*

1. *the quantum rotation number for joint eigenvalues near c can be algorithmically computed from the joint spectrum of the quantum system near c;*
2. *the quantum rotation number for joint eigenvalues at $\mathscr{O}(\hbar)$-distance of c converges, as $\hbar \to 0$ to the classical rotation number of Λ;*
3. *therefore, the joint spectrum of the quantum system completely determines the rotation numbers of the classical system.*

In this process, we are lead to introducing the notions of *asymptotic lattices* and their *good labellings* in Section 3.3, and develop their analysis, independently of any quantization theory. Then, an important part of the paper (Section 3.6) is devoted to the proof that such good labellings can be algorithmically detected from the asymptotic lattices (Theorems 3.42 and 3.46), which finally leads to the proof of Theorem 1.1 in Corollary 3.47.

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2 The Classical Rotation Number

We first recall the notion of rotation number for integrable Hamiltonian systems. A general discussion can also be found in [4]. Let H be a two-degree of freedom completely integrable Hamiltonian on a four-dimensional symplectic manifold M: there exists a smooth function f on M such that $\{f, H\} = 0$ and the differentials df and dH are almost everywhere independent. From the dynamical viewpoint, the Hamiltonian H defines a dynamical system through its Hamiltonian vector field $\mathcal{X}_H$, and the function f is a constant of motion. On the geometric side, we have a foliation of the phase space M by (possibly singular) Lagrangian leaves given by the common level sets of H and f.

2.1 Möbius transformations

Let us assume that the map $F := (f, H) : M \to \mathbb{R}^2$ is proper, and let c be a regular value of F. By the action-angle theorem [35] (see [17] for a more modern proof), the level set $F^{-1}(c)$ is a finite union of two-dimensional tori, called *Liouville tori*; and near each Liouville torus, there exist action-angle coordinates $(I_1, I_2, \theta_1, \theta_2) \in \text{neigh}(\{0\} \times \{0\} \times \mathbb{T}^2, \mathbb{R}^2 \times \mathbb{T}^2)$ such that

$$H = g(I_1, I_2), \tag{1}$$

for some smooth function g, and the symplectic form of M is $dI_1 \wedge d\theta_1 + dI_2 \wedge d\theta_2$. More precisely, the action-angle theorem states that there is a local diffeomorphism:

$$G : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, c)$$

such that, in the new coordinates, $F = G(I_1, I_2)$. We shall always assume (as we may) that the actions (I_1, I_2) are *oriented* with respect to (f, H), which means that $\det dG(0, 0) > 0$. In particular, dg does not vanish, which enables the following definition:

Definition 2.1 *The* rotation number *of H relative to the oriented action variables $I := (I_1, I_2)$ on the Liouville torus $\Lambda \subset M$ is the projective number*

$$[w_I](\Lambda) := [\partial_1 g(I(\Lambda)) : \partial_2 g(I(\Lambda))] \in \mathbb{R}P^1.$$

Probably the easiest dynamical interpretation of $[w_{I_1,I_2}](\Lambda)$ is the following. One deduces from (1) that the Hamiltonian vector field of H, which is tangent to Λ, has the form

$$\mathcal{X}_H = \partial_1 g \frac{\partial}{\partial \theta_1} + \partial_2 g \frac{\partial}{\partial \theta_2}.$$

Therefore, the flow of $\mathcal{X}_H$ is a "straight line" winding quasi-periodically on the affine torus Λ in the coordinates (θ_1, θ_2), and the rotation number $[\partial_1 g : \partial_2 g]$ is simply the direction of the trajectory. The map $\Lambda \mapsto (\partial_1 g(I(\Lambda)), \partial_2 g(I(\Lambda)))$ is generally referred to as the *frequency map* of the system. It is also convenient (and usual) to define the rotation number as the ratio:

$$w_I(\Lambda) := \frac{\partial_1 g(I(\Lambda))}{\partial_2 g(I(\Lambda))} \in \overline{\mathbb{R}} \tag{2}$$

with values in the 1-point compactification of the real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. The diffeomorphism $\overline{\mathbb{R}} \to \mathbb{R}P^1, z \mapsto [z]$ is defined by $[z] := [z : 1]$ if $z \in \mathbb{R}$ and $[z] := [1 : 0]$ if $z = \infty$, and $[w_I](\Lambda) = [w_I(\Lambda)]$.

Note that the rotation number is not well defined by H only: it depends on the choice of actions, which in turn might depend on the choice of constant of motion f. In this paper, we shall always assume that f is given, as part of the data of the integrable Hamiltonian H; the question of the consequences of the choice of f is interesting and apparently not widely spread; we hope to return to this problem in a future paper.

Thus, we assume that (f, H) is fixed. We formalize the result on the change of action variables $I \to I'$ in the next lemma for further reference.

Lemma 2.2 *If $I' = (I'_1, I'_2)$ is another set of action variables, then there is a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ such that*

$$dI := A\, dI', \tag{3}$$

and the new rotation number related to I' is

$$[w_{I'}] = {}^t A \circ [w_I]\ , \tag{4}$$

where $\circ$ denotes the natural action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{R}P^1$ (elements of $\mathbb{R}P^1$ are viewed here as equivalence classes of vectors in $\mathbb{R}^2$). If instead we write w_I as the quotient (2), then $w_{I'}$ is obtained by the Möbius transformation:

$$w_{I'} = \frac{a w_I + c}{b w_I + d}. \tag{5}$$

An important fact is that the *rationality* of w_I is well defined (i.e. preserved by Möbius transformations). This is easy to understand from the dynamical viewpoint, since a rational rotation number is equivalent to a periodic Hamiltonian flow for the vector field $\mathcal{X}_H$ on the Liouville torus, which is of course independent of the choice of action-angle coordinates. Actually, the orbit of a (projective) rational rotation number under (projective) Möbius transformations is the whole set of (projective) rational numbers: if $[w_I] = [p : q]$ where (p, q) are either co-prime integers, or $(1, 0)$, or $(0, 1)$, then we can find integers u, v such that $pu + qv = 1$. Let $A = \begin{pmatrix} q & u \\ -p & v \end{pmatrix}$. The transformed rotation number is $[w_{I'}] = [0 : 1]$.

In fact, there exist more general notions of (ir)rationality that are also preserved by Möbius transformations. An interesting class is given by non quadratic irrationals $\mathbb{R} \setminus \mathbb{Q}_{(2)}$, where we denote by $\mathbb{Q}_{(2)}$ the set of algebraic numbers of degree 2, *i.e.* numbers $x \in \mathbb{R}$ such that $(x^2, x, 1)$ are linearly dependent over $\mathbb{Q}$. Of course, $\mathbb{Q}_{(2)}$ contains $\mathbb{Q}$, and it is clear that if $x \in \mathbb{Q}_{(2)}$ and $a \in \mathbb{Q}$, then ax, $x + a$ and $1/x$ (if $x \neq 0$) also belong to $\mathbb{Q}_{(2)}$. Hence Möbius transforms preserve $\mathbb{Q}_{(2)}$. We shall let $\overline{\mathbb{Q}}_{(2)} = \mathbb{Q}_{(2)} \cup \infty$, and denote by $[\mathbb{Q}_{(2)}] := [\overline{\mathbb{Q}}_{(2)}]$ the corresponding set in $\mathbb{R}P^1$, which is again SL$(2,\mathbb{Z})$-invariant.

Let B_r be the set of regular Liouville tori. If F has connected level sets, B_r can be identified with the open subset in $\mathbb{R}^2$ of regular values of F. In general, it follows from the action-angle theorem that B_r is a smooth covering above the open set of regular values of F in $\mathbb{R}^2$.

Definition 2.3 *A function* $[w] : W \to \mathbb{R}P^1$ *is called a* rotation number *for the system* (f, H) *on the open set* $W \subset B_r$ *if for every* $\Lambda \in W$*, there exist a neighborhood* U *of* Λ *in* W *and a set of action variables* $I := (I_1, I_2)$ *on* U *such that* $[w] = [w_I]$ *on* U*.*

(We will use the same terminology for a function $w : W \to \overline{\mathbb{R}}$.) As a consequence, a rotation number is always a smooth function. In KAM theory, it is customary to require a "nondegeneracy" assumption, roughly that $[w]$ is "far from constant"; here, we will use a similar, albeit weaker, assumption.

Definition 2.4 *An open set* $V \subset B_r$ *will be called* degenerate *for* (f, H) *if a (and hence any) rotation number* $[w]$ *on* V *is locally constant with values in* $[\mathbb{Q}_{(2)}]$*.*

Proof. We need to prove that this definition does not depend on the choice of a rotation number $[w]$. Assume that $[w]$ is locally constant on V with values in $[\mathbb{Q}_{(2)}]$. Let $\Lambda \in V$, let U be a connected open neighborhood of Λ where $[w] = [w_I]$. Let $[w']$ be another rotation number for (f, H) on V. Then, up to taking a smaller open set U, one can assume that $[w'] = [w_{I'}]$ on U, for some choice of actions I' on U. It follows from the Möbius transform (4) that $[w_{I'}]$, and hence $[w']$, must be constant on U. Moreover, by the remark above, this constant must belong to $[\mathbb{Q}_{(2)}]$. □

Definition 2.5 *The* non-degenerate support *of* (f, H) *is the complement set in* B_r *of the union of all degenerate open sets for* (f, H)*.*

A rotation number exists near any $\Lambda \in B_r$, but it may not exist on the whole of B_r; however if we restrict the non-degenerate support to an open set where a rotation number w does exist, the following characterizations are elementary (due to the smoothness of w, and the fact that $\mathbb{Q}_{(2)}$ is countable):

Proposition 2.6 *Let* K *be the non-degenerate support of* (f, H)*; let* $[w] : W \to \mathbb{R}P^1$ *be a rotation number. Then* $K \cap W$ *is characterized by any of the following equivalent properties:*

1. $K \cap W$ *is the closure in* W *of the set of* Λ*'s such that* $[w](\Lambda) \notin [\mathbb{Q}_{(2)}]$*.*
2. $K \cap W$ *is the union of the support of the differential* $d[w]$ *with all the sets with non-empty interior where* $[w]$ *is constant, and the constant is not in* $[\mathbb{Q}_{(2)}]$*.*

Remark 2.7 Let $E \in \mathbb{R}$ be a regular value of the Hamiltonian H. The submanifold $\Sigma_E := H^{-1}(E)$ is foliated by the level sets of f. Regular level sets of $f\restriction_{\Sigma_E}$ correspond to Liouville tori; they form smooth one-dimensional families. The restriction of the rotation number to this family is usually called the *rotation function*. It is important in many situations to know whether this function is a local diffeomorphism. This property (the so-called "isoenergetic KAM condition", see for instance [1]) is invariant under Möbius transformations. By Proposition 2.6, the set of tori satisfying the isoenergetic KAM condition is a subset of the non-degenerate support of the system. △

The relevance of the non-degenerate support in the inverse question is indicated by the following proposition.

Proposition 2.8 *Let W be a connected open subset of the non-degenerate support of (f, H). Let $[w]$ and $[w']$ be two rotation numbers defined on W. Assume that $[w]$ and $[w']$ coincide on a neighborhood of a point $\Lambda \in W$. Then they coincide on all of W.*

Proof. Without loss of generality, we can assume that W is relatively compact. We may then find a finite covering of W by open sets U_j, $j = 1, \ldots, N$, such that on each U_j there exist two sets of action coordinates I_j and I'_j with $[w] = [w_{I_j}]$ and $[w'] = [w'_{I'_j}]$; moreover, one can assume that $\Lambda \in U_1$ and $[w] = [w']$ on U_1. We use the following lemma (proved below):

Lemma 2.9 *Let U_1 be an open subset of the non-degenerate support of (f, H), and assume that there are action coordinates I and I' on U_1 such that $[w_I] = [w_{I'}]$. Then $dI' = \pm dI$.*

The Lemma implies that $dI_1 = \pm dI'_1$. Now let $j \in \{2, \ldots, N\}$ be such that $U_1 \cap U_j \neq \varnothing$. The lemma implies in the same way that $dI_j = \pm dI'_j$ on the open set $U_1 \cap U_j$. Formula (3) shows that the presheaf of action variables modulo constants is flat (transition functions A are constant); as a consequence, one must have $dI_j = \pm dI'_j$ on the whole open set U_j. This in turn implies that $[w] = [w']$ on U_j as well, by Formula (4). Thus, by connectedness, we obtain that $dI_j = \pm dI'_j$ for every j, and $[w] = [w']$ on all U_j's, and hence on W. □

Proof *of Lemma 2.9*. Let $w : \Lambda \mapsto w(\Lambda) \in \overline{\mathbb{R}}$ be the function defined as the common value of $w_I(\Lambda)$ and $w_{I'}(\Lambda)$, for $\Lambda \in U_1$. By the Möbius transformation formula (5), we have $w = \dfrac{aw + c}{bw + d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. Thus,

$$bw^2 + (d - a)w - c = 0. \tag{6}$$

By Proposition 2.6, we have only two possible cases: either $d_\Lambda w \neq 0$, or there exists a neighborhood of Λ on which $w = w_0$ is constant, with $w_0 \notin \overline{\mathbb{Q}}_{(2)}$. In the latter case, it follows directly from the definition of $\mathbb{Q}_{(2)}$ that the coefficients of the quadratic equation (6) must vanish:

$$b = d - a = -c = 0.$$

This implies $A = \pm\text{Id}$. In the former case, there exists a local 1-dimensional submanifold Γ through Λ on which $dw \neq 0$ (submersion theorem). The restriction w_Γ can be taken as a coordinate, say x, on Γ. Thus, for x in a open interval, one has

$$bx^2 + (d - a)x - c = 0.$$

This, of course, implies that $b = d - a = -c = 0$, and we conclude again that $A = \pm\text{Id}$. □

2.2 Rotation Number and Monodromy

We have seen that rotation numbers exist in a neighborhood of any regular Liouville torus. A natural question arises, whether it is possible to define a rotation number on the whole set of regular tori, B_r. In fact, since the rotation number depends on the choice of local action coordinates, it is naturally related to the so-called Hamiltonian monodromy of the system. We show in this section the relation between these two objects, which explains how to define a global rotation number. We were not able to locate this statement in the literature, however it is implicitly underlying Cushman's argument for the non-triviality of the monodromy of the Spherical Pendulum reported in [17]; see also [12]. In the presence of global $\mathbb{S}^1$ actions, this argument can be generalized to the co-called "fractional monodromy" [5].

In the realm of classical integrable systems (of any number of degrees of freedom n), the Hamiltonian monodromy was discovered by Duistermaat in [17]. It is the obstruction to the existence of global action coordinates (modulo constants). In topological terms, it is merely the $\text{SL}(n, \mathbb{Z})$-holonomy of the flat bundle over the set of regular tori of the system whose fiber is the homology of the corresponding Liouville torus. Since then, a number of references have explained this monodromy and its relationships with the many interesting geometric data involved. Nowadays, the prominent object which contains the Duistermaat monodromy is the integral affine structure on the set of regular tori. It shows up for instance in Mirror Symmetry questions, see [30]. The relationships with quantum integrable systems was pointed out in [13] and proved in [43]. See also [45] for an overview of various aspects of monodromy and semiclassical analysis.

In this paper, we restrict to two degree of freedom integrable systems. The Hamiltonian monodromy is a homomorphism:

$$\mu : \pi_1(B_r) \to \text{SL}(2, \mathbb{Z}),$$

defined up to a global conjugacy. Here $\pi_1(B_r)$ is the fundamental group of the set of regular tori B_r. Since B_r may not be simply connected, we introduce a simply connected covering

$$\pi : \tilde{B}_r \to B_r.$$

An element $\tilde{\Lambda} \in \tilde{B}_r$ can be seen as a homotopy class of a path from a fixed torus Λ_0 to $\pi(\Lambda)$. Recall that a loop γ in B_r acts fiberwise on the covering $\tilde{B}_r$ by concatenation of paths: given

$\tilde{\Lambda} \in \pi^{-1}(\gamma(0))$, we define $\gamma \cdot \tilde{\Lambda}$ to be (the homotopy class of) $\tilde{\Lambda} \circ \gamma$, *i.e.* the path corresponding to $\tilde{\Lambda}$ followed by γ.

The following proposition shows that, because of monodromy, the rotation number should be seen as a function on $\tilde{B}_r$, *i.e.* a *multivalued* function on B_r.

Proposition 2.10 *Given a regular torus $\Lambda \in B_r$ for the integrable system (f, H), a choice of action variables $I = (I_1, I_2)$ near Λ, and $\tilde{\Lambda} \in \pi^{-1}(\Lambda)$, there exists a smooth function $[\tilde{w}] : \tilde{B}_r \to \mathbb{R}P^1$ such that:*

1. *for any simply connected open set $U \subset B_r$, the function $[w] : U \to \mathbb{R}P^1$ defined by*
$$[\tilde{w}] = [w] \circ \pi$$
is a rotation number for the system (f, H) (Definition 2.3).
2. *near $\tilde{\Lambda}$, $[\tilde{w}] = [w_I] \circ \pi$.*

Moreover, let $K \subset B_r$ be the non-degenerate support of the system, and assume that $\Lambda \in K$. Let W be a connected open subset of K containing Λ. Then the restriction of $[\tilde{w}]$ to $\pi^{-1}(W)$ is unique, and we have, for any loop $\gamma \in \pi_1(W)$,
$$\gamma \cdot [\tilde{w}] = {}^t\mu(\gamma) \circ [\tilde{w}],$$
where $\gamma \cdot [\tilde{w}] := \left(\tilde{\Lambda}' \mapsto [\tilde{w}](\gamma \cdot \tilde{\Lambda}')\right)$.

Proof. Since $\tilde{B}_r$ is simply connected, we can find "global action variables" on it, *i.e.* a map $\tilde{I} : \tilde{B}_r \to \mathbb{R}^2$ such that for each small open ball $\tilde{U}$ on $\tilde{B}_r$, the restriction of $\tilde{I}$ to $\tilde{U}$ descends to action variables $I_{\tilde{U}}$ for the initial integrable system (f, H) on $U = \pi(\tilde{U})$. Moreover, we may assume that $I_{\tilde{U}_0} = (I_1, I_2)$ if $\tilde{U}_0$ is a neighborhood of the initial torus $\tilde{\Lambda}$. For each $\tilde{U}$, we define $[\tilde{w}]$ on $\tilde{U}$ by
$$[\tilde{w}] = [w_{I_{\tilde{U}}}] \circ \pi \ , \tag{7}$$
where $[w_{I_{\tilde{U}}}]$ given by Definition 2.1. The fact that $[\tilde{w}]$ is a smooth, single-valued function on $\tilde{B}_r$ follows from the transition formula (4), where by definition here all transition maps A are the identity matrix, because $I_{\tilde{U}} = I_{\tilde{U}'}$ on $\pi(\tilde{U}) \cap \pi(\tilde{U}')$. This proves the existence of $[\tilde{w}]$ satisfying *1*. and *2*.

The uniqueness statement is just a consequence of the unique continuation principle of Proposition 2.6, and can be detailed as follows. Let $\Lambda \in K$, let W be a connected open subset of K containing Λ, and let $[\tilde{w}']$ satisfy *1*. and *2*. Let $\tilde{\Lambda}' \in \tilde{B}_r$; we may connect $\tilde{\Lambda}$ to $\tilde{\Lambda}'$ by a path in $\tilde{B}_r$ and cover this path by open subsets $\pi^{-1}(U_1), \ldots, \pi^{-1}(U_N)$, where each U_j is simply connected, and $\tilde{\Lambda} \in U_1$. If follows from *2*. that $[\tilde{w}]$ and $[\tilde{w}']$ coincide near $\tilde{\Lambda}$. Let $[w_1]$ and $[w_1']$ be the corresponding rotation numbers defined by *1*. on U_1. They coincide near Λ (because π is a local diffeomorphism), and hence they coincide on U_1 by Proposition 2.6. We may now choose $\tilde{\Lambda}_1 \in \pi^{-1}(U_1 \cap U_2)$ and repeat the argument; we obtain that $[\tilde{w}]$ and $[\tilde{w}']$ coincide on U_2. Continuing this way, we finally get $[\tilde{w}](\tilde{\Lambda}') = [\tilde{w}'](\tilde{\Lambda}')$, and hence $[\tilde{w}] = [\tilde{w}']$ on B_r.

Let us now turn to the last statement of the Proposition. To the action variables (I_1, I_2) corresponds a unique basis (δ_1, δ_2) of $H_1(\Lambda, \mathbb{Z})$: δ_j is the cycle obtained by a trajectory of the Hamiltonian flow of I_j. Since μ is the holonomy of the $H_1(\cdot, \mathbb{Z})$-bundle over B_r, the parallel transport of (δ_1, δ_2) above the closed loop γ yields a new basis of $H_1(\Lambda, \mathbb{Z})$ equal to $\mu(\gamma)(\delta_1, \delta_2)$, which corresponds to the action variables $\mu(\gamma)(I_1, I_2)$, up to some constant in $\mathbb{R}^2$. In other words, $d\tilde{I}(\gamma \cdot \tilde{\Lambda}) = \mu(\gamma) \circ d\tilde{I}(\tilde{\Lambda})$. The transition formula (4) says

$$[w_{\mu(\gamma)(I_1,I_2)}] = {}^t\mu(\gamma) \circ [w_{(I_1,I_2)}].$$

But, because of uniqueness, one can assume that $[\tilde{w}]$ is defined by (7); hence $[\tilde{w}](\gamma \cdot \tilde{\Lambda}) = [w_{\mu(\gamma)(I_1,I_2)}](\Lambda)$, which yields the result. □

2.3 The semitoric case

A semitoric system, in a broad sense, is a two degree of freedom integrable Hamiltonian system with a global $\mathbb{S}^1$ symmetry; more precisely, the Hamiltonian H commutes with a smooth function J whose Hamiltonian flow $t \mapsto \varphi_J^t$ is 2π-periodic, except at fixed points. Under some additional hypothesis (which ensures, for instance, that the level sets of the energy-momentum map (J, H) are connected), such systems have been introduced in [47] and classified in [37, 38]. Recently, the hypothesis have been generalized, allowing to include famous examples like the Spherical Pendulum, see [36, 27].

In the present work, we don't need this classification, and hence we will use the terminology "semitoric" is a very general acceptance:

Definition 2.11 *An integrable system $F = (J, H)$ on a 4-dimensional symplectic manifold M will be called* semitoric *if the map $F : M \to \mathbb{R}^2$ is proper, and the function J is a momentum map for an effective Hamiltonian $\mathbb{S}^1$-action on M.*

The existence of such a global symmetry has the interesting consequence that the rotation number for the system $F = (J, H)$ is defined in a more natural way, and can be interpreted as an angle in $\mathbb{R}/\mathbb{Z}$:

Definition 2.12 *If Λ is a Liouville torus for the semitoric system (J, H), the (semitoric)* rotation number *$w(\Lambda) \in \mathbb{R}/\mathbb{Z}$ is defined as follows. Take a point $A \in \Lambda$ and let $\mathbb{S}^1_A$ be the orbit of this point under the flow of J. Consider the H-flow of A and denote by A' the first return point of the trajectory starting from A on the orbit $\mathbb{S}^1_A$. (See Figure 1.) Then $2\pi w(\Lambda)$ is the time necessary to flow, under the action of J, from A to A':*

$$\varphi_J^{2\pi w(\Lambda)}(A) = A'.$$

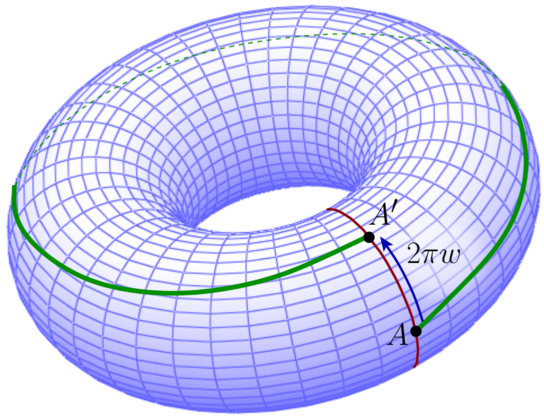

Figure 1: The semitoric rotation number.

This definition is of course not new. It was popularized in particular by Arnold in the treatment of perturbations of integrable systems, see for instance [2, Section 3 G]. In order to state in which sense this new rotation number coincides with the general rotation number from Definition 2.1 (more specifically Equation (2)), we recall the following basic fact from the theory of semitoric systems (see for instance [36, Lemma 5.1]):

Lemma 2.13 *Let (J, H) be a semitoric system, and let Λ be a Liouville torus. Then there exists a set of oriented action integrals near Λ of the form (J, I_2).*

It will be convenient to call such sets of action integrals "semitoric action variables".

Proposition 2.14 *Let (J, H) be a semitoric system, and let Λ be a Liouville torus. Let I be semitoric action variables near Λ. Let $w(\Lambda)$ be the semitoric rotation number of Definition 2.12, and let $w_I(\Lambda)$ be the rotation number from Equation (2). Then*

$$w(\Lambda) = (w_I(\Lambda) \mod \mathbb{Z}).$$

Proof. Since $I = (J, J_2)$, we have $\mathcal{X}_H = a_1\mathcal{X}_J + a_2\mathcal{X}_{I_2}$, for some smooth functions $a_j = a_j(\Lambda)$, such that $w_I(\Lambda) = a_1(\Lambda)/a_2(\Lambda)$. Notice that, since $\mathcal{X}_H$ and $\mathcal{X}_J$ must be linearly independent on Λ, we must have $a_2(\Lambda) \neq 0$. Hence w_I is a standard real number.

From Definition 2.12, there exists a positive time $\tau = \tau(\Lambda)$ such that the flow of $\mathcal{X}_H$ at time τ followed by the flow of $\mathcal{X}_J$ at time $-2\pi w(\Lambda)$ sends A to itself. In other words, the time-1 flow of

$$\mathcal{X}_{\text{per}} := \tau(\Lambda)\mathcal{X}_H - 2\pi w(\Lambda)\mathcal{X}_J \tag{8}$$

is the identity on Λ. By the theory of action-angle variables, $\mathcal{X}_{\text{per}}$ is the Hamiltonian vector field of the Hamiltonian

$$H_{\text{per}} := \int_\gamma \alpha,$$

where α is a primitive of the symplectic form in a neighborhood of Λ, and the cycle γ is the homology class of a periodic trajectory of H_{per} on Λ. Moreover, the action of $\mathcal{X}_{\text{per}}$ must be effective, otherwise there would be a periodic trajectory of $\mathcal{X}_{\text{per}}$ of period $1/2$, which would imply by (8) that the image of A by the flow of $\mathcal{X}_H$ at time $\tau(\Lambda)/2$ would return to the $\mathbb{S}^1$-orbit of A, contradicting the definition of $\tau(\Lambda)$. Hence the pair (J, H_{per}) is a set of action variables near Λ, which implies $H_{\text{per}} = \pm 2\pi I_2 + 2\pi k J$ for some $k \in \mathbb{Z}$. From (8) we get

$$\tau(\Lambda)\mathcal{X}_H = \pm 2\pi \mathcal{X}_{I_2} + 2\pi(w(\Lambda) - k)\mathcal{X}_J.$$

The assumption that $I = (J, I_2)$ is oriented means $\frac{\partial I_2}{\partial H} > 0$, which implies that only the sign $+2\pi I_2$ can occur, and hence we get $w_I(\Lambda) = w(\Lambda) - k$.

In concrete examples, computing the rotation number can be a hard task. But, even in the purely classical theory, it is an important piece of information about the system, related to the "tennis-racket effect" [12], and is intimately related to the symplectic invariants classifying semitoric systems, as demonstrated in the case of the spherical pendulum by [19].

□

3 The Quantum Rotation Number

It is very natural to try and define a "quantum rotation number" by replacing energies by "quantized energies". That this is indeed possible will follow from the mathematical version of the semiclassical Bohr-Sommerfeld rules, that we recall below. But first of all, we need to introduce the quantum version of the image of the map $F = (f, H)$, which is the joint spectrum [9].

3.1 Joint spectrum of commuting operators

Let P be a (possibly unbounded) selfadjoint operator acting on a Hilbert space $\mathcal{H}$. The spectral theorem constructs from P its so-called spectral measure, which is a projector-valued measure on $\mathbb{R}$. The support of the spectral measure is the spectrum of P, denoted by $\sigma(P)$. The spectrum of P in an interval $[E_1, E_2]$ is called discrete if any $\lambda \in \sigma(P)$ is isolated in $\sigma(P) \cap [E_1, E_2]$, and for any $\epsilon > 0$ small enough, the spectral projector associated with the interval $[\lambda - \epsilon, \lambda + \epsilon]$ has finite rank.

The selfadjoint operators $P_1, \ldots, P_k$ are said to pairwise commute if their spectral measures commute, which we denote by $[P_j, P_k] = 0$. In this case, one can define the joint spectral measure, and by definition the *joint spectrum* of $(P_1, \ldots, P_k)$ is the support (in $\mathbb{R}^k$) of this measure. We shall denote it by $\Sigma(P_1, \ldots, P_k)$. The joint spectrum is called discrete if all $\lambda \in \Sigma(P_1, \ldots, P_k)$ are isolated and, for any small enough compact neighborhood K of λ, the joint spectral projector on K has finite rank.

3.2 Bohr-Sommerfeld quantization rules for quantum integrable systems

Let P be a semiclassical $\hbar$-pseudo-differential operator on $X = \mathbb{R}^n$, or on a compact manifold X, where the semiclassical parameter $\hbar$ varies in an interval $]0, \hbar_0]$ for some $\hbar_0 > 0$. More precisely, in the $\mathbb{R}^n$ case, we will assume that P is the semiclassical Weyl quantization of a symbol in $S(m)$ where m is an order function in the sense of [15, Definition 7.4]: there exist $C > 0, N > 0$ such that

$$1 \leq m(X) \leq C\langle X - Y\rangle^N m(Y) \qquad \forall X, Y \in \mathbb{R}^{2n},$$

where $\langle X\rangle := (1 + \|X\|^2)^{1/2}$. For instance, one can take $m(X) := \langle X\rangle^d$, for some $d \in \mathbb{R}$. A function $a = a(x, \xi; \hbar) \in C^\infty(\mathbb{R}^{2n})$ belongs to the class $S(m)$ if, by definition,

$$\forall \alpha \in \mathbb{N}^{2n}, \forall \hbar \in (0, \hbar_0], \quad |\partial^\alpha a(x, \xi; \hbar)| \leq C_\alpha m(x, \xi),$$

and, in this paper, we shall always assume that such a symbol a admits an asymptotic expansion $a \sim \sum_{j\geq 0} \hbar^j a_j$ in non-negative integral powers of $\hbar$ in this topology: for any $N > 0$,

$$a(x, \xi; \hbar) - \sum_{j=0}^{N} \hbar^j a_j(x, \xi) \in \hbar^{N+1} S(m). \tag{9}$$

We then say that a pseudo-differential operator P belongs to $S(m)$ (using the same notation, since there will be in general no ambiguity) if its Weyl symbol belongs to $S(m)$. The first term a_0 in (9) is called the principal symbol of P. Other classes of pseudo-differential operators can be used, see [6], but we stick to $S(m)$ for its ease of use and ample documentation (see for instance [15, 34, 52]).

In the manifold case, we will assume that P belongs to the Kohn-Nirenberg class $S^d(X)$, ($d \in \mathbb{R}$), which means that in local coordinates, after cut-off in the position variable $x \in X$, P can be written as a pseudo-differential operator with a symbol $a(x, \xi; \hbar)$ such that (see for instance [52, Chapter 14])

$$\forall (\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n, \forall \hbar \in (0, \hbar_0], \quad \left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; \hbar)\right| \leq C_{\alpha,\beta} \langle \xi\rangle^{d-|\beta|}.$$

In this case, to unify notations, we call $m(x, \xi) := \langle \xi\rangle^d$ the corresponding order function, and denote $S(m) := S^d(X)$.

The operator P is said to be elliptic at infinity in $S(m)$ if $P \in S(m)$ and $|a(x, \xi; \hbar)| \geq c m(x, \xi)$ for some $c > 0$ and for large (x, ξ) in the $\mathbb{R}^n$ case; and similarly $|a_0(x, \xi)| \geq c\langle \xi\rangle^d$ in the manifold case.

A *quantum integrable system* is the data of n commuting selfadjoint pseudo-differential operators $P_1, \ldots, P_n$ in $S(m)$, such that the map of the principal symbols $F := (p_1, \ldots, p_n)$ defines a classical integrable system in T^*X. We shall always assume that F is proper; in case the order function m is unbounded, we make the stronger assumption that the operator $Q := P_1^2 + \cdots + P_n^2$

is elliptic at infinity in $S(m^2)$. Note that, for pseudo-differential operators, the commutation property $[P_j, P_k] = 0$ is equivalent to the weak commutation (see [6])

$$\forall u \in C_0^\infty(\mathbb{R}^n), \quad P_j P_k u = P_k P_j u.$$

In this case, for any $f \in C_0^\infty(\mathbb{R}^n)$, the operator $f(P_1, \dots, P_n)$ (obtained via the joint spectral measure) is a pseudo-differential operator in $S(m^{-N})$ for any $N \in \mathbb{N}$ ([15, Chapter 8]).

Typical examples of quantum integrable systems (with unbounded symbols) are given by two degree of freedom Schrödinger operators with an axi-symmetric potential; these can occur either on $\mathbb{R}^2$ [6], or on a Riemannian surface of revolution [9, 11], and can be used to obtain efficient numerical methods, see [3].

Proposition 3.1 ([6]) *If $P_1, \dots, P_n$ is a quantum integrable system, satisfying the above hypothesis, then its joint spectrum $\Sigma(P_1, \dots, P_n)$ is discrete when $\hbar$ is small enough.*

Proof. Let $\lambda := (\lambda_1, \dots, \lambda_n) \in \Sigma(P_1, \dots, P_n)$, and let $K \subset \mathbb{R}$ be a compact neighborhood of λ. Since F is proper, the map $f \circ F$ has compact support for any $f \in C_0^\infty(\mathbb{R}^n)$. Hence the pseudo-differential operator $f(P_1, \dots, P_n)$ is compact for small $\hbar$. Choosing $f \equiv 1$ on a neighborhood of K, we see that the spectral projector $1_K(P_1, \dots, P_n) = f(P_1, \dots, P_n)1_K(P_1, \dots, P_n)$ is compact; hence its rank is finite. ☐

The following result, which is a mathematical justification of the old Bohr-Sommerfeld rule from quantum mechanics, states that the joint spectrum of a quantum integrable system takes the form of an approximate lattice in any neighborhood of a regular value of F. It was first obtained by Colin de Verdière in the homogeneous setting [11], and then by Charbonnel in the semiclassical setting [6]. A purely microlocal approach was proposed in [44] (see also [46]). When $n = 1$ and P is a Schrödinger operator, explicit methods for second order ODEs can be employed, see [49].

Theorem 3.2 ([11, 6]) *Let $P_1, \dots, P_n$ is a quantum integrable system satisfying the above hypothesis. Let $c \in \mathbb{R}^n$ be a regular value of the principal symbol map $F = (p_1, \dots, p_n)$. Assume that the fiber $F^{-1}(c)$ is connected. Then we can describe the joint spectrum around c as follows.*

1. ***(joint eigenvalues have multiplicity one)*** *There exists a non empty open ball $B \subset \mathbb{R}^n$ (independent of $\hbar$) around c and $\hbar_0 > 0$ such that for any $\lambda \in \Sigma(P_1, \dots, P_n) \cap B$ and for all $\hbar < \hbar_0$, the joint spectral projector of $(P_1, \dots, P_n)$ onto the ball $B(\lambda, \hbar^2)$ has rank 1.*

2. ***(the joint spectrum is a deformed lattice)*** *There exists a bounded open set $U \subset \mathbb{R}^n$ and a smooth map $G_\hbar : U \to \mathbb{R}^n$ admitting an asymptotic expansion in the C^∞ topology:*

$$G_\hbar = G_0 + \hbar G_1 + \hbar^2 G_2 + \cdots$$

such that the joint eigenvalues in $\Sigma(P_1, \dots, P_n) \cap B$ are given by the quantities

$$\lambda = G_\hbar(\hbar k_1, \dots, \hbar k_n) + \mathscr{O}(\hbar^\infty), \tag{10}$$

where $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ *is such that* $G_\hbar(\hbar k_1, \ldots, \hbar k_n) \in B$*, and the* $\mathscr{O}(\hbar^\infty)$ *remainder is uniform as* $\hbar \to 0$*. Moreover,* G_0 *is a diffeomorphism from* U *onto a neighborhood of* $\overline{B}$ *given by the action-angle theorem:*

$$F = G_0(I_1, \ldots, I_n), \tag{11}$$

where $(I_1, \ldots, I_n)$ *is a set of action variables for the classical system, defined in a neighborhood of* $F^{-1}(c)$*, and* U *is a neighborhood of* $I_c = I(F^{-1}(c))$*.*

It is important to notice that, contrary to the classical case, one can not always assume that the actions $(I_1, \ldots, I_n)$ take values in a neighborhood of the origin (in other words, I_c is not necessarily zero). In fact I_c is given by the integrals of the Liouville 1-form of the cotangent bundle T^*X over a basis of cycles of the torus $F^{-1}(c)$.

Remark 3.3 That this statement can be generalized to multiple connected components of $F^{-1}(c)$ is "well known to specialists"; however to our knowledge that generalization cannot be found in the literature. We haven't included it here because it will be important for us to make the connectedness hypothesis throughout the paper. However, we expect it to become necessary for the future study of integrable systems with hyperbolic singularities, which are very common in the spectroscopy of small molecules (eg. LiNC/NCLi, see [29]), and related to the so-called fractional monodromy, see for instance [20]. △

Remark 3.4 We have not attempted here to extend our analysis to commuting Berezin-Toeplitz operators on prequantizable symplectic manifolds (see [32] for a nice introduction to Berezin-Toeplitz quantization). Thanks to the work of Charles [7], we believe that most of our results should be adaptable to that situation. △

3.3 Asymptotic lattices and good labelling

Sets of points in $\mathbb{R}^n$ that are described by an equation of the type of (10) can be called 'asymptotic lattices', see [43]. Indeed, when $\hbar$ is small enough, the map $G_\hbar : U \to \mathbb{R}^n$ is a diffeomorphism onto its image, and hence it sends the local lattice $\hbar\mathbb{Z}^n \cap U$ one-to-one into the joint spectrum $\Sigma(P_1, \ldots, P_n) \cap B \mod \mathscr{O}(\hbar^\infty)$. This also implies that for any strictly smaller ball $\tilde{B} \Subset B$, the map obtained by the restriction to lattice points

$$G_\hbar : \hbar\mathbb{Z}^2 \cap G_\hbar^{-1}(\tilde{B}) \to \Sigma(P_1, \ldots, P_n) \cap \tilde{B} \mod \mathscr{O}(\hbar^\infty)$$

is onto. The goal of this section is to make these observations precise by introducing the formal definition of asymptotic lattices and deriving some of their properties; this analysis partly builds on and extends [43].

Definition 3.5 *Let* $B \subset \mathbb{R}^n$ *be a simply connected bounded open set. Let* $\mathcal{L}_\hbar \subset B$ *be a discrete subset of* B *depending on the small parameter* $\hbar \in \mathcal{I}$*, where* $\mathcal{I} \subset \mathbb{R}_+^*$ *is a set of positive real numbers admitting* 0 *as an accumulation point. We say that* $(\mathcal{L}_\hbar, \mathcal{I}, B)$ *is an* asymptotic lattice *if the following two statements hold:*

1. *there exist* $\hbar_0 > 0$, $\epsilon_0 > 0$*, and* $N_0 \geq 1$ *such that for all* $\hbar \in \mathcal{I} \cap]0, \hbar_0]$*,*

$$\hbar^{-N_0} \min_{\substack{(\lambda_1,\lambda_2)\in\mathcal{L}_\hbar^2 \\ \lambda_1\neq\lambda_2}} \|\lambda_1 - \lambda_2\| \geq \epsilon_0;$$

2. *there exist a bounded open set* $U \subset \mathbb{R}^n$ *and a family of smooth maps* $G_\hbar : U \to \mathbb{R}^n$*, for* $\hbar \in \mathcal{I}$*, admitting an asymptotic expansion in the* $C^\infty(U)$ *topology:*

$$G_\hbar = G_0 + \hbar G_1 + \hbar^2 G_2 + \cdots \tag{12}$$

with $G_j \in C^\infty(U; \mathbb{R}^n)$*, for all* $j \geq 0$*. Moreover,* G_0 *is an orientation preserving diffeomorphism from* U *onto a neighborhood of* $\overline{B}$*, and*

$$G_\hbar(\hbar\mathbb{Z}^n \cap U) = \mathcal{L}_\hbar + \mathscr{O}(\hbar^\infty) \quad \textit{inside } B,$$

by which we mean: there exists a sequence of positive numbers $(C_N)_{N\in\mathbb{N}}$*, such that for all* $\hbar \in \mathcal{I}$*, the following holds.*

   (a) *For all* $\lambda \in \mathcal{L}_\hbar$ *there exists* $k \in \mathbb{Z}^n$ *such that* $\hbar k \in U$ *and*

$$\forall N \in \mathbb{N}, \qquad \|\lambda - G_\hbar(\hbar k)\| \leq C_N \hbar^N. \tag{13}$$

   (b) *For every open set* $\tilde{U}_0 \Subset G_0^{-1}(B)$*, there exists* $\tilde{\hbar}_0 > 0$ *such that for all* $\hbar \in \mathcal{I} \cap]0, \tilde{\hbar}_0]$*, for all* $k \in \mathbb{Z}^n$ *with* $\hbar k \in \tilde{U}_0$*, there exists* $\lambda \in \mathcal{L}_\hbar$ *such that* (13) *holds.*

Such a map $G_\hbar$ *will be called an* asymptotic chart *for* $\mathcal{L}_\hbar$*. We will also denote it by* $(G_\hbar, U)$*.*

Notice that the orientation preserving requirement is not a true restriction; indeed, one may pick an element $A \in \mathrm{GL}(n, \mathbb{Z})$ with $\det A = -1$, and then $(G_\hbar \circ A, A^{-1}U)$ is a good asymptotic chart if and only if $(G_\hbar, U)$ is an "orientation reversing asymptotic chart".

For shortening notation, we shall often simply call $\mathcal{L}_\hbar$ an asymptotic lattice, instead of the triple $(\mathcal{L}_\hbar, \mathcal{I}, B)$. Of course the simplest asymptotic lattice is the square lattice $\hbar\mathbb{Z}^n \cap B$, with chart $G_\hbar = \mathrm{Id}$. In general, if $G_\hbar$ is any map satisfying (12) with G_0 a diffeomorphism onto a neighborhood of $\overline{B}$, then $G_\hbar(\hbar\mathbb{Z}^n \cap U) \cap B$ is an asymptotic lattice.

With this definition, we may now rephrase Theorem 3.2 as follows.

Theorem 3.6 *With the hypothesis of Theorem 3.2, near any regular value* $c \in \mathbb{R}^n$ *with connected fiber, the joint spectrum of the quantum integral system* $(P_1, \ldots, P_n)$*, with* $\hbar \in \mathcal{I} =]0, \hbar_0]$ *is an asymptotic lattice.*

Let us state some elementary properties of asymptotic charts, for further reference. For subsets A, B of $\mathbb{R}^n$, we use the notation $A \Subset B$ to mean that $\overline{A}$ is compact and contained in B. For a map $G : U \to \mathbb{R}^n$, where $U \subset \mathbb{R}^d$ is an open set, we denote by $G' : U \to \mathcal{L}(\mathbb{R}^d; \mathbb{R}^n)$ the full derivative (or linear tangent map) of G.

Lemma 3.7 *Let $(\mathcal{L}_\hbar, \mathcal{I}, B)$ be an asymptotic lattice, and let $(G_\hbar, U)$ be an asymptotic chart for it. Let $\hbar_0 > 0$. The following properties of $G_\hbar$ hold.*

1. *All derivatives of $G_\hbar$ are bounded on any compact subset of U, uniformly for $\hbar \leq \hbar_0$:*

$$\forall \tilde{U} \Subset U, \forall \alpha \in \mathbb{N}^n, \exists C_{\tilde{U},\alpha} > 0, \forall \hbar \in \mathcal{I} \cap]0, \hbar_0], \quad \sup_{\tilde{U}} \|\partial^\alpha G_\hbar\| \leq C_{\tilde{U},\alpha}.$$

2. *For any $\tilde{U} \Subset U$, there exist non-negative constants M, M' such that*

$$\forall \hbar \in \mathcal{I} \cap]0, \hbar_0], \qquad \sup_{\tilde{U}} \|G_\hbar - G_0\| \leq \hbar M, \quad \sup_{\tilde{U}} \|G'_\hbar - G'_0\| \leq \hbar M'. \tag{14}$$

3. *For any $\tilde{U} \Subset U$, for any $N \geq 0$, there exists a non-negative constant R_N such that, for all $\hbar_1, \hbar_2 \in \mathcal{I} \cap]0, \hbar_0]$,*

$$\sup_{\tilde{U}} \|G_{\hbar_2} - G_{\hbar_1}\| \leq R_N(|\hbar_2 - \hbar_1| + \hbar_1^N + \hbar_2^N).$$

4. *For any $\tilde{U} \Subset U$, there exist non-negative constants L_0, L_1 such that, for all $\hbar \in \mathcal{I} \cap]0, \hbar_0]$, $\forall \xi_1, \xi_2 \in \tilde{U}$, if the segment $[\xi_1, \xi_2]$ is contained in $\tilde{U}$ then*

$$\|G_\hbar(\xi_2) - G_\hbar(\xi_1)\| \leq L_0 \|\xi_2 - \xi_1\| \tag{15}$$

$$\|G_\hbar(\xi_2) - G_\hbar(\xi_1) - G'_\hbar(\xi_1) \cdot (\xi_2 - \xi_1)\| \leq L_1 \|\xi_2 - \xi_1\|^2. \tag{16}$$

5. *The map $G_\hbar$ is formally invertible: there exists a smooth map $F_\hbar$, defined on a neighborhood of $\overline{B}$, that admits an asymptotic expansion*

$$F_\hbar = F_0 + \hbar F_1 + \hbar^2 F_2 + \cdots \tag{17}$$

such that

$$F_\hbar \circ G_\hbar = \mathrm{Id} + \mathscr{O}(\hbar^\infty), \quad \text{and} \quad G_\hbar \circ F_\hbar = \mathrm{Id} + \mathscr{O}(\hbar^\infty). \tag{18}$$

In particular, $F_0 = G_0^{-1}$.

6. *For any $\tilde{U} \Subset U$, there exists $\hbar_0 > 0$ such that for all $\hbar \in \mathcal{I} \cap]0, \hbar_0]$, the restriction of $G_\hbar$ to $\tilde{U}$ is invertible onto its image. Moreover, its inverse $G_\hbar^{-1}$ is smooth and admits an asymptotic expansion in powers of $\hbar$ equal to that of $F_\hbar$ in (17).*

7. *Given $\xi \in U$, the map $\hbar \mapsto G_\hbar(\xi)$ is not necessarily continuous. However, there exists a smooth map $\tilde{G} \in C^\infty([-\hbar_0, \hbar_0]) \times U)$ such that, for all $\xi \in U$, $\forall \hbar \in \mathcal{I} \cap]0, \hbar_0]$, $G_\hbar(\xi) = \tilde{G}(\hbar, \xi) + R_\hbar(\xi)$, and $R_\hbar = \mathscr{O}(\hbar^\infty)$ in the $C^\infty(U)$ topology. In particular, $\tilde{G}(0, \xi) = G_0(\xi)$.*

8. If $\tilde{G}_\hbar = G_\hbar + \mathscr{O}(\hbar^\infty)$ in $C^\infty(U)$, then $\tilde{G}_\hbar$ is also an asymptotic chart for $(\mathcal{L}_\hbar, \mathcal{I}, B)$.

Proof. Items 1, 2 and 3 directly follow from the asymptotic expansion (12). Taylor's formula and Item 1 gives Item 4. Item 5 is a standard consequence of (12): the functions F_j can be defined by induction, using Taylor expansions, and $F_\hbar$ is obtained by the Borel lemma. Item 7 also follows from a Borel summation of the formal series $\sum \hbar^k G_k$. To prove Item 6, observe that near any $\xi \in U$, if $\hbar$ is small enough, $G'_\hbar(\xi)$ is invertible due to the invertibility of G'_0 and (14). Hence on a compact subset of U, we may find $\hbar_0$ such the local inversion theorem applies if $\hbar < \hbar_0$, showing that $G_\hbar$ is a local diffeomorphism for all $\hbar \in \mathcal{I} \cap]0, \hbar_0]$. Thus, there exists $\delta > 0$ such that for any $\xi_0 \in \tilde{U}$, the restriction of $G_\hbar$ to the ball $B(\xi_0, \delta)$ is a diffeomorphism onto its image. If follows from (18) that there exists $\hbar_0 > 0$ depending only on $G_\hbar$ and $\tilde{U}$ such that

$$\forall \hbar \in \mathcal{I} \cap]0, \hbar_0], \quad \forall \xi \in \tilde{U}, \quad \|F_\hbar \circ G_\hbar(\xi) - \xi\| \leq \delta/3.$$

Hence $G_\hbar(\xi_1) = G_\hbar(\xi_2)$ implies $\|\xi_1 - \xi_2\| \leq 2\delta/3$, which in turn implies $\xi_1 = \xi_2$, proving the injectivity of $G_\hbar$ on $\tilde{U}$, which gives the result. Notice that, by choosing $\tilde{U}$ large enough, we can always ensure that $G_\hbar(\tilde{U})$ contains a neighborhood of $\overline{B}$. □

Remark 3.8 As usual in semiclassical analysis, equality of sets "modulo $\mathscr{O}(\hbar^\infty)$" has to be taken with care. Given an asymptotic lattice $(\mathcal{L}_\hbar, \mathcal{I}, B)$, because of boundary effects, it is not true that the Hausdorff distance between $\mathcal{L}_\hbar$ and $B \cap G_\hbar(\hbar\mathbb{Z}^n \cap U)$ is $\mathscr{O}(\hbar^\infty)$. Consider the following example: $n = 1$, $B =]0, 1[$, $\mathcal{I} =]0, 1]$, $\mathcal{L}_\hbar = \hbar\mathbb{Z} \cap B$. One can check that $(\mathcal{L}_\hbar, \mathcal{I}, B)$ is an asymptotic lattice with chart $G_\hbar = \mathrm{Id} : U =]-1, 2[\to U$. By virtue of Item 8 of Lemma 3.7, the map $\tilde{G}_\hbar(\xi) = \xi + e^{-1/\hbar}$ is again an asymptotic chart for $\mathcal{L}_\hbar$. However, if $\hbar$ is small enough, $\min G_\hbar(\hbar\mathbb{Z} \cap U) \cap B = e^{-1/\hbar}$ while $\min \mathcal{L}_\hbar = \hbar$. So, in this case, the Hausdorff distance is $\hbar - e^{-1/\hbar}$, which is not $\mathscr{O}(\hbar^\infty)$. △

Remark 3.9 It follows from Item 7 of Lemma 3.7 that, up to changing U for a smaller $U' \Subset U$, one can always choose an asymptotic chart $G_\hbar$ that is smooth with respect to $\hbar$. This, in turns, improves Item 3 by suppressing the term $\hbar_1^N + \hbar_2^N$. △

We now turn to the description of the asymptotic lattices themselves. Using an asymptotic chart, we can put an integer label on each point of the lattice:

Lemma 3.10 *If $G_\hbar$ is an asymptotic chart for $\mathcal{L}_\hbar$, then there exists a family of maps $k_\hbar : \mathcal{L}_\hbar \mapsto \mathbb{Z}^n$, $\hbar \in \mathcal{I}$, which is unique for $\hbar$ small enough, and such that* (13) *holds with $k = k_\hbar(\lambda)$,* i.e. *$\hbar k_\hbar(\lambda) \in U$ and*

$$\forall \lambda \in \mathcal{L}_\hbar, \quad \forall N \geq 0, \quad \|G_\hbar(\hbar k_\hbar(\lambda)) - \lambda\| \leq C_N \hbar^N. \tag{19}$$

Moreover this map is injective.

Proof. From Item 6 of Lemma 3.7, let $\hbar_0 > 0$ and $\tilde{U} \Subset U$ be such that, for all $\hbar \leq \hbar_0$, $G_\hbar : \tilde{U} \to B_1$ is invertible onto B_1, a fixed bounded open neighborhood of $\overline{B}$. Let $\lambda \in \mathcal{L}_\hbar$. If k and $\tilde{k}$ in $\mathbb{Z}^n$ satisfy (13) (for some arbitrary family $(C_N)_{N\geq 0}$), then $G_\hbar(\hbar k)$ and $G_\hbar(\hbar\tilde{k})$ belong to $B + B(0, C_N \hbar^N)$. We can assume that $\hbar_0$ is small enough so that the latter is contained in B_1. Thus, $\hbar k$ and $\hbar \tilde{k}$ belong to $\tilde{U}$. Moreover, $\|G_\hbar(\hbar k) - G_\hbar(\hbar \tilde{k})\| \leq 2C_N \hbar^N$, which entails $\|\hbar k - \hbar \tilde{k}\| \leq 2L_F C_N \hbar^N$, where L_F is a uniform upper bound for the Lipschitz norm of $G_\hbar^{-1}$ on $\tilde{U}$. Choose any $N_1 > 1$ and take $\hbar_0$ small enough so that

$$2L_F C_{N_1} \hbar_0^{N_1 - 1} < 1\,; \tag{$\hbar$-20}$$

we obtain, for any $\hbar \leq \hbar_0$, $\|k - \tilde{k}\| < 1$ and hence $k = \tilde{k}$. This shows that, for any $\hbar \leq \hbar_0$, any $\lambda \in \mathcal{L}_\hbar$ is associated with a unique $k \in \mathbb{Z}^n$ such that (13) holds. We call this map $k_\hbar$. Notice that $k_\hbar = \frac{1}{\hbar} G_\hbar^{-1} \upharpoonright_{\mathcal{L}_\hbar} + \mathscr{O}(\hbar^\infty)$. In order to prove injectivity, assume $k_\hbar(\lambda_1) = k_h(\lambda_2)$. From (13) again we now get $\|\lambda_1 - \lambda_2\| \leq 2C_N \hbar^N$ for all N. Let (ϵ_0, N_0) be the constants defined by item 1 of Definition 3.5; if

$$2C_N \hbar^N < \epsilon_0 \hbar^{N_0}\,, \tag{$\hbar$-21}$$

which will happen if one chooses $N > N_0$ and $\hbar_0$ small enough, we conclude that $\lambda_1 = \lambda_2$. □

Remark 3.11 Note that a consequence of the above lemma (and its proof) is that, eventually, Condition 1 in the definition of asymptotic lattices (Definition 3.5) holds with $N_0 = 1$, provided $\hbar_0$ is small enough. Indeed, let $\lambda_1, \lambda_2 \in \mathcal{L}_\hbar$ and let $k_i = k_\hbar(\lambda_i)$, $i = 1, 2$: for all $N \geq 0$, $\|\lambda_i - G_\hbar(\hbar k_i)\| \leq C_N \hbar^N$. Then if $\|\lambda_1 - \lambda_2\| < \epsilon \hbar$ for some $\epsilon > 0$, we obtain

$$\|G_\hbar(\hbar k_1) - G_\hbar(\hbar k_2)\| < 2C_N \hbar^N + \epsilon \hbar,$$

and hence $\|\hbar k_1 - \hbar k_2\| < L_F(2C_N \hbar^{N-1} + \epsilon)\hbar$. Therefore, we may choose any $N > 1$, (and, as above, any $N_1 > 1$, $N_2 > N_0$), and conclude by the injectivity of $k_\hbar$ that $\lambda_1 = \lambda_2$ as soon as ϵ and $\hbar_0$ verify

$$\hbar_0^{N_1 - 1} < \frac{1}{2L_F C_{N_1}}\,, \quad \hbar_0^{N_2 - N_0} < \frac{\epsilon_0}{2C_{N_2}}\,, \quad \text{and} \quad \epsilon + 2C_N \hbar_0^{N-1} \leq \frac{1}{L_F}\,. \tag{$\hbar$-22}$$

In other words, if ϵ and $\hbar_0$ satisfy ($\hbar$-22), then

$$\forall \hbar \in \mathcal{I} \cap\,]0, \hbar_0] \quad \forall \lambda \in \mathcal{L}_\hbar, \quad B(\lambda, \epsilon\hbar) \cap \mathcal{L}_\hbar = \{\lambda\}\,. \tag{23}$$

In order to simplify the statement of our results, we could fix from now on $N_1 = N = 2$ and $N_2 = N_0 + 1$, and ($\hbar$-22) would be satisfied if we took $\epsilon = \frac{1}{3L_F}$ and

$$\hbar_0 = \min\left(\frac{\epsilon_0}{4C_{N_0+1}}\,;\, \frac{1}{6L_F C_2}\right).$$

However, we believe that in some situations, having the possibility to optimize such estimates by taking large N's can be useful. △

Remark 3.12 A completely similar argument shows that for all $\lambda \in \mathcal{L}_\hbar$, the integer vector $k_\hbar(\lambda)$ defined in Lemma 3.10 is the unique element $k \in \mathbb{Z}^n$ such that $\hbar k \in U$ and

$$\|\lambda - G_\hbar(k\hbar)\| \leq \epsilon\hbar$$

as soon as $L_F(C_N\hbar^{N-1}+\epsilon)<1$ (which is implied by ($\hbar$-22)). Of course, this, in turn, implies the much better estimate (19). $\triangle$

Definition 3.13 *The map $k_\hbar$ from Lemma 3.10 will be called a* good labelling *of $\mathcal{L}_\hbar$.*

Let $\tilde{B}\Subset B$. Then a good labelling is surjective onto $(\frac{1}{\hbar}G_0^{-1}(\tilde{B}))\cap\mathbb{Z}^n$, in a uniform way: by Item 2b of Definition 3.5 with $\tilde{U}_0=G_0^{-1}(\tilde{B})$, there exists $\tilde{\hbar}_0>0$ such that, for all $\hbar\leq\tilde{\hbar}_0$ there exists $\lambda\in\mathcal{L}_\hbar$ with $\lambda=G_\hbar(\hbar k)+\mathscr{O}(\hbar^\infty)$, and hence by Lemma 3.10, $k_\hbar(\lambda)=k$, as soon as $\hbar_0$ satisfies ($\hbar$-20).

It follows that the set $\mathcal{L}_\hbar$ is always "dense in B as $\hbar\to 0$", by which we mean the following:

Lemma 3.14 *For any $c\in B$, there exists a family $(\lambda_\hbar)_{\hbar\in\mathcal{I}}$ with $\lambda_\hbar\in\mathcal{L}_\hbar$ such that*

$$\lambda_\hbar=c+\mathscr{O}(\hbar).$$

Proof. If $k(\hbar):=\lfloor\frac{\xi}{\hbar}\rfloor$ (the vector obtained by taking the integer part of all the components of $\frac{\xi}{\hbar}$), where $\xi:=G_0^{-1}(c)$, then $\|\hbar k(\hbar)-\xi\|\leq\hbar$. Thus, if $\tilde{B}\Subset B$ is a neighborhood of c, then $\hbar k(\hbar)\in G_0^{-1}(\tilde{B})$ for $\hbar$ small enough, and we may define $\lambda_\hbar$ to be the point in $\mathcal{L}_\hbar$ associated with the label $k(\hbar)$. Since $\|G_\hbar(\hbar k(\hbar))-c\|\leq L_0\|\hbar k(\hbar)-\xi\|+\|G_\hbar(\xi)-c\|\leq L_0\hbar+M\hbar$, where M is defined in (14), we get, for any $N\geq 0$, $\|\lambda_\hbar-c\|\leq(L_0+M)\hbar+C_N\hbar^N$. $\square$

Equipped with a good labelling (or, equivalently, with an asymptotic chart), an asymptotic lattice possesses the interesting feature that its individual points inherit a well-defined "smooth" evolution as $\hbar$ varies, in the following sense. Let $\lambda_\hbar\in\mathcal{L}_\hbar$. If $\hbar_0$ is small enough, then by the injectivity of Lemma 3.10 and (23), for any $\hbar\leq\hbar_0$, $\lambda_\hbar$ is the unique closest point to $G_\hbar(\hbar k_\hbar(\lambda_\hbar))$ in $\mathcal{L}_\hbar$. Thus, we may now fix the integer $k_0=k_{\hbar_0}(\lambda_{\hbar_0})$ and consider the evolution of the corresponding point $k_\hbar^{-1}(k_0)\in\mathcal{L}_\hbar$ as $\hbar$ varies close to $\hbar_0$. Although this evolution may not be continuous, it is $\mathscr{O}(\hbar^\infty)$-close to the smooth map $\hbar\mapsto\tilde{G}(\hbar,\hbar k_0)$, where $\tilde{G}$ is the smooth representative introduced in Lemma 3.7, Item 7. Notice that the typical behaviour of the point $G_\hbar(\hbar k_0)$, as $\hbar\to 0$, is to *leave* the neighborhood where the chart $G_\hbar$ is meaningful. Indeed, $\hbar k_0\to 0\in\mathbb{R}^n$, and there is no reason why 0 should belong to U. This is analogous to the well-known phenomenon that occurs for eigenvalues of operators depending on a parameter, when we restrict our attention to a fixed spectral window. As a way to disregard this "drift", it will be useful to define the notion of a good labelling "modulo a constant", as follows.

Definition 3.15 *A map $\bar{k}_\hbar:\mathcal{L}_\hbar\to\mathbb{Z}^n$ is called a* linear labelling *for $\mathcal{L}_\hbar$ if there exists a family $(\kappa_\hbar)_{\hbar\in]0,\hbar_0]}$ of vectors in $\mathbb{Z}^n$ such that $\bar{k}_\hbar+\kappa_\hbar$ is a good labelling for $\mathcal{L}_\hbar$.*

Of course any good labelling is a linear labelling; we shall see in Section 3.6 below that linear labellings may be easier to construct.

Another important property of asymptotic lattices is that they can be equipped with an "asymptotic $\mathbb{Z}^n$-action" in the following sense. Fix a good labelling $k_\hbar$ for $\mathcal{L}_\hbar$. Let $\tilde{B}\Subset B$ be open, and let $\kappa\in\mathbb{Z}^n$ be a fixed integral vector. Let $\hat{B}$ be an open set such that $\tilde{B}\Subset\hat{B}\Subset B$. If $\hbar$ is small enough (depending on $\tilde{B}$, $\hat{B}$, and κ), for any $\lambda \in \mathcal{L}_\hbar \cap \tilde{B}$, we have $\hbar(k_\hbar(\lambda) + \kappa) \in G_0^{-1}(\hat{B})$; therefore, there exists a unique point in $\mathcal{L}_\hbar$, denoted by $\lambda \underset{k_\hbar}{+} \kappa$, such that

$$k_\hbar(\lambda \underset{k_\hbar}{+} \kappa) = k_\hbar(\lambda) + \kappa. \tag{24}$$

This property characterizes linear labellings, as follows.

Proposition 3.16 *Let $\mathcal{L}_\hbar$ be an asymptotic lattice in the open set B with good labelling $k_\hbar$. Let $\tilde{B} \Subset B$ be open and connected. Let a map $\tilde{k}_\hbar : \mathcal{L}_\hbar \to \mathbb{Z}^n$, defined for $\hbar \in \mathcal{I}$, commute with translations in the following sense: for any $\kappa \in \mathbb{Z}^n$, there exists $\hbar_\kappa > 0$ such that*

$$\forall \hbar \leq \hbar_\kappa, \quad \forall \lambda \in \mathcal{L}_\hbar \cap \tilde{B}, \qquad \tilde{k}_\hbar(\lambda \underset{k_\hbar}{+} \kappa) = \tilde{k}_\hbar(\lambda) + \kappa. \tag{25}$$

Then $\tilde{k}_\hbar$ is a linear labelling for $(\mathcal{L}_\hbar \cap \tilde{B}, \mathcal{I}, \tilde{B})$ (associated with the good labelling $k_\hbar$ restricted to $\tilde{B}$).

In order to prove this proposition, we first show that $\mathcal{L}_\hbar \cap \tilde{B}$ is 'connected by lattice paths'.

Lemma 3.17 *Let $\mathcal{L}_\hbar$, $\hbar \in \mathcal{I}$, be an asymptotic lattice in an open set B. Let $\tilde{B} \Subset \hat{B} \Subset B$, where $\tilde{B}$ and $\hat{B}$ are open and $\tilde{B}$ is connected. Given a good labelling $k_\hbar$ for $\mathcal{L}_\hbar$, there exists $\hbar_0 > 0$ such that, given any $\hbar \in \,]0, \hbar_0] \cap \mathcal{I}$, and given any pair of points $z_1, z_2 \in \mathcal{L}_\hbar \cap \tilde{B}$, there exists a finite sequence $(\epsilon^{(j)})_{j \in \{1,\dots,N\}}$ in $\{-1, 0, 1\}^n$, and a finite sequence of points $\lambda_j \in \mathcal{L}_\hbar \cap \hat{B}$, $j = 0, \dots, N$, such that*

$$\lambda_0 = z_1\,, \quad \forall j = 0, \dots, N-1, \quad \lambda_{j+1} = \lambda_j \underset{k_\hbar}{+} \epsilon^{(j+1)}\,, \quad \textit{and} \quad \lambda_N = z_2\,. \tag{26}$$

Proof. Let $(G_\hbar, U)$ be an asymptotic chart for $\mathcal{L}_\hbar$ associated with $k_\hbar$ (see Lemma 3.10). Let $c \in \tilde{B}$. Let $\tilde{V} \subset U$ be an open cube centered at $\xi = G_0^{-1}(c)$. Let us first prove the 'lattice-connectedness' property (26) for $\tilde{B} = G_0(\tilde{V})$. Let $\hat{V} \subset U$ be another open cube containing $\tilde{V}$. It follows from the existence and the asymptotic expansion of $G_\hbar^{-1}$ (Item 6 of Lemma 3.7) that $G_\hbar^{-1}(\tilde{B}) \subset \hat{V}$, if $\hbar$ is small enough. Therefore, if $V \subset U$ is another open cube containing $\hat{V}$, Lemma 3.10 implies that, for $\hbar$ small enough, $\hbar k_\hbar(\lambda) \subset V$ for any $\lambda \in \mathcal{L}_\hbar \cap \tilde{B}$, and $k_\hbar^{-1}((\hbar^{-1} V) \cap \mathbb{Z}^n) \subset \hat{B}$. In $V \cap \hbar\mathbb{Z}^n$, any two points ξ_1, ξ_2 can obviously be joined by acting by the canonical basis of the lattice. Applying $k_\hbar^{-1}$, this shows the required property (26). The number N of steps in (26) can be as large as $\mathcal{O}(1/\hbar)$, but the set of involved κ's in $\{-1, 0, 1\}^n$ being finite, we do get an $\hbar_0 > 0$ for which the result is uniform for all $\hbar \in \mathcal{I} \cap \,]0, \hbar_0]$.

If $\tilde{B} \subset B$ is a general connected open set, with compact closure contained in B, we can cover it by a finite number of deformed cubes of the form $G_0(\tilde{V}) \Subset \hat{B}$, as above. If $\hbar$ is small enough, the intersection of two such deformed cubes is either empty or contains a point in $\mathcal{L}_\hbar$. Therefore the connectedness property holds for the union of two deformed cubes with a common point, and by induction for the union of all deformed cubes, hence for $\tilde{B}$. □

Proof *of Proposition 3.16.* Let $\hbar_0 = \min_{\kappa\in\{-1,0,1\}^n} \hbar_\kappa$. We may also assume that $\hbar_0$ is small enough so that Lemma 3.17 holds. Hence, applying (25) to (26) we obtain $\tilde{k}_\hbar(\lambda_{j+1}) = \tilde{k}_\hbar(\lambda_j) + \epsilon^{(j+1)}$, and hence

$$\tilde{k}_\hbar(z_2) = \tilde{k}_\hbar(z_1) + \sum_{j=1}^{N} \epsilon^{(j)} .$$

On the other hand, if we consider the good labelling $k_\hbar$, the translation invariance property (24) gives similarly

$$k_\hbar(z_2) = k_\hbar(z_1) + \sum_{j=1}^{N} \epsilon^{(j)} .$$

Therefore, $\tilde{k}_\hbar(z_2) - k_\hbar(z_2) = \tilde{k}_\hbar(z_1) - k_\hbar(z_1)$, for all $z_1, z_2 \in \tilde{B}$ and $\hbar \leq \hbar_0$; so $\tilde{k}_\hbar - k_\hbar$ is equal to a constant $K(\hbar) \in \mathbb{Z}^n$ on $\tilde{B}$. Hence $\tilde{k}_\hbar = k_\hbar - K(\hbar)$ is a linear labelling for $\mathcal{L}_\hbar$ on $\tilde{B}$. □

Theorem 3.2 says that the joint spectrum of a quantum integrable system is an asymptotic lattice near any regular point of the momentum map; in other words, the joint spectrum possesses "good quantum numbers" $(k_1, \ldots, k_n)$, which is formalized in Definition 3.13 as the existence of a good labelling. This property was used in [43] to define the notion of *quantum monodromy*.

In this paper, we shall need to go one step further, namely we will ask the question: can one construct a good labelling from the joint spectrum only? Indeed, a good labelling is not unique, according to the following, straightforward, result:

Lemma 3.18 *If $G_\hbar$ is an asymptotic chart for $\mathcal{L}_\hbar$, defined on an open set U, then for any orientation preserving linear transformation with integer coefficients $A \in \mathrm{SL}(n, \mathbb{Z})$, the map $\tilde{G}_\hbar := G_\hbar \circ A$ is another asymptotic chart for $\mathcal{L}_\hbar$, defined on $A^{-1}U$, and corresponding to the new good labelling $\tilde{k}_\hbar = A^{-1} \circ k_\hbar$.*

Proposition 3.19 *If $\bar{k}_\hbar^{(1)}$ and $\bar{k}_\hbar^{(2)}$ are two linear labellings for an asymptotic lattice $(\mathcal{L}_\hbar, \mathcal{I}, B)$, then for any connected open set $\tilde{B} \Subset B$, there exists a unique matrix $A \in \mathrm{SL}(n, \mathbb{Z})$, independent of $\hbar$, a family $(\kappa_\hbar)_{\hbar\in\mathcal{I}}$ in $\mathbb{Z}^n$, and $\hbar_0 > 0$ such that*

$$\forall \hbar \in \mathcal{I} \cap [0, \hbar_0], \qquad \bar{k}_\hbar^{(2)} = A \circ \bar{k}_\hbar^{(1)} + \kappa_\hbar \qquad \text{on} \quad \mathcal{L}_\hbar \cap \tilde{B}.$$

Proof. (a) Let $G_\hbar^{(1)}$ and $G_\hbar^{(2)}$ be asymptotic charts associated with the linear labellings $\bar{k}_\hbar^{(1)}$ and $\bar{k}_\hbar^{(2)}$, respectively, and let $k_\hbar^{(1)}$ and $k_\hbar^{(2)}$ be the corresponding good labellings. Let $\lambda_0 \in \mathcal{L}_\hbar$ and choose it as an "origin" of the linear labellings: up to changing their constant term, one can assume that $\bar{k}_\hbar^{(1)}(\lambda_0) = \bar{k}_\hbar^{(2)}(\lambda_0) = 0 \in \mathbb{Z}^n$. Let $\mathcal{B} = (e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{Z}^n$, and let $(\lambda_1^{(j)}, \ldots, \lambda_n^{(j)})$, $j = 1, 2$, be the corresponding points in $\mathcal{L}_\hbar$ for the two labellings, namely:

$$\bar{k}_\hbar^{(j)}(\lambda_i^{(j)}) = e_i. \tag{27}$$

To make sure these points do exist, one can impose λ_0 to stay in a $\mathscr{O}(\hbar)$ neighborhood of a fixed point $c \in \tilde{B}$; then by Item 6 of Lemma 3.7, $(G_\hbar^{(j)})^{-1}(\lambda_0)$ will stay in an $\mathscr{O}(\hbar)$ neighborhood of $\xi^{(j)} := (G_0^{(j)})^{-1}(c)$. Hence, by (19), the same holds for $\hbar k_\hbar^{(j)}(\lambda_0)$. Denote $k_0^{(j)} := k_\hbar^{(j)}(\lambda_0)$. Let $\tilde{U}_0^{(j)} \Subset (G^{(j)})^{-1}(B)$ containing $\xi^{(j)}$, and let $\hbar$ be small enough so that $\tilde{U}_0^{(j)}$, for $j = 1, 2$, contains $\hbar k_0^{(j)} + \hbar\mathcal{B}$. By Item 2b of Definition 3.13, the points $\lambda_i^{(j)}$ are now well-defined by (27), because $\bar{k}_\hbar^{(j)}(\lambda_i^{(j)}) = \bar{k}_\hbar^{(j)}(\lambda_i^{(j)}) - \bar{k}_\hbar^{(j)}(\lambda_0) = k_\hbar^{(j)}(\lambda_i^{(j)}) - k_\hbar^{(j)}(\lambda_0)$, and hence $\lambda_i^{(j)}$ is defined by

$$\hbar k_\hbar^{(j)}(\lambda_i^{(j)}) = \hbar k_0^{(j)} + \hbar e_i \in \tilde{U}_0. \tag{28}$$

(b) Now, let $v_i^{(j)} = (\lambda_i^{(j)} - \lambda_0)$, $i = 1, \ldots, n$, be the corresponding "basis vectors" of the asymptotic lattice. By the uniform Taylor formula (16), and (14),

$$v_i^{(1)} = \hbar(G_\hbar^{(1)})'(\hbar k_0^{(1)}) \cdot e_i + \mathscr{O}(\hbar^2) = \hbar(G_0^{(1)})'(\xi^{(1)}) \cdot e_i + \mathscr{O}(\hbar^2)\,. \tag{29}$$

The point $\lambda_i^{(1)}$ can also be labelled by $k_\hbar^{(2)}$; let

$$z_i := k_\hbar^{(2)}(\lambda_i^{(1)}) - k_0^{(2)} \in \mathbb{Z}^n. \tag{30}$$

Remember that z_i, contrary to e_i, depends on $\hbar$. Similarly, we have

$$v_i^{(1)} = \hbar(G_0^{(2)})'(\xi^{(2)}) \cdot z_i + \mathscr{O}(\hbar^2)\,,$$

and hence

$$e_i = [(G_0^{(1)})'(\xi^{(1)})]^{-1}(G_0^{(2)})'(\xi^{(2)}) \cdot z_i + \mathscr{O}(\hbar)\,. \tag{31}$$

Let $d_2 = d_2(\hbar) := \det(z_1, \ldots, z_n) \in \mathbb{N}$. Let $\delta_{1,2} := \det\Big([(G_0^{(1)})'(\xi^{(1)})]^{-1}(G_0^{(2)})'(\xi^{(2)})\Big)$. From (31), $1 = \delta_{1,2}d_2 + \mathscr{O}(\hbar)$. Now repeat the argument with exchanging the two labellings: we obtain a new integer $d_1 = d_1(\hbar) \in \mathbb{N}$ such that $1 = \delta_{2,1}d_1 + \mathscr{O}(\hbar)$, with $\delta_{2,1} = \delta_{1,2}^{-1}$. Hence $d_1 d_2 = 1 + \mathscr{O}(\hbar)$, which implies $d_1 d_2 = 1$ if $\hbar$ is small enough, and hence $d_2 = d_1 = 1$. This shows that $(z_1, \ldots, z_n)$ is an oriented $\mathbb{Z}$-basis of $\mathbb{Z}^n$; let $A \in \mathrm{SL}(n, \mathbb{Z})$ be the corresponding matrix. Equation (31) gives $[(G_0^{(2)})'(\xi^{(2)})]^{-1}(G_0^{(1)})'(\xi^{(1)}) = A + \mathscr{O}(\hbar)$, and since the left-hand side does not depend on $\hbar$, A must converge towards it as $\hbar \to 0$, but since A has integer coefficients, A must be constant for $\hbar$ small enough, and we have

$$A = [(G_0^{(2)})'(\xi^{(2)})]^{-1}(G_0^{(1)})'(\xi^{(1)}) \in \mathrm{SL}(n, \mathbb{Z})\,. \tag{32}$$

From (30) and (28), we have

$$k_\hbar^{(2)}(\lambda_i^{(1)}) = A \cdot e_i + k_0^{(2)} = A \cdot k_\hbar^{(1)}(\lambda_i^{(1)}) - A \cdot k_0^{(1)} + k_0^{(2)}\,,$$

which says that

$$k_\hbar^{(2)}(\lambda) = A \circ k_\hbar^{(1)}(\lambda) + \kappa_h \tag{33}$$

with $\kappa_\hbar := k_0^{(2)} - A \cdot k_0^{(1)}$, *when* λ *is restricted to the set* $\{\lambda_1^{(1)}, \dots \lambda_n^{(1)}\}$. It remains to extend this equality to the whole set $\mathcal{L}_\hbar$.

(c) We cannot directly adapt the argument, replacing $\lambda_i^{(1)}$ by *any* $\lambda \in \mathcal{L}_\hbar$, because the corresponding integral vector z_i from (30) would be unbounded (typically, of size $1/\hbar$), which would make the Taylor formula unusable. Instead, we need a connectedness argument, as in Proposition 3.16. First, remark that we may now replace in (28) the vector e_i by a linear combination $e := \sum n_i e_i$, where n_i are bounded integers, defining a lattice point μ by

$$\hbar k_\hbar^{(1)}(\mu) = \hbar k_0^{(1)} + \hbar e \in \tilde{U}_0 \,.$$

Let $v := \mu - \lambda_0$. From

$$v = \hbar (G_0^{(1)})'(\xi^{(2)}) \cdot e + \mathscr{O}(\hbar^2) \,,$$

and comparing with (29), we see that $v = \sum n_i v_i^{(1)} + \mathscr{O}(\hbar)$. Letting $z := k_\hbar^{(2)}(\mu) - k_0^{(2)}$, we see from

$$v = \hbar (G_0^{(2)})'(\xi^{(2)}) \cdot z + \mathscr{O}(\hbar^2) \,,$$

that $z = \sum n_i z_i + \mathscr{O}(\hbar)$. Since the integers n_i are bounded, we must have $z = \sum n_i z_i$ for $\hbar$ small enough. This gives

$$k_\hbar^{(2)}(\mu) = z + k_0^{(2)} = A \cdot e + k_0^{(2)} \,.$$

Therefore, the equality (33) still holds for μ, *i.e.* for all points of the form $\lambda = \lambda_0 \underset{k_\hbar^{(1)}}{+} n$, where n is uniformly bounded in $\mathbb{Z}^n$. Let us restrict such n to belong to $\{-1, 0, 1\}^n$. This means that the set of points satisfying (33) is invariant under the action of $\{-1, 0, 1\}^n$. From Lemma 3.17, this set must be $\mathcal{L}_\hbar \cap \tilde{B}$. □

We now want to extend this discussion to good labellings. In order to control the "constant term", we shall need a new assumption.

Definition 3.20 *An asymptotic lattice is called* $\hbar$-continuous *if the set* $\mathcal{I}$*, which the small parameter* $\hbar$ *belongs to, is such that the set* $\{\frac{1}{\hbar_1} - \frac{1}{\hbar_2}; \quad \hbar_1, \hbar_2 \in \mathcal{I}\}$ *accumulates at zero, namely:*

For all $\epsilon > 0$*, there exists* $\hbar_0 > 0$ *such that for all* $\hbar_1 \in \overline{\mathcal{I}} \setminus \{0\}$ *with* $\hbar_1 < \hbar_0$*, there exists* $\hbar_2 \in \mathcal{I}$*,* $\hbar_2 < \hbar_1$ *such that*

$$\frac{1}{\hbar_2} - \frac{1}{\hbar_1} < \epsilon. \tag{34}$$

Of course, this property is satisfied if $\mathcal{I} = (0, \hbar_0]$, because the map $\hbar \to \frac{1}{\hbar}$ is continuous. It also holds if the closure $\overline{\mathcal{I}}$ contains $[0, \hbar_0]$ for some $\hbar_0 > 0$. However, it is not satisfied if $\mathcal{I} = \{1/k; \quad k \in \mathbb{N}^*\}$, which is typical in geometric quantization. For $\mathcal{I} = \{1/k^\beta; \quad k \in \mathbb{N}^*\}$, the property is satisfied if and only if $\beta \in]0, 1[$.

The following proposition can be seen as the quantum analogue of Lemma 2.2.

Proposition 3.21 *Let* $k_\hbar$ *and* $\tilde{k}_\hbar$ *be two good labellings for the* $\hbar$*-continuous asymptotic lattice* $\mathcal{L}_\hbar$*. Then there exists a unique orientation preserving transformation* $\tau \in \mathrm{GA}_{\mathbb{Z}}^+(n, \mathbb{Z})$*, independent of* $\hbar$*, and* $\hbar_0 > 0$*, such that* $\tilde{k}_\hbar = \tau \circ k_\hbar$ *for all* $\hbar \leq \hbar_0$*,* $\hbar \in \mathcal{I}$*.*

Here $\mathrm{GA}^+_{\mathbb{Z}}(n,\mathbb{Z})$ is the orientation preserving integral affine group $\mathrm{GA}^+_{\mathbb{Z}}(n,\mathbb{Z}) := \mathrm{SL}(n,\mathbb{Z}) \ltimes \mathbb{Z}^n$.

Proof. We could start this proof by applying Proposition 3.19. However, it turns out that, in the case of $\hbar$-continuous asymptotic lattices, a more direct and elementary argument is available. Let $G_\hbar$ and $\tilde{G}_\hbar$ be the corresponding asymptotic charts (Equation (13)). Let $\tilde{F}_\hbar$ be a formal inverse of $\tilde{G}_\hbar$, as in the proof of Lemma 3.10, and define $K_\hbar := \tilde{F}_\hbar \circ G_\hbar$. Let $c \in B$, and let $\xi = G_0^{-1}(c)$. Let $k = k(\hbar)$ be a family in $\mathbb{Z}^n$ such that $\hbar k(\hbar) = \xi + \mathscr{O}(\hbar)$ as $\hbar \to 0$. From (13), there exists another family $\tilde{k} = \tilde{k}(\hbar) \in \mathbb{Z}$ such that

$$K_\hbar(\hbar k) = \hbar\tilde{k} + \mathscr{O}(\hbar^\infty). \tag{35}$$

Since $K_\hbar$ admits a C^∞ asymptotic expansion $K_\hbar = K_0 + \hbar K_1 + \cdots$, a uniform Taylor expansion gives, for any fixed integer vector $v \in \mathbb{Z}^n$,

$$K_\hbar(\hbar(k+v)) - K_\hbar(\hbar k) = \hbar K_\hbar'(\xi)\cdot v + \mathscr{O}(\hbar^2) = \hbar K_0'(\xi)\cdot v + \mathscr{O}(\hbar^2)\ .$$

Hence from (35) we get, letting $\hbar \to 0$,

$$K_0'(\xi)\cdot v \in \mathbb{Z}\,.$$

This shows that the matrix $K_0'(\xi)$ has integer entries. Swapping $G_\hbar$ and $\tilde{G}_\hbar$, and repeating the argument, we obtain that the inverse of $K_0'(\xi)$ has integer entries as well, meaning that $K_0'(\xi) \in \mathrm{GL}(n,\mathbb{Z})$. In particular, $\xi \mapsto K_0'(\xi)$ must be a constant matrix $A \in \mathrm{GL}(n,\mathbb{Z})$ in the connected open set $G_0^{-1}(B)$, which in turn implies that $K_0(\xi) = A\xi + \alpha$, for some $\alpha \in \mathbb{R}^n$. Thus, the maps G_0 and $\tilde{G}_0$ must differ by an affine transformation in $\mathrm{GA}(n,\mathbb{Z})$:

$$\tilde{G}_0^{-1} \circ G_0(\xi) = A\xi + \alpha.$$

Let us assume, by contradiction, that $\alpha \neq 0$. We have

$$\begin{aligned}\tilde{k} &= \frac{1}{\hbar}K_0(\hbar k) + K_1(\hbar k) + \mathscr{O}(\hbar)\\ &= Ak + \frac{\alpha}{\hbar} + K_1(\xi) + R(\hbar),\end{aligned} \tag{36}$$

with $R(\hbar) = \mathscr{O}(\hbar)$ (because $\hbar k = \xi + \mathscr{O}(\hbar)$). Let $\ell(\hbar) := \tilde{k}(\hbar) - Ak(\hbar)$. Let $\hbar_0 \in \mathcal{I}$ be small enough so that $\sup_{\hbar\leq\hbar_0}\|R(\hbar)\| < \frac{1}{4}$ and so that the $\hbar$-continuity property (34) holds for $\epsilon = \frac{1}{4|\alpha|}$. Let

$$\mathcal{I}_0 := \{\hbar \in \mathcal{I} \quad | \quad \hbar \leq \hbar_0, \quad \ell(\hbar) = \ell(\hbar_0)\},$$

and let $\hbar_1 := \inf \mathcal{I}_0$. Since $\ell(\hbar) \sim \frac{\alpha}{\hbar}$, it must be unbounded as $\hbar \to 0$, which implies that $\hbar_1 > 0$.

Case 1: $\hbar_1 \in \mathcal{I}_0$**.** Then, by $\hbar$-continuity, one can find $\hbar_2 \in \mathcal{I}$, $\hbar_2 < \hbar_1$, such that $\frac{1}{\hbar_2} - \frac{1}{\hbar_1} \le \frac{1}{2|\alpha|}$. Since $\hbar_2 \notin \mathcal{I}_0$ and $\ell(\hbar) \in \mathbb{Z}^n$, we have

$$|\ell(\hbar_1) - \ell(\hbar_2)| \ge 1.$$

From (36) we get

$$|\ell(\hbar_1) - \ell(\hbar_2)| < |\alpha| \left(\frac{1}{\hbar_2} - \frac{1}{\hbar_1} \right) + \frac{1}{2} < 1,$$

a contradiction.

Case 2: $\hbar_1 \notin \mathcal{I}_0$**.** Let $\eta := \frac{\hbar_1^2}{4|\alpha|}$. By definition of $\hbar_1$, there exists $\hbar_2 \in \mathcal{I}_0$ such that $\hbar_1 < \hbar_2 \le \hbar_1 + \eta$. We have

$$\hbar_2 \le \hbar_1 + \eta = \hbar_1 (1 + \frac{\hbar_1}{4\,|\alpha|}) \le \frac{\hbar_1}{1 - \frac{\hbar_1}{4|\alpha|}}.$$

Hence $\frac{1}{\hbar_1} - \frac{1}{\hbar_2} \le \frac{1}{4|\alpha|}$. Since $\hbar_1 \in \overline{\mathcal{I}}$, by $\hbar$-continuity one can find $\hbar_3 \in \mathcal{I}$, $\hbar_3 < \hbar_1$, such that $\frac{1}{\hbar_3} - \frac{1}{\hbar_1} \le \frac{1}{4|\alpha|}$. Hence $\frac{1}{\hbar_1} - \frac{1}{\hbar_3} \le \frac{1}{2|\alpha|}$ and we may conclude as in Case 1 that

$$|\ell(\hbar_1) - \ell(\hbar_3)| < |\alpha| \left(\frac{1}{\hbar_1} - \frac{1}{\hbar_3} \right) + \frac{1}{2} < 1,$$

while $|\ell(\hbar_1) - \ell(\hbar_3)| \ge 1$, a contradiction.

Consequently we must have $\alpha = 0$. Thus we can write

$$\ell(\hbar) = K_1(\xi) + \mathscr{O}(\hbar),$$

which implies that $\ell(\hbar)$ converges to $K_1(\xi)$ as $\hbar \to 0$ and hence that $K_1(\xi) \in \mathbb{Z}^n$, forcing it to be a constant $\ell \in \mathbb{Z}^n$ on the connected open set $G_0^{-1}(B)$. We obtain

$$\tilde{k} = Ak + \ell + R(\hbar),$$

thus $R(\hbar) \in \mathbb{Z}^n$. Since $R(\hbar) = \mathscr{O}(\hbar)$, we must have, for $\hbar < \hbar_0$ small enough, $R(\hbar) = 0$, which finishes the proof of the proposition. □

Proposition 3.21 shows that, given a set $\mathcal{L}_\hbar \subset B$, the set of good labellings on subsets $\tilde{B} \subset B$, *i.e.* for which $(\mathcal{L}_\hbar \cap \tilde{B}, \mathcal{I}, \tilde{B})$ is an asymptotic lattice, form a flat sheaf over B. As a consequence, if B is simply connected and B is covered by open subsets on which $\mathcal{L}_\hbar$ admit a good labelling, then $\mathcal{L}_\hbar$ admits a good labelling on B itself.

The $\hbar$-continuity property has a natural interpretation in terms of the uniform continuity of individual points in $\mathcal{L}_\hbar$, as $\hbar$ varies. Recall from the discussion before Definition 3.15 that the choice of a good labelling defines an $\hbar$-evolution of each individual point in $\mathcal{L}_\hbar$. If $\mathcal{L}_\hbar$ is $\hbar$-continuous, Proposition 3.21 implies that this evolution is in fact intrinsic to $\mathcal{L}_\hbar$ itself. However, this is not the case in general. The heuristics are very simple: given $\hbar \in \mathcal{I}$, if the closest element to $\hbar$ in $\mathcal{I}$ is of the form $\hbar + \delta$, then the lattice point corresponding to $G_\hbar(k\hbar)$ gets displaced by a distance of order $\mathscr{O}(k\delta)$. If $k\delta$ is of the same order as the mean spacing between points (*i.e.* $\mathscr{O}(\hbar)$), then there will be a confusion between the evolution of this point with that of its closest neighbors on $\mathcal{L}_\hbar$. To avoid this confusion, we need $k\delta \ll \hbar$; since k is in general of order $1/\hbar$, this means $\delta \ll \hbar^2$, which precisely gives the $\hbar$-continuity condition (34).

Let us now give a precise statement of this, which will be needed for the inverse problem in the next section.

Proposition 3.22 *Let $(\mathcal{L}_\hbar, \mathcal{I}, B)$ be an asymptotic lattice. There exists $\hbar_0 > 0, \delta > 0$, and $\epsilon > 0$ such that, if $\hbar_1, \hbar_2 \in \mathcal{I} \cap]0, \hbar_0]$ satisfy:*

$$\left| \frac{1}{\hbar_1} - \frac{1}{\hbar_2} \right| < \epsilon$$

then the following holds. Fix $\lambda_1 \in \mathcal{L}_{\hbar_1}$ and let $\lambda_2 \in \mathcal{L}_{\hbar_2}$ be defined by

$$\lambda_2 = k_{\hbar_2}^{-1}(k_{\hbar_1}(\lambda_1)),$$

for some good labelling $k_\hbar$. Then

$$\mathcal{L}_{\hbar_i} \cap B(\lambda_j, \delta\hbar_i) = \{\lambda_i\} \qquad \forall (i,j) \in \{1,2\}^2. \tag{37}$$

Here $B(\lambda_j, \delta\hbar_i)$ denotes the Euclidean ball centered at λ_j, of radius $\delta\hbar_i$.

In other words, we fix $\lambda_1 \in \mathcal{L}_{\hbar_1}$, and we consider the evolution of $\mathcal{L}_\hbar$ as $\hbar$ moves from $\hbar_1$ to $\hbar_2$; then the closest element to λ_1 in $\mathcal{L}_{\hbar_2}$ is unique and is precisely the natural evolution of λ_1 obtained by fixing its integer label $k_{\hbar_1}(\lambda_1)$.

Proof. First of all, by Remark 3.11 we may choose $\delta_0 > 0$ such that $\mathcal{L}_\hbar \cap B(\lambda, \delta_0\hbar) = \{\lambda\}$ for all $\hbar < \hbar_0$ and all $\lambda \in \mathcal{L}_\hbar$. In particular, (37) holds when $i = j$ for any $\delta \le \delta_0$. Without loss of generality we may assume $\hbar_2 < \hbar_1$. Let $(G_\hbar, U)$ be an asymptotic chart associated with the good labelling $k_\hbar$. Let $k_1 = k_{\hbar_1}(\lambda_1)$. We have

$$\lambda_2 = G_{\hbar_2}(\hbar_2 k_1) + \mathscr{O}(\hbar_2^\infty) = G_{\hbar_2}(\hbar_2 k_1) + \mathscr{O}(\hbar_1^\infty)\,,$$

and hence $\lambda_2 - \lambda_1 = G_{\hbar_2}(\hbar_2 k_1) - G_{\hbar_1}(\hbar_1 k_1) + \mathscr{O}(\hbar_1^\infty)$. Let $\tilde{U} \Subset U$ be such that $G_0^{-1}(\tilde{U})$ contains $\overline{B}$. Taking $\hbar_0$ small enough, we may assume that $G_\hbar$ is invertible on $\tilde{U}$ for all $\hbar \in \mathcal{I} \cap]0, \hbar_0]$, see Lemma 3.7, Item 6. Therefore $\hbar_1 k_1 \in B$ and hence is bounded. Thus, there exists $M > 0$ such that

$$\|\hbar_2 k_1 - \hbar_1 k_1\| \le \epsilon \hbar_1 \hbar_2 k_1 \le \epsilon M \hbar_2.$$

Hence we may apply Item 4 of Lemma 3.7 and get $\|G_{\hbar_2}(\hbar_2 k_1) - G_{\hbar_2}(\hbar_1 k_1)\| \le L_0 \epsilon M \hbar_2$. Using now Item 3 of the same lemma, we obtain

$$\begin{aligned} \|\lambda_2 - \lambda_1\| &\le L_0 \epsilon M \hbar_2 + C(|\hbar_2 - \hbar_1| + \hbar_1^2 + \hbar_2^2) + \mathscr{O}(\hbar_1^\infty) \\ &\le L_0 \epsilon M \hbar_2 + C(\epsilon + 2)\hbar_1^2 + \mathscr{O}(\hbar_1^\infty). \end{aligned} \tag{38}$$

Since $\hbar_1 \leq \hbar_2(1 + \epsilon\hbar_1)$, we see that, if ϵ and $\hbar_0$ are small enough, the right-hand side of (38) is less than $\delta_0\hbar_2/3$. Choosing finally $\delta = \delta_0/2$, we see that (37) must hold. Indeed, if $\mu_1 \in \mathcal{L}_{\hbar_1}$ and $\mu_1 \neq \lambda_1$, then $\|\mu_1 - \lambda_1\| \geq \delta_0\hbar_1$, hence $\|\mu_1 - \lambda_2\| \geq \|\mu_1 - \lambda_1\| - \|\lambda_2 - \lambda_1\| \geq \delta_0\hbar_1 - \delta_0\hbar_2/3 > \delta\hbar_1$, and similarly if $\mu_2 \in \mathcal{L}_{\hbar_2}$ and $\mu_2 \neq \lambda_2$, then $\|\mu_2 - \lambda_1\| \geq \delta_0\hbar_2 - \delta_0\hbar_2/3 > \delta\hbar_2$. □

We conclude this section by making precise the relationship between the natural evolution of $\hbar$-continuous asymptotic lattices given by Proposition 3.22 above, the "drift" mentioned before Definition 3.15, and the asymptotic expansion of the lattice points in $\hbar$.

Proposition 3.23 *Let $(\mathcal{L}_\hbar, \mathcal{I}, B)$ be an asymptotic lattice. Let $c \in B$, and let $\lambda_\hbar \in \mathcal{L}_\hbar$ be such that $\lambda_\hbar = c + \mathscr{O}(\hbar)$. Let $k_\hbar$ be a good labelling for this lattice. There there exists $\alpha_c \in \mathbb{R}^n$, $A_c \in \mathbf{M}_n(\mathbb{R})$, and $\beta_c \in \mathbb{R}^n$ such that*

$$\lambda_\hbar = \alpha_c + \hbar(A_c \cdot k_\hbar(\lambda_\hbar) + \beta_c) + \mathscr{O}(\hbar^2). \tag{39}$$

Moreover,

$$\alpha_c = c - G_0'(\xi_0) \cdot \xi_0, \quad A_c = G_0'(\xi_0), \quad \beta_c = G_1(\xi_0),$$

where $G_\hbar = G_0 + \hbar G_1 + \cdots$ is the asymptotic chart associated with $k_\hbar$, and $\xi_0 = G_0^{-1}(c)$.

Proof. Let $k = k_\hbar(\lambda_\hbar)$ for simplifying notation. Using a uniform Taylor formula (16), we have

$$G_\hbar(\hbar k) = G_0(\xi_0) + G_0'(\xi_0) \cdot (\hbar k - \xi_0) + \hbar G_1(\xi_0) + \mathscr{O}(\|\hbar k - \xi_0\|^2) + \hbar\mathscr{O}(\|\hbar k - \xi_0\|) + \mathscr{O}(\hbar^2).$$

Using the boundedness of $G_\hbar^{-1}$ (Item 5 or 6 of Lemma 3.7), we have

$$\|\hbar k - \xi_0\| = \mathscr{O}(\|G_\hbar^{-1}(\hbar k) - c\|).$$

Since $\lambda_\hbar = G_\hbar(\hbar k) + \mathscr{O}(\hbar^\infty)$, this gives

$$\lambda_\hbar = \alpha_c + \hbar(A_c \cdot k_\hbar(\lambda_\hbar) + \beta_c + \mathscr{O}(\|\lambda_\hbar - c\|)) + \mathscr{O}(\|\lambda_\hbar - c\|^2) + \mathscr{O}(\hbar^2), \tag{40}$$

which establishes the result. □

Assume now that the asymptotic lattice $\mathcal{L}_\hbar$ is $\hbar$-continuous; we see from (39) that the natural evolution of a point $\lambda \in \mathcal{L}_\hbar$, as $\hbar$ varies slightly, which corresponds to freezing the integer k, is to move "as if it wanted to tend to α_c"; and, in general $\alpha_c \neq c$.

Definition 3.24 *Let $(\mathcal{L}_\hbar, \mathcal{I}, B)$ be an $\hbar$-continuous asymptotic lattice. Let $c \in B$ and $\xi_0 := G_0^{-1}(c)$. We call the quantity*

$$\delta_c := G_0'(\xi_0) \cdot \xi_0 \in \mathbb{R}^2$$

the drift *of the asymptotic lattice at c. It does not depend on the choice of a good labelling.*

Proof. Let show that the drift is indeed well defined. Let $\tilde{G}_\hbar$ be another asymptotic chart of $\mathcal{L}_\hbar$. Since $\mathcal{L}_\hbar$ is $\hbar$-continuous, we may apply Proposition 3.21, and obtain $\tau \in \mathrm{GA}_{\mathbb{Z}}^+(n,\mathbb{Z})$ such that $G_\hbar(\xi) = \tilde{G}_\hbar(\tau(\xi)) + \mathscr{O}(\hbar^\infty)$, for all $\xi \in G_0^{-1}(B)$. In particular $G_0(\xi) = \tilde{G}_0(\tau(\xi))$. Taking the derivatives we get

$$\forall v \in \mathbb{R}^n, \quad G_0'(\xi_0)\cdot v = \tilde{G}_0'(\tau(\xi_0))\cdot \tau(v).$$

Evaluating at $v = \xi_0$, we get

$$G_0'(\xi_0)\cdot \xi_0 = \tilde{G}_0'(\tilde{\xi}_0)\cdot \tilde{\xi}_0,$$

where $\tilde{\xi}_0 = \tilde{G}_0^{-1}(c)$. □

Since the drift is well-defined, we can expect to find a way to recover it directly from the asymptotic lattice, without choosing a particular good labelling. This is the content of the following lemma.

Proposition 3.25 *Let $(\mathcal{L}_\hbar, \mathcal{I}, B)$ be an $\hbar$-continuous asymptotic lattice. Let $c \in B$, and let $\lambda_\hbar \in \mathcal{L}_\hbar$ be such that $\lambda_\hbar = c + \mathscr{O}(\hbar)$. For $\hbar_1, \hbar_2 \in \mathcal{I}$, let $\lambda(\hbar_1,\hbar_2)$ be a closest element to $\lambda_{\hbar_1}$ in $\mathcal{L}_{\hbar_2}$. We define the divided difference*

$$\delta(\hbar_1,\hbar_2) := \hbar_1 \frac{\lambda_{\hbar_1} - \lambda(\hbar_1,\hbar_2)}{\hbar_1 - \hbar_2}\,. \tag{41}$$

For any $N > 2$, there exists positive numbers $\hbar_0$, ε_0, and $C > 0$ such that for all $\varepsilon \le \varepsilon_0$, $\hbar_1 \in \mathcal{I}$, $\hbar_1 \le \hbar_0$ and $\hbar_2 \in \mathcal{I}$ such that

$$\hbar_2 < \hbar_1\,, \quad \hbar_1 - \hbar_2 \ge \hbar_1^N\,, \quad \textit{and} \quad \frac{1}{\hbar_2} - \frac{1}{\hbar_1} < \varepsilon\,, \tag{42}$$

then $\lambda(\hbar_1,\hbar_2)$ is unique, and

$$|\delta(\hbar_1,\hbar_2) - \delta_c| \le C\hbar_1\,,$$

where δ_c is the drift of $\mathcal{L}_\hbar$ at c.

As we shall see below (Lemma 3.26), for an $\hbar$-continuous asymptotic lattice, it is always possible to find couples $(\hbar_1,\hbar_2)$ satisfying the requirements (42) while being arbitrarily small. Hence, the conclusion of this lemma is that we can recover the drift δ_c from the limit of the divided difference (41), as $\hbar_1 \to 0$.

Proof *of Proposition 3.25***.** Let $k_\hbar$ be a good labelling for $\mathcal{L}_\hbar$, and $G_\hbar$ an associated asymptotic chart. From Proposition 3.22 there exist $\delta > 0$, $\varepsilon_0 > 0$ and $\hbar_0 > 0$ such that for all $\hbar_1, \hbar_2 \in \mathcal{I} \cap]0,\hbar_0]$ with $\left|h_1^{-1} - \hbar_2^{-1}\right| < \varepsilon_0$, the element $\lambda(\hbar_1,\hbar_2)$ is uniquely defined, and $\|\lambda(\hbar_1,\hbar_2) - \lambda_{\hbar_1}\| < \delta\hbar_2$. Moreover, $k_{\hbar_1}(\lambda_{\hbar_1}) = k_{\hbar_2}(\lambda(\hbar_1,\hbar_2))$. For ease of notation, let us denote by k this integer. We have

$$\lambda_{\hbar_1} = G_{\hbar_1}(\hbar_1 k) + \mathscr{O}(\hbar_1^\infty), \qquad \lambda(\hbar_1,\hbar_2) = G_{\hbar_2}(\hbar_2 k) + \mathscr{O}(\hbar_2^\infty). \tag{43}$$

If follows from Item 2 of Lemma 3.7 that $c = G_0(\hbar_1 k) + \mathscr{O}(\hbar_1)$, and hence, because $\hbar_2 \in (\frac{\hbar_1}{1+\hbar_1\epsilon}, \hbar_1)$, there exists a constant $C > 0$ such that

$$\|\hbar_1 k - \xi_0\| \le C\hbar_1, \quad \text{ and } \|\hbar_2 k - \xi_0\| \le C\hbar_1,$$

where $\xi_0 = G_0^{-1}(c)$. On the other hand, from Item 3 of Lemma 3.7, there exists $C_N > 0$ such that

$$\|G_{\hbar_1}(\hbar_1 k) - G_{\hbar_2}(\hbar_1 k)\| \le C_N(|\hbar_1 - \hbar_2| + \hbar_1^N + \hbar_2^N).$$

(Note that we could have got rid of the terms $\hbar_1^N + \hbar_2^N$ by choosing an asymptotic chart that is smooth in $\hbar$, see Remark 3.9). Hence, since $\hbar_1 - \hbar_2 \ge \hbar_1^N$,

$$\hbar_1 \frac{\|G_{\hbar_1}(\hbar_1 k) - G_{\hbar_2}(\hbar_1 k)\|}{\hbar_1 - \hbar_2} \le 3C_N \hbar_1 \tag{44}$$

From Item 4 of Lemma 3.7, we can write

$$G_{\hbar_2}(\hbar_1 k) - G_{\hbar_2}(\hbar_2 k) = (\hbar_1 - \hbar_2) G'_{\hbar_2}(\hbar_2 k) \cdot (k) + \mathscr{O}(|\hbar_1 - \hbar_2|^2 \|k\|^2),$$

which, in view of (44), and using that $\hbar_j k = \xi_0 + \mathscr{O}(\hbar_1)$ for $j = 1, 2$, we obtain

$$\begin{aligned} \hbar_1 \frac{G_{\hbar_1}(\hbar_1 k) - G_{\hbar_2}(\hbar_2 k)}{\hbar_1 - \hbar_2} &= G'_{\hbar_2}(\hbar_2 k) \cdot (\hbar_1 k) + \mathscr{O}(\hbar_1 |\hbar_1 - \hbar_2| \|k\|^2) \\ &= G'_{\hbar_2}(\xi_0) \cdot (\xi_0) + \mathscr{O}(\hbar_1) + \mathscr{O}(\hbar_1 \epsilon \hbar_1 \hbar_2 \|k\|^2) \\ &= \delta_c + \mathscr{O}(\hbar_2) + \mathscr{O}(\hbar_1) + \mathscr{O}(\epsilon \hbar_2 \|\xi_0\|^2) \\ &= \delta_c + \mathscr{O}(\hbar_1). \end{aligned}$$

Finally, using (43) and the fact that $\hbar_1 - \hbar_2 \ge \hbar_1^N$, this yields the desired estimate. □

The following lemma shows that the requirement (42) can always be met.

Lemma 3.26 *Assume $\mathcal{I}$ is $\hbar$-continuous. For any $\epsilon > 0$, for any $N > 2$, there exists $\hbar_0 > 0$ such that for all $\hbar_1 \in \mathcal{I} \cap]0, \hbar_0]$, there exists $\hbar_2 \in \mathcal{I}$ such that*

$$\hbar_1 < \hbar_2, \quad \frac{1}{\hbar_2} - \frac{1}{\hbar_1} < \epsilon \quad \text{ and } \hbar_1 - \hbar_2 \ge \hbar^N.$$

Proof. By contradiction, assume that the statement of the lemma does not hold. Let $\hbar_3 := \inf\{\hbar \in \mathcal{I}; \hbar < \hbar_1, \frac{1}{\hbar} - \frac{1}{\hbar_1} < \epsilon\}$. By Definition 3.20 the set in question is not empty, hence $\hbar_3 \ge 0$, and $\hbar_3 \in [\frac{\hbar_1}{1+\hbar_1\epsilon}, \hbar_1]$. By the negation of the lemma, one must have $\hbar_1 - \hbar_3 < \hbar_1^N$, and hence $\frac{1}{\hbar_3} - \frac{1}{\hbar_1} < \hbar_1^{N-2}(1 + \hbar_0\epsilon)$. Applying Definition 3.20 (with ϵ replaced by $\frac{\epsilon}{2}$) to this $\hbar_3 \in \overline{\mathcal{I}} \backslash \{0\}$, we obtain $\hbar_4 \in \mathcal{I}$ with $\hbar_4 < \hbar_3$ and $\frac{1}{\hbar_4} - \frac{1}{\hbar_3} < \frac{\epsilon}{2}$. Hence $\frac{1}{\hbar_4} - \frac{1}{\hbar_1} < \frac{\epsilon}{2} + \hbar_0^{N-2}(1 + \hbar_0\epsilon)$. If $\hbar_0$ was small enough, the right-hand side is less than ϵ, thus $\hbar_4$ contradicts the definition of $\hbar_3$. □

3.4 Quantum rotation number

In this section, we define a spectral quantity from the joint spectrum of a quantum integrable system, which will be the natural analogue of the rotation number $w_I(\Lambda)$. As we shall see, making this quantity a purely spectral invariant is not obvious, because, in the same way as the classical rotation number $w_I(\Lambda)$ depended on the choice of the action variables I (Definition 2.1), the quantum rotation number will depend on the choice of a good labelling.

Let us consider a two degree of freedom quantum integrable system $(\hat{J}, \hat{H})$ (see Section 3.2, with $n = 2$, $P_1 = \hat{J}, P_2 = \hat{H}$), with proper classical momentum map $F = (J, H) : M \to \mathbb{R}^2$ (here (J, H) is the principal symbol of $(\hat{J}, \hat{H})$). Let $c \in F(M)$ be a regular value of F, and assume that $F^{-1}(c)$ is connected. From Theorem 3.6, we know that the joint spectrum $\Sigma_\hbar$ is an asymptotic lattice near c.

Definition 3.27 *Let $\lambda \mapsto (j, k)$ be a good labelling for the joint spectrum $\Sigma_\hbar$ in a ball $B \subset \mathbb{R}^2$. Let $\tilde{B} \Subset B$ be an open set. Let $\lambda \subset \Sigma \cap \tilde{B}$, and let (j, k) be the corresponding labels. Denote*

$$\lambda =: (J_{j,k}(\hbar), E_{j,k}(\hbar)) .$$

We define the quantum rotation number*, for the quantum numbers (j, k), to be*

$$[\hat{w}_\hbar](j, k) := [E_{j+1,k}(\hbar) - E_{j,k}(\hbar) : E_{j,k+1}(\hbar) - E_{j,k}(\hbar)] \in \mathbb{R}P^1. \tag{45}$$

The restriction to the smaller open subset $\tilde{B}$ ensures that the labels $(j + 1, k)$ and $(j, k + 1)$ do correspond to joint eigenvalues in B, when $\hbar < \hbar_0$ and $\hbar_0$ is small enough. As in the classical case (Equation (2)), it is often more convenient to think of the rotation number as an element of the 1-point compactification $\overline{\mathbb{R}}$,

$$\hat{w}_\hbar(j, k) := \frac{E_{j+1,k}(\hbar) - E_{j,k}(\hbar)}{E_{j,k+1}(\hbar) - E_{j,k}(\hbar)} \in \overline{\mathbb{R}}. \tag{46}$$

The following result shows that, once a good labelling is known, the classical rotation number can be recovered from the quantum rotation number, in the semiclassical limit.

Theorem 3.28 *Let $c \in \mathbb{R}^2$ be a regular value of F, and assume that the Liouville torus $\Lambda := F^{-1}(c)$ is connected. Let $\lambda_\hbar$ be a joint eigenvalue in $\Sigma_\hbar$ such that*

$$\lambda_\hbar = c + \mathscr{O}(\hbar).$$

Let $G_\hbar$ be an asymptotic chart for $\Sigma_\hbar$ in a neighborhood of c, let (j, k) be the corresponding labels for $\lambda_\hbar$ and let $I := G_0^{-1} \circ F$ be the associated action variables (see Equation (11)). Then

$$[\hat{w}_\hbar](j, k) = [w_I(\Lambda)] + \mathscr{O}(\hbar).$$

The term $\mathscr{O}(\hbar)$ is relative to the topology of $\mathbb{R}P^1$ as homeomorphic to the circle, and is uniform if c varies in a compact subset. The existence of $\lambda_\hbar$ in the statement of the theorem is guaranteed by Lemma 3.14.

Proof. Let $g_\hbar$ be the second component of $G_\hbar$, so that $E_{j,k}(\hbar) = g_\hbar(\hbar j, \hbar k) + \mathscr{O}(\hbar^\infty)$. We have the C^∞ asymptotic expansion

$$g_\hbar(\xi) = g_0(\xi) + \hbar g_1(\xi) + \cdots$$

where g_0 is the second component of G_0; thus, $H = g_0(I)$. By definition

$$[w_I](\Lambda) = [\partial_1 g_0(I(\Lambda)) : \partial_2 g_0(I(\Lambda))].$$

A Taylor formula (Item 4 of Lemma 3.7) gives

$$\begin{aligned} E_{j+1,k}(\hbar) - E_{j,k}(\hbar) &= g_\hbar(\hbar(j+1), \hbar k) - g_\hbar(\hbar j, \hbar k) + \mathscr{O}(\hbar^\infty) \\ &= \hbar \partial_1 g_\hbar(\hbar j, \hbar k) + \mathscr{O}(\hbar^2) \\ &= \hbar \partial_1 g_0(\hbar j, \hbar k) + \mathscr{O}(\hbar^2), \end{aligned}$$

because $(\hbar j, \hbar k)$ is bounded. In fact, when $\hbar \to 0$, $(\hbar j, \hbar k) \to G_0^{-1}(c) = I(\Lambda)$; hence $E_{j+1,k}(\hbar) - E_{j,k}(\hbar) = \hbar\partial_1 g_0(I(\Lambda)) + \mathscr{O}(\hbar^2)$. Similarly, $E_{j,k+1}(\hbar) - E_{j,k}(\hbar) = \hbar\partial_2 g_0(I(\Lambda)) + \mathscr{O}(\hbar^2)$, which yields

$$\hat{w}_\hbar(j,k) = [\partial_1 g_0(I(\Lambda)) + \mathscr{O}(\hbar) : \partial_2 g_0(I(\Lambda)) + \mathscr{O}(\hbar)],$$

which gives the result, because $(\partial_1 g_0(I(\Lambda)), \partial_2 g_0(I(\Lambda))) \neq (0,0)$. □

3.5 Quantum rotation number for semitoric systems

If $F = (J, H)$ is a classical semitoric system, we have shown in Section 2.3 that the rotation number is well defined as an angle in $\mathbb{R}/\mathbb{Z}$. We will show the quantum analogue here, namely that for a quantum semitoric system, the rotation number can be defined with no ambiguity on the choice of a good labelling.

Definition 3.29 ([39]) *A quantum integrable system $(\hat{J}, \hat{H})$ is called semitoric if the corresponding classical system (J, H) given by the principal symbols is semitoric.*

Remark 3.30 It would be very interesting to have a purely spectral characterization of a quantum semitoric system. The quantum analogue of the Hamiltonian $\mathbb{S}^1$ action should be reflected in the fact that the spectrum of $\hat{J}$ coming from a bounded region of the joint spectrum is close to an arithmetic sequence of the type $\alpha + \hbar(j + \mu)$, $j \in \mathbb{Z}$ (see Proposition 3.31 below).

However the fact that a point $c \in \mathbb{R}^2$ is a regular value of F seems more delicate to obtain in a purely spectral way. Moreover, the semitoric hypothesis also impacts the singularity types at

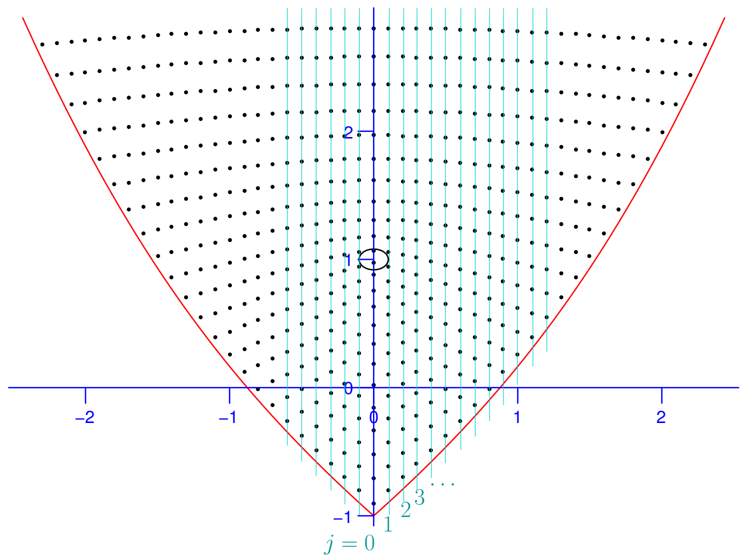

Figure 2: Joint spectrum of the Spherical Pendulum. Joint eigenvalues are organized along vertical lines indexed by j.

the boundary of the image of $F = (J, H)$ (see Proposition 3.35). We don't address these issues here. Instead, we assume the semitoric nature of the system, and from this we try and recover the rotation number. △

In the semitoric case, Proposition 3.23 can be improved.

Proposition 3.31 *Let $(\hat{J}, \hat{H})$ be a semitoric quantum integrable system defined for $\hbar \in]0, \hbar_0]$. Let $c \in \mathbb{R}^2$ be a regular value of F and assume that $F^{-1}(c)$ is connected, where $F = (J, H)$ is the classical momentum map. Then there exist a ball B around c, $\alpha \in \mathbb{R}$, $\mu \in \mathbb{R}$, and a good labelling $\lambda \mapsto (j, k)$ of the joint spectrum $\Sigma_\hbar$ in B such that, uniformly for $\lambda = (J_{j,k}(\hbar), E_{j,k}(\hbar)) \in \Sigma_\hbar \cap B$,*

$$J_{j,k}(\hbar) = \alpha + \hbar(j + \mu + \mathscr{O}(\lambda - c)) + \mathscr{O}(\hbar^2). \tag{47}$$

Loosely speaking, this proposition says that, in a small ball around c, the joint spectrum of a semitoric system is organized along regularly spaced vertical lines $J = \alpha + \hbar(j + \mu)$, and the quantum number $j \in \mathbb{Z}$ labels these lines. See Figure 2. If J is proper (and $\hat{J}$ has no subprincipal symbol), the result follows from the cluster structure of the spectrum of pseudo-differential operators with periodic characteristics, see for instance [18, 16, 10, 22]. In our case, we do not impose the properness of J and there is no restriction on subprincipal symbols; the result is still valid because we restrict the joint spectrum to a small B (it would not hold for the usual spectrum of $\hat{J}$ alone).

Lemma 3.32 *Under the assumptions of Proposition 3.31, let B be a bounded, simply connected open subset of regular values around c, such that $(\Sigma_\hbar,]0, \hbar_0], B)$ is an asymptotic lattice. Then* *this asymptotic lattice admits a* semitoric asymptotic chart, i.e. *an asymptotic chart* $G_\hbar \sim G_0 + \hbar G_1 + \cdots$ *such that the first component* $G_0^{(1)}$ *of* $G_0 = (G_0^{(1)}(\xi_1,\xi_2), G_0^{(1)}(\xi_1,\xi_2)) : U \to \mathbb{R}^2$ *satisfies, for all* $(\xi_1,\xi_2) \in U$,

$$\mathrm{d}G_0^{(1)}(\xi_1,\xi_2) = \mathrm{d}\xi_1. \tag{48}$$

Proof. Let $G_\hbar$ be an asymptotic chart for $\Sigma_\hbar$ near c, see (10) and Definition 3.5, and let $I = (I_1, I_2)$ be the corresponding action coordinates: $F = G_0 \circ I$. Since (J, H) is semitoric, there exist oriented action coordinates near $\Lambda := F^{-1}(c)$ of the form (J, J_2). Hence by Lemma 2.2 there is an affine transformation $\tau \in \mathrm{GA}(n, \mathbb{Z})$ such that $(I_1, I_2) = \tau(J, J_2) = A(J, J_2) + z$, where $A \in \mathrm{SL}(2,\mathbb{Z})$ and $z \in \mathbb{R}^2$. Then the map $\hat{G}_\hbar(\xi) := G_\hbar(A\xi)$ is an asymptotic chart by Lemma 3.18 and we have:

$$\mathrm{d}\hat{G}_0 = \mathrm{d}G_0 \circ A = \mathrm{d}G_0 \circ \mathrm{d}\tau = \mathrm{d}(G_0 \circ \tau)\,.$$

Since $G_0(\tau(J, J_2)) = (J, H)$, this implies $\mathrm{d}(G_0 \circ \tau)^{(1)} = \mathrm{d}\xi_1$, and gives the result. $\square$

Equation (48) means that, up to a constant, the chart $G_\hbar$ is associated with semitoric action variables (Lemma 2.13).

Proof *of Proposition 3.31***.** Let us consider the proof of Proposition 3.23. In view of Lemma 3.32, if we project (40) on the first component, the term $\mathscr{O}(\|\lambda_\hbar - c\|^2)$ disappears, because the first component of G_0 is an affine map. This gives the required estimate. $\square$

Remark 3.33 Comparing with Proposition 3.23, we see that the number α in (49) is equal to $c^{(1)} - \xi_0^{(1)}$, *i.e.* the first component of $c - \xi_0$. Thus $\xi_0^{(1)}$ is the first component of the *drift* of the joint spectrum (Definition 3.24). Recall that $\xi_0^{(1)}$ is also the value of the action integral along the $\mathbb{S}^1$-cycle on the torus Λ_c. $\triangle$

Theorem 3.34 *Let* $(\hat{J}, \hat{H})$ *be a semitoric quantum integrable system, with momentum map* $F = (J, H)$. *Let* $c \in \mathbb{R}^2$ *be a regular value of* F, *and assume that the Liouville torus* $\Lambda := F^{-1}(c)$ *is connected. Let* $G_\hbar = G_0 + \hbar G_1 + \cdots$ *be an asymptotic chart for* $\Sigma_\hbar$ *near* c, *such that the associated good labelling* $\lambda \mapsto (j, k)$ *satisfies Equation* (47). *Then* (48) *holds,* i.e. *the first component of* G_0 *satisfies, for all* (ξ_1, ξ_2) *near* $G_0^{-1}(c)$*:*

$$\mathrm{d}G_0^{(1)}(\xi_1,\xi_2) = \mathrm{d}\xi_1.$$

Moreover, let $\lambda_\hbar$ *be a joint eigenvalue in* $\Sigma_\hbar$ *such that*

$$\lambda_\hbar = c + \mathscr{O}(\hbar)\,.$$

Then the corresponding quantum rotation number (Equation (46)*) satisfies:*

$$\hat{w}_\hbar(j,k) = w(\Lambda) + \mathscr{O}(\hbar) \mod \mathbb{Z}\,,$$

where $w(\Lambda)$ *is the semitoric rotation number in the sense of Definition 2.12.*

Proof. By assumption, the first component of $\lambda_\hbar$ has the asymptotic expansion:

$$J_{j,k}(\hbar) = \alpha + \hbar(j + \mu + \mathscr{O}(\lambda - c)) + \mathscr{O}(\hbar^2)\,, \tag{49}$$

where $\alpha, \mu \in \mathbb{R}$ do not depend on $\hbar$. From Theorem 3.28, we have

$$[\hat{w}_\hbar](j,k) = [w_I(\Lambda)] + \mathscr{O}(\hbar), \tag{50}$$

where $I = G_0^{-1} \circ F$. From Lemma 3.32, we introduce a semitoric asymptotic chart $\hat{G}_\hbar := G_\hbar \circ A$, where $A \in \mathrm{SL}(2,\mathbb{Z})$ is such that $\hat{G}_\hbar$ satisfies

$$\mathrm{d}\hat{G}_0^{(1)}(\xi_1, \xi_2) = \mathrm{d}\xi_1. \tag{51}$$

From the proof of Proposition 3.31, $\hat{G}_\hbar$ satisfies Equation (47): there exists α', μ' in $\mathbb{R}$ such that, $\forall \hbar < \hbar_0$, $J_{j,k}(\hbar) = \alpha' + \hbar(j' + \mu' + \mathscr{O}(\lambda - c)) + \mathscr{O}(\hbar^2)$; here $j' = aj + bk$, where a, b are the integers forming the fist line of A^{-1}. Comparing with Equation (49), we obtain, since $\lambda_\hbar - c = \mathscr{O}(\hbar)$,

$$\alpha' + \hbar(aj + bk + \mu') = \alpha + \hbar(j + \mu) + \mathscr{O}(\hbar^2)\,,$$

hence there exists a constant $C > 0$ such that, for all $\hbar \in \mathcal{I}$,

$$\left|\alpha' - \alpha + \hbar\big((a-1)j + bk + \mu' - \mu\big)\right| \leq C\hbar^2. \tag{52}$$

This holds for all $\hbar$-dependent couples $(j,k) = (j(\hbar), k(\hbar))$ that label a joint eigenvalue $\lambda_\hbar \in \Sigma_\hbar$ near c. If u, v are given integers, independent of $\hbar$, then the joint eigenvalue labelled by $(j(\hbar) + u, k(\hbar) + v)$ differs from $\lambda_\hbar$ by $\mathscr{O}(\hbar)$. Hence (52) must hold for this new label as well, and writing the triangle inequality we obtain, for all fixed $(u,v) \in \mathbb{Z}^2$,

$$\hbar\,|(a-1)u + bv| \leq 2C_{u,v}\hbar^2.$$

Choosing $(u,v) = (0,1)$ or $(1,0)$, as soon as $2C_{u,v}\hbar < 1$, this implies $a = 1$ and $b = 0$ (and hence $\alpha = \alpha'$ and $\mu' = \mu$ from (52)). This entails that the matrix A takes the form $A = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$. Hence Equation (48) follows from (51). This proves that, up to a constant, the action variables I are in fact semitoric. Thus, from Proposition 2.14 we obtain

$$w(\Lambda) = w_I(\Lambda) \mod \mathbb{Z},$$

which, together with (50) finishes the proof of the Theorem (recall that in the semitoric case, the direction $[w_I(\Lambda)]$ can never be vertical). ☐

In order to illustrate this result, we have produced a numerical comparison between classical and quantum rotation numbers, in the case of the spherical pendulum (an axisymmetric Schrödinger operator on the sphere $\mathbb{S}^2$), see Figure 3.

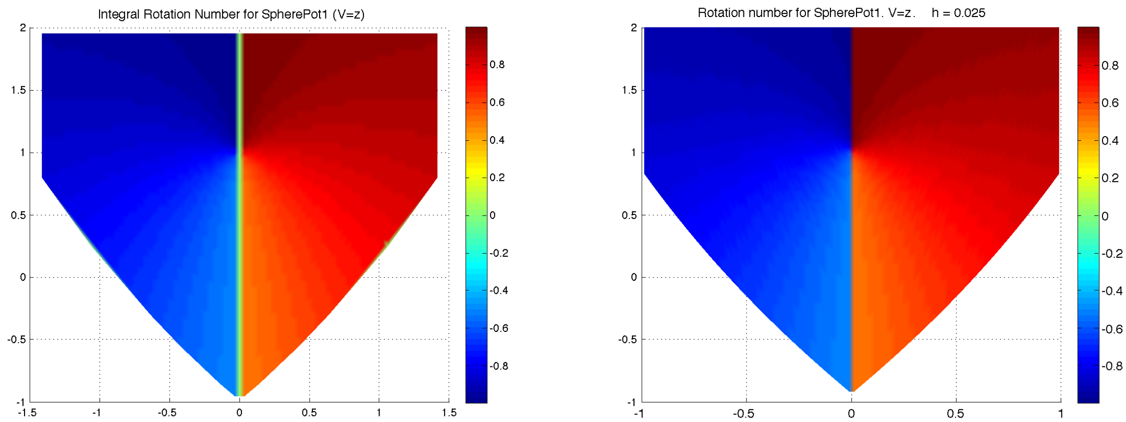

Figure 3: Classical (left) and quantum (right) rotation numbers for an axisymmetric Schrödinger operator on the sphere $\mathbb{S}^2$ with potential $V = z$ *(quantum spherical pendulum)*.

Theorem 3.34 will be used in Section 3.6.2 to obtain, algorithmically, a good labelling for the joint spectrum of a quantum semitoric system, based on the fact that the "global label j" corresponding to the integer in (47) is easy to detect. However, even for a semitoric system, assigning another quantum number k to each joint eigenvalue can be more delicate. While it is in principle always possible near any regular value c of the moment map (J, H), thanks to Theorem 3.6, there is no global recipe: due to a possibly non-trivial monodromy, a good labelling for the joint spectrum may simply not globally exist [43]. There is, however, a common situation where a natural second quantum number shows up: consider, for a fixed j, the corresponding vertical spectral band V_j. If the spectrum inside V_j is either bounded from above or from below, and if this bound corresponds to an elliptic singularity — which is generically the case —, then labelling k in increasing order from the bounded side will provide a good labelling. To understand this, let $c = (x, y)$ be a rank-one elliptic singularity of F, where x is a regular value of J. This means that the restriction of H to the submanifold $J^{-1}(x)$ admits a Morse-Bott non-degenerate critical point m with $H(m) = y$. Let us assume that $F^{-1}(c)$ is connected; by the theory of non-degenerate singularities of integrable systems [51], this fiber must then be a circle of critical points (the corresponding point in the reduced space $J^{-1}(x)/\mathbb{S}^1$ is a standard Morse non-degenerate critical point). Letting x vary in a small neighborhood, we thus obtain a smooth cylinder of critical points in M, whose critical values form a smooth curve through c in $\mathbb{R}^2$, which belongs to the boundary of $F(M)$. This situation is called a *simple J-transversal elliptic singularity*. From the viewpoint of the energy H, there are two situations, where c is either a local maximum or minimum of H restricted to $J^{-1}(x)$. For simplicity we shall only deal with the minimum case (which we call 'positive' in the statement below). This is the case of the Spherical Pendulum, Figure 2; of course the maximum case is completely similar.

Proposition 3.35 *Let $(\hat{J}, \hat{H})$ be a semitoric quantum integrable system with principal symbols*

(J, H). *Let* $c \in \mathbb{R}^2$ *be a simple* positive J*-transversal elliptic critical value of* F. *Then, for* $\hbar$ *small enough, the joint eigenvalues* $\lambda \in \Sigma_\hbar$ *in a neighborhood of* c *belong to the union of disjoint vertical bands* V_j *given by the equation*

$$x = \alpha + \hbar(j + \mu + \mathscr{O}(\lambda - c)) + \mathscr{O}(\hbar^2), \qquad j \in \mathbb{Z} \tag{53}$$

where $\alpha, \mu \in \mathbb{R}$ *are fixed. In each vertical band* V_j*, the* y *coordinates of the joint eigenvalues are distinct and bounded from below. We label them in increasing* y*-order by a non-negative integer* $k \in \mathbb{N}$*. Then, for any regular value* c' *of* F*, close to* c*, the labels* (j, k) *form a good labelling of* $\Sigma_\hbar$ *near* c'*.*

Contrary to the rest of the article, for this result the standard action-angle theorem (and its semi-classical version) is not enough. We need to resort to the microlocal study of the spectrum near a simple transversally elliptic singularity, which was done in [46, Theorem 5.2.4].

Theorem 3.36 ([46]) *Let* $(\hat{J}, \hat{H})$ *be a quantum integrable system, with momentum map* $F = (J, H)$*, and let* c *be a simple transversally elliptic critical value of* F*. Then the joint spectrum* $\Sigma_\hbar$ *in a neighborhood of* c *(independent of* $\hbar$*) can be described as follows:*

1. *joint eigenvalues have multiplicity one, in the sense of item 1. of Theorem 3.2;*
2. $\Sigma_\hbar$ *is an "asymptotic half lattice" in the sense that item 2. of Theorem 3.2 holds:*

$$\lambda = G_\hbar(\hbar k_1, \hbar k_2) + \mathscr{O}(\hbar^\infty), \tag{54}$$

when replacing $(k_1, k_2) \in \mathbb{Z}^2$ *by* $(k_1, k_2) \in \mathbb{Z} \times \mathbb{N}$*, and replacing Equation* (11) *by*

$$F = G_0(\xi_1, q_2) \tag{55}$$

where $q_2(x, \xi) = (x_2^2 + \xi_2^2)/2$*. Here the local coordinates* $(x_1, \xi_1, x_2, \xi_2) \in T^*\mathbb{S}^1 \times \mathbb{R}^2$ *near* $\mathbb{S}^1 \times \{\beta\} \times (0, 0)$ *for some* $\beta \in \mathbb{R}$*, describing a neighborhood of the circle* $F^{-1}(c)$*, are symplectic, and* G_0 *is a local diffeomorphism from* $(\mathbb{R}^2, (\beta, 0))$ *to a neighborhood of* $\overline{B}$*.*

Proof *of Proposition 3.35***.** It follows from (55) that $J = g(\xi_1, q_2)$ for a smooth g. Since J, ξ_1 and q_2 all have 2π-periodic flows, there must exist integers a, b such that $dJ = ad\xi_1 + bdq_2$. Since the J-action is effective, a and b must be co-prime. The hypothesis of J-transversality implies that $a \neq 0$. Thus $\xi_1 = a^{-1}J - ba^{-1}q_2 + \text{const}$, and the same argument implies that $a^{-1} \in \mathbb{Z}$ and hence $a = \pm 1$. Up to composing by the symplectomorphism $(x_1, \xi_1) \mapsto (-\xi_1, x_1)$ (and replacing k_1 by $-k_1$ in (54)), one may assume that $a = 1$. Thus $(\xi_1, q_2) = \tau(J, q_2)$, where $\tau \in \mathrm{GA}(n, \mathbb{Z})$ with linear part $A = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}$. Arguing as in the proof of Proposition 3.31, we let $\hat{G}_\hbar(x, y) := G_\hbar(A(x, y))$, and the joint spectrum is described as

$$\lambda = G_\hbar(\hbar k_1, \hbar k_2) + \mathscr{O}(\hbar^\infty) = \hat{G}_\hbar(\hbar(j, k_2)) + \mathscr{O}(\hbar^\infty),$$

with $k_1 = j - bk_2$. Then $\hat{G}_\hbar = \hat{G}_0 + \hbar\hat{G}_1 + \cdots$, with

$$\hat{G}_0(x, y) = G_0 \circ A(x, y) = G_0(x - by + \alpha, y),$$

for some constant $\alpha \in \mathbb{R}$. But since the first component of G_0 is g and $J = g(\xi_1, q_2) = g(J - bq_2, q_2)$, we deduce that for all (x, y) near $A^{-1}(\beta, 0) = (\beta, 0)$,

$$\hat{G}_0(x, y) = (x + \alpha, f(x, y))$$

for some smooth function f with $\partial_y f \neq 0$. We conclude as in the proof of Proposition 3.31 that the first component of λ takes the form

$$J_{j,k_2}(\hbar) = \alpha + \hbar j + \hbar\mu + \hbar\mathscr{O}(\lambda - c) + \mathscr{O}(\hbar^2).$$

Hence we obtain the description of the vertical bands B_j in (53), which are disjoint if $|\lambda - c|$ is small enough (independently of $\hbar$) and $\hbar$ itself is small enough. Finally, if j is fixed, the second component of the joint eigenvalues is given by

$$E_{j,k_2}(\hbar) = f(\hbar j, \hbar k_2) + \hbar\mathscr{O}(\lambda - c) + \mathscr{O}(\hbar^2).$$

Thus, $E_{j,k_2+1} - E_{j,k_2} = \hbar\partial_y f(c) + \hbar\mathscr{O}(\lambda - c) + \mathscr{O}(\hbar^2)$. For a fixed value of J, H is assumed to be minimal at the elliptic critical point. Since $q_2 \geq 0$, we must have $\partial_{q_2} H \geq 0$ when $q_2 = 0$, which implies $\partial_y \hat{h} > 0$. Hence $E_{j,k_2+1} - E_{j,k_2} \geq \hbar/C$ for some constant $C > 0$.

It remains to prove that (j, k_2) is a good labelling away from the critical value. Since $(k_1, k_2) = A(j, k_2)$, this is equivalent to showing that (k_1, k_2) is a good labelling. Since item *1.* in Definition 3.5 is already given by Theorem 3.36, we must only prove item *2.* Let c' be a regular value of F. Let B be a ball around c in which the description of the joint spectrum by Theorem 3.36 is valid:

$$\lambda = G_\hbar(\hbar k_1, \hbar k_2) + \mathscr{O}(\hbar^\infty),$$

where $(k_1, k_2) \in \mathbb{Z} \times \mathbb{N}$, and $G_\hbar$ is defined in a neighborhood of $(\beta, 0)$. Let $B' \subset B$ be a ball of regular values around c'. We claim that $G_\hbar\restriction_{G_0^{-1}(B')}$ is an asymptotic chart for $\Sigma_\hbar$ in B'. Indeed, let $(\beta', \gamma') := G_0^{(-1)}(c')$; necessarily $\gamma' > 0$ because c' is regular. Now assume that $(k_1, k_2) \in \mathbb{Z}^2$ is such that $\hbar(k_1, k_2) \in G_0^{-1}(B')$; then $\hbar k_2$ is close to γ', and in particular, $k_2 > 0$. Hence, by (54), there exists a joint eigenvalue in $\lambda \in B'$ such that $\lambda = G_\hbar(\hbar k_1, \hbar k_2) + \mathscr{O}(\hbar^\infty)$. Therefore, according to Definition 3.5 (item *2.*), $G_\hbar$ is an asymptotic chart for $\Sigma_\hbar$ in B'. $\square$

3.6 Labelling algorithms

In order to obtain a fully satisfactory inverse spectral result, we need to recover a good labelling from the joint spectrum. Once a good labelling is known, the quantum rotation number can be computed via (45), and its asymptotics reveal the classical rotation number thanks to Theorem 3.28.

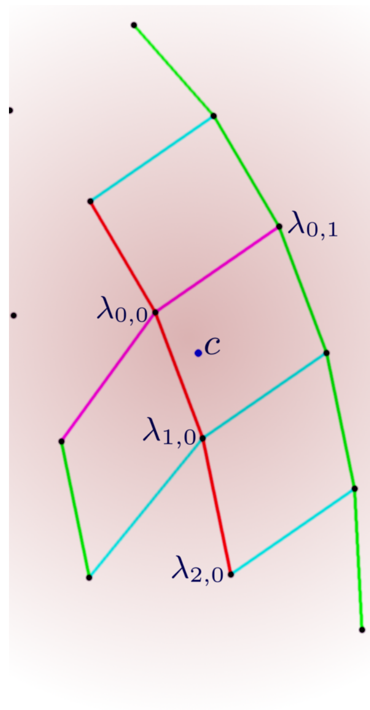

Figure 4: The labelling algorithm

As was shown in Section 3.5, a good labelling is easy to find for quantum semitoric systems with elliptic critical values. In the general case however, the question is more delicate. In this final section, we consider an arbitrary asymptotic lattice $(\mathcal{L}_\hbar, \mathcal{I}, B)$ and investigate the problem of constructing a good labelling for it.

3.6.1 An algorithm for fixed $\hbar$

The core part of our labelling algorithm can be executed for any fixed value of $\hbar \in \mathcal{I}$. Actually, it can be applied to any finite set $\mathcal{L}_\hbar \subset B$. If $c \in B$, we will use the expression "choose a closest point to c" to indicate that we have to choose a point in $\mathcal{L}_\hbar$ that minimizes the distance to c. After such a point has been picked up and labelled, we remove it from $\mathcal{L}_\hbar$; thus, subsequent calls to "choose a closest point to..." implicitly indicates that this point should be chosen within $\mathcal{L}_\hbar$ minus all the already labelled points. This is of course always possible as long as this new set is not empty.

We write the algorithm in the two-dimensional case, which is enough for our purposes. We believe that the general case can be dealt with in a similar way. If (n, m) is a label for a point $\lambda \in \mathcal{L}_\hbar$, we shall denote this point $\lambda = \lambda_{n,m}$. See Figure 4. The complete algorithm consists in the twelve following steps.

1. Choose an open subset $B_0 \Subset B$, and fix $c \in B_0$.

2. Choose a closest point to c. Label it as $(0,0)$.

3. Choose a closest point to $\lambda_{0,0}$ (in the set $\mathcal{L}_\hbar \setminus \{\lambda_{0,0}\}$). Label it as $(1,0)$.

4. Choose a closest point to $2\lambda_{1,0} - \lambda_{0,0} = \lambda_{1,0} + (\lambda_{1,0} - \lambda_{0,0})$ and label it as $(2,0)$.

   Continuing in this fashion (if $\lambda_{n-1,0}$ is chosen, take $\lambda_{n,0}$ to be a closest point to $\lambda_{n-1,0} + (\lambda_{n-1,0} - \lambda_{n-2,0})$), label points $\lambda_{n,0}$, $n > 0$, until the next point lies outside of B_0.

   Label $\lambda_{n,0}$ for negative n in the same way, starting from the closest point to $2\lambda_{0,0} - \lambda_{1,0}$.

5. Choose a closest point to $\lambda_{0,0}$ not already labeled and label it as $(0,1)$.

6. Label a closest point to $\lambda_{0,1} + (\lambda_{1,0} - \lambda_{0,0})$ as $(1,1)$.

7. Use the points $\lambda_{0,1}$, $\lambda_{1,1}$ to repeat the process in step 4, labelling as many points $\lambda_{n,1}$ as possible (if $\lambda_{n-1,1}$ is chosen, take $\lambda_{n,1}$ to be a closest point to $\lambda_{n-1,1} + (\lambda_{n-1,1} - \lambda_{n-2,1})$).

8. Label a closest point to $2\lambda_{0,1} - \lambda_{0,0}$ as $\lambda_{0,2}$. Repeat steps 6-7 to label all points $\lambda_{n,2}$.

9. Continuing as above, label all points $\lambda_{n,m}$, $m > 0$ which lie in the given neighborhood.

10. Label a closest point to $2\lambda_{0,0} - \lambda_{0,1}$ as $(0,-1)$.

11. Repeat steps 6,7,8,9 with negative m indices.

12. Finally, if the determinant of the vectors $(\lambda_{1,0} - \lambda_{0,0}, \lambda_{0,1} - \lambda_{0,0})$ is negative, switch the labelling $\lambda_{n,m} \mapsto \lambda_{m,n}$ (in order to make it oriented).

Let us prove now some properties of this algorithm when $(\mathcal{L}_\hbar, \mathcal{I}, B)$ is a given asymptotic lattice. We use the notation of Definition 3.5; in particular, $G_\hbar : U \to \mathbb{R}^n$ is an asymptotic chart, and $k_\hbar$ is the corresponding good labelling. Of course, both of them are *a priori* unknown. Let $B_0 \Subset B$ be an open subset containing c, and let $\tilde{B}_0$ be an open subset such that $B_0 \Subset \tilde{B}_0 \Subset B$. Let $\tilde{\hbar}_0$ be given by Item 2b of Definition 3.5 with $\tilde{U}_0 := G_0^{-1}(\tilde{B}_0)$. Because the results of this section are made to be directly implementable on a computer, we shall try to write all estimates as explicitly as possible. Let $\tilde{U} \Subset U$ and $\hbar_0 \in]0, \tilde{\hbar}_0]$ be such that $(G_\hbar)\restriction_{\tilde{U}}$ is invertible onto a neighborhood of $\overline{B}$ for all $\hbar \in \mathcal{I} \cap]0, \hbar_0]$, see Item 6 of Lemma 3.7. From now on in this section, every $\hbar$ is tacitly assumed to belong to $\mathcal{I} \cap]0, \hbar_0]$.

Let $U_0 = G_0^{-1}(B_0)$ and $U_\hbar = G_\hbar^{-1}(B_0)$. Then $U_0 \Subset G_0^{-1}(B)$ and for all $\hbar \leq \hbar_0$, $U_\hbar \Subset \tilde{U}$. It follows from the asymptotic expansion of $G_\hbar^{-1}$ that there exists $C > 0$ such that

$$\forall \hbar \leq \hbar_0, \quad U_\hbar \subset U_0 + B(0, C\hbar)\,,$$

and hence for any $R > 0$, there exists $\hbar_0^{[0]} \in]0, \hbar_0]$ such that

$$\forall \hbar \leq \hbar_0^{[0]}, \quad U_\hbar + B(0, R\hbar) \subset \tilde{U}_0 \quad \text{and} \quad U_\hbar + B(0, R\hbar) \subset \tilde{U}. \tag{$\hbar$-56}$$

As in Lemma 3.10, let L_F be an upper bound on the Lipschitz constant of $G_\hbar^{-1}$ on a neighborhood of $\overline{B}$. Choose $\epsilon \in (0, \frac{1}{L_F})$, and let $\hbar_0^{[1]} \in]0, \hbar_0]$ be small enough to verify ($\hbar$-22), so that (23) holds.

Note that, with the exception of the three points $\lambda_{0,0}$, $\lambda_{1,0}$, and $\lambda_{0,1}$, all points are constructed by the following process:

(i) Choose a point $\mu_1 \in B_0 \cap \mathcal{L}_\hbar$.

(ii) Choose a "vector" $\vec{v} = \mu_0 - \mu_{-1}$ (the difference between two previously constructed points μ_0 and $\mu_{-1} \in B_0 \cap \mathcal{L}_\hbar$).

(iii) Identify a closest point $\mu_2 \in \mathcal{L}_\hbar$ to $\mu_1 + \vec{v}$.

The following lemma shows that this process is uniformly well-defined if these points lie in ball of size $\mathscr{O}(\hbar)$ and $\hbar_0$ is small enough. Then, Step (iii) amounts to picking up the natural parallel transport defined in (24):

$$\mu_2 = \mu_1 \underset{k_\hbar}{+} (k_\hbar(\mu_0) - k_\hbar(\mu_{-1})).$$

Lemma 3.37 *Given $\epsilon \in (0, \frac{1}{L_F})$, there exists $L > 0$ such that the following holds. Choose $\hbar \leq \hbar_0^{[1]}$ and four points $\mu_i \in \mathcal{L}_\hbar$ satisfying (i)-(iii) above, and let $k_i := k_\hbar(\mu_i)$ be the corresponding multi-integers in $\mathbb{Z}^2$, for $i \in \{-1, 0, 1, 2\}$. Let $\rho > 0$ be such that μ_1 and μ_{-1} belong to the ball $B(\mu_0, \rho\hbar)$. If $\hbar \leq \hbar_0^{[0]}$ defined by ($\hbar$-56) with some $R > 2L_F\rho$, and if for some $N > 2$, $\hbar$ satisfies the inequalities*

$$3C_N\hbar^{N-1} < \frac{R}{L_F} - 2\rho \quad \text{and} \quad \hbar\big(L(\rho + C_N\hbar^{N-1})^2 + 4C_N\hbar^{N-2}\big) \leq \epsilon\,, \tag{$\hbar$-57}$$

where C_N is defined in (19), then μ_2 is unique and $k_2 = k_1 + k_0 - k_{-1}$.

Proof. We wish to consider the point $\lambda_2 \in \mathcal{L}_\hbar$ whose label is $k_1 + k_0 - k_{-1}$. First of all we show that $\hbar(k_1 + k_0 - k_{-1}) \in \tilde{U}_0$. For all $\lambda \in \mathcal{L}_\hbar$, we get from (19)

$$\left\|\hbar k - G_\hbar^{-1}(\lambda)\right\| \leq L_F C_N \hbar^N\,.$$

Therefore, $\hbar k_j$, for $j \in \{-1, 0, 1\}$, belongs to $U_\hbar + B(0, L_F C_N\hbar^N) \subset \tilde{U}$ provided $L_F C_N \hbar^{N-1} < R$, which is a consequence of the first inequality in ($\hbar$-57). Thus, we may apply $G_\hbar^{-1}$ and use (19) to obtain, for $j \in \{-1, 1\}$,

$$\|\hbar k_0 - \hbar k_j\| \leq L_F\,\|\mu_0 - \mu_j\| + 2L_F C_N \hbar^N < 2L_F(\hbar\rho + C_N\hbar^N)\,. \tag{58}$$

Using the inequality for $j = -1$ we get $\hbar(k_1 + k_0 - k_{-1}) \in U_\hbar + B\big(0, L_F(3C_N\hbar^N + 2\hbar\rho)\big)$. From ($\hbar$-56) and the first inequality in ($\hbar$-57), we get $\hbar(k_1 + k_0 - k_{-1}) \in \tilde{U}_0$. Since $U_0 \Subset G_0^{-1}(B)$, by Item 2b of Definition 3.5 we may define $\lambda_2 := k_\hbar^{(-1)}(k_1 + k_0 - k_{-1}) \in \mathcal{L}_\hbar$, *i.e.*, for all $N \geq 0$,

$$\|\lambda_2 - G_\hbar(\hbar(k_1 + k_0 - k_{-1}))\| \leq C_N \hbar^N\,.$$

We have, using a Taylor expansion at $\hbar k_1$, see (16),

$$\begin{aligned}\|\lambda_2 - (\mu_1 + \vec{v})\| &\leq \|G_\hbar(\hbar(k_1 + k_0 - k_{-1})) - (\mu_1 + \vec{v})\| + C_N \hbar^N \\ &\leq \|G_\hbar(\hbar k_1) - \mu_1 + G'_\hbar(\hbar k_1) \cdot (\hbar k_0 - \hbar k_{-1}) - \vec{v}\| + L_1 \|\hbar k_0 - \hbar k_{-1}\|^2 + C_N \hbar^N \\ &\leq \|G'_\hbar(\hbar k_1) \cdot (\hbar k_0 - \hbar k_{-1}) - \vec{v}\| + L_1 \|\hbar k_0 - \hbar k_{-1}\|^2 + 2C_N \hbar^N .\end{aligned}$$

We can also write $\|\vec{v} - G_\hbar(\hbar k_0) + G_\hbar(\hbar k_{-1})\| \leq 2C_N \hbar^N$. By Taylor expanding at $\hbar k_0$,

$$\|G_\hbar(\hbar k_0) - G_\hbar(\hbar k_{-1}) - G'_\hbar(\hbar k_0) \cdot (\hbar k_0 - \hbar k_{-1})\| \leq L_1 \|\hbar k_0 - \hbar k_{-1}\|^2 .$$

Hence

$$\|\lambda_2 - (\mu_1 + \vec{v})\| \leq \|(G'_\hbar(\hbar k_0) - G'_\hbar(\hbar k_1)) \cdot (\hbar k_0 - \hbar k_{-1})\| + 2L_1 \|\hbar k_0 - \hbar k_{-1}\|^2 + 4C_N \hbar^N .$$

If follows from Item 1 of Lemma 3.7 that there exists a constant $L_2 > 0$ such that

$$\forall [\xi_1, \xi_2] \subset \tilde{U}, \forall v \in \mathbb{R}^n, \quad \|G'_\hbar(\xi_2) \cdot v - G'_\hbar(\xi_1) \cdot v\| \leq L_2 \|\xi_2 - \xi_1\| \|v\| .$$

This gives

$$\|(G'_\hbar(\hbar k_0) - G'_\hbar(\hbar k_1)) \cdot (\hbar k_0 - \hbar k_{-1})\| \leq L_2 \|\hbar k_0 - \hbar k_1\| \|\hbar k_0 - \hbar k_{-1}\| ,$$

and hence

$$\|\lambda_2 - (\mu_1 + \vec{v})\| \leq \|\hbar k_0 - \hbar k_{-1}\| (2L_1 \|\hbar k_0 - \hbar k_{-1}\| + L_2 \|\hbar k_0 - \hbar k_1\|) + 4C_N \hbar^N .$$

Using (58) again,

$$\|\lambda_2 - (\mu_1 + \vec{v})\| < \hbar^2 \big(4(2L_1 + L_2) L_F^2 (\rho + C_N \hbar^{N-1})^2 + 4C_N \hbar^{N-2}\big) .$$

Hence, for $\hbar$ small enough: precisely, as soon as

$$\hbar \big(4(2L_1 + L_2) L_F^2 (\rho + C_N \hbar^{N-1})^2 + 4C_N \hbar^{N-2}\big) \leq \epsilon ,$$

we may apply (23) to see that the closest point to $(\mu_1 + \vec{v})$ in $\mathcal{L}_\hbar$ must be λ_2, which proves the lemma with $L = 4(2L_1 + L_2) L_F^2$. ☐

Of course, as in the end of Remark 3.11, one can simplify estimates by choosing specific values, for instance $\epsilon = \frac{1}{3L_F}$, $R = 3L_F \rho$, and then we may replace ($\hbar$-57) by the stronger assumption that there exist $N > 1$ such that

$$C_N \hbar^{N-1} \leq \min(\frac{\rho}{3} \,;\, \frac{\epsilon}{8}) \quad \text{and } \hbar \leq \frac{\epsilon}{4L\rho^2} .$$

Corollary 3.38 *With $\lambda_{m,n}$ the collection of points constructed in the algorithm without Step 12, let $k_{n,m} = k_\hbar(\lambda_{m,n})$. Set*

$$\vec{z}_1 = k_{1,0} - k_{0,0} \quad \textit{and} \quad \vec{z}_2 = k_{0,1} - k_{0,0}\,. \tag{59}$$

There exists $\hbar_0^{[2]} > 0$ (see Remark 3.39 below) such that for all $\hbar \leq \hbar_0^{[2]}$, we have

$$k_{n,m} - k_{0,0} = n\,\vec{z}_1 + m\,\vec{z}_2\,. \tag{60}$$

Moreover, for all $N \geq 1$,

$$\|\lambda_{1,0} - \lambda_{0,0}\| \leq \hbar\rho_0\,, \quad \|\lambda_{0,1} - \lambda_{0,0}\| \leq \hbar\rho_0\,, \quad \textit{with} \quad \rho_0 := L_0 + 2C_N\hbar^{N-1}\,, \tag{61}$$

where L_0 is an upper bound on the Lipschitz constant of $G_\hbar$ on $\tilde{U}$, see (15), and

$$\|\vec{z}_1\| \leq \tilde{\rho}\,, \quad \|\vec{z}_2\| \leq \tilde{\rho}\,, \quad \textit{with} \quad \tilde{\rho} := L_F(L_0 + 4C_N\hbar^{N-1})\,. \tag{62}$$

After Step 12, $\vec{z}_1$ and $\vec{z}_2$ are possibly swapped.

Proof. Let $\lambda_1 = k_\hbar^{(-1)}(k_{0,0} + \vec{e}_1)$, where $\vec{e}_1 = (1,0)$. Necessarily, $\lambda_1 \neq \lambda_{0,0}$; hence, by Step 3, $\|\lambda_1 - \lambda_{0,0}\| \geq \|\lambda_{1,0} - \lambda_{0,0}\|$. Since (using (13)),

$$\|\lambda_1 - \lambda_{0,0}\| \leq \|G_\hbar(\hbar(k_{0,0} + \vec{e}_1)) - G_\hbar(\hbar k_{0,0})\| + 2C_N\hbar^N \leq L_0\hbar + 2C_N\hbar^N\,; \tag{63}$$

this shows the first inequality in (61). We now proceed with Step 4 of the algorithm by applying Lemma 3.37 to the triple $(\lambda_{0,0}, \lambda_{1,0}, \lambda_{1,0})$, with $\rho = \rho_0$, and thus obtain a unique $\lambda_{2,0} = \lambda_{0,0} \underset{k_\hbar}{+} \vec{z}_1$. From (63) we immediately obtain

$$\begin{aligned}\hbar\|\vec{z}_1\| &\leq L_F\|G_\hbar(\hbar k_{1,0}) - G_\hbar(\hbar k_{0,0})\| \leq L_F\|\lambda_{1,0} - \lambda_{0,0}\| + 2L_FC_N\hbar^N \\ &\leq L_FL_0\hbar + 4L_FC_N\hbar^N = \hbar\tilde{\rho}\,.\end{aligned}$$

This proves the first inequality of (62). In order to repeat the application of Lemma 3.37 to the triple $(\lambda_{1,0}, \lambda_{2,0}, \lambda_{2,0})$ we need to estimate $\|\lambda_{2,0} - \lambda_{1,0}\|$. Since $k_{2,0} = k_{1,0} + \vec{z}_1$, we get

$$\|\lambda_{2,0} - \lambda_{1,0}\| \leq L_0\hbar\,\|\vec{z}_1\| + 2C_N\hbar^N \leq \hbar\rho_1 \tag{64}$$

with $\rho_1 := L_0\tilde{\rho} + 2C_N\hbar^{N-1}$. Thus we may apply Lemma 3.37 with $\rho = \rho_1$, and obtain that $\lambda_{3,0}$ is labelled by $k_{0,3} = k_{0,0} + 2\vec{z}_1$. Therefore, we can estimate $\|\lambda_{3,0} - \lambda_{2,0}\| \leq \hbar\rho_1$ exactly as in (64). Repeating this process with the same $\rho = \rho_1$, we complete Step 4 to obtain all $\lambda_{n,0}$ as long as they belong to B_0, and obtain, for all n, $\|\lambda_{n,0} - \lambda_{n-1,0}\| \leq \hbar\rho_1$ and

$$k_{n,0} - k_{0,0} = n\,\vec{z}_1, \quad \textit{i.e.} \quad \lambda_{n,0} = \lambda_{0,0} \underset{k_\hbar}{+} n\vec{z}_1\,. \tag{65}$$

Next, we consider $\lambda_{(0,1)}$ from Step 5. If λ_1 was not labelled in Step 4, we have $\|\lambda_1 - \lambda_{0,0}\| \geq \|\lambda_{0,1} - \lambda_{0,0}\|$, which leads to the same estimates of as above, namely (61) and (62) hold. If, on the contrary, λ_1 was labelled in Step 4, then by (65) there exists a non-zero integer n such that $\vec{e}_1 = n\vec{z}_1$. Thus the point $\lambda_2 = k_\hbar(k_{0,0} + \vec{e}_2)$ was not labelled in Step 4. Therefore $\|\lambda_2 - \lambda_{0,0}\| \geq \|\lambda_{0,1} - \lambda_{0,0}\|$ and we the above estimates still hold, with the same ρ, which proves (61) and (62). Therefore, we may continue to follow the algorithm and use Lemma 3.37 at each step, proving the corollary. $\square$

Remark 3.39 Let us investigate how small $\hbar_0^{[2]}$ should be for Corollary 3.38 to hold. We do not try to have an optimal bound, but rather to check what the geometric constraints are. First, we have the upper bound $\hbar_0^{[1]}$ defined by ($\hbar$-22), which ensures that $k_\hbar$ is well defined and one-to-one. This one is quite weak in principle, because we are free to increase the exponent of $\hbar$ to make it smaller. Then we apply Lemma 3.37 with $\rho = \rho_0$, which gives another upper bound given by ($\hbar$-57), where we are free to choose another $N > 2$. Essentially, this means $\hbar \lesssim \frac{\epsilon}{L\rho_0} \sim \frac{1}{L_0(L_1+L_2)L_F^3}$. Another application of Lemma 3.37 with $\rho = \rho_1 \sim L_F L_0^2$ gives a new bound ($\hbar$-57), which, roughly speaking, imposes $\hbar \lesssim \frac{1}{L_0^2(L_1+L_2)L_F^4}$, which is *a priori* stronger than the previous one, at least if $L_F, L_0 \geq 1$ and we neglect the term $C_N\hbar^N$. However, at each application of the lemma we are free to optimize by choosing a different N. In both cases one needs to select $R > 2L_F\rho$, which gives yet another bound by adjusting $\hbar_0^{[0]} \sim o(1/R)$, see ($\hbar$-56). Given R, this last bound only depends on the size of the domain B_0 within B; we can improve it if necessary by choosing a smaller B_0. $\triangle$

We now consider in more details the construction of the first three points $(\lambda_{0,0}, \lambda_{1,0}, \lambda_{0,1})$ near c. Naturally, we assume that $\hbar$ is small enough so that $\mathcal{L}_\hbar$ contains at least three points. Our aim is to prove that $\vec{z}_1$ and $\vec{z}_2$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$. After $\lambda_{0,0}$ is chosen, $\vec{v}_1 = \lambda_{1,0} - \lambda_{0,0}$ and $\vec{v}_2 = \lambda_{0,1} - \lambda_{0,0}$ are chosen by a minimization process. We have to be careful with the fact that the vectors $\vec{z}_i$ do not satisfy the same minimization properties because $G'_\hbar(\hbar k_{0,0})$ is not orthogonal, in general.

Lemma 3.40 *There exists $\hbar_0^{[3]} > 0$ (given by ($\hbar$-66) and ($\hbar$-69) below) such that if $\hbar \leq \hbar_0^{[3]}$, then the entries of $\vec{z}_1$ are co-prime integers.*

Proof. Assume that there exists $\vec{z}_0 \in \mathbb{Z}^2 \setminus \{0\}$ and $n \in \mathbb{N}^*$ such that $\vec{z}_1 = n\vec{z}_0$, where n may depend on $\hbar$. Let us first show that $\hbar(k_{0,0} + \vec{z}_0) \in \tilde{U}_0$ (notice that $\tilde{U}_0$ is not assumed to be convex). From Lemma 3.14, for any N we have

$$\|c - \lambda_{0,0}\| \leq (M + L_0)\hbar + C_N\hbar^N.$$

Likewise, if we let $k_{0,0} := k_\hbar(\lambda_{0,0})$ and $\xi := G_0^{-1}(c)$, then $\xi \in \tilde{B}_0$ and we have

$$\begin{aligned}
\|\xi - \hbar k_{0,0}\| &\leq L_F \left(\|G_\hbar(\xi) - \lambda_{0,0}\| + \|\lambda_{0,0} - G_\hbar(\hbar k_{0,0})\|\right) \\
&\leq L_F \left(M\hbar + \|c - \lambda_{0,0}\| + C_N\hbar^N\right) \\
&\leq L_F(2M + L_0 + 2C_N\hbar^{N-1})\hbar.
\end{aligned}$$

Therefore, using (62),

$$\|\hbar(k_{0,0}+\vec{z}_0)-\xi\| \leq \|\hbar k_{0,0}-\xi\| + \hbar\,\|\vec{z}_0\| \leq L_F(2M+L_0+2C_N\hbar^{N-1})\hbar + \hbar\tilde{\rho}/n$$

Since $\tilde{B}_0$ is open and independent of $\hbar$, there is $r_0 > 0$ such that $B(\xi, r_0) \subset \tilde{U}_0$. Assume $\hbar(L_F(2M+L_0+2C_N\hbar^{N-1})+\tilde{\rho}) < r_0$, *i.e.*

$$\hbar L_F(2M+2L_0+6C_N\hbar^{N-1}) < r_0\,. \tag{$\hbar$-66}$$

Then $\hbar(k_{0,0}+\vec{z}_0) \in B(\xi, r_0) \subset \tilde{U}_0$. We may now let $\mu := k_\hbar^{-1}(\hbar(k_{0,0}+\vec{z}_0))$ be the corresponding element in $\mathcal{L}_\hbar$. By (16), for any $\vec{z} \in \mathbb{R}^2$ such that $\|\vec{z}\| \leq \|\vec{z}_1\|$,

$$\|G_\hbar(\hbar k_{0,0}+\hbar\vec{z}) - G_\hbar(\hbar k_{0,0}) - G'_\hbar(\hbar k_{0,0})\cdot(\hbar\vec{z})\| \leq L_1\,\|\hbar\vec{z}\|^2 \tag{67}$$

and hence

$$\begin{aligned}\|\lambda_{1,0}-\lambda_{0,0}\| &= \|G'_\hbar(\hbar k_{0,0})\cdot(\hbar\vec{z}_1)\| + \sigma_1\\ \|\mu-\lambda_{0,0}\| &= \|G'_\hbar(\hbar k_{0,0})\cdot(\hbar\vec{z}_0)\| + \sigma_0\,,\end{aligned} \tag{68}$$

with $|\sigma_j| \leq L_1\,\|\hbar\vec{z}_1\|^2 + 2C_N\hbar^N$, for $j=0,1$. From the algorithm we know that $\|\lambda_{1,0}-\lambda_{0,0}\| \leq \|\mu-\lambda_{0,0}\|$ and therefore

$$(n-1)\,\|G'_\hbar(\hbar k_{0,0})\cdot(\hbar\vec{z}_0)\| \leq \sigma_0-\sigma_1.$$

From Lemma 3.7, there exists $\Gamma > 0$ independent of $\hbar$ such that $\left\|G'^{-1}_\hbar\right\| \geq \Gamma$ on $\tilde{U}$, and hence

$$(n-1) \leq \frac{\sigma_0-\sigma_1}{\Gamma\,\|\hbar\vec{z}_0\|} \leq \frac{2L_1\hbar^2\tilde{\rho}^2+4C_N\hbar^N}{\Gamma\hbar}\,.$$

Thus, if

$$\hbar(2L_1\tilde{\rho}^2+4C_N\hbar^{N-2}) < \Gamma, \tag{$\hbar$-69}$$

then we must have $n=1$. □

Lemma 3.41 *For all $\hbar \leq \hbar_0^{[3]}$, the vectors $\vec{z}_1$ and $\vec{z}_2$ (59) defined in Corollary 3.38 are linearly independent.*

Proof. If $\vec{z}_1$ and $\vec{z}_2$ are colinear, there exists $\sigma \in \mathbb{Q}$ such that $\vec{z}_2 = \sigma\vec{z}_1$. Since the coefficients of $\vec{z}_1$ are co-prime, we must have $\sigma = n \in \mathbb{Z}$. Writing, $k_{0,1} = k_{0,0}+\vec{z}_2 = k_{0,0}+n\vec{z}_1$, we obtain from Corollary 3.38 that $\lambda_{0,1} = \lambda_{n,0}$, which contradicts Step 5 of the algorithm. □

We arrive at the main result of this section.

Theorem 3.42 *There exists $\hbar_0^{[4]} > 0$ (defined in (3.6.1)) such that $\vec{z}_1$ and $\vec{z}_2$ defined in Corollary 3.38 form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$ for any $\hbar \leq \hbar_0^{[4]}$.*

Proof. Let $D = D(\hbar) := \det(\vec{z}_1, \vec{z}_2)$. We know that $D \neq 0$ by the previous lemma. We want now to prove that $D = \pm 1$. By Pick's formula, the number of integral points in the closed convex hull of the three points $k_{0,0}$, $k_{1,0}$ and $k_{0,1}$ (other than these vertices), where points in the boundary count half, is $\frac{|D|-1}{2}$. Hence if $|D| \geq 2$, there is at least one such integral point, call it $\ell \in \mathbb{Z}^2$, set $\vec{z}_3 = \ell - k_{0,0}$, and let $\mu = k_\hbar^{-1}(\hbar\ell)$ be the corresponding element of $\mathcal{L}_\hbar$.

(i) Since the components of $\vec{z}_1$ are co-prime by Lemma 3.40, $\vec{z}_3$ cannot be colinear to $\vec{z}_1$, because it then would be equal to it, and ℓ would be a vertex of the triangle. Therefore, by Corollary 3.38, μ cannot be one of the $\lambda_{n,0}$. This implies

$$\|\lambda_{0,1} - \lambda_{0,0}\| \leq \|\mu - \lambda_{0,0}\| . \tag{70}$$

(ii) Set $\vec{u}_i = G'_h(\hbar k_{0,0}) \cdot \vec{z}_i$, $i = 1, 2, 3$. As $\vec{z}_3$ is a convex linear combination of $\vec{z}_1$ and $\vec{z}_2$, the same holds between $\vec{u}_3$, and $\vec{u}_1$, $\vec{u}_2$. Using (67), we see that there exists a constant $C > 0$ that can be made explicit (namely $C \geq 2L_1\tilde{\rho} + 2C_N\hbar^{N-2}$) such that the ball $B(\mu, \hbar^2 C)$ intersects the interior of the triangle $(\lambda_{0,0}, \lambda_{1,0}, \lambda_{0,1})$. Let $\tilde{\mu}$ be a point of this intersection. Since $\|\vec{z}_j - \vec{z}_3\| \geq 1$ for $j = 1, 2$, because these are non zero integer vectors, the point $\tilde{\mu}$ must stay away from a ball of size $\delta\hbar$ of the vertices, for some $\delta \in]0, 1/L_F[$ independent of $\hbar$ and which can also be made explicit.

(iii) We now need some elementary triangle estimates in order to bound from below the distance $\|\lambda_{1,0} - \lambda_{0,0}\| - \|\tilde{\mu} - \lambda_{0,0}\|$. By construction of the algorithm, $\|\lambda_{1,0} - \lambda_{0,0}\| \leq \|\lambda_{0,1} - \lambda_{0,0}\|$. Hence the angle at the vertex $\lambda_{0,1}$ is strictly less than $\frac{\pi}{2}$, which implies that the orthogonal projection H of $\lambda_{0,0}$ onto the line $(\lambda_{1,0}\lambda_{0,1})$ is located on the strict half line starting at $\lambda_{0,1}$ and containing $\lambda_{1,0}$. Thus there exists $r > 0$ be such that the ball $B(\lambda_{0,1}, r)$ does not contain H (see Figure 3.6.1). In

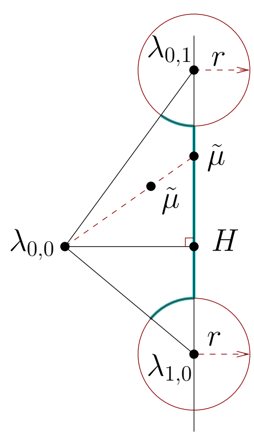

Figure 5: The point $\tilde{\mu}$ is pushed onto the thickened path.

fact any $r \leq \|\lambda_{1,0} - \lambda_{0,1}\|/2$ fulfills this requirement. We choose $r = \min(\|\lambda_{1,0} - \lambda_{0,1}\|/2, \delta\hbar)$ so that, in addition, $\tilde{\mu} \notin B(\lambda_{0,1}, r) \cup B(\lambda_{1,0}, r)$. Let us push $\tilde{\mu}$ along the ray $(\lambda_{0,0}\mu)$ until it reaches either the boundary of one of the balls $B(\lambda_{0,1}, r)$ or $B(\lambda_{1,0}, r)$, or the opposite edge of the triangle, $E := (\lambda_{1,0}\lambda_{0,1})$. We may call $\tilde{\mu}$ again this new point. Doing this, the distance $d := \|\tilde{\mu} - \lambda_{0,0}\|$ can not decrease. When $\tilde{\mu}$ is on the boundary of a ball, it is clear that d increases as we move $\tilde{\mu}$ on the circle towards the edge E. So, for any configuration of $\tilde{\mu}$, there is a position of $\tilde{\mu}$ on E that produces a larger or equal distance d. Now it is easy to study the variation of $d = d(x)$ as a function of the abscissa x of $\tilde{\mu}$ along E. Taking $x = 0$ for the point H and $x =: b > 0$ at $\lambda_{0,1}$, and using $d(x) = \sqrt{x^2 + h^2}$, with $h = \|H - \lambda_{0,0}\|$, we obtain for $x \in (0, b - r)$ that d is increasing and

$$d(b) - d(x) \geq \frac{rb}{2d(b)} \geq \frac{r^2}{2d(b)}.$$

If the point $\lambda_{1,0}$ has a negative abscissa $c < 0$, the distance d will have a local maximum also at c, and we can repeat the argument above to get, when $|c| > r$,

$$\forall x \in (c + r, 0), \qquad d(c) - d(x) \geq \frac{r\,|c|}{2d(c)} \geq \frac{r^2}{2d(c)}.$$

Recall that $d(c) = \|\lambda_{1,0} - \lambda_{0,0}\| \leq \|\lambda_{0,1} - \lambda_{0,0}\| = d(b)$. Thus,

$$\forall x \in (c + r, b - r), \qquad d(b) - d(x) \geq \frac{r^2}{2d(b)}\,.$$

This finally gives

$$\|\lambda_{0,1} - \lambda_{0,0}\| - \|\tilde{\mu} - \lambda_{0,0}\| \geq \frac{r^2}{2\,\|\lambda_{0,1} - \lambda_{0,0}\|}\,. \tag{71}$$

(iv) From (70) we can write

$$\begin{aligned}\|\lambda_{0,1} - \lambda_{0,0}\| &\leq \|\mu - \lambda_{0,0}\| \leq \|\tilde{\mu} - \lambda_{0,0}\| + C\hbar^2 \\ &\leq \|\lambda_{0,1} - \lambda_{0,0}\| - \frac{r^2}{2\,\|\lambda_{0,1} - \lambda_{0,0}\|} + C\hbar^2 \quad \text{by (71)}.\end{aligned}$$

Therefore

$$\frac{r^2}{2\,\|\lambda_{0,1} - \lambda_{0,0}\|} \leq C\hbar^2.$$

Since $\|k_{0,1} - k_{1,0}\| \geq 1$, we have $\|\lambda_{0,1} - \lambda_{1,0}\| \geq \hbar/L_F - 2C_N\hbar^N$, while (61) says $\|\lambda_{0,1} - \lambda_{0,0}\| \leq L_0\hbar + 2C_N\hbar^N$. We obtain

$$\frac{\min((\frac{1}{L_F} - 2C_N\hbar^{N-1})^2, \delta^2)}{2(L_0 + 2C_N\hbar^{N-1})} \leq C\hbar$$

which is of course impossible if $\hbar$ is small enough, namely if one takes $\hbar \leq \hbar_0^{[4]}$ with

$$2C_N(\hbar_0^{[4]})^{N-1} < \frac{1}{L_F} \quad \text{and} \quad \delta < \frac{1}{L_F} - 2C_N(\hbar_0^{[4]})^{N-1}$$
$$\text{and} \quad \hbar_0^{[4]} < \frac{\delta^2}{2C(L_0 + 2C_N(\hbar_0^{[4]})^{N-1})} \,. \qquad (\hbar\text{-}72)$$

For all $\hbar \leq \hbar_0^{[4]}$, we conclude that $|D| = 1$. This concludes the proof of the theorem. □

Remark 3.43 In order to have a rough idea of the size of the various bounds on $\hbar$, following up Remark 3.39, we may neglect the terms $C_N\hbar^N$ and get from ($\hbar$-66) and ($\hbar$-69) and the approximation $\Gamma \sim 1/L_0$ that $\hbar_0^{[3]} \lesssim \min\left(\hbar_0^{[2]}\, ;\, \frac{r_0}{2L_F(M+L_0)}\, ;\, \frac{1}{2L_0^3L_1L_F^2}\right)$. Then, from (3.6.1) and $C \sim 2L_0^2L_1L_F^2$, we get $\hbar_0^{[4]} \lesssim \min\left(\hbar_0^{[3]}\, ;\, \frac{1}{4L_0^3L_1L_F^4}\right)$. △

Definition 3.44 *When the vectors* $(\vec{z}_1, \vec{z}_2)$ *form a* $\mathbb{Z}$*-basis of* $\mathbb{Z}^2$*, the triple* $(\lambda_{0,0}, \lambda_{1,0}, \lambda_{0,1})$ *will be called an* affine basis *of* $\mathcal{L}_\hbar$ *at* $\lambda_{0,0}$.

The algorithm does not necessarily label *all* points of $\mathcal{L}_\hbar \cap B_0$. Indeed, by construction, the set of produced labels (n, m) is of the form

$$\mathcal{E} := \{(n, m) \in \mathbb{Z}^2; \quad m_{\text{min}} \leq m \leq m_{\text{max}}; \quad n_{\text{min}}(m) \leq n \leq n_{\text{max}}(m)\}\,,$$

where m_{min}, m_{max}, and $n_{\text{min}}(m)$ and $n_{\text{max}}(m)$, for $m \in \{m_{\text{min}}, \ldots, m_{\text{max}}\}$, may depend on $\hbar$. For a given m, $n_{\text{max}}(m)$ is the smallest positive integer produced by Step 4 such that $\lambda_{n,m} \in B_0$ and $\lambda_{n+1,m} \notin B_0$. The integer $n_{\text{min}}(m)$ is defined in a similar way, m_{max} is the smallest positive integer produced by Step 8 such that $\lambda_{0,m} \in B_0$ but $\lambda_{0,m+1} \notin B_0$, and m_{min} is defined in a similar way. Thus, there is no reason why a set of the form $\mathcal{E}$ would fill up $\mathcal{L}_\hbar \cap B_0$ entirely. However, since $\mathcal{E}$ will fill up all integral points of any convex set $V \subset \tilde{U}_0$, the algorithm is guaranteed to label all points of $\mathcal{L}_\hbar$ in *some* $\hbar$-independent open subset containing c. It would be interesting to improve the algorithm in order to make sure that it explores the whole connected component of B_0.

Note that, after the last step (orientation test) of the algorithm, the basis $\vec{v}_1$, $\vec{v}_2$ is made direct, so is the case for the basis $\vec{z}_1$, $\vec{z}_2$ because by convention $\det G_0'(\xi) > 0$. It follows from Theorem 3.42 and Corollary 3.38 that for each $\hbar$ small enough, we can define a matrix $Z_\hbar \in \mathrm{SL}(2, \mathbb{Z})$ such that $Z_\hbar(\vec{e}_1, \vec{e}_2) = (\vec{z}_1, \vec{z}_2)$, and the "labelling" $\lambda_{n,m} \mapsto (n, m)$ of the algorithm is such that

$$\lambda_{n,m} = G_\hbar(\hbar(k_{0,0} + n\vec{z}_1 + m\vec{z}_2)) + \mathscr{O}(\hbar^\infty) = G_\hbar \circ Z_\hbar(\hbar(\kappa + n\vec{e}_1 + m\vec{e}_2)) + \mathscr{O}(\hbar^\infty)$$

where $\kappa = Z_\hbar^{-1}k_{0,0} \in \mathbb{Z}^2$. However, this does not produce a linear labelling for $\mathcal{L}_\hbar$ because in general the map $\hbar \to Z_\hbar$ will not be constant (and hence, not continuous), even for arbitrary small values of $\hbar$. In order to produce a linear labelling, we should find a way to "detect" the matrix $Z_\hbar$, which will allow to correct the initial algorithm and make it smooth in $\hbar$. This is the aim of the following section.

3.6.2 A semitoric algorithm

If a quantum system is known to be semitoric, the joint spectrum is an asymptotic lattice with a special property given by Proposition 3.31. In this case, we expect a good algorithm not to provide *any* labelling, but rather a *semitoric labelling*, *i.e.* associated with a semitoric chart, see Lemma 3.32.

Let (j, k) be the semitoric labelling of Theorem 3.34. From (47), we know that for any $\epsilon \in (0, \frac{1}{2})$, there exists a ball B' around c such that, if $\hbar_0$ is small enough, $\Sigma_\hbar \cap B'$ is contained in a union of disjoint vertical strips of width $2\epsilon\hbar$:

$$\Sigma_\hbar \cap B' \subset \bigcup_{j\in\mathbb{Z}} V_j(\hbar), \tag{73}$$

where

$$V_j(\hbar) = [\alpha + \hbar(j + \mu) - \hbar\epsilon, \alpha + \hbar(j + \mu) + \hbar\epsilon] \times \mathbb{R} \subset \mathbb{R}^2. \tag{74}$$

The precise size of B' depends on the variations of the subprincipal term G_1 of a semitoric asymptotic chart $G_\hbar \sim G_0 + \hbar G_1 + \cdots$.

Now, from the data of $\mathcal{L}_\hbar$, perform the generic algorithm of the previous section, in order to obtain a good labelling $k := (k_1, k_2)$ of the asymptotic lattice. Let $(\lambda_{0,0}, \lambda_{1,0}, \lambda_{0,1})$ be the corresponding affine basis of $\mathcal{L}_\hbar$. Let V be the vertical strip of width $\hbar^{3/2}$, vertically centered at $\lambda_{0,0}$, and define $\mu \in \mathcal{L}_\hbar$ as the nearest point to $\lambda_{0,0}$ located in V and *above* $\lambda_{0,0}$. The existence of a semitoric chart (Lemma 3.32) and the decomposition (73)-(74) imply that μ exists in an $\mathscr{O}(\hbar)$ neighborhood of $\lambda_{0,0}$, belongs to the strip V_{j_0} that contains $\lambda_{0,0}$, and is unique. Therefore, we know from the analysis of the general algorithm (Corollary 3.38) that there exists bounded co-prime integers $(n, m) \in \mathbb{Z}^2$ such that

$$k_\hbar(\mu) - k_{0,0} = n\,\vec{z}_1 + m\,\vec{z}_2\,.$$

In practice, the integers (n, m) can be found by expressing $(\mu - \lambda_{0,0})/\hbar$ on the basis (v_1, v_2) and then rounding its coefficients to their nearest integers. Finally, we choose $(u', v') \in \mathbb{Z}^2$ such that $u'v - v'u = 1$, and define $\mu' \in \mathcal{L}_\hbar$ by

$$k_\hbar(\mu') = n'\,\vec{z}_1 + m'\,\vec{z}_2\,.$$

We obtain in this way a new affine basis of $\mathcal{L}_\hbar$ given by

$$(\mu' - \lambda_{0,0}, \mu - \lambda_{0,0})\,. \tag{75}$$

Proposition 3.45 *The good labelling obtained by using the affine basis (75) in performing Steps 6 to 12 of the general algorithm of Section 3.6.1 is associated with a semitoric asymptotic chart.*

Proof. The uniqueness of μ ensures that its label, in a semitoric chart $\hat{G}_\hbar$ associated with a labelling $\hat{k}_\hbar$, is equal to $\hat{k}_\hbar(\lambda_{0,0})+(0,1)$. Hence the label of μ' must be of the form $\hat{k}_\hbar(\lambda_{0,0})+(1,\ell)$, for some $\ell \in \mathbb{Z}$. Hence the good labelling produced by the algorithm will be associated with the chart $\hat{G}_\hbar \circ A$, with $A = \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix}$. This matrix A preserves the semitoric property of the chart. □

3.6.3 An algorithm for a sequence of values of $\hbar$

In our way to reconstructing a good labelling from the data of the sets $\mathcal{L}_\hbar$ and of a window B with a distinguished point $c \in B$, we introduce a "algorithm with uniform labelling" on a decreasing sequence $\hbar_j \in \mathcal{I}$, $j \geq 1$, tending to 0. The algorithm is inductive: once the labelling $k_{\hbar_j} : \Lambda_{\hbar_j} \ni \lambda^{(j)} \mapsto (n^{(j)}, m^{(j)}) \in \mathbb{Z}^2$ is known, together with a matrix $S_j \in \mathrm{M}_2(\mathbb{Z})$, it will produce the labelling $k_{\hbar_{j+1}} : \Lambda_{\hbar_{j+1}} \to \mathbb{Z}^2$ (and the matrix S_{j+1}). Thus, it theoretically defines $k_{\hbar_j}$ for all j. Of course, in practice, if one wants to obtain the labelling $k_{\hbar_j}$ for a specific j, it is enough to stop at the step j.

The algorithm works by running the previous algorithm of Section 3.6.1 with all values $\hbar_j$, for all $j = 1, \ldots$, and self-adjusting the resulting labelling for each j. In order to have a more efficient implementation, if we know in advance that we want to stop at a specific step $j = j_{\text{stop}}$, it is in fact not necessary to compute the full labellings for the values of j less than j_{stop}; for these values, it is enough to find the correct "affine basis", which corresponds to Steps 1 to 5 of the algorithm of Section 3.6.1. Thus, the new algorithm, with exit test at $j = j_{\text{stop}}$, works as follows.

a) Choose an open subset $B_0 \Subset B$, and fix $c \in B_0$. Let $S_0 = \mathrm{Id} \in \mathrm{M}_2(\mathbb{R})$. Let $j = 1$.

b) Apply steps 1 to 5 of the algorithm of Section 3.6.1 with $\hbar = \hbar_j$. This defines points $\lambda^{(j)}_{n,m}$. In particular, we have an origin $\lambda^{(j)}_{0,0}$, and the first generating vectors $v^{(j)}_1 = \lambda^{(j)}_{1,0} - \lambda^{(j)}_{0,0}$ and $v^{(j)}_2 = \lambda^{(j)}_{0,1} - \lambda^{(j)}_{0,0}$.

c) Define $T_j \in \mathrm{M}_2(\mathbb{R})$ to be the matrix formed by the column vectors:

$$T_j := (\hbar_j^{-1}\, v^{(j)}_1,\ \hbar_j^{-1}\, v^{(j)}_2).$$

If T_j is not invertible or $j = 1$, increase j by one, let $S_j := S_{j-1}$, and go back to Step *b*). If T_j is invertible, make it oriented as in Step 12 of the previous algorithm.

d) Define $A_j := T_j^{-1} T_{j-1}$, and let $A^\sharp_j \in \mathrm{M}_2(\mathbb{Z})$ be the matrix obtained by rounding the entries of A_j to their "nearest integer" (in the usual, unique way). If $\det A^\sharp_j \neq 1$, define $S_j := S_{j-1}$. If $\det A^\sharp_j = 1$, define

$$S_j := S_{j-1}\, (A^\sharp_j)^{-1}\,.$$

e) In case of a concrete implementation, if $j < j_{\text{stop}}$, increase j by one and go back to Step *b*). Otherwise, finish the previous algorithm, *i.e.* perform Steps 6 to 12. Let $\lambda \mapsto (n, m)$ be the resulting labelling.

f) The new labelling of $\mathcal{L}_{\hbar_j}$ is the map $\lambda^{(j)} \mapsto (\tilde{n}, \tilde{m})$ given by a linear transformation acting on the labelling λ of the previous step according to

$$\begin{pmatrix} \tilde{n} \\ \tilde{m} \end{pmatrix} = S_j \begin{pmatrix} n \\ m \end{pmatrix} .$$

In other words, $\lambda_{n,m} = \lambda^{(j)}_{S_j(m,n)} = \lambda^{(j)}_{\tilde{m},\tilde{n}}$.

Theorem 3.46 *Let $(\mathcal{L}_\hbar, \mathcal{I}, B)$ be an asymptotic lattice, where $B \subset \mathbb{R}^2$. Let $\hbar_j \in \mathcal{I}$, $j \geq 1$, be a decreasing sequence tending to 0. Then the previously described algorithm produces a linear labelling of the asymptotic lattice $(\mathcal{L}_\hbar, \mathcal{I}', B)$, where $\mathcal{I}' = \{\hbar_j, j \in \mathbb{N}^*\}$.*

Proof. We interpret the newly introduced objects with respect to an asymptotic chart $G_\hbar$ (which is known to exist, but is unknown). Let us denote by Z_j the matrix formed by the column vectors $z_1^{(j)} = \vec{z}_1$ and $z_2^{(j)} = \vec{z}_2$ defined in Corollary 3.38(59) for $\hbar = \hbar_j$ (initial algorithm):

$$Z_j := (z_1^{(j)},\ z_2^{(j)}).$$

Formula (16), in view of (62), gives

$$T_j = G_0'(\xi)\, Z_j + \mathscr{O}(\hbar_j)\, .$$

By Theorem 3.42, Z_j is unimodular and hence T_j is invertible (with positive determinant) if $\hbar_j$ is small enough, which happens for all $j \geq j_0$, for some j_0. Thus,

$$A_j := T_j^{-1} T_{j-1} = Z_j^{-1} Z_{j-1} + \mathscr{O}(\hbar_{j-1})\, .$$

Therefore, if j_0 is large enough, we obtain for all $j \geq j_0$,

$$A_j^\sharp = Z_j^{-1} Z_{j-1} \quad \text{and} \quad \det A_j^\sharp = 1\, .$$

Set $\tilde{Z}_j = Z_j\, S_j^{-1}$. The matrix $\tilde{Z}_j$ has integer coefficients and satisfies by definition of the new labelling for $\hbar = \hbar_j$

$$\tilde{Z}_j \begin{pmatrix} \tilde{n} \\ \tilde{m} \end{pmatrix} = Z_j \begin{pmatrix} n \\ m \end{pmatrix} .$$

We check that the sequence $\tilde{Z}_j$ is stationary as $j \to \infty$:

$$\tilde{Z}_j = Z_j\, S_j^{-1} = Z_j\, A_j^\sharp\, S_{j-1}^{-1} = Z_j\, Z_j^{-1} Z_{j-1}\, S_{j-1}^{-1} = Z_{j-1}\, S_{j-1}^{-1} = \cdots = \tilde{Z}_{j_0}\, .$$

By (60), in terms of the good labelling $k_\hbar$ associated with $G_\hbar$

$$k_\hbar(\tilde\lambda^{(j)}_{\tilde n,\tilde m}) = \tilde n\,\vec z_1 + \tilde m\,\vec z_2 + k^{(j)}_{0,0},$$

i.e.

$$\tilde Z_{j_0}^{-1}\,k_\hbar(\tilde\lambda^{(j)}_{n,m}) = n\,\vec e_1 + m\,\vec e_2 + \tilde Z_{j_0}^{-1}\,k^{(j)}_{0,0}.$$

Thus we have constructed a linear labelling $\lambda \to (n,m)$ for $(\mathcal{L}_\hbar, \mathcal{I}', B)$, where (n,m) is such that $\lambda = \tilde\lambda^{(j)}_{n,m}$, associated with the asymptotic chart $\tilde G_\hbar := G_\hbar \circ \tilde Z_{j_0}$. □

3.7 The inverse problem for the rotation number

We can finally apply our algorithms to the initial question, because having a linear labelling is actually enough for recovering the rotation number. In Theorem 3.28 we have obtained the classical rotation number as an $\mathscr{O}(h)$ limit of the quantum rotation number; this suggests that one can actually expect that the recovery process is robust with respect to smaller (namely, $\mathscr{O}(\hbar^2)$) perturbation of the asymptotic lattice. Previous results for special cases of pseudo-differential operators show that this expectation is very natural [48, 33].

Corollary 3.47 ((Theorem 1.1)) *Let $(\hat J, \hat H)$ be a quantum integrable system with principal symbols $F := (J, H)$. Let $c \in \mathbb{R}^2$ be a regular value of F, such that the fiber $F^{-1}(c)$ is compact and connected. Then from the knowledge of the family of joint spectra $(\Sigma_\hbar(\hat J, \hat H))_{\hbar\in\mathcal{I}'}$ modulo $\mathscr{O}(\hbar^2)$, where $\hbar$ varies in a sequence $\mathcal{I}' = (\hbar_j)_{j\in\mathbb{N}^*}$ accumulating at zero, one can recover the classical rotation number $[w]$ in a neighborhood of c modulo the natural action of Möbius transformations (Lemma 2.2).*

Proof. Recall from Definition 2.1 that we need to recover the rotation number $w_I(b)$ for all points b in a neighborhood of c, where I is a fixed set of action variables, which should not depend on b.

We first assume that $(\Sigma_\hbar := \Sigma_\hbar(\hat J, \hat H))_{\hbar\in\mathcal{I}'}$ is given exactly, without any $\mathscr{O}(\hbar^2)$ error term. By Theorem 3.6, there exists a ball B around c, contained in the set of regular values of F, and $\hbar_0 > 0$, such that $(\Sigma_\hbar \cap B,]0, \hbar_0[, B)$ is an asymptotic lattice.

Let $\bar k_\hbar$ be a linear labelling of a neighborhood $\tilde B \Subset B$ of c constructed by the algorithm of Theorem 3.46. Let $b \in \tilde B$ and let $\lambda \in \mathcal{L}_\hbar$ be such that $\lambda = b + \mathscr{O}(\hbar)$ (for instance, choose a closest point to b). Let $(\bar n, \bar m) = \bar k_\hbar(\lambda)$ be the corresponding label obtained from the algorithm, *i.e.* $\lambda = \lambda_{\bar n,\bar m}$.

Let $k_\hbar$ be the good labelling associated with $\bar k_\hbar$ (it is not known from the algorithm). There exist $\hbar$-dependent integers n_0, m_0 such that for any $\lambda \in \Sigma_\hbar$, if we let $(n,m) = k_\hbar(\lambda)$ and $(\bar n, \bar m) = \bar k_\hbar(\lambda)$, then

$$(n,m) = (\bar n, \bar m) + (n_0, m_0). \tag{76}$$

Recall from (46) that the quantum rotation number $\hat w_\hbar(n,m)$ is by definition

$$\hat w_\hbar(n,m) = \frac{E_{n+1,m}(\hbar) - E_{n,m}(\hbar)}{E_{n,m+1}(\hbar) - E_{n,m}(\hbar)}\,,$$

where for all n, m, we denote by $E_{n,m}(\hbar)$ the first component of $\lambda = k_\hbar^{-1}(n, m) \in \Sigma_\hbar$. Hence

$$\hat{w}_\hbar(n, m) = \frac{\pi_1 \lambda_{\bar{n}+1,\bar{m}} - \pi_1 \lambda_{\bar{n},\bar{m}}}{\pi_1 \lambda_{\bar{n},\bar{m}+1} - \pi_1 \lambda_{\bar{n},\bar{m}}},$$

where $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the first factor; note that this quantity can be computed directly from the algorithm. It just remains to apply Theorem 3.28: for the action variables $I = G_0^{-1}(F)$, defined in a fixed neighborhood of the torus $\Lambda_c = F^{-1}(c)$, containing $\Lambda_b = F^{-1}(b)$, the classical rotation number is

$$[w_I(\Lambda_b)] = [\hat{w}_\hbar(n, m)] + \mathscr{O}(\hbar)$$

and hence can be recovered in the limit as $\hbar \to 0$, $\hbar \in \mathcal{I}'$.

It remains to investigate the effect of the error term $\mathscr{O}(\hbar^2)$ in the knowledge of the joint spectrum. By (23), this will not affect, if $\hbar$ is small enough, the choice of all points of the form μ_2 in Lemma 3.37, that is, once the affine basis $(\lambda_{0,0}, \lambda_{1,0}, \lambda_{0,1})$ is chosen (Definition 3.44). In contrast, the choice of this affine basis *can* depend on the error term, because we are minimizing distances of order $\hbar$. Fortunately, this error won't affect estimates like (67), provided we accept to make the constant L_1 larger, which is harmless. Thus, the perturbed triple $(\lambda'_{0,0}, \lambda'_{1,0}, \lambda'_{0,1})$ is still an affine basis for $\hbar$ small enough, and the rest of the algorithm (Lemma 3.37) goes through, leading to a labelling $\lambda'_{\bar{n},\bar{m}}$ of the perturbed joint spectrum near c. Let $\lambda_{\bar{n},\bar{m}}$ be the unique point in $\Sigma_\hbar$ that is $\mathscr{O}(\hbar^2)$-close to $\lambda'_{\bar{n},\bar{m}}$, using (23) again. We have $\lambda_{\bar{n},\bar{m}} = b + \mathscr{O}(\hbar)$, and because of that uniqueness, $\lambda_{\bar{n},\bar{m}} \mapsto (\bar{n}, \bar{m})$ is a linear labelling of $\Sigma_\hbar$. As a above, we can now introduce the good labelling for which (76) holds. The resulting "perturbed quantum rotation number" will be computed as

$$\hat{w}'_\hbar(n, m) = \frac{\pi_1 \lambda'_{\bar{n}+1,\bar{m}} - \pi_1 \lambda'_{\bar{n},\bar{m}}}{\pi_1 \lambda'_{\bar{n},\bar{m}+1} - \pi_1 \lambda'_{\bar{n},\bar{m}}} = \frac{\pi_1 \lambda_{\bar{n}+1,\bar{m}} - \pi_1 \lambda_{\bar{n},\bar{m}}}{\pi_1 \lambda_{\bar{n},\bar{m}+1} - \pi_1 \lambda_{\bar{n},\bar{m}}} + \mathscr{O}(\hbar^2).$$

Hence, as above, we introduce the corresponding asymptotic chart $G_\hbar$ and conclude that

$$[w_I(\Lambda_b)] = [\hat{w}'_\hbar(n, m)] + \mathscr{O}(\hbar),$$

with $I = G_0^{-1} \circ F$. □

We may now turn to the global problem which, for simplicity, we state without the $\mathscr{O}(\hbar^2)$ perturbation.

Theorem 3.48 *Let $(\hat{J}, \hat{H})$ be a quantum integrable system with principal symbols $F := (J, H)$. Let B_r be the set of regular values of F; we assume that all fibers $F^{-1}(c)$ are compact and connected. Then, from the joint spectrum $\Sigma_\hbar$ of $(\hat{J}, \hat{H})$, one can construct a map $\hat{\omega} : \tilde{B}_r \to \mathbb{R}P^1$ such that the following holds.*

1. *If $[\tilde{w}]$ is the globalized rotation number defined in Proposition 2.10, then*

$$\hat{\omega} = A \circ [\tilde{w}] + \mathscr{O}(\hbar)\,,$$

for some fixed $A \in \mathrm{SL}(2,\mathbb{Z})$. In particular, the globalized rotation number $[\tilde{w}]$ can be recovered from the joint spectrum, modulo a global Möbius transformation.

2. *Let W be a connected open subset of the non-degenerate support of (J,H) (Definition 2.5). Given any rotation number $[w]$ on W (Definition 2.3), there exists a constant $A \in \mathrm{SL}(2,\mathbb{Z})$ such that*

$$\hat{\omega} = A \circ [w] + \mathscr{O}(\hbar)\,.$$

Proof. We construct the map $\hat{\omega}$ by a Čech cohomology argument. Fix $c_0 \in B_r$, and let $\gamma : [0,1] \to B_r$ be a path starting at $\gamma(0) = c_0$. Let $c = \gamma(1)$. Let $\lambda \in \Sigma_\hbar$ be a nearest point to c, so that, by Lemma 3.14,

$$\lambda = c + \mathscr{O}(\hbar)\,.$$

Applying Theorem 3.46 we get a neighborhood B_0 of c_0 equipped with a linear labelling $\bar{k}_{0,\hbar}$ of $(\Sigma_\hbar \cap B_0, \mathcal{I}', B_0)$. Cover the image $\gamma([0,1])$ by a finite union of small balls $B_1, \ldots, B_N$, such that $c \in B_N$, $B_i \cap B_{i+1} \neq \varnothing$, for all $i = 0, \ldots, N$, and such that on each B_i, the algorithm of Theorem 3.46 produces a linear labelling $\bar{k}_{i,\hbar}$ of $(\Sigma_\hbar \cap B_i, \mathcal{I}', B_i)$. From Proposition 3.19 applied to the restrictions of $\bar{k}_{i,\hbar}$ and $\bar{k}_{i+1,\hbar}$ on the asymptotic lattice $\Sigma_\hbar \cap B_i \cap B_{i+1}$, there exists a unique matrix $A_i \in \mathrm{SL}(2,\mathbb{Z})$ and a family $(\varkappa_{i,\hbar})_{\hbar \in \mathcal{I}'}$ in $\mathbb{Z}^2$ such that, for $\hbar$ small enough,

$$\bar{k}_{i+1,\hbar} = A_i \circ \bar{k}_{i,\hbar} + \varkappa_{i,\hbar} \quad \text{on} \quad \Sigma_\hbar \cap B_i \cap B_{i+1}\,. \tag{77}$$

We define

$$\hat{\omega}(\gamma) = [\hat{w}_N](A_0^{-1} \circ A_1^{-1} \circ \cdots A_{N-1}^{-1} \circ \bar{k}_{N,\hbar}(\lambda))\,, \tag{78}$$

where $[\hat{w}_N]$ is the quantum rotation number associated with $\bar{k}_{N,\hbar}$. It is constructed from the joint spectrum as in Corollary 3.47. Note that the transition matrices A_i can be detected from the algorithm by comparing the affine basis described in Theorem 3.42.

By Lemma 3.19, the sheaf that assigns to a point $c \in B_r$ a linear labelling (given by the algorithm) on a small neighborhood of c has constant transition functions, when $\hbar$ is small enough, modulo the addition of $\hbar$-families in $\mathbb{Z}^2$. This ensures that the cocyle condition for the linear part of the transition functions is satisfied, and hence that the definition in (78) is invariant by homotopy transformation of the path γ with fixed endpoints, provided the initial labelling $\bar{k}_{0,\hbar}$ is fixed. Thus, it defines a map

$$\hat{\omega} : \tilde{B}_r \to \mathbb{R}P^1\,.$$

This map will be modified by a global $\mathrm{SL}(2,\mathbb{Z})$ transformation if one changes $\bar{k}_{0,\hbar}$. *A priori*, the smallness of $\hbar$ for which (78) is defined depends on γ; however, if γ stays in a compact region, one can use a fixed, finite covering by small balls on which the algorithm applies, and hence obtain a uniform $\hbar_0 > 0$ for which (78) holds for all $\hbar \leq \hbar_0$.

From (77) we get, using (32),

$$A_i = [(G'_{i+1,0}(\xi_{i+1})]^{-1}(G_{i,0})'(\xi_i)\,. \tag{79}$$

with $\xi_i := G_{i,0}^{-1}(c_i)$.

Let $\tilde{I} : \tilde{B}_r \to \mathbb{R}^2$ be the global action variable used in Proposition 2.10 to define $[\tilde{w}]$. Along the path γ, the balls B_i can be lifted to open sets $\tilde{B}_i \subset \tilde{B}_r$ on which π_i, the restriction of $\pi : \tilde{B}_r \to B_r$ is a diffeomorphism. Since both $\tilde{I} \circ \pi_i^{-1}$ and $I_i := G_{i,0}^{-1} \circ F$ are action variables above B_i, we must have

$$\tilde{I} \circ \pi_i^{-1} = Z_i \circ I_i\,,$$

for some $Z_i \in \mathrm{SL}(2,\mathbb{Z})$, which is of course independent of $\hbar$. Using (79), we obtain $A_i = Z_{i+1}^{-1} \circ Z_i$, for all $i = 0, \ldots, N-1$. Thus, $A_{N-1} \cdots A_1 A_0 = Z_N^{-1} Z_0$. For $\hbar$ small enough, $Z_0^{-1} Z_N \bar{k}_{N,\hbar}(\lambda)$ stays in the image of the linear labelling $\bar{k}_{N,\hbar}$; hence we can write

$$\begin{aligned}[\tilde{w}](\gamma) &= [\hat{w}_N](Z_0^{-1} Z_N \bar{k}_{N,\hbar}(\lambda)) \\ &= [w_{Z_0^{-1} Z_N I_N}(\Lambda)] + \mathscr{O}(\hbar) \quad \text{by Theorem 3.28} \\ &= {}^t Z_0 \circ [w_{Z_N I_N}(\Lambda)] + \mathscr{O}(\hbar) \quad \text{by Lemma 2.2} \\ &= {}^t Z_0 \circ [\tilde{w}](\gamma) + \mathscr{O}(\hbar)\end{aligned}$$

because by definition, $[\tilde{w}]\restriction_{\tilde{B}_N} = [w_{\tilde{I}\circ\pi_N^{-1}}] = [w_{Z_N I_N}]$. This finishes the proof of Item 1. Item 2 now directly follows from Item 1 and the uniqueness part of Proposition 2.10. □

Finally, let us consider the semitoric case. In the presence of an elliptic singularity we may combine Theorem 3.34 with Proposition 3.35 to obtain the following.

Corollary 3.49 *Let $(\hat{J}, \hat{H})$ be a semitoric quantum integrable system with principal symbols $F = (J, H)$. Let $c \in \mathbb{R}^2$ be a simple J-transversal elliptic critical value of F. Then the joint spectrum $(\Sigma_\hbar)_{\hbar\in\mathcal{I}}$ in a neighborhood B of c, where $\mathcal{I}$ is a set accumulating at zero, completely determines the rotation numbers $w(\Lambda)$ for all Liouville tori $\Lambda \subset F^{-1}(B)$.*

If, however, we don't have access to an elliptic singularity, we cannot make use of Proposition 3.35. But we can always resort to the semitoric algorithm of Section 3.6.2 (Proposition 3.45), which gives:

Corollary 3.50 *Let $(\hat{J}, \hat{H})$ be a semitoric quantum integrable system with principal symbols $F = (J, H)$. Let $c \in \mathbb{R}^2$ be regular value of F. Then the joint spectrum $(\Sigma_\hbar)_{\hbar\in\mathcal{I}}$ in a neighborhood B of c, where $\mathcal{I}$ is a set accumulating at zero, completely determines the rotation numbers $w(\Lambda)$ for all Liouville tori $\Lambda \subset F^{-1}(B)$.*

Here again, the detection of the rotation number is robust with respect to an $\mathscr{O}(\hbar^2)$ perturbation of the joint spectrum.

Remark 3.51 Quantum integrable systems, as defined in Section 3, come from pseudo-differential operators, and hence are defined for an interval of values of $\hbar \in]0, \hbar_0]$. For this reason, they are obviously $\hbar$-continuous, and one can apply Proposition 3.25 to detect their drift. However, the detection of the rotation number in Corollary 3.47 does not use the $\hbar$-continuity. As a consequence, it can in principle be applied to quantum integrable systems defined by Berezin-Toeplitz operators on compact, prequantizable symplectic manifolds, using the Bohr-Sommerfeld theory developed in [7]. $\triangle$

We hope that the formalism of asymptotic lattices, that we have tried to develop here in a precise way, independently of any particular quantization scheme, should help attacking other inverse problems for quantum integrable systems (and in particular semitoric systems). It may also prove useful for the geometric quantization of Lagrangian fibrations, extending the first order Bohr-Sommerfeld quantization that is often used instead. However, in order to get a more complete picture of asymptotic lattices *vs.* joint spectra, one should include singularities of the moment map into the picture.

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