ON THE SEMICLASSICAL LAPLACIAN WITH MAGNETIC FIELD HAVING SELF-INTERSECTING ZERO SET

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ABSTRACT. This paper is devoted to the spectral analysis of the Neumann realization of the 2D magnetic Laplacian with semiclassical parameter $h > 0$ in the case when the magnetic field vanishes along a smooth curve which crosses itself inside a bounded domain. We investigate the behavior of its eigenpairs in the limit $h \to 0$. We show that each crossing point acts as a potential well, generating a new decay scale of $h^{3/2}$ for the lowest eigenvalues, as well as exponential concentration for eigenvectors around the set of crossing points. These properties are consequences of the nature of associated model problems in $\mathbb{R}^2$ for which the zero set of the magnetic field is the union of two straight lines. In this paper we also analyze the spectrum of model problems when the angle between the two straight lines tends to 0.

1. INTRODUCTION

1.1. **The magnetic Laplacian.** Let Ω be a bounded, smooth and simply connected open set of $\mathbb{R}^2$, and $\mathbf{A} \in \mathscr{C}^\infty(\overline{\Omega}, \mathbb{R}^2)$ be a regular potential vector. For $h > 0$, we consider the self-adjoint operator

$$\mathscr{P}_{h,\Omega}^{\mathbf{A}} = (-ih\nabla + \mathbf{A})^2,$$

with domain

$$\mathsf{Dom}(\mathscr{P}_{h,\Omega}^{\mathbf{A}}) = \{u \in H^2(\Omega), (-ih\nabla + \mathbf{A})u \cdot \nu = 0 \quad \text{on} \quad \partial\Omega\},$$

where ν is the outward pointing normal at the boundary of Ω.

The operator $\mathscr{P}_{h,\Omega}^{\mathbf{A}}$ has compact resolvent and is associated with the quadratic form $\mathscr{Q}_{h,\Omega}^{\mathbf{A}}$ defined on the form domain $\mathsf{Dom}(\mathscr{Q}_{h,\Omega}^{\mathbf{A}}) = H^1(\Omega)$ by

$$(1.1) \qquad \mathscr{Q}_{h,\Omega}^{\mathbf{A}}(u) = \int_\Omega |(-ih\nabla + \mathbf{A})u(\mathsf{x})|^2 \, d\mathsf{x}.$$

Here $\mathsf{x} = (x_1, x_2)$ denotes Cartesian coordinates in $\mathbb{R}^2$. We have the gauge invariance

$$(1.2) \qquad e^{-i\phi/h}(-ih\nabla + \mathbf{A})^2 e^{i\phi/h} = (-ih\nabla + \mathbf{A} + \nabla\phi)^2,$$

for any $\phi \in H^2(\Omega)$. Therefore, the spectrum of $\mathscr{P}_{h,\Omega}^{\mathbf{A}}$ only depends on the magnetic field

$$(1.3) \qquad \mathbf{B} = \nabla \times \mathbf{A} = \partial_2 A_1 - \partial_1 A_2.$$

Notation 1.1. We denote by $\lambda_n(h)$ the n-th eigenvalue (with multiplicity) of $\mathscr{P}_{h,\Omega}^{\mathbf{A}}$. In all of the paper, $\mathfrak{S}(\mathscr{P})$ denotes the spectrum of any operator $\mathscr{P}$.

We are interested in the behavior of the eigenvalues $\lambda_n(h)$ and their associated eigenfunctions in the semiclassical limit $h \to 0$ for special configurations of the magnetic field $\mathbf{B}$.

1.2. **Motivations and context.** The spectral analysis of the magnetic Laplacian comes from the theory of superconductivity in which the magnetic Laplacian appears in the study of the third critical field of the Ginzburg-Landau functional (see for instance [34] and also the books [15] and [32], and the references therein). The regime when h goes to 0 (called the semiclassical limit) is equivalent to the strong magnetic field limit which is often involved in applications. In this paper, we restrict to dimension two.

1.2.1. *Overview of the literature.* In the past two decades, most of the contributions dealt with non-vanishing magnetic fields. We can refer for instance to the works by Bolley & Helffer [5], Bauman, Phillips & Tang [3], del Pino, Felmer & Sternberg [13], Helffer & Morame [21], Bonnaillie-Noël [6], Lu & Pan [25], Raymond [29], Bonnaillie-Noël & Dauge [7], Bonnaillie-Noël & Fournais [8], Raymond & Vu-Ngoc [33].

The present paper is devoted to the case of vanishing magnetic fields. Such an investigation was initially motivated by a paper of Montgomery [27], followed by the contributions of Helffer & Morame [20], Helffer & Kordyukov [18, 16], and Dombrowski & Raymond [14]. The aforementioned papers do not investigate the case when the zero set of the magnetic field **B** intersects the boundary. This was the purpose of the work by Pan & Kwek [28] and [26]. We can find in [28] a one term asymptotics of the first eigenvalue $\lambda_1(h)$. The paper [26] establishes a sharper result by giving an explicit control of the remainder as well as expansions of all the eigenvalues and eigenfunctions, as the semiclassical parameter h goes to 0, under suitable assumptions when the zero set of **B** does not self-intersect.

1.2.2. *When the zero set of* **B** *self-intersects.* In the present paper, we want to include non-degenerate quadratic cancellations inside the domain, which is a new configuration in the investigations about vanishing magnetic fields.

Assumption 1.2. Let
$$\Gamma = \left\{ \mathsf{x} \in \overline{\Omega} : \mathbf{B}(\mathsf{x}) = 0 \right\},$$
and assume that $\Gamma \neq \emptyset$. We work under the following assumptions
  i) The set
$$\Sigma = \left\{ \mathsf{x} \in \Gamma, \nabla \mathbf{B}(\mathsf{x}) = 0 \right\}$$
  is non-empty, finite and such that $\partial \Omega \cap \Sigma = \emptyset$.
  ii) For any $\mathsf{x} \in \Sigma$, the Hessian matrix $\mathsf{Hess}\, \mathbf{B}(\mathsf{x})$ of the magnetic field at the point x has two non-zero eigenvalues with opposite signs.
  iii) The set $\Gamma \cap \partial \Omega$ is finite, and in each of these intersection points, Γ is non tangent to $\partial \Omega$.

Note that assumptions i)-ii) imply that the set Γ is a simple curve in a neighborhood of each of its intersection points with $\partial \Omega$. Moreover, the set Σ is made of isolated points which are locally the intersection point of two smooth curves.

Notation 1.3. Choose $\mathsf{x}_0 \in \Sigma$.
  i) $\mathbf{B}^{\mathsf{x}_0}$ denotes the second order Taylor expansion of the magnetic field at x_0. So
$$\mathbf{B}^{\mathsf{x}_0}(\mathsf{x}) = \tfrac{1}{2}(\mathsf{x} - \mathsf{x}_0)\, \mathsf{Hess}\, \mathbf{B}(\mathsf{x}_0)(\mathsf{x} - \mathsf{x}_0)^\top.$$
  ii) $\mathbf{A}^{\mathsf{x}_0}$ denotes the Taylor expansion of the magnetic potential $\mathbf{A}$ to the third order at the point x_0. Thus $\mathbf{B}^{\mathsf{x}_0} = \nabla \times \mathbf{A}^{\mathsf{x}_0}$.
  iii) Let $\alpha(\mathsf{x}_0)$, $\beta(\mathsf{x}_0)$ be the eigenvalues of $\tfrac{1}{2}\, \mathsf{Hess}\, \mathbf{B}(\mathsf{x}_0)$, agreeing that $|\alpha(\mathsf{x}_0)| \leq |\beta(\mathsf{x}_0)|$. Set
$$\varepsilon(\mathsf{x}_0) = \sqrt{|\alpha(\mathsf{x}_0)|/|\beta(\mathsf{x}_0)|} \quad \text{and} \quad \Xi(\mathsf{x}_0) = |\beta(\mathsf{x}_0)|,$$

so that in a suitable local system of orthogonal coordinates $\mathsf{y} = (s, t)$ centered at x_0

$$\mathbf{B}^{\mathsf{x}_0}(\mathsf{x}) = \alpha(\mathsf{x}_0)s^2 + \beta(\mathsf{x}_0)t^2 = -\beta(\mathsf{x}_0)\big(\varepsilon^2(\mathsf{x}_0)s^2 - t^2\big)\,.$$

Hence the zero set of $\mathbf{B}^{\mathsf{x}_0}$ has the equation $\varepsilon^2(\mathsf{x}_0)s^2 - t^2 = 0$: It is the union of the two lines $\{t = \pm\varepsilon(\mathsf{x}_0)s\}$. These lines are the two tangents to the set Γ at the point x_0.

The main novelty in this paper is related to the presence of Σ and to the role of the following family of model operators indexed by ε and acting on $L^2(\mathbb{R}^2)$, defined as

$$(1.4) \qquad \mathscr{X}_\varepsilon = \left(D_\sigma - \frac{\tau^3}{3} + \varepsilon^2\sigma^2\tau \right)^2 + D_\tau^2, \qquad (\sigma, \tau) =: \mathsf{Y} \in \mathbb{R}^2\,,$$

with $D_\sigma = -i\partial_\sigma$ and $D_\tau = -i\partial_\tau$. Note that the magnetic field associated with the operator $\mathscr{X}_\varepsilon$ is $\mathbf{B}(\sigma, \tau) = -\tau^2 + \varepsilon^2\sigma^2$. That is why the operator $\mathscr{X}_\varepsilon$ for $\varepsilon = \varepsilon(\mathsf{x}_0)$ will serve as a model magnetic operator at point x_0.

As a consequence of [19, 22], the operator $\mathscr{X}_\varepsilon$ has a compact resolvent in $\mathbb{R}^2$. Then it follows that the eigenfunctions of $\mathscr{X}_\varepsilon$ have an exponential decay (see [1], [2], [15, Theorem B.5.1] and the example in [30, p. 100]). To sum up:

Proposition 1.4. *Let $\varepsilon > 0$. The spectrum of the operator $\mathscr{X}_\varepsilon$ is formed by a non-decreasing unbounded sequence of positive eigenvalues denoted by $(\varkappa_n(\varepsilon))_{n\geq 1}$. Moreover, for any eigenfunction Ψ_ε of $\mathscr{X}_\varepsilon$, there exists $c > 0$ such that $\mathrm{e}^{c|Y|}\Psi_\varepsilon \in L^2(\mathbb{R}^2)$.*

1.3. **Semiclassical expansions of the magnetic eigenvalues.** The model operators $\mathscr{X}_\varepsilon$ have the following homogeneity property, due to their magnetic potential of degree 3. By rescaling, we find immediately:

Lemma 1.5. *Set $\mathbf{A}_\varepsilon := (-\frac{\tau^3}{3} + \varepsilon^2\sigma^2\tau, 0)$ the magnetic potential of $\mathscr{X}_\varepsilon$. Let $(\varkappa, \Psi)$ be a normalized eigenpair of $\mathscr{X}_\varepsilon$. Setting for $h > 0$ and $\Xi > 0$*

$$\psi_h(\mathsf{y}) = \Xi^{1/4}h^{-1/4}\Psi(\Xi^{1/4}h^{-1/4}\mathsf{Y})$$

we obtain that $(h^{3/2}\,\Xi^{1/2}\varkappa,\,\psi_h)$ is a normalized eigenpair for the semiclassical magnetic operator $(-ih\nabla + \Xi\mathbf{A}_\varepsilon)^2$ on $\mathbb{R}^2$.

After the scale h for non-vanishing magnetic fields, the scale $h^{4/3}$ for magnetic field vanishing at order 1, we note the apparition of the new scale $h^{3/2}$.

Since for each crossing point $\mathsf{x}_0 \in \Sigma$ the operator $(-ih\nabla + \Xi(\mathsf{x}_0)\mathbf{A}_{\varepsilon(\mathsf{x}_0)})^2$ is unitarily equivalent to the "tangent" magnetic operator $(-ih\nabla + \mathbf{A}^{\mathsf{x}_0})^2$, we can guess that the behavior of the low lying spectrum of the operator $\mathscr{P}_{h,\Omega}^{\mathbf{A}}$ corresponds to the low lying spectrum of the operator

$$(1.5) \qquad \mathscr{P}_h^{\mathbf{A},\Sigma} := \bigoplus_{\mathsf{x}\in\Sigma}\big(-ih\nabla + \Xi(\mathsf{x})\mathbf{A}_{\varepsilon(\mathsf{x})}\big)^2,$$

on $L^2(\mathbb{R}^2)^{\sharp\Sigma}$, where $\sharp\Sigma$ is the cardinal of the finite set Σ. Lemma 1.5 then gives that

$$\mathfrak{S}(\mathscr{P}_h^{\mathbf{A},\Sigma}) = \coprod_{\mathsf{x}\in\Sigma} h^{3/2}\,\Xi(\mathsf{x})^{1/2}\mathfrak{S}(\mathscr{X}_{\varepsilon(\mathsf{x})}).$$

This leads to introduce, for all $\mathsf{x} \in \Sigma$ and all $n \in \mathbb{N}^*$, the enumeration of the eigenvalues

$$(1.6a) \qquad \Lambda_n^{\mathsf{x}} = \Xi(\mathsf{x})^{1/2}\varkappa_n(\varepsilon(\mathsf{x}))\,,$$

and the ordered set

$$(1.6b) \qquad \big\{\Lambda_{\mathfrak{n}}^{\mathbf{B}}, \mathfrak{n} \in \mathbb{N}^*\big\} = \coprod_{\mathsf{x}\in\Sigma}\big\{\Lambda_n^{\mathsf{x}}, n \in \mathbb{N}^*\big\}\,,$$

for which the same value can possibly appear several times (for instance, if we have $\Lambda_n^{\times} = \Lambda_{n'}^{\times'}$ for $(n, \times) \neq (n', \times')$). Note in particular that the smallest element of this set is given by

$$(1.7) \qquad \Lambda_1^{\mathbf{B}} = \min_{\times \in \Sigma} \, \Xi(\times)^{1/2} \varkappa_1(\varepsilon(\times)),$$

where we recall that $\varkappa_1(\varepsilon)$ is the first eigenvalue of $\mathcal{X}_\varepsilon$ and $\varepsilon(\times)$ is defined in Notation 1.3.

We are ready to state the main two results relating to the low lying spectrum of the operator $\mathscr{P}_{h,\Omega}^{\mathbf{A}}$. The first result provides a localized estimate from below of the energy functional $\mathcal{Q}_{h,\Omega}^{\mathbf{A}}$ (defined in (1.1)) and exponential decay estimates for the eigenvectors of $\mathscr{P}_{h,\Omega}^{\mathbf{A}}$. The second result states an asymptotic expansion for the eigenvalues of $\mathscr{P}_{h,\Omega}^{\mathbf{A}}$.

1.3.1. *Estimate from below of the energy and Agmon estimates.* Let the function $\times \mapsto \mathrm{d}(\times, \Sigma)$ be the Euclidean distance between the point $\times \in \overline{\Omega}$ and the set Σ.

Theorem 1.6. *Under Assumption* 1.2, *for any exponent* $d \in (\frac{3}{16}, \frac{1}{4})$, *there exist* c_d, $\mathrm{C}_d > 0$, $h_d > 0$, *such that for all* $u \in \mathsf{Dom}(\mathcal{Q}_{h,\Omega}^{\mathbf{A}})$ *and all* $h \in (0, h_d)$,

$$(1.8a) \qquad \mathcal{Q}_{h,\Omega}^{\mathbf{A}}(u) \geq \int_\Omega \mathscr{I}_{h,d}^{\Sigma}(\times) \, |u(\times)|^2 \, \mathrm{d}\times,$$

with

$$(1.8b) \qquad \mathscr{I}_{h,d}^{\Sigma}(\times) = \begin{cases} \Lambda_1^{\mathbf{B}} h^{3/2} - \mathrm{C}_d \, h^{\min\{2-2d, \, 3/4+4d\}} & \text{if} \quad \mathrm{d}(\times, \Sigma) \leq h^d \\ \mathrm{c}_d \, h^{4/3+2d/3} & \text{if} \quad \mathrm{d}(\times, \Sigma) > h^d, \end{cases}$$

where $\Lambda_1^{\mathbf{B}}$ *is defined in* (1.7). *Note that the exponents present in the definition of* $\mathscr{I}_{h,d}^{\Sigma}$ *satisfy*

$$(1.9) \qquad \min\{2-2d, \tfrac{3}{4}+4d\} > \tfrac{3}{2} \quad \text{and} \quad \tfrac{4}{3} + \tfrac{2d}{3} < \tfrac{3}{2}.$$

Agmon estimates are an *a priori* result of exponential decay of the eigenfunctions. The following theorem states that the eigenfunctions associated with eigenvalues of order $h^{4/3+2d/3}$ are localized near Σ.

Theorem 1.7. *Let* $L > 0$ *and* $d \in (\frac{3}{16}, \frac{1}{4})$. *There exist* $C, h_d > 0$ *such that for all* $h \in (0, h_d)$, *and all eigenpair* $(\lambda(h), \psi_h)$ *of* $\mathscr{P}_{h,\Omega}^{\mathbf{A}}$ *with* $\lambda(h) \leq L h^{3/2}$, *we have*

$$(1.10) \qquad \int_\Omega \mathrm{e}^{2\mathrm{d}(\times,\Sigma)/h^d} |\psi_h(\times)|^2 \, \mathrm{d}\times + h^{-3/2} \mathcal{Q}_{h,\Omega}^{\mathbf{A}}\big(\mathrm{e}^{\mathrm{d}(\cdot,\Sigma)/h^d} \psi_h\big) \leq C \|\psi_h\|^2.$$

In fact estimate (1.10) would also hold for the relaxed condition $\lambda(h) \leq L h^{4/3+2d/3}$ if $L < \mathrm{c}_d$.

1.3.2. *Expansion of lowest eigenvalues.* Let us recall that

$$\lambda_1(h) \leq \lambda_2(h) \leq \ldots \leq \lambda_n(h) \ldots$$

denote the increasing sequence of the eigenvalues of the operator $\mathscr{P}_{h,\Omega}^{\mathbf{A}}$ while

$$\Lambda_1^{\mathbf{B}} \leq \Lambda_2^{\mathbf{B}} \leq \ldots \leq \Lambda_n^{\mathbf{B}} \ldots$$

denote the elements defined in (1.6a)-(1.6b) from the eigenvalues of the model operators $\mathcal{X}_\varepsilon$.

Theorem 1.8. *Under Assumption* 1.2, *for all* $N \in \mathbb{N}^*$, *there exist* $C_N > 0$ *and* $h_0 > 0$ *such that, for all* $h \in (0, h_0)$ *and all* $n = 1, \ldots, N$

$$|\lambda_n(h) - h^{3/2} \Lambda_n^{\mathbf{B}}| \leq C_N h^{7/4}.$$

In Section 3, an extended version of this theorem is proved, providing full expansions of all lowest eigenvalues $\lambda_n(h)$, see Theorem 3.6.

1.4. **Low lying eigenvalues of the magnetic cross in the small angle limit.** When ε tends to 0, the angle between the two lines $\{\tau = \pm\varepsilon\sigma\}$ tends to 0. It is interesting to understand the behavior of $\varkappa_n(\varepsilon)$ in such a limit. One could naively expect that, when ε goes to 0, $\inf\mathfrak{S}(\mathscr{X}_\varepsilon)$ goes to $\inf\mathfrak{S}(\mathscr{M}^{[2]})$ where

$$\mathscr{M}^{[2]} = D_\tau^2 + \left(D_\sigma - \frac{\tau^3}{3}\right)^2 \quad \text{acting on } L^2(\mathbb{R}^2).$$

The operator $\mathscr{M}^{[2]}$ is sometimes called Montgomery operator of order two. By Fourier transform, we have

$$\inf\mathfrak{S}(\mathscr{M}^{[2]}) = \inf_{\xi\in\mathbb{R}}\left(\inf\mathfrak{S}(\mathscr{M}_\xi^{[2]})\right), \quad \text{where } \mathscr{M}_\xi^{[2]} = D_\tau^2 + \left(\xi - \frac{\tau^3}{3}\right)^2 \text{ acting on } L^2(\mathbb{R}).$$

The quantity $\inf\mathfrak{S}(\mathscr{M}^{[2]})$ has been numerically estimated, see [23, Table 1].

Actually, the limit $\varepsilon \to 0$ is singular: The operator $\mathscr{X}_\varepsilon$ is partially semiclassical. Indeed, after the scaling $\sigma = \varepsilon^{-1}s$, $\tau = t$, the operator $\mathscr{X}_\varepsilon$ becomes

$$(1.11) \qquad D_t^2 + \left(\varepsilon D_s - \frac{t^3}{3} + s^2 t\right)^2 \quad \text{with} \quad D_t = -i\partial_t, \quad D_s = -i\partial_s.$$

The operator (1.11) is the Weyl quantization $\mathsf{Op}_\varepsilon^{\mathsf{w}}(\mathsf{X}_{\alpha,\xi})$ of the symbol $\mathsf{X}_{\alpha,\xi}$ where

$$(1.12) \qquad \mathsf{X}_{\alpha,\xi} = D_t^2 + \left(\xi - \frac{t^3}{3} + \alpha^2 t\right)^2$$

is a self-adjoint operator acting on $L^2(\mathbb{R})$ depending on the two real parameters α and ξ. In the small angle regime, the spectral analysis of the operator $\mathscr{X}_\varepsilon$ is related to the spectral analysis of the family $(\mathsf{X}_{\alpha,\xi})_{(\alpha,\xi)\in\mathbb{R}^2}$. The asymptotic expansions of the first eigenvalues of $\mathscr{X}_\varepsilon$ is related to the"band function" ϱ_1 defined by the ground state energy of $\mathsf{X}_{\alpha,\xi}$

$$\varrho_1(\alpha,\xi) = \min\mathfrak{S}(\mathsf{X}_{\alpha,\xi}), \quad (\alpha,\xi)\in\mathbb{R}^2,$$

see for instance [31] and [9], where such operators and reductions are considered. One of the requirements to apply the theory developed in [9] (and that relates ϱ_1 to the eigenvalue asymptotic expansions when $\varepsilon \to 0$ for the operator $\mathscr{X}_\varepsilon$) is that ϱ_1 has a minimum.

Theorem 1.9. *The function ϱ_1 reaches its infimum S_0 in $\mathbb{R}^2$, so*

$$\mathrm{S}_0 = \min_{(\alpha,\xi)\in\mathbb{R}^2} \varrho_1(\alpha,\xi),$$

and this minimum is reached on the set

$$\left\{(\alpha,\xi)\in\mathbb{R}^2, \quad |\xi| \le \frac{2}{3}|\alpha|^3\right\}.$$

By construction, $\mathrm{S}_0 \le \inf\mathfrak{S}(\mathscr{M}^{[2]})$. The numerical simulations that we have performed provide the upper bound

$$(1.13) \qquad \mathrm{S}_0 \le \varrho_1(\alpha_0, 0) \simeq 0.4941 \quad \text{for} \quad \alpha_0 = 0.786.$$

We note that the value given in [23, Table 1] for $\inf\mathfrak{S}(\mathscr{M}^{[2]})$ is $\simeq 0.66$, corresponding to the value $\xi = 0$ of the Fourier parameter. Hence we have obtained the strict inequality

$$\mathrm{S}_0 < \inf\mathfrak{S}(\mathscr{M}^{[2]}).$$

Our numerical results for the band function ϱ_1, see Figures 1 and 2, suggest that the minimum S_0 is attained at the sole two values $\pm(\alpha_0, 0)$ of (α,ξ).

The following result gives the convergence of the eigenvalues $\varkappa_n(\varepsilon)$ of the model operator $\mathscr{X}_\varepsilon$ as $\varepsilon \to 0$.

Theorem 1.10. *For all $n \geq 1$, there exist $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$*

$$|\varkappa_n(\varepsilon) - \mathrm{S}_0| \leq C\varepsilon.$$

1.5. **Organization of the paper.** In Section 2, we prove the preliminary Theorems 1.6 and 1.7. In Section 3, we establish the full asymptotic expansions of the eigenvalues and eigenfunctions. Section 4 is devoted to the proofs of Theorems 1.9 and 1.10 and to the presentation of numerical simulations concerning the band function ϱ_1 and the eigenpairs of $\mathscr{X}_\varepsilon$ as $\varepsilon \to 0$.

2. BOUNDS FROM BELOW AND EXPONENTIAL DECAY

In this section we prove the preliminary bounds from below (1.8a)-(1.8b) and the Agmon estimates (1.10). We will use several times a perturbation formula for the magnetic Laplacian which we first state.

2.1. **Perturbation of the magnetic potential.** Let $\mathbf{A}^\star$ be another smooth magnetic potential defined on $\overline{\Omega}$. In practice $\mathbf{A}^\star$ will be the Taylor expansion of $\mathbf{A}$ at some point x_0 and to various orders (2, 3 or 4). By expanding the square, we get

$$(2.1) \qquad \mathcal{Q}_{h,\Omega}^{\mathbf{A}}(u) = \mathcal{Q}_{h,\Omega}^{\mathbf{A}^\star}(u) + 2\,\mathrm{Re}\left\langle(-ih\nabla + \mathbf{A}^\star)u, (\mathbf{A} - \mathbf{A}^\star)u\right\rangle + \|(\mathbf{A} - \mathbf{A}^\star)u\|^2.$$

This yields $\mathcal{Q}_{h,\Omega}^{\mathbf{A}}(u) \geq \mathcal{Q}_{h,\Omega}^{\mathbf{A}^\star}(u) - 2\mathcal{Q}_{h,\Omega}^{\mathbf{A}^\star}(u)^{1/2}\|(\mathbf{A} - \mathbf{A}^\star)u\|$ by Cauchy-Schwarz inequality, leading to the parametric estimate (based on the inequality $2ab \leq \eta a^2 + \eta^{-1}b^2$, for all $\eta > 0$)

$$(2.2) \qquad \forall \eta > 0, \quad \mathcal{Q}_{h,\Omega}^{\mathbf{A}}(u) \geq (1 - \eta)\mathcal{Q}_{h,\Omega}^{\mathbf{A}^\star}(u) - \eta^{-1}\|(\mathbf{A} - \mathbf{A}^\star)u\|^2.$$

Such a lower bound is used, for instance, in the seminal paper [20, p. 51], and in the book [15, Chap. 8].

2.2. **Lower bound.** The proof of (1.8a) is based on a quadratic partition of unity. Let us recall that such a partition is given for each relevant $h > 0$ by a finite collection of smooth cutoff functions (χ_j^h) satisfying on Ω

$$(2.3) \qquad\qquad\qquad \sum_j |\chi_j^h|^2 = 1.$$

Then we have the following well-known localization formula (see [11])

$$(2.4) \qquad\qquad \mathcal{Q}_{h,\Omega}^{\mathbf{A}}(u) = \sum_j \mathcal{Q}_{h,\Omega}^{\mathbf{A}}(\chi_j^h u) - h^2 \sum_j \|u\,|\nabla(\chi_j^h)|\|_{L^2(\Omega)}^2.$$

We introduce three sets $\Sigma^{[1]}$, $\Sigma^{[2]}(h)$ and $\Sigma^{[3]}(h)$ covering Ω.

Notation 2.1. Let $\mathscr{B}(\mathsf{x}_0, r)$ denote the open ball of center x_0 and radius r and

$$\mathscr{B}(\Sigma, r) = \bigcup_{\mathsf{x} \in \Sigma} \mathscr{B}(\mathsf{x}, r).$$

Let $R_1 > 0$ such that $\mathscr{B}(\Sigma, 3R_1) \subset \Omega$. Choose $d \in (\frac{3}{16}, \frac{1}{4})$ and introduce

  (i) $\Sigma^{[1]} = \Omega \setminus \overline{\mathscr{B}}(\Sigma, R_1)$.
  (ii) $\Sigma^{[2]}(h) = \mathscr{B}(\Sigma, 2R_1) \setminus \overline{\mathscr{B}}(\Sigma, \frac{1}{2}h^d)$.
  (iii) $\Sigma^{[3]}(h) = \mathscr{B}(\Sigma, h^d)$.

Then we consider a partition of unity composed of three cutoff functions $(\mathfrak{X}^{[1]}, \mathfrak{X}^{h,[2]}, \mathfrak{X}^{h,[3]})$ associated with this covering of Ω in the sense that

$$|\mathfrak{X}^{[1]}|^2 + |\mathfrak{X}^{h,[2]}|^2 + |\mathfrak{X}^{h,[3]}|^2 = 1,$$

and

$$\operatorname{supp} \mathfrak{X}^{[1]} \subset \Sigma^{[1]}, \quad \operatorname{supp} \mathfrak{X}^{h,[2]} \subset \Sigma^{[2]}(h), \quad \operatorname{supp} \mathfrak{X}^{h,[3]} \subset \Sigma^{[3]}(h).$$

The distance between $\partial\Sigma^{[2]}(h) \cap \Sigma^{[1]}(h)$ and $\partial\Sigma^{[1]}(h) \cap \Sigma^{[2]}(h)$ is R_1. The distance between $\partial\Sigma^{[3]}(h)$ and $\partial\Sigma^{[2]}(h) \cap \Sigma^{[3]}(h)$ is $\frac{1}{2}h^d$. Hence we can choose the cutoff functions so that

$$(2.5) \qquad |\nabla\mathfrak{X}^{[1]}|^2 \le K_{\mathsf{loc}}, \quad |\nabla\mathfrak{X}^{h,[2]}|^2 \le K_{\mathsf{loc}}h^{-2d}, \quad |\nabla\mathfrak{X}^{h,[3]}|^2 \le K_{\mathsf{loc}}h^{-2d}.$$

with a constant K_{loc} independent of h and d. Combined with the localization formula (2.4) associated with the partition $(\mathfrak{X}^{[1]}, \mathfrak{X}^{h,[2]}, \mathfrak{X}^{h,[3]})$, the estimates (2.5) yield for all $u \in H^1(\Omega)$,

$$(2.6) \qquad \mathscr{Q}^{\mathbf{A}}_{h,\Omega}(u) \ge \mathscr{Q}^{\mathbf{A}}_{h,\Omega}(\mathfrak{X}^{[1]}u) + \mathscr{Q}^{\mathbf{A}}_{h,\Omega}(\mathfrak{X}^{h,[2]}u) + \mathscr{Q}^{\mathbf{A}}_{h,\Omega}(\mathfrak{X}^{h,[3]}u) - K_{\mathsf{loc}}h^{2-2d}\|u\|^2_{L^2(\Omega)},$$

We set $u_1 = \mathfrak{X}^{[1]}u$, $u_2 = \mathfrak{X}^{h,[2]}u$ and $u_3 = \mathfrak{X}^{h,[3]}u$, and are going to bound each $\mathscr{Q}^{\mathbf{A}}_{h,\Omega}(u_k)$ from below (for $k = 1, 2, 3$).

Lemma 2.2 (Lower bound on $\Sigma^{[1]}$). *There exist c_1 and $h_0 > 0$ such that for all $0 < h < h_0$*

$$\mathscr{Q}^{\mathbf{A}}_{h,\Omega}(u_1) \ge c_1 h^{4/3}\|u_1\|^2.$$

Proof. This result is a consequence of [26, Theorem 1.9]. □

Lemma 2.3 (Lower bound on $\Sigma^{[2]}(h)$). *There exist c_2 and $h_0 > 0$ such that for all $0 < h < h_0$*

$$\mathscr{Q}^{\mathbf{A}}_{h,\Omega}(u_2) \ge c_2 h^{4/3+2d/3}\|u_2\|^2.$$

Proof. Set $\rho = \frac{1}{4}$. We introduce a second partition of unity $(\chi^h_j)_{j\in\mathscr{J}}$ on $\Sigma^{[2]}(h)$ associated with a family of balls $\mathscr{B}(\mathsf{x}_j, d_j h^\rho)$ for some constants d_j. We first cover the zero set $\Gamma \cap \Sigma^{[2]}(h)$ and choose the centers $\mathsf{x}_j \in \Gamma$ with $d_j = 1$ so that the distances between consecutive points along Γ is $\simeq \frac{1}{2}h^\rho$. Hence, setting

$$\widetilde{\Sigma^{[2]}}(h) = \Sigma^{[2]}(h) \setminus \bigcup_{\mathsf{x}_j\in\Gamma} \mathscr{B}(\mathsf{x}_j, h^\rho)$$

we obtain that the distance between Γ and $\widetilde{\Sigma^{[2]}}(h)$ is larger than $\frac{1}{2}h^\rho$. We then cover $\widetilde{\Sigma^{[2]}}(h)$ with balls $\mathscr{B}(\mathsf{x}_j, d_j h^\rho)$ choosing $d_j = \frac{1}{4}$ and the mutual distances between the centers bounded from below by $\frac{1}{8}h^\rho$. Finally we can choose the functions χ^h_j so that $|\nabla\chi^h_j|^2 \le C_{2,\mathsf{loc}}h^{2\rho}$ and the localization formula (2.4) yields

$$(2.7) \qquad \mathscr{Q}^{\mathbf{A}}_{h,\Omega}(u_2) \ge \sum_j \mathscr{Q}^{\mathbf{A}}_{h,\Omega}(\chi^h_j u_2) - C_{2,\mathsf{loc}}h^{2-2\rho}\|u_2\|^2.$$

We have, by [26, Lemma 2.3] (for the case $(\ell) = (3)$ of Table 2.1),

$$(2.8) \qquad \mathscr{Q}^{\mathbf{A}}_{h,\Omega}(\chi^h_j u_2) \ge \left(\frac{1}{2}\mathrm{M}_0 |\nabla\mathbf{B}(\mathsf{x}_j)|^{2/3} h^{4/3} - 2Ch^{6\rho}\right)\|\chi^h_j u_2\|^2 \qquad (\mathsf{x}_j \in \Gamma).$$

As a consequence of the non degeneracy of $\nabla\mathbf{B}$ on Γ outside Σ (namely $\nabla\mathbf{B}| \ne 0$ on $\Gamma\backslash\Sigma$) and the non degeneracy of $\operatorname{Hess}\mathbf{B}$ on Σ (meaning that the eigenvalues of the Hessian matrix are not equal to 0, according to Assumption 1.2) there exists a positive constant $C(\mathbf{B})$ such that

$$|\nabla\mathbf{B}(\mathsf{x})| \ge C(\mathbf{B})\,\mathrm{d}(\mathsf{x}, \Sigma), \quad \forall \mathsf{x} \in \Omega.$$

Since $\mathrm{d}(\mathsf{x}_j, \Sigma)$ is larger than $\frac{1}{2}h^d$ by construction, from (2.8) we get

$$\mathscr{Q}^{\mathbf{A}}_{h,\Omega}(\chi^h_j u_2) \ge \left(\frac{1}{4}\mathrm{M}_0 C(\mathbf{B})^{2/3} h^{4/3+2d/3} - 2Ch^{6\rho}\right)\|\chi^h_j u_2\|^2.$$

We note that, with $\rho = \frac{1}{4}$ and $d < \frac{1}{4}$, the exponent $6\rho = \frac{3}{2}$ is (strictly) larger than $\frac{4}{3} + \frac{2}{3}d$. Therefore, for h small enough

$$(2.9) \qquad \mathcal{Q}^{\mathbf{A}}_{h,\Omega}(\chi^h_j u_2) \geq \frac{1}{8} \mathrm{M}_0 C(\mathbf{B})^{2/3} h^{4/3+2d/3} \|\chi^h_j u_2\|^2 \qquad (\mathsf{x}_j \in \Gamma).$$

When x_j does not belong to Γ, in the ball $\mathscr{B}(\mathsf{x}_j, d_j h^\rho)$ we use the lower bound (see [15, Lemma 1.4.1])

$$(2.10) \qquad \mathcal{Q}^{\mathbf{A}}_{h,\Omega}(\chi^h_j u_2) \geq \inf_{\mathsf{x} \in \mathscr{B}(\mathsf{x}_j, d_j h^\rho)} |\mathbf{B}(\mathsf{x})| \, h \, \|\chi^h_j u_2\|^2 \qquad (\mathsf{x}_j \notin \Gamma).$$

For any $\mathsf{x} \in \mathscr{B}(\mathsf{x}_j, d_j h^\rho)$, its distance to Γ is larger than $\frac{1}{2} h^\rho$ by construction. Let $\mathsf{g} \in \Gamma$ be such that $\mathrm{d}(\mathsf{x}, \Gamma) = \mathrm{d}(\mathsf{x}, \mathsf{g})$. Then $\mathrm{d}(\mathsf{g}, \Sigma)$ is larger than $\frac{1}{2} h^d$. As a consequence of the Morse lemma

$$|\mathbf{B}(\mathsf{x})| \geq C'(\mathbf{B}) \, \mathrm{d}(\mathsf{x}, \mathsf{g}) \, \mathrm{d}(\mathsf{g}, \Sigma) \geq C h^{\rho+d}.$$

Hence, with (2.10) we obtain

$$(2.11) \qquad \mathcal{Q}^{\mathbf{A}}_{h,\Omega}(\chi^h_j u_2) \geq C h^{1+\rho+d} \|\chi^h_j u_2\|^2 \qquad (\mathsf{x}_j \notin \Gamma).$$

With $\rho = \frac{1}{4}$ and $d < \frac{1}{4}$, the exponent $1 + \rho + d$ is $< \frac{4}{3} + \frac{2}{3}d$. Therefore, for h small enough, as a result of (2.9) and (2.11) we have, for a positive constant c independent of h

$$\mathcal{Q}^{\mathbf{A}}_{h,\Omega}(\chi^h_j u_2) \geq c h^{4/3+2d/3} \|\chi^h_j u_2\|^2 \qquad (\forall j \in \mathscr{J}).$$

With (2.7) this yields

$$\mathcal{Q}^{\mathbf{A}}_{h,\Omega}(u_2) \geq c h^{4/3+2d/3} \sum_j \|\chi^h_j u_2\|^2 - C_{2,\mathsf{loc}} h^{2-2\rho} \|u_2\|^2$$

$$\geq c_2 h^{4/3+2d/3} \|u_2\|^2,$$

since $2 - 2\rho = \frac{3}{2} > 4/3 + 2d/3$ for any $d < \frac{1}{4}$. The lemma is proved. $\qquad \square$

Lemma 2.4 (Lower bound on $\Sigma^{[3]}$)**.** *There exist C_3 and $h_0 > 0$ such that for all $0 < h < h_0$*

$$\mathcal{Q}^{\mathbf{A}}_{h,\Omega}(u_3) \geq \left(\Lambda^{\mathbf{B}}_1 h^{3/2} - C_3 h^{3/4+4d} \right) \|u_3\|^2.$$

Proof. For h small enough the set $\Sigma^{[3]}$ is the union of the balls $\mathscr{B}(\mathsf{x}_0, h^d)$, with x_0 spanning Σ. It suffices to consider each point x_0 separately. We denote by $u^{\mathsf{x}_0}_3$ the restriction of u_3 to $\mathscr{B}(\mathsf{x}_0, h^d)$ and use the perturbative lower bound (2.2) with $\mathbf{A}^\star = \mathbf{A}^{\mathsf{x}_0}$ the third order Taylor expansion of $\mathbf{A}$ at x_0:

$$\forall \eta > 0, \quad \mathcal{Q}^{\mathbf{A}}_{h,\Omega}(u^{\mathsf{x}_0}_3) \geq (1-\eta)\mathcal{Q}^{\mathbf{A}^{\mathsf{x}_0}}_{h,\Omega}(u^{\mathsf{x}_0}_3) - \eta^{-1} \|(\mathbf{A} - \mathbf{A}^{\mathsf{x}_0})u^{\mathsf{x}_0}_3\|^2.$$

Recall from the introduction that the magnetic field associated with $\mathbf{A}^{\mathsf{x}_0}$ is $\mathbf{B}^{\mathsf{x}_0}$, and that the magnetic operator $(-ih\nabla + \mathbf{A}^{\mathsf{x}_0})^2$ is unitarily equivalent to $(-ih\nabla + \Xi(\mathsf{x}_0)\mathbf{A}_{\varepsilon(\mathsf{x}_0)})^2$ whose first eigenvalue is $h^{3/2}\Lambda^{\mathsf{x}_0}_1$. Besides,

$$|\mathbf{A}(\mathsf{x}) - \mathbf{A}^{\mathsf{x}_0}(\mathsf{x})| \leq C h^{4d}, \qquad \forall \mathsf{x} \in \mathscr{B}(\mathsf{x}_0, h^d).$$

So we have

$$\forall \eta > 0, \quad \mathcal{Q}^{\mathbf{A}}_{h,\Omega}(u^{\mathsf{x}_0}_3) \geq \left((1-\eta)h^{3/2}\Lambda^{\mathsf{x}_0}_1 - \eta^{-1}Ch^{8d} \right) \|u^{\mathsf{x}_0}_3\|^2.$$

Choosing $\eta = h^{4d-3/4}$ to equalize the remainders, we deduce the inequality $\mathcal{Q}^{\mathbf{A}}_{h,\Omega}(u^{\mathsf{x}_0}_3) \geq \left(h^{3/2}\Lambda^{\mathsf{x}_0}_1 - Ch^{3/4+4d} \right) \|u^{\mathsf{x}_0}_3\|^2$. Taking the infimum over all the points of the finite set Σ gives the result. $\qquad \square$

We can now conclude with the proof of Theorem 1.6.

Proof of Theorem 1.6. We gather the estimates provided by Lemmas 2.2–2.4 and combine them with the localization estimate (2.6) and obtain that $\mathcal{Q}^{\mathbf{A}}_{h,\Omega}(u)$ is bounded from below by

$$c_1 h^{4/3}\|u_1\|^2 + c_2 h^{4/3+2d/3}\|u_2\|^2 + \left(\Lambda^{\mathbf{B}}_1 h^{3/2} - C_3 h^{3/4+4d}\right)\|u_3\|^2 - K_{\mathsf{loc}} h^{2-2d}\|u\|^2.$$

Since $d < \frac{1}{4}$, then $h^{2-2d} \ll h^{4/3+2d/3}$ and the lower bound above can be replaced by

$$ch^{4/3+2d/3}(\|u_1\|^2 + \|u_2\|^2) + \left(\Lambda^{\mathbf{B}}_1 h^{3/2} - C_3 h^{3/4+4d} - K_{\mathsf{loc}} h^{2-2d}\right)\|u_3\|^2$$
$$\geq ch^{4/3+2d/3}\|(1-\chi)u\|^2 + \left(\Lambda^{\mathbf{B}}_1 h^{3/2} - C h^{\min\{3/4+4d,2-2d\}}\right)\|\chi u\|^2$$

with χ the characteristic function of the set $\{\mathsf{x} \in \Omega,\ \mathrm{d}(\mathsf{x},\Sigma) < h^d\}$. The theorem is proved. $\quad\square$

2.3. Agmon estimates.

Let $d \in (\frac{3}{16}, \frac{1}{4})$ and $L > 0$. We consider an eigenpair $(\lambda(h), \psi_h)$ of $\mathscr{P}^{\mathbf{A}}_{h,\Omega}$ such that $\lambda(h) \leq Lh^{3/2}$. For all function $\Phi \in W^{1,\infty}(\Omega)$, we have $\mathrm{e}^{\Phi}\psi_h \in \mathsf{Dom}(\mathcal{Q}^{\mathbf{A}}_{h,\Omega})$ and

$$(2.12) \qquad \mathcal{Q}^{\mathbf{A}}_{h,\Omega}(\mathrm{e}^{\Phi}\psi_h) = \lambda(h)\|\mathrm{e}^{\Phi}\psi_h\|^2 + h^2\left\||\nabla\Phi|\,\mathrm{e}^{\Phi}\psi_h\right\|^2.$$

Defining $\Phi(\mathsf{x}) = \mathrm{d}(\mathsf{x},\Sigma)h^{-d}$, we have

$$|\nabla\Phi(\mathsf{x})|^2 = h^{-2d}.$$

Hence

$$(2.13) \qquad h^2\||\nabla\Phi|\,\mathrm{e}^{\Phi}\psi_h\|^2 \leq h^{2-2d}\|\mathrm{e}^{\Phi}\psi_h\|^2.$$

We denote

$$Z^{\mathsf{near}}_h = \{\mathsf{x} \in \Omega,\ \mathrm{d}(\mathsf{x},\Sigma) \leq h^d\} \quad \text{and} \quad Z^{\mathsf{far}}_h = \{\mathsf{x} \in \Omega,\ \mathrm{d}(\mathsf{x},\Sigma) > h^d\}.$$

We introduce δ such that $\delta = \min\{3/4 + 4d, 2 - 2d\} - \frac{3}{2}$. Since $d \in (\frac{3}{16}, \frac{1}{4})$, the number δ is positive. Using (1.8a)-(1.8b) we obtain

$$\mathcal{Q}^{\mathbf{A}}_{h,\Omega}(\mathrm{e}^{\Phi}\psi_h) \geq \mathsf{c}_d\, h^{4/3+2d/3}\left\|\mathrm{e}^{\Phi}\psi_h\right\|^2_{L^2(Z^{\mathsf{far}}_h)} + \left(\Lambda^{\mathbf{B}}_1 h^{3/2} - \mathsf{C}_d\, h^{3/2+\delta})\right)\left\|\mathrm{e}^{\Phi}\psi_h\right\|^2_{L^2(Z^{\mathsf{near}}_h)}$$
$$\geq \mathsf{c}_d\, h^{4/3+2d/3}\left\|\mathrm{e}^{\Phi}\psi_h\right\|^2_{L^2(Z^{\mathsf{far}}_h)} - \mathsf{C}_d\, h^{3/2+\delta}\left\|\mathrm{e}^{\Phi}\psi_h\right\|^2_{L^2(Z^{\mathsf{near}}_h)}.$$

Combining the latter inequality with (2.12) and (2.13), we obtain (using the fact that $\lambda(h) \leq Lh^{3/2}$)

$$\left(\mathsf{c}_d\, h^{4/3+2d/3} - Lh^{3/2} - h^{2-2d}\right)\|\mathrm{e}^{\Phi}\psi_h\|^2_{L^2(Z^{\mathsf{far}}_h)} \leq (Lh^{3/2} + h^{2-2d} + \mathsf{C}_d\, h^{3/2+\delta})\|\mathrm{e}^{\Phi}\psi_h\|^2_{L^2(Z^{\mathsf{near}}_h)}$$

from which we immediately deduce that for h small enough

$$\left\|\mathrm{e}^{\Phi}\psi_h\right\|^2_{L^2(Z^{\mathsf{far}}_h)} \leq \left\|\mathrm{e}^{\Phi}\psi_h\right\|^2_{L^2(Z^{\mathsf{near}}_h)}.$$

Since by construction $|\Phi|$ is bounded by 1 on Z^{near}_h, there holds

$$\left\|\mathrm{e}^{\Phi}\psi_h\right\|^2_{L^2(Z^{\mathsf{near}}_h)} \leq \|\psi_h\|^2_{L^2(Z^{\mathsf{near}}_h)},$$

we finally get $\|\mathrm{e}^{\Phi}\psi_h\|^2_{L^2(\Omega)} \leq \|\psi_h\|^2_{L^2(\Omega)}$, which implies the Agmon estimates (1.10).

3. ASYMPTOTIC EXPANSIONS OF EIGENVALUES

In this section, we prove Theorem 1.8 in two steps. First, the localization around each crossing point $\mathsf{x}_0 \in \Sigma$, and, second, a perturbation argument for the magnetic potential around each crossing point. We conclude the section by stating a full asymptotic expansion for eigenvalues and a sketch of the proof.

3.1. **Preliminaries.** We will use several times an argument based on reciprocal quasimodes between two operators. We state this in a general form.

Lemma 3.1. *Let $\mathscr{P}$ and $\mathscr{P}^\star$ two positive operators with discrete spectra associated with sesquilinear forms a and $a^\star$, respectively. Let (μ_n) and $(\mu_n^\star)$ be the increasing sequences of their eigenvalues (counted with multiplicity). Let (φ_n) be an associated orthonormal basis of eigenvectors for $\mathscr{P}$. Let N be a positive integer and assume that for each $n = 1, \ldots, N$, there exists $\varphi_n^\star \in \mathsf{Dom}(a^\star)$ such that*

(3.1a) $$\langle \varphi_n^\star, \varphi_m^\star \rangle = \langle \varphi_n, \varphi_m \rangle + \nu_{n,m} \quad and \quad a^\star(\varphi_n^\star, \varphi_m^\star) = a(\varphi_n, \varphi_m) + \mu_{n,m}$$

and set

(3.1b) $$\nu = \max_{n,m} |\nu_{n,m}| \quad and \quad \mu = \max_{n,m} |\mu_{n,m}|$$

Assume that $\nu < \frac{1}{N}$. Then

(3.2) $$\mu_n^\star \leq \frac{\mu_n + n\mu}{1 - n\nu}, \quad n = 1, \ldots, N.$$

Proof. Let $M \leq N$. By the min-max formula,

$$\mu_M^\star \leq \max_{\varphi \in \mathrm{span}\{\varphi_n^\star,\, n=1,\ldots,M\}} \frac{a^\star(\varphi, \varphi)}{\langle \varphi, \varphi \rangle}$$

We write φ as a sum $\sum_n \gamma_n \varphi_n^\star$. Then

$$a^\star(\varphi, \varphi) = \sum_{n,m} \gamma_n \bar{\gamma}_m a^\star(\varphi_n^\star, \varphi_m^\star)$$

$$= \sum_n |\gamma_n|^2 \mu_n + \sum_{n,m} \gamma_n \bar{\gamma}_m \mu_{n,m} \leq (\mu_M + M\mu) \sum_n |\gamma_n|^2.$$

Likewise

$$\langle \varphi, \varphi \rangle = \sum_{n,m} \gamma_n \bar{\gamma}_m \langle \varphi_n^\star, \varphi_m^\star \rangle$$

$$= \sum_n |\gamma_n|^2 + \sum_{n,m} \gamma_n \bar{\gamma}_m \nu_{n,m} \geq (1 - M\nu) \sum_n |\gamma_n|^2.$$

Whence formula (3.2) $\qquad\qquad\square$

We will also need a simple but useful consequence of Agmon estimates.

Lemma 3.2. *Assume that the family of function $(\psi_h)_{h>0}$ satisfies (for some γ and $\delta > 0$) the estimate*

$$\int_{\mathbb{R}^2} e^{\gamma|\mathsf{y}|/h^\delta} |\psi_h(\mathsf{y})|^2 \, d\mathsf{y} \leq C\|\psi_h\|^2,$$

for h small enough and C independent of h. Then for all $m > 0$, there exists C_m such that for $h > 0$ small enough

$$\int_{\mathbb{R}^2} |\mathsf{y}|^m |\psi_h(\mathsf{y})|^2 \, d\mathsf{y} \leq C_m h^{m\delta} \|\psi_h\|^2.$$

Proof. It suffices to write

$$\int_{\mathbb{R}^2} |\mathsf{y}|^m |\psi_h(\mathsf{y})|^2 \, d\mathsf{y} \leq \max_{\rho > 0} \rho^m e^{-\gamma\rho/h^\delta} \int_{\mathbb{R}^2} e^{\gamma|\mathsf{y}|/h^\delta} |\psi_h(\mathsf{y})|^2 \, d\mathsf{y}$$

and notice that $\max_{\rho>0} \rho^m e^{-\gamma\rho/h^\delta} = \max_{\rho>0}(h^\delta \rho)^m e^{-\gamma\rho} = h^{m\delta} \max_{\rho>0} \rho^m e^{-\gamma\rho}.$ $\qquad\square$

3.2. **Localization.** Introduce a positive radius r_0 such that the collection of balls $\mathscr{B}(\mathsf{x}_0, r_0)$, $\mathsf{x}_0 \in \Sigma$, are pairwise disjoint. Then consider the collection of operators $\mathscr{P}^{\mathbf{A}}_{h, \mathscr{B}(\mathsf{x}_0, r_0)}$ for $\mathsf{x}_0 \in \Sigma$. The main result of this section is:

Lemma 3.3. *Recall that $\lambda_n(h)$ is the increasing sequence of eigenvalues of $\mathscr{P}^{\mathbf{A}}_{h, \Omega}$ (counted with multiplicity). Denote by $\lambda^{\mathsf{loc}}_n(h)$ the increasing sequence of eigenvalues of*

$$\bigoplus_{\mathsf{x}_0 \in \Sigma} \mathscr{P}^{\mathbf{A}}_{h, \mathscr{B}(\mathsf{x}_0, r_0)} \, .$$

Let $d \in (\frac{3}{16}, \frac{1}{4})$ and let N be a positive integer. If $\lambda^{\mathsf{loc}}_N(h) \leq Lh^{3/2}$ for $h > 0$ small enough, then there exist C_N and $h_N > 0$ such that for all $h \in (0, h_N)$

$$|\lambda_n(h) - \lambda^{\mathsf{loc}}_n(h)| \leq C_N \, \mathrm{e}^{-r_0/h^d}, \quad \forall \, n = 1, \dots, N.$$

Proof. Choose for each $\mathsf{x}_0 \in \Sigma$ a smooth cut-off function χ_{x_0} with support in $\mathscr{B}(\mathsf{x}_0, r_0)$ and equal to 1 on $\mathscr{B}(\mathsf{x}_0, \frac{1}{2} r_0)$.

i) Use Lemma 3.1 with $\mathscr{P}^{\star} = \mathscr{P}^{\mathbf{A}}_{h, \Omega}$ and $\mathscr{P} = \oplus_{\mathsf{x}_0 \in \Sigma} \mathscr{P}^{\mathbf{A}}_{h, \mathscr{B}(\mathsf{x}_0, r_0)}$ (acting on $\oplus_{\mathsf{x}_0 \in \Sigma} L^2(\mathscr{B}(\mathsf{x}_0, r_0))$): For each eigenvector φ_n of $\mathscr{P}$, one crossing point $\mathsf{x}_0 \in \Sigma$ is selected and we set

$$\varphi^{\star}_n = \chi_{\mathsf{x}_0} \varphi_n \quad \text{defined on} \quad \Omega.$$

Relying on the assumption that $\lambda^{\mathsf{loc}}_n(h) \leq Lh^{3/2}$ for $n = 1, \dots, N$, we may apply the Agmon estimates (1.10) to the operator $\mathscr{P}^{\mathbf{A}}_{h, \mathscr{B}(\mathsf{x}_0, r_0)}$. This yields conditions (3.1a)-(3.1b) considering $\nu = \mu = C\mathrm{e}^{-r_0/h^d}$. Hence $\lambda_n(h) \leq \lambda^{\mathsf{loc}}_n(h) + C_N \, \mathrm{e}^{-r_0/h^d}$.

ii) Conversely, we swap the roles of the two operators: $\mathscr{P} = \mathscr{P}^{\mathbf{A}}_{h, \Omega}$ and $\mathscr{P}^{\star} = \oplus_{\mathsf{x}_0 \in \Sigma} \mathscr{P}^{\mathbf{A}}_{h, \mathscr{B}(\mathsf{x}_0, r_0)}$. For each eigenvector φ_n of $\mathscr{P}$, we consider

$$\varphi^{\star}_n := \left(\chi_{\mathsf{x}_0} \varphi_n \right)_{\mathsf{x}_0 \in \Sigma} \quad \text{defined on} \quad \coprod_{\mathsf{x}_0 \in \Sigma} \mathscr{B}(\mathsf{x}_0, r_0) \, .$$

Note that, as a consequence of the previous step of the proof, we have that $\lambda_n(h) \leq 2Lh^{3/2}$ for all $n = 1, \dots, N$. As above, we conclude with the help of Agmon estimates for the operator $\mathscr{P}^{\mathbf{A}}_{h, \Omega}$ that $\lambda^{\mathsf{loc}}_n(h) \leq \lambda_n(h) + C_N \, \mathrm{e}^{-r_0/h^d}$. The lemma is proved. $\qquad\square$

3.3. **Taylor approximation of a localized operator.** With Lemma 3.3 at hand, we can assume that $\Omega = \mathscr{B}(\mathsf{x}_0, r_0)$. Thus Σ is reduced to one element, x_0. Recall that $\mathbf{A}^{\mathsf{x}_0}$ denotes the third order Taylor expansion of $\mathbf{A}$ at the point x_0. We are going to consider the operator $\mathscr{P}^{\mathbf{A}^{\mathsf{x}_0}}_h := (-ih\nabla + \mathbf{A}^{\mathsf{x}_0})^2$ posed on $\mathbb{R}^2$. We have seen in the introduction that the eigenvalues of $\mathscr{P}^{\mathbf{A}^{\mathsf{x}_0}}_h$ are the $h^{3/2}\Lambda^{\mathsf{x}_0}_n$, see (1.6a), and that its eigenvectors $\psi^{\mathsf{x}_0}_{h,n}$ are scaled from the eigenvectors of $\mathscr{X}_\varepsilon$ with $\varepsilon = \varepsilon(\mathsf{x}_0)$ and $\Xi = \Xi(\mathsf{x}_0)$. As a consequence of the exponential decay of the eigenvectors of $\mathscr{X}_\varepsilon$ (Proposition 1.4) and the scaling provided by Lemma 1.5, we find that, for some positive constant γ

$$(3.3) \qquad \int_{\mathbb{R}^2} \mathrm{e}^{2\gamma|\mathsf{y}|/h^{1/4}} |\psi^{\mathsf{x}_0}_{h,n}(\mathsf{y})|^2 \, \mathrm{d}\mathsf{y} + h^{-3/2} \mathscr{Q}^{\mathbf{A}^{\mathsf{x}_0}}_h \left(\mathrm{e}^{\gamma|\mathsf{y}|/h^{1/4}} \psi^{\mathsf{x}_0}_{h,n} \right) \leq C \|\psi^{\mathsf{x}_0}_{h,n}\|^2 \, .$$

Lemma 3.4. *With $\Omega = \mathscr{B}(\mathsf{x}_0, r_0)$, we denote by $\lambda^{\mathsf{x}_0}_n(h)$ the eigenvalues of $\mathscr{P}^{\mathbf{A}}_{h, \Omega}$. The eigenvalues of $\mathscr{P}^{\mathbf{A}^{\mathsf{x}_0}}_h$ are given by $h^{3/2}\Lambda^{\mathsf{x}_0}_n$. For any $d_0 < \frac{1}{4}$ and any positive integer N, there exist C_N and $h_N > 0$ such that for all $h \in (0, h_N)$*

$$(3.4) \qquad h^{3/2}\Lambda^{\mathsf{x}_0}_n - C_N \, h^{3/2+d_0} \leq \lambda^{\mathsf{x}_0}_n(h) \leq h^{3/2}\Lambda^{\mathsf{x}_0}_n + C_N \, h^{7/4}, \quad n = 1, \dots, N.$$

Proof. The proof combines Lemma 3.1 with the perturbation identity (2.1). We still use the cut-off function χ_{x_0} as in the proof of Lemma 3.3.

i) Use Lemma 3.1 with $\mathscr{P} = \mathscr{P}_h^{\mathbf{A}^{\mathsf{x}_0}}$ and $\mathscr{P}^\star = \mathscr{P}_{h,\Omega}^{\mathbf{A}}$. For each eigenvector φ_n of $\mathscr{P}$, we consider the quasimode $\varphi_n^\star = \chi_{\mathsf{x}_0}\varphi_n$ for $\mathscr{P}_h^{\mathbf{A}^{\mathsf{x}_0}}$. The localization error is exponentially decaying thanks to (3.3) but the principal part of the discrepancy for the quasimodes arises from the difference between $\mathbf{A}$ and $\mathbf{A}^{\mathsf{x}_0}$. Thus the bound ν in (3.1b) satisfies (for some $r_0 > 0$)

$$\nu \le C\mathrm{e}^{-r_0/h^{1/4}}.$$

For estimate the diagonal terms $\mu_{n,n}$ in (3.1a), we use the identity (2.1) for $\mathbf{A}^\star = \mathbf{A}^{\mathsf{x}_0}$. Then

$$\mathscr{Q}_{h,\Omega}^{\mathbf{A}}(\chi_{\mathsf{x}_0}\psi_{h,n}^{\mathsf{x}_0}) = \mathscr{Q}_h^{\mathbf{A}^{\mathsf{x}_0}}(\chi_{\mathsf{x}_0}\psi_{h,n}^{\mathsf{x}_0}) + 2\,\mathrm{Re}\left\langle(-ih\nabla + \mathbf{A}^{\mathsf{x}_0})\chi_{\mathsf{x}_0}\psi_{h,n}^{\mathsf{x}_0}, (\mathbf{A} - \mathbf{A}^{\mathsf{x}_0})\chi_{\mathsf{x}_0}\psi_{h,n}^{\mathsf{x}_0}\right\rangle$$
$$+ \|(\mathbf{A} - \mathbf{A}^{\mathsf{x}_0})\chi_{\mathsf{x}_0}\psi_{h,n}^{\mathsf{x}_0}\|^2.$$

Hence the difference $\mu_{n,n} := \mathscr{Q}_{h,\Omega}^{\mathbf{A}}(\chi_{\mathsf{x}_0}\psi_{h,n}^{\mathsf{x}_0}) - \mathscr{Q}_h^{\mathbf{A}^{\mathsf{x}_0}}(\chi_{\mathsf{x}_0}\psi_{h,n}^{\mathsf{x}_0})$ is estimated by

$$|\mu_{n,n}| \le 2\big(\mathscr{Q}_h^{\mathbf{A}^{\mathsf{x}_0}}(\chi_{\mathsf{x}_0}\psi_{h,n}^{\mathsf{x}_0})\big)^{1/2}\|(\mathbf{A} - \mathbf{A}^{\mathsf{x}_0})\chi_{\mathsf{x}_0}\psi_{h,n}^{\mathsf{x}_0}\| + \|(\mathbf{A} - \mathbf{A}^{\mathsf{x}_0})\chi_{\mathsf{x}_0}\psi_{h,n}^{\mathsf{x}_0}\|^2$$

Using the Agmon estimates (3.3) and Lemma 3.2 with $\delta = \frac{1}{4}$ and $m = 8$, we find

$$|\mu_{n,n}| \le C(h^{3/4}h + h^2).$$

The reasonning is similar for $\mu_{n,m}$, $n \neq m$. Hence the right part of inequalities (3.4).

ii) For the left part of (3.4), we swap the roles of $\mathscr{P}_h^{\mathbf{A}^{\mathsf{x}_0}}$ and $\mathscr{P}_{h,\Omega}^{\mathbf{A}}$. The sole difference consists in Agmon estimates for the truncated eigenvectors of $\mathscr{P}_{h,\Omega}^{\mathbf{A}}$. Thanks to the right part of (3.4), Agmon estimates (1.10) hold and then Lemma 3.2 with $\delta = d \in \left(\frac{3}{16}, \frac{1}{4}\right)$ and $m = 8$, from which we deduce

$$|\mu_{n,n}| \le C(h^{3/4}h^{4d} + h^{8d}).$$

Choosing $d = \frac{3}{16} + \frac{d_0}{4}$, we obtain the left part of (3.4). $\qquad\square$

3.4. Expansion of eigenvalues.
Putting together Lemmas 3.3 and 3.4, we can see that Lemma 3.4 provides the bound $\lambda_n^{\mathsf{loc}}(h) \le Lh^{3/2}$ which validates the application of Lemma 3.3. Therefore, we have now proved:

Lemma 3.5. *Under Assumption 1.2, for any $d < \frac{1}{4}$ and any positive integer N, there exist $C_N > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$ and all $n = 1, \ldots, N$*

$$(3.5) \qquad h^{3/2}\Lambda_n^{\mathbf{B}} - C_N h^{3/2+d} \le \lambda_n(h) \le h^{3/2}\Lambda_n^{\mathbf{B}} + C_N h^{7/4}, \quad n = 1, \ldots, N.$$

To go further, we are going to exhibit, for each n, series expansions of eigenpairs. Owing to the exponential localization given by Lemma 3.3, it suffices to restrict the construction to any chosen localized operator $\mathscr{P}_{h,\mathscr{B}(\mathsf{x}_0,r_0)}^{\mathbf{A}}$. In order to alleviate notations, we will remove the mention of x_0 in general, and work in the Cartesian coordinates y for which the crossing point is at the origin. Then the domain Ω is the ball $\mathscr{B}(0, r_0)$ and the magnetic field cancels to the order 2 at 0. After a possible change of gauge, we can assume that the magnetic potential $\mathbf{A}$ cancels to the order 3 at 0. We write its Taylor formal series as

$$\mathbf{A} \sim \sum_{j \ge 0} \mathbf{A}_j \quad \text{where} \quad \mathbf{A}_j \text{ is polynomial and homogeneous of degree } 3 + j.$$

The first nonzero term is $\mathbf{A}_0$ formerly denoted by $\mathbf{A}^{\mathsf{x}_0}$. We retrieve the principal part of $\mathscr{P}_{h,\Omega}^{\mathbf{A}}$ at 0, and its natural expansion in powers of $h^{1/4}$ by considering, via the change of variables $\mathsf{Y} = \mathsf{y}/h^{1/4}$:

$$h^{-3/2}\mathscr{P}_{h,\Omega}^{\mathbf{A}}[\mathsf{y}, \nabla_{\mathsf{y}}] = h^{-3/2}\mathscr{P}_{h,\Omega/h^{1/4}}^{\mathbf{A}}[h^{1/4}\mathsf{Y}, h^{-1/4}\nabla_{\mathsf{Y}}].$$

We expand the right hand side as a formal series of operators $\sum_j h^{j/4}\mathcal{L}_j[Y, \nabla_Y]$ defined on $\mathbb{R}^2$. We have

$$\sum_j h^{j/4}\mathcal{L}_j[Y, \nabla_Y] = h^{-3/2}\Big(-ih^{3/4}\nabla_Y + \sum_{j\geq 0} h^{(3+\ell)/4}\mathbf{A}_j(Y)\Big)^2.$$

Hence the series $\sum_j h^{j/4}\mathcal{L}_j$ starts with $\mathcal{L}_0$ given by

$$\mathcal{L}_0 = \big(-i\nabla_Y + \mathbf{A}_0(Y)\big)^2$$

and the other terms $\mathcal{L}_j$ for $j \geq 1$ are partial differential operators of degree 1 with polynomial coefficients. The main term $\mathcal{L}_0$ is isospectral to $(-i\nabla + \Xi(\mathsf{x}_0)\mathbf{A}_\varepsilon(\mathsf{x}_0))^2$, see (1.5), and its eigenvalues are given by the $\Lambda_n^{\mathsf{x}_0}$ ($n \in \mathbb{N}^*$), see (1.6a).

Choose a normalized eigenpair of $\mathcal{L}_0$, which we denote by (ℓ_0, Ψ_0). We look for

$$\ell_j \in \mathbb{R} \quad \text{and} \quad \Psi_j \in \mathsf{Dom}(\mathcal{L}_0), \quad j = 1, 2, \ldots$$

solving $\big(\sum_j h^{j/4}\mathcal{L}_j\big)\big(\sum_k h^{k/4}\Psi_k\big) = \big(\sum_j h^{j/4}\ell_j\big)\big(\sum_k h^{k/4}\Psi_k\big)$ in the sense of formal series, i.e., solving the sequence of equations,

$$\sum_{j=0}^m \mathcal{L}_j\Psi_{m-j} = \sum_{j=0}^m \ell_j\Psi_{m-j} \quad m = 1, 2, \ldots.$$

We write the first equation (for $m = 1$) as

$$(\mathcal{L}_0 - \ell_0)\Psi_1 = \ell_1\Psi_0 - \mathcal{L}_1\Psi_0,$$

and the next ones as

$$(\mathcal{L}_0 - \ell_0)\Psi_m = \ell_m\Psi_0 - \mathcal{L}_m\Psi_0 + \sum_{j=1}^{m-1}(\ell_j - \mathcal{L}_j)\Psi_{m-j}.$$

If ℓ_0 is a *simple* eigenvalue of $\mathcal{L}_0$, the solution of such a sequence of equations is classical, resulting from the Fredholm alternative for the self-adjoint operator $\mathcal{L}_0$. For instance, we get

$$(3.6) \qquad\qquad \ell_1 = \langle\mathcal{L}_1\Psi_0, \Psi_0\rangle$$

if $m = 1$. If ℓ_0 is a *multiple* eigenvalue of $\mathcal{L}_0$, we cannot choose a priori an associated eigenvector, but have to work in the whole associated eigenspace E_{ℓ_0}. Then identity (3.6) is replaced by an eigen-equation for a finite dimensional hermitian matrix acting on the eigenspace E_{ℓ_0}. The process can be pursed as well, see [12] for details on this procedure. The terms Ψ_m belong to the domain of $\mathcal{L}_0$ and have, furthermore, exponential decay. Setting for $m \geq 1$

$$\ell^{[m]}(h) = h^{3/2}\sum_{j=0}^m h^{j/4}\ell_j \quad \text{and} \quad \psi_h^{[m]}(\mathsf{y}) = \chi_{\mathsf{x}_0}(\mathsf{y})\sum_{j=0}^m h^{j/4}\Psi_j(\mathsf{y}/h^{1/4})$$

we have constructed a quasimode for $\mathscr{P}_{h,\Omega}^{\mathbf{A}}$ to the order $h^{3/2+(m+1)/4}$, i.e.

$$\mathscr{P}_{h,\Omega}^{\mathbf{A}}(\psi_h^{[m]}) = \ell^{[m]}(h)\psi_h^{[m]} + \rho_h^{[m+1]} \quad \text{with} \quad \|\rho_h^{[m+1]}\| \leq C_m h^{3/2+(m+1)/4}\|\psi_h^{[m]}\|.$$

Combining this with (3.5), we deduce by the spectral theorem that there holds

Theorem 3.6. *Under Assumption* 1.2, *for all integers $N \geq 1$ and $M \geq 0$, there exist coefficients*

$$\Lambda_{n,m}^{\mathbf{B}} \in \mathbb{R}^+ \quad \text{for} \quad 1 \leq n \leq N, \ 0 \leq m \leq M, \quad \text{with} \quad \Lambda_{n,0}^{\mathbf{B}} = \Lambda_n^{\mathbf{B}}$$

and constants $C_{N,M} > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$ and all $n = 1, \ldots, N$

$$\Big|\lambda_n(h) - h^{3/2}\sum_{m=0}^M h^{m/4}\Lambda_{n,m}^{\mathbf{B}}\Big| \leq C_N h^{3/2+(M+1)/4}.$$

Of course, Theorem 1.8 is a particular case of the above statement if we choose $M = 0$.

4. SMALL ANGLE LIMIT

The first part of this section is devoted to theoretical results on the band function $(\alpha, \xi) \mapsto \varrho_1(\alpha, \xi)$ and to their numerical illustration. In the second part, we rely on these results to prove the convergence of eigenvalues of $\mathscr{X}_\varepsilon$ in the small angle limit and present the computations of their first eigenstates for a set of small values of ε.

4.1. **Operator symbol and band function.**
Here we study the behavior of the first eigenvalue $\varrho_1(\alpha, \xi)$ of the operator symbol $\mathsf{X}_{\alpha,\xi}$, for $(\alpha, \xi) \in \mathbb{R}^2$

$$\mathsf{X}_{\alpha,\xi} = D_t^2 + \left(\xi + \alpha^2 t - \frac{t^3}{3} \right)^2,$$

acting on $L^2(\mathbb{R})$, see (1.12).

4.1.1. *Preliminaries.*
Let us introduce the potential $\mathsf{V}_{\alpha,\xi}$ of $\mathsf{X}_{\alpha,\xi}$ and its generating polynomial $\mathsf{P}_{\alpha,\xi}$:

$$\mathsf{P}_{\alpha,\xi}(t) = \xi + \alpha^2 t - \frac{t^3}{3} \quad \text{and} \quad \mathsf{V}_{\alpha,\xi}(t) = \left(\mathsf{P}_{\alpha,\xi}(t) \right)^2.$$

$$\mathsf{V}_{\alpha,\xi}(t) = \xi^2 + 2\xi\alpha^2 t - 2\xi \frac{t^3}{3} + \alpha^4 t^2 - 2\alpha^2 \frac{t^4}{3} + \frac{t^6}{9}$$

The potential $\mathsf{V}_{\alpha,\xi}$ depends smoothly on the parameters (α, ξ) and is confining for each value of (α, ξ). So there holds

Proposition 4.1. *For all $(\alpha, \xi) \in \mathbb{R}^2$, the operator $\mathsf{X}_{\alpha,\xi}$ has a compact resolvent and the family $(\mathsf{X}_{\alpha,\xi})$ with $(\alpha, \xi) \in \mathbb{R}^2$ is analytic (of type (B) according to Kato theory, see [24]).*

Since the $\mathsf{X}_{\alpha,\xi}$ are Sturm-Liouville operators, we obtain

Corollary 4.2. *For all (α, ξ), the eigenvalue $\varrho_1(\alpha, \xi)$ is simple and depends analytically of α and ξ. The associated eigenfunctions do not vanish and the unique normalized and positive eigenfunction $u_{\alpha,\xi}$ associated with $\varrho_1(\alpha, \xi)$ depends analytically of (α, ξ).*

As a consequence we have the following "Feynman-Hellmann" formulas.

Corollary 4.3. *For all (α, ξ) we have the following identities*

$$(\partial_\alpha \varrho_1)(\alpha, \xi) = 4\alpha \int_{\mathbb{R}} \left(\xi + \alpha^2 t - \frac{t^3}{3} \right) t u_{\alpha,\xi}^2(t) \, dt,$$

$$(\partial_\xi \varrho_1)(\alpha, \xi) = 2 \int_{\mathbb{R}} \left(\xi + \alpha^2 t - \frac{t^3}{3} \right) u_{\alpha,\xi}^2(t) \, dt.$$

The potential $\mathsf{V}_{\alpha,\xi}$ has the following obvious symmetry properties: $\mathsf{V}_{\alpha,\xi}(t) = \mathsf{V}_{-\alpha,\xi}(t)$ and $\mathsf{V}_{\alpha,-\xi}(t) = \mathsf{V}_{\alpha,\xi}(-t)$. Hence the band function $\varrho_1(\alpha, \xi)$ is even with respect to each of the two variables α and ξ, so its analysis can be restricted to the first quadrant $\{\alpha \geq 0, \xi \geq 0\}$. The following lemma gives an expression of the roots of the generating polynomial $\mathsf{P}_{\alpha,\xi}$, depending on the sign of its discriminant.

Lemma 4.4. *For all $\alpha \geq 0$ and all $\xi \geq 0$, denote by $t_k(\alpha, \xi)$ the three roots of the polynomial $\mathsf{P}_{\alpha,\xi}$, agreeing that*

$$\operatorname{Re} t_1(\alpha, \xi) \leq \operatorname{Re} t_2(\alpha, \xi) \leq \operatorname{Re} t_3(\alpha, \xi).$$

Then, if $(\alpha, \xi) \neq (0, 0)$, $t_3(\alpha, \xi)$ is a positive simple real root. More precisely we have

- *For $\xi < \frac{2}{3}\alpha^3$, the polynomial $\mathsf{P}_{\alpha,\xi}$ admits three distinct real roots (that we denote $t_1(\alpha,\xi) < t_2(\alpha,\xi) < t_3(\alpha,\xi)$) given (for all $k \in \{1,2,3\}$) by*

$$(4.1a) \qquad t_k(\alpha,\xi) = j^{3-k}\sqrt[3]{\frac{1}{2}\left(3\xi + i\sqrt{4\alpha^6 - 9\xi^2}\right)} + j^{k-3}\sqrt[3]{\frac{1}{2}\left(3\xi - i\sqrt{4\alpha^6 - 9\xi^2}\right)},$$

where j is the complex number defined by $j = e^{2i\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.
- *For $\xi = \frac{2}{3}\alpha^3$, the polynomial $\mathsf{P}_{\alpha,\xi}$ admits a simple real root and a double real root respectively given by*

$$(4.1b) \qquad t_3(\alpha,\xi) = 2\alpha \text{ and } t_2(\alpha,\xi) = t_1(\alpha,\xi) = -\alpha.$$

- *For $\xi > \frac{2}{3}\alpha^3$, the polynomial $\mathsf{P}_{\alpha,\xi}$ admits a unique real root given by*

$$(4.1c) \qquad t_3(\alpha,\xi) = \sqrt[3]{\frac{1}{2}\left(3\xi + \sqrt{-4\alpha^6 + 9\xi^2}\right)} + \sqrt[3]{\frac{1}{2}\left(3\xi - \sqrt{-4\alpha^6 + 9\xi^2}\right)}.$$

The next result shows that the minimum cannot be reached on the set $\{\xi \geq \frac{2}{3}\alpha^3 > 0\}$. Its proof uses Corollary 4.3 and the fact that, if $\xi \geq \frac{2}{3}\alpha^3 > 0$, the polynomial $t \mapsto (t - t_3)\mathsf{P}_{\alpha,\xi}(t)$ is negative.

Proposition 4.5. *For all (α,ξ) such that $\xi \geq \frac{2}{3}\alpha^3 > 0$, we have*

$$(\partial_\alpha \varrho_1)(\alpha,\xi) - 2\alpha t_3(\alpha,\xi)(\partial_\xi \varrho_1)(\alpha,\xi) < 0.$$

In particular, there is no critical point on the set $\{\xi \geq \frac{2}{3}\alpha^3 > 0\}$.

4.1.2. *Behavior of the band function at infinity.* Now, the remaining part of this section is devoted to prove that $\varrho_1(\alpha,\xi)$ tends to infinity as $|\alpha| + |\xi|$ tends to infinity, namely

$$(4.2) \qquad \lim_{|\alpha|+|\xi|\to\infty} \varrho_1(\alpha,\xi) = \infty.$$

Note that Proposition 4.5 combined with (4.2) implies Theorem 1.9.

To prove (4.2), we split (for each $R > 1$) the region

$$\mathsf{A}_R := \{(\alpha,\xi) \in \mathbb{R}^2, \quad \alpha \geq 0, \ \xi \geq 0, \ \text{and} \ \alpha + \xi > R\}$$

into the three subregions

$$(4.3a) \qquad \mathsf{A}_R^\circ := \mathsf{A}_R \cap \{(\alpha,\xi) \in \mathbb{R}^2, \quad \alpha \in [0,1] \quad \xi > \tfrac{2}{3}\alpha^3\}$$

$$(4.3b) \qquad \mathsf{A}_R^\sharp := \mathsf{A}_R \cap \{(\alpha,\xi) \in \mathbb{R}^2, \quad \alpha \in [1,\infty) \quad \xi > \tfrac{2}{3}\alpha^3\}$$

$$(4.3c) \qquad \mathsf{A}_R^\flat := \mathsf{A}_R \cap \{(\alpha,\xi) \in \mathbb{R}^2, \quad \alpha \in [1,\infty) \quad \xi \leq \tfrac{2}{3}\alpha^3\}$$

and are going to prove the next lemma.

Lemma 4.6. *We denote $\mathsf{Q}_{\alpha,\xi}$ the quadratic form associated with the operator $\mathsf{X}_{\alpha,\xi}$. There exists constants $R > 1$ and $B > 0$ such that for all $\psi \in \mathscr{C}_0^\infty(\mathbb{R})$ the following lower bounds hold*

$$(4.4a) \qquad \mathsf{Q}_{\alpha,\xi}(\psi) \geq B\,\xi^{2/3}\|\psi\|_{L^2(\mathbb{R})}^2, \quad \forall(\alpha,\xi) \in \mathsf{A}_R^\circ$$

$$(4.4b) \qquad \mathsf{Q}_{\alpha,\xi}(\psi) \geq B\,\xi^{2/9}\|\psi\|_{L^2(\mathbb{R})}^2, \quad \forall(\alpha,\xi) \in \mathsf{A}_R^\sharp$$

$$(4.4c) \qquad \mathsf{Q}_{\alpha,\xi}(\psi) \geq B\,\alpha^{2/3}\|\psi\|_{L^2(\mathbb{R})}^2, \quad \forall(\alpha,\xi) \in \mathsf{A}_R^\flat$$

We can see that

$$\xi^{2/9} > \tfrac{1}{2}\xi^{2/9} + \tfrac{1}{3}\alpha^{2/3} \ \text{on} \ \mathsf{A}_R^\circ \cup \mathsf{A}_R^\sharp \quad \text{and} \quad \alpha^{2/3} > \tfrac{1}{2}(\alpha^{2/3} + \xi^{2/9}) \ \text{on} \ \mathsf{A}_R^\flat.$$

Therefore the bounds (4.4) imply (4.2).

Proof of (4.4a). We recall that, on the set $\{\xi > \frac{2}{3}\alpha^3 \geq 0\}$, the polynomial $\mathsf{P}_{\alpha,\xi}$ admits a unique real root, denoted $t_3(\alpha,\xi)$. Using that $\mathsf{P}_{\alpha,\xi}(t_3(\alpha,\xi)) = 0$, we immediately check that we have the factorization

$$(4.5) \qquad \mathsf{P}_{\alpha,\xi}(t) = -\mathsf{N}_{\alpha,\xi}(t)\big(t - t_3(\alpha,\xi)\big) \quad \text{with} \quad \mathsf{N}_{\alpha,\xi}(t) = \frac{1}{3}t^2 + \frac{t_3(\alpha,\xi)}{3}t + \frac{\xi}{t_3(\alpha,\xi)}.$$

The factor $\mathsf{N}_{\alpha,\xi}(t)$ is positive for all t when $\xi > \frac{2}{3}\alpha^3$ and we have $\partial_t \mathsf{N}_{\alpha,\xi}(t) = 0$ for $t = -\frac{t_3(\alpha,\xi)}{2}$. Therefore, for all $\xi > \frac{2}{3}\alpha^3$, we have the lower bound

$$\mathsf{N}_{\alpha,\xi}(t) \geq -\frac{t_3(\alpha,\xi)^2}{12} + \frac{\xi}{t_3(\alpha,\xi)}.$$

Then, the quotient $\mathsf{Q}_{\alpha,\xi}(\psi)/\|\psi\|^2_{L^2(\mathbb{R})}$ is bounded from below by the ground state energy of the operator

$$D_t^2 + \left(-\frac{t_3(\alpha,\xi)^2}{12} + \frac{\xi}{t_3(\alpha,\xi)}\right)^2 (t - t_3(\alpha,\xi))^2 \quad \text{on} \quad \mathbb{R}.$$

By considering the expression of $t_3(\alpha,\xi)$ given in (4.1c), we get, uniformly in $\alpha \in [0,1]$,

$$t_3(\alpha,\xi) = (3\xi)^{1/3} + \mathcal{O}_{\xi\to+\infty}(\xi^{-1/3}).$$

Hence, there exist constants $B > 0$ and $K > 0$ such that for all $\xi > K$ (with $\xi > \frac{2}{3}\alpha^3$ and $0 \leq \alpha \leq 1$)

$$\left(-\frac{t_3(\alpha,\xi)^2}{12} + \frac{\xi}{t_3(\alpha,\xi)}\right) \geq B\,\xi^{2/3}.$$

For all $\xi > K$, $\frac{\mathsf{Q}_{\alpha,\xi}(\psi)}{\|\psi\|^2_{L^2(\mathbb{R})}}$ is bounded from below by the ground state energy of

$$D_t^2 + B^2\xi^{4/3}(t - t_3(\alpha,\xi))^2 \quad \text{on} \quad \mathbb{R}.$$

By translation and homogeneity, we get (using the harmonic oscillator):

$$(4.6) \qquad\qquad\qquad \mathsf{Q}_{\alpha,\xi}(\psi) \geq B\,\xi^{2/3}\|\psi\|^2_{L^2(\mathbb{R})}.$$

This concludes the proof of the estimate (4.4a). $\square$

Preliminaries for the proof of (4.4b) *and* (4.4c). For the proof of estimates (4.4b) and (4.4c), we use a quadratic partition of unity (χ_1, χ_2) on $\mathbb{R}$ in order to isolate the root $t_3(\alpha,\xi)$ from the other two roots of $\mathsf{P}_{\alpha,\xi}$. For this we choose two real numbers α_- and α_+ such that

$$\alpha_- = \gamma_-\alpha, \quad \alpha_+ = \gamma_+\alpha \quad \text{with} \quad 0 \leq \gamma_- < \gamma_+$$

and we take the two functions χ_1 and χ_2 such that $\chi_1^2 + \chi_2^2 = 1$ on $\mathbb{R}$ and

$$\chi_1(t) = \begin{cases} 1 & \text{on} \quad (-\infty, \alpha_-] \\ 0 & \text{on} \quad [\alpha_+, +\infty) \end{cases} \quad \text{and} \quad \chi_2(t) = \begin{cases} 0 & \text{on} \quad (-\infty, \alpha_-] \\ 1 & \text{on} \quad [\alpha_+, +\infty) \end{cases}$$

with the control of their derivatives

$$\sup_{t\in\mathbb{R}} |\chi_j'(t)| \leq \frac{K_{\text{loc}}}{\alpha_+ - \alpha_-}, \quad \text{for all } j \in \{1, 2\}.$$

The localization formula (see (2.4)) gives, for all function ψ in the form domain,

$$(4.7) \qquad \mathsf{Q}_{\alpha,\xi}(\psi) = \mathsf{Q}_{\alpha,\xi}(\chi_1\psi) + \mathsf{Q}_{\alpha,\xi}(\chi_2\psi) - \|\psi\chi_1'\|^2_{L^2(\mathbb{R})} - \|\psi\chi_2'\|^2_{L^2(\mathbb{R})}.$$

Whence

$$(4.8) \qquad\qquad \mathsf{Q}_{\alpha,\xi}(\psi) \geq \mathsf{Q}_{\alpha,\xi}(\chi_1\psi) + \mathsf{Q}_{\alpha,\xi}(\chi_2\psi) - \frac{K_{\text{loc}}^2}{(\alpha_+ - \alpha_-)^2}\|\psi\|^2_{L^2(\mathbb{R})}.$$

We let $\Omega^{(1)} = (-\infty, \alpha_+)$ and $\Omega^{(2)} = (\alpha_-, +\infty)$ and denote $\psi_1 = \chi_1\psi$ and $\psi_2 = \chi_2\psi$. We will work out a lower bound of the quadratic form on each of these subdomains.

On both subdomains, using the factorization (4.5), we start from the expression of $\mathsf{X}_{\alpha,\xi}$ as

$$(4.9) \qquad \mathsf{X}_{\alpha,\xi} = D_t^2 + \mathsf{V}_{\alpha,\xi} \quad \text{with} \quad \mathsf{V}_{\alpha,\xi} = \left(\mathsf{N}_{\alpha,\xi}(t)\right)^2 \left(t - t_3(\alpha,\xi)\right)^2.$$

On $\Omega^{(1)}$, we bound from below the potential $\mathsf{V}_{\alpha,\xi}$ as

$$(4.10) \qquad \mathsf{V}_{\alpha,\xi} \geq B_1(\alpha,\xi) \left(\mathsf{N}_{\alpha,\xi}(t)\right)^2 \quad \text{with} \quad B_1(\alpha,\xi) = \min_{t \leq \alpha_+} \left(t - t_3(\alpha,\xi)\right)^2.$$

Therefore, the quotient $\mathsf{Q}_{\alpha,\xi}(\psi_1)/\|\psi_1\|^2_{L^2(\mathbb{R})}$ is bounded from below by the ground state energy of

$$(4.11) \qquad D_t^2 + B_1(\alpha,\xi) \left(\mathsf{N}_{\alpha,\xi}(t)\right)^2 \quad \text{on} \quad \mathbb{R}.$$

Using the canonical form of the factor $\mathsf{N}_{\alpha,\xi}$

$$(4.12) \qquad \mathsf{N}_{\alpha,\xi}(t) = \frac{1}{3}\left(t + t_3(\alpha,\xi)\right)^2 - \frac{t_3(\alpha,\xi)^2}{12} + \frac{\xi}{t_3(\alpha,\xi)}.$$

we find by translation and scaling that the operator (4.11) is isospectral to the operator

$$(4.13) \qquad (\tfrac{4}{9})^{1/3} B_1(\alpha,\xi)^{1/3} \left(D_\tau^2 + \left(\frac{\tau^2}{2} - \eta\right)^2\right)^2 \quad \text{on} \quad \mathbb{R},$$

for a suitable real number η. We know from [17] that the ground state energy $\gamma(\eta)$ of the operator $D_\tau^2 + \left(\frac{\tau^2}{2} - \eta\right)^2$ as a function of $\eta \in \mathbb{R}$ reaches its minimum (for a positive value η_0 of η). Therefore this minimum is positive. We denote it by M_0. Finally we bound $\mathsf{Q}_{\alpha,\xi}(\psi_1)$ from below as follows

$$(4.14) \qquad \mathsf{Q}_{\alpha,\xi}(\psi_1) \geq (\tfrac{4}{9})^{1/3} B_1(\alpha,\xi)^{1/3} \mathrm{M}_0 \|\psi_1\|^2_{L^2(\mathbb{R})}.$$

On $\Omega^{(2)}$, we swap the roles of the two factors in $\mathsf{P}_{\alpha,\xi}$ and obtain the lower bound:

$$(4.15) \qquad \mathsf{V}_{\alpha,\xi} \geq B_2(\alpha,\xi) \left(t - t_3(\alpha,\xi)\right)^2 \quad \text{with} \quad B_2(\alpha,\xi) = \min_{t \geq \alpha_-} \left(\mathsf{N}_{\alpha,\xi}(t)\right)^2.$$

By translation and homogeneity, we get via the harmonic oscillator:

$$(4.16) \qquad \mathsf{Q}_{\alpha,\xi}(\psi_2) \geq B_2(\alpha,\xi)^{1/2} \|\psi_2\|^2_{L^2(\mathbb{R})}.$$

Finally, combining (4.14) and (4.16) with (4.8) we find

$$(4.17) \quad \mathsf{Q}_{\alpha,\xi}(\psi) \geq \left(\min\left\{(\tfrac{4}{9})^{1/3} B_1(\alpha,\xi)^{1/3} \mathrm{M}_0, \; B_2(\alpha,\xi)^{1/2}\right\} - \frac{K_{\mathsf{loc}}^2}{(\alpha_+ - \alpha_-)^2}\right) \|\psi\|^2_{L^2(\mathbb{R})}.$$

It remains to choose α_+ and α_- so that we can find suitable lower bounds for the constants $B_1(\alpha,\xi)$ and $B_2(\alpha,\xi)$. This will be done finding upper and lower bounds for $t_3(\alpha,\xi)$. $\qquad \square$

Proof of (4.4b). In the region $\mathsf{A}^\sharp$, we have $\xi > \frac{2}{3}\alpha^3$ (and $\alpha > 1$). With the aim of finding bounds for $t_3(\alpha,\xi)$ we calculate the derivative of expression (4.1c) with respect to α:

$$\partial_\alpha t_3(\alpha,\xi) = -\frac{4\alpha^5}{\sqrt{-4\alpha^6 + 9\xi^2}} \left(\frac{1}{2}\left(3\xi + \sqrt{-4\alpha^6 + 9\xi^2}\right)\right)^{-2/3}$$

$$+ \frac{4\alpha^5}{\sqrt{-4\alpha^6 + 9\xi^2}} \left(\frac{1}{2}\left(3\xi - \sqrt{-4\alpha^6 + 9\xi^2}\right)\right)^{-2/3}.$$

We can see that the modulus of the second term is larger than the modulus of the first term. Hence the positivity of the derivative $\partial_\alpha t_3(\alpha, \xi)$. Therefore

$$t_3(0, \xi) \le t_3(\alpha, \xi) \le t_3\left((\tfrac{3}{2}\xi)^{1/3}, \xi\right), \quad \forall \xi > 0.$$

With (4.1c) (and using again that $\xi > \tfrac{2}{3}\alpha^3$), we deduce

$$2^{1/3}\alpha < (3\xi)^{1/3} \le t_3(\alpha, \xi) \le 2^{2/3}(3\xi)^{1/3}.$$

We choose

$$\alpha_- = 0 \quad \text{and} \quad \alpha_+ = \alpha.$$

Thus, we get for the constants $B_1(\alpha, \xi)$ and $B_2(\alpha, \xi)$ appearing in (4.10) and (4.15):

$$B_1(\alpha, \xi) = \min_{t \le \alpha_+} \left(t - t_3(\alpha, \xi)\right)^2 \ge \left(\alpha_+ - (3\xi)^{1/3}\right)^2 \ge (3\xi)^{2/3}(1 - 2^{-1/3})^2$$

and

$$B_2(\alpha, \xi) = \min_{t \ge \alpha_-} \left(\mathsf{N}_{\alpha, \xi}(t)\right)^2$$

$$= \min_{t \ge \alpha_-} \left(\frac{1}{3}t^2 + \frac{t_3(\alpha, \xi)}{3}t + \frac{\xi}{t_3(\alpha, \xi)}\right)^2 \ge \left(\frac{\xi}{t_3(\alpha, \xi)}\right)^2 \ge 2^{-4/3}3^{-2/3}\xi^{4/3}.$$

Then (4.17) yields

$$\mathsf{Q}_{\alpha, \xi}(\psi) \ge \left(\min\left\{C_1\xi^{2/9}, \, C_2\xi^{2/3}\right\} - K_{\mathsf{loc}}^2\alpha^{-2}\right)\|\psi\|_{L^2(\mathbb{R})}^2.$$

Since $\alpha \ge 1$, this clearly implies (4.4b) if ξ is large enough. $\quad\square$

Proof of (4.4c). In the region $\mathsf{A}^\flat$, $\tfrac{2}{3}\alpha^3 \ge \xi$ and the polynomial $\mathsf{P}_{\alpha, \xi}$ has three real roots. We note that $\mathsf{P}_{\alpha, \xi}(t) \to -\infty$ as $t \to +\infty$ and $\mathsf{P}_{\alpha, \xi}(t) \to +\infty$ as $t \to -\infty$. We check that

$$\mathsf{P}_{\alpha, \xi}(-\sqrt{3}\alpha) = \xi > 0, \qquad \mathsf{P}_{\alpha, \xi}(-\alpha) = \xi - \tfrac{2}{3}\alpha^3 \le 0,$$

$$\mathsf{P}_{\alpha, \xi}(\sqrt{3}\alpha) = \xi > 0, \qquad \mathsf{P}_{\alpha, \xi}(2\alpha) = \xi - \tfrac{2}{3}\alpha^3 \le 0.$$

This implies that

$$-\sqrt{3}\alpha < t_1(\alpha, \xi) \le -\alpha \le t_2(\alpha, \xi) < \sqrt{3}\alpha < t_3(\alpha, \xi) \le 2\alpha.$$

Now we choose

$$\alpha_- = \tfrac{1}{2}\alpha \quad \text{and} \quad \alpha_+ = \alpha,$$

and we get for the constants $B_1(\alpha, \xi)$ and $B_2(\alpha, \xi)$:

$$B_1(\alpha, \xi) = \min_{t \le \alpha_+} \left(t - t_3(\alpha, \xi)\right)^2 \ge \left(\alpha_+ - \sqrt{3}\alpha\right)^2 = \alpha^2(1 - \sqrt{3})^2$$

and

$$B_2(\alpha, \xi) = \min_{t \ge \alpha_-} \left(\frac{1}{3}t^2 + \frac{t_3(\alpha, \xi)}{3}t + \frac{\xi}{t_3(\alpha, \xi)}\right)^2 \ge \left(\frac{\alpha^2}{12}\right)^2 = \frac{1}{144}\alpha^4.$$

Then (4.17) yields

$$\mathsf{Q}_{\alpha, \xi}(\psi) \ge \left(\min\left\{C_1\alpha^{2/3}, \, C_2\alpha^2\right\} - K_{\mathsf{loc}}^2\alpha^{-2}\right)\|\psi\|_{L^2(\mathbb{R})}^2,$$

which implies (4.4c). $\quad\square$

The proof of Lemma 4.6 is now achieved, hence Theorem 1.9 is proved.

4.1.3. *Numerical simulations.* We have computed an approximation of the band function ϱ_1 on a grid Σ of values of (α, ξ) covering the square $(-2, 2)^2$. The grid points of Σ are (α_k, ξ_l) with $\alpha_k = -2 + k/100$ and $\xi_l = -2 + l/100$, for $k, l \in \{0, \ldots, 400\}$. In Figure 1, we plot the level lines of ϱ_1 above the grid Σ and in Figure 2, we plot the same band function restricted on the axis $\xi = 0$.
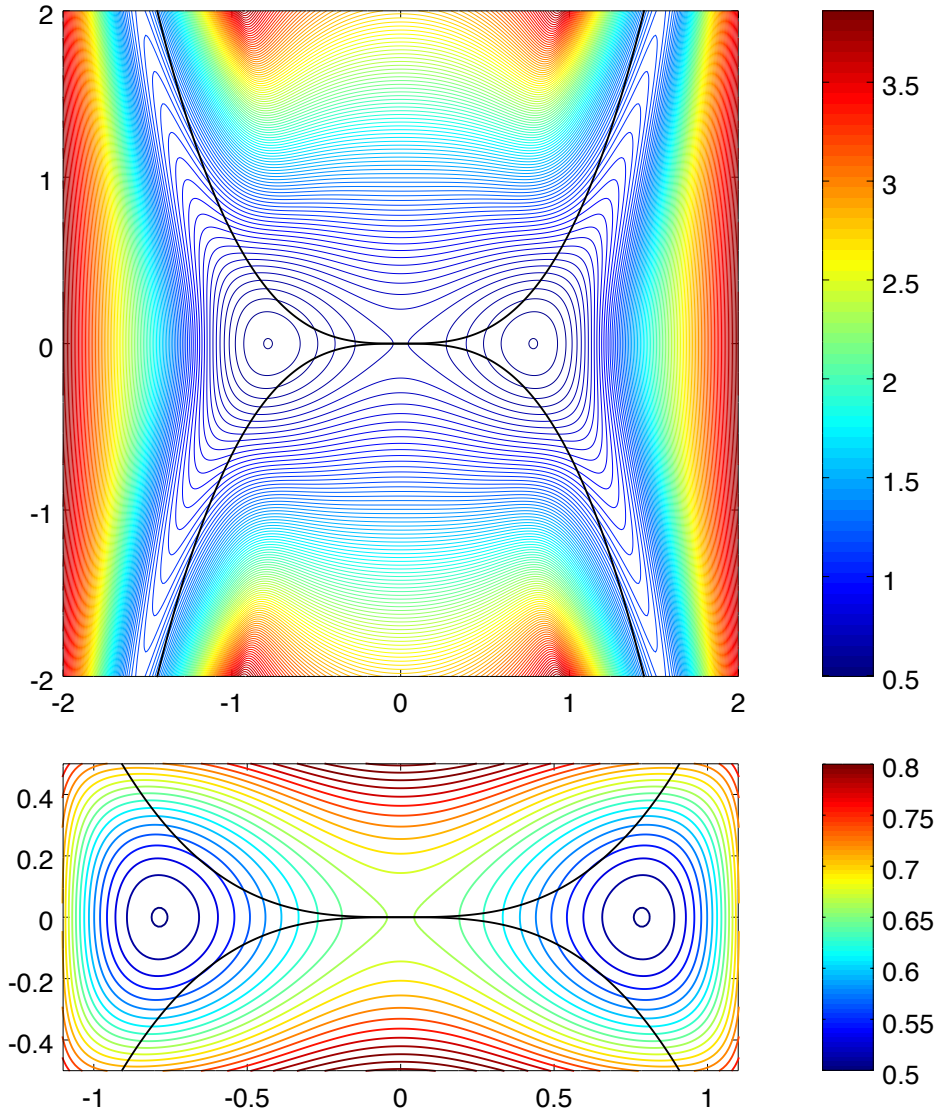

FIGURE 1. Contour lines for the numerical values of $\varrho_1(\alpha, \xi)$ on the grid Σ, with a zoom in a neighborhood of the two points where the minimum is attained. The curves $\{\alpha \mapsto \xi = \pm\frac{2}{3}\alpha^3\}$ are also represented.

The computations for Figures 1 and 2 are performed by the finite element method[1] and are based on a Galerkin projection on the interval $(-5, 5) \ni t$ with natural boundary conditions at the ends, discretized by 10 elements with polynomial degree 10. With this number of elements the degree 10 saturates the double precision, see Table 1. Enlarging the domain yields numbers

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[1]All our computations are preformed with the FEM library XLiFE++ under a GNU GPL licence.
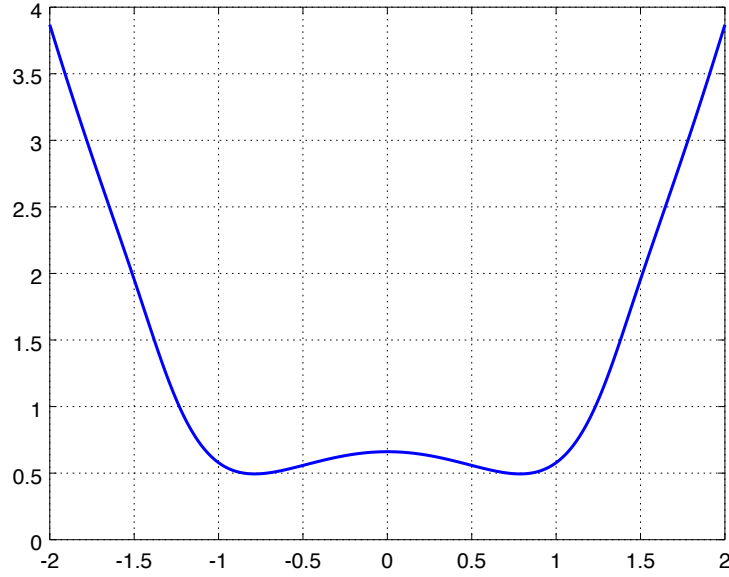
FIGURE 2. Numerical values for $\varrho_1(\alpha, \xi)$ on the axis $\xi = 0$.

with the same first 13 digits, which proves that $(-5, 5)$ is large enough to capture the numerical support of the first eigenvectors when (α, ξ) belongs to $[-2, 2]^2$.

According to the analysis of [23], the bottom of the spectrum $\inf \mathfrak{S}(\mathscr{M}^{[2]})$ of the Montgomery operator of order two coincides with $\varrho_1(0, 0)$. We can see in Figure 2, that $\alpha = 0$ is a local *maximum* of the function $\alpha \mapsto \varrho_1(\alpha, 0)$. Table 1 provides the value 0.660952004868639 for $\varrho_1(0, 0)$ (with presumably 13 correct digits), and the value 0.786 for α_0 with 3 correct digits. Refining the sampling of α by 101 values in the interval $[0.786, 0.787]$ yields

$$\alpha_0 = 0.78628 \quad \text{and} \quad \varrho_1(\alpha_0, 0) = 0.49410921120.$$

Q	$\varrho_1(0, 0)$	α_0	$\varrho_1(\alpha_0, 0)$
1	0.716813090776313	0.794	0.549407920248045
2	0.665333352584016	0.790	0.495300498319300
3	0.660969098915948	0.786	0.494298816339735
4	0.660960180256631	0.786	0.494116056132206
5	0.660952197968529	0.786	0.494109730708665
6	0.660952010967773	0.786	0.494109338690037
7	0.660952005398424	0.786	0.494109316007370
8	0.660952004871061	0.786	0.494109315475798
9	0.660952004869326	0.786	0.494109315436505
10	0.660952004868639	0.786	0.494109315435604
11	0.660952004868671	0.786	0.494109315435619
12	0.660952004868692	0.786	0.494109315435650

TABLE 1. Computed values of $\varrho_1(\alpha_0, 0)$ and $\varrho_1(\alpha_0, 0)$ with polynomial degree Q on 10 elements in $(-5, 5)$. The numerical value of α_0 is also provided.

4.2. **Asymptotic analysis in the small angle limit.** In this section, we prove Theorem 1.10. The presentation is mostly inspired by [9, Section 2] and we just highlight the most important steps and differences.

4.2.1. *Changes of variables.* We are interested in the behavior of the first eigenpair of $\mathscr{X}_\varepsilon$ (defined in (1.4)) as $\varepsilon \to 0$. To investigate this, we perform two changes of variables. First, the scaling

$$(4.18) \qquad (\sigma, \tau) \longrightarrow (s, t) \quad \text{so that} \quad s = \varepsilon\sigma, \quad t = \tau,$$

brings the spectral analysis of the operator $\mathscr{X}_\varepsilon$ to the following unitarily equivalent operator

$$(4.19) \qquad \mathscr{L}_\varepsilon = D_t^2 + \left(\varepsilon D_s + s^2 t - \frac{t^3}{3} \right)^2 .$$

Second, we localize in s around a point α_0 such that there exists a value ξ_0 for which ϱ_1 reaches its minimum S_0 in (α_0, ξ_0).

By the new change of variable

$$(4.20) \qquad (s, t) \longrightarrow (\mathfrak{s}, \mathfrak{t}) \quad \text{so that} \quad s = \alpha_0 + \varepsilon^{1/2}\mathfrak{s}, \quad t = \mathfrak{t},$$

and a gauge transform, the operator $\mathscr{L}_\varepsilon$ becomes

$$(4.21) \qquad \mathfrak{L}_\varepsilon = D_\mathfrak{t}^2 + \left(\xi_0 + \varepsilon^{1/2} D_\mathfrak{s} + \varepsilon\mathfrak{s}^2\mathfrak{t} + 2\varepsilon^{1/2}\alpha_0\mathfrak{s}\mathfrak{t} + \alpha_0^2\mathfrak{t} - \frac{\mathfrak{t}^3}{3} \right)^2 .$$

Thus the above three operators have the same eigenvalues

$$\mathfrak{S}(\mathscr{X}_\varepsilon) = \mathfrak{S}(\mathscr{L}_\varepsilon) = \mathfrak{S}(\mathfrak{L}_\varepsilon).$$

Proposition 4.7. *For all $N \geq 1$, there exist $C_N > 0$ and $\varepsilon_N > 0$ such that for all $\varepsilon \in (0, \varepsilon_N)$ there exist at least N eigenvalues (counted with multiplicity) of the operator $\mathscr{X}_\varepsilon$ contained in the ball of radius $C_N \varepsilon$ centered at S_0.*

Proof. We can write

$$\mathfrak{L}_\varepsilon = \mathfrak{L}_0 + \varepsilon^{1/2}\mathfrak{L}_1 + \mathscr{O}(\varepsilon),$$

with

$$\mathfrak{L}_0 = \mathsf{X}_{\alpha_0, \xi_0} \quad \text{and} \quad \mathfrak{L}_1 = 2\left(\xi_0 + \alpha_0^2\mathfrak{t} - \frac{\mathfrak{t}^3}{3} \right)(D_\mathfrak{s} + 2\alpha_0\mathfrak{s}\mathfrak{t}).$$

We are looking for quasimodes in the form

$$\psi = \psi_0 + \varepsilon^{\frac{1}{2}}\psi_1 \quad \text{and} \quad \varkappa = \varkappa_0 + \varepsilon^{\frac{1}{2}}\varkappa_1,$$

such that $\mathfrak{L}_\varepsilon\psi = \varkappa\psi + \mathscr{O}(\varepsilon)$ is satisfied. Gathering the terms in ε^0 we get the equation

$$\mathsf{X}_{\alpha_0, \xi_0}\psi_0 = \varkappa_0\psi_0 .$$

Thus $\varkappa_0$ is in the spectrum of $\mathsf{X}_{\alpha_0, \xi_0}$ and that ψ_0 is an associated eigenfunction. We choose

$$\varkappa_0 = \mathrm{S}_0,$$

and we take ψ_0 (unitary) in the following form

$$(4.22) \qquad \psi_0(\mathfrak{s}, \mathfrak{t}) = f_0(\mathfrak{s})\, u_0(\mathfrak{t}) ,$$

with $u_0 = u_{\alpha_0, \xi_0}$ and f_0 in the Schwartz class.

Gathering the terms in $\varepsilon^{1/2}$, we get the equation

$$(4.23) \qquad (\mathfrak{L}_0 - \mathrm{S}_0)\psi_1 = -(\mathfrak{L}_1 - \varkappa_1)\psi_0 .$$

We have

$$\mathfrak{L}_1\psi_0(\mathfrak{s}, \mathfrak{t}) = \left(\xi_0 + \alpha_0^2\mathfrak{t} - \frac{\mathfrak{t}^3}{3}\right)\left(2u_{\alpha_0,\xi_0}(\mathfrak{t})\,(D_\mathfrak{s}f_0)(\mathfrak{s}) + 4\alpha_0\mathfrak{t}\,u_{\alpha_0,\xi_0}(\mathfrak{t})\,\mathfrak{s}f_0(\mathfrak{s})\right).$$

At this point, we introduce the notation

$$(\partial_\alpha u)_{\alpha_0,\xi_0} = (\partial_\alpha u_{\alpha,\xi})_{(\alpha,\xi)=(\alpha_0,\xi_0)} \quad \text{and} \quad (\partial_\xi u)_{\alpha_0,\xi_0} = (\partial_\xi u_{\alpha,\xi})_{(\alpha,\xi)=(\alpha_0,\xi_0)}\,.$$

Taking the derivative of the equation $\mathsf{X}_{\alpha,\xi}u_{\alpha,\xi} = \varrho_1(\alpha,\xi)\,u_{\alpha,\xi}$ with respect to α and ξ, we obtain

$$(4.24\text{a}) \qquad \left(\mathsf{X}_{\alpha,\xi} - \varrho_1(\alpha,\xi)\right)(\partial_\alpha u)_{\alpha,\xi} = \left(\partial_\alpha\varrho_1(\alpha,\xi) - 4\alpha\mathfrak{t}\left(\xi_0 + \alpha_0^2\mathfrak{t} - \frac{\mathfrak{t}^3}{3}\right)\right)u_{\alpha,\xi}$$

$$(4.24\text{b}) \qquad \left(\mathsf{X}_{\alpha,\xi} - \varrho_1(\alpha,\xi)\right)(\partial_\xi u)_{\alpha,\xi} = \left(\partial_\alpha\varrho_1(\alpha,\xi) - 2\left(\xi_0 + \alpha_0^2\mathfrak{t} - \frac{\mathfrak{t}^3}{3}\right)\right)u_{\alpha,\xi}\,.$$

In particular, for (α,ξ) equal to the critical point (α_0,ξ_0) we find that

$$(\mathfrak{L}_0 - \mathsf{S}_0)(\partial_\alpha u)_{\alpha_0,\xi_0} = -4\alpha\mathfrak{t}\left(\xi_0 + \alpha_0^2\mathfrak{t} - \tfrac{\mathfrak{t}^3}{3}\right)u_{\alpha_0,\xi_0}$$

$$(\mathfrak{L}_0 - \mathsf{S}_0)(\partial_\xi u)_{\alpha_0,\xi_0} = -2\left(\xi_0 + \alpha_0^2\mathfrak{t} - \tfrac{\mathfrak{t}^3}{3}\right)u_{\alpha_0,\xi_0}.$$

Thus we have obtained explicit solutions of equation (4.23) as:

$$(4.25) \qquad \psi_1(\mathfrak{s}, \mathfrak{t}) = (D_\mathfrak{s}f_0)(\mathfrak{s})\,(\partial_\alpha u)_{\alpha_0,\xi_0}(\mathfrak{t}) + \mathfrak{s}f_0(\mathfrak{s})\,(\partial_\xi u)_{\alpha_0,\xi_0}(\mathfrak{t}) \quad \text{with} \quad \varkappa_1 = 0\,.$$

Therefore, for any function ψ in the form

$$\psi = \psi_0 + \varepsilon^{1/2}\psi_1,$$

where ψ_0 and ψ_1 are respectively given by in (4.22) and (4.25), we have

$$\|\left(\mathfrak{L}_\varepsilon - \mathsf{S}_0\right)\psi\| \le C\varepsilon\|\psi\|\,.$$

In the construction procedure of ψ_0 and ψ_1, the function f_0 is left undetermined. Therefore the above estimate holds on finite dimensional spaces of arbitrary dimensions, which ends the proof of the proposition. □

4.2.2. *Localization estimates.* The following lemma is crucial to follow the strategy in [9, Section 2]. Note in particular that [9, Assumption 1.7] is not obviously checked in the present context.

Lemma 4.8. *There exists $C_2 > 0$ such that, for all $\varepsilon \in (0,1)$, and all ψ in the form domain of $\mathscr{X}_\varepsilon$,*

$$(4.26) \qquad \|m_\varepsilon(\sigma,\tau)^{1/4}\psi\|^2 \le C_2\left(\langle\mathscr{X}_\varepsilon\psi,\psi\rangle + \|\psi\|^2\right),$$

where

$$m_\varepsilon(\sigma,\tau) = |\varepsilon^2\sigma^2 - \tau^2| + 2|\sigma|\varepsilon^2 + 2|\tau|\,.$$

Proof. The lemma follows from [19, Théorème (1.1)]. We only have to check the assumptions of the theorem and the uniformity with respect to ε. Here the magnetic field has one component $B = \varepsilon^2\sigma^2 - \tau^2$. We take the integer r used there as 1. The condition [19, (1.9)] is trivially satisfied for $C_1 = 8$

$$2\varepsilon^2 + 2 \le C_1(m(\sigma,\tau) + 1)\,.$$

Then [19, (1.11)] tells us that

$$\forall\psi \in \mathscr{C}_0^\infty(\mathbb{R}^2), \quad \|m(\sigma,\tau)^{1/4}\psi\|^2 \le C_2\left(\langle\mathscr{X}_\varepsilon\psi,\psi\rangle + \|\psi\|^2\right).$$

A careful check of the proof of [19, Théorème (1.1)], involving only commutators of analytic vector fields with respect to ε, shows that C_2 does not depend on $\varepsilon \in (0, 1)$. The extension of this estimate to the form domain follows by density. $\qquad\square$

We can reformulate Lemma 4.8 in terms of the operator $\mathscr{L}_\varepsilon$ and deduce the following.

Proposition 4.9. *There exists $C_2 > 0$ such that, for all $\varepsilon \in (0, 1)$, and all ψ in the form domain of $\mathscr{L}_\varepsilon$,*

$$(4.27) \qquad \| \left[|s^2 - t^2| + 2|t| \right]^{1/4} \psi \|^2 \leq C_2 \left(\langle \mathscr{L}_\varepsilon \psi, \psi \rangle + \|\psi\|^2 \right) .$$

In particular, for all $\mathrm{S}_0^* > \mathrm{S}_0$ there exists $R_* > 0$ such that, for all $\varepsilon \in (0, 1)$,

$$\mathscr{L}_\varepsilon^{\mathsf{Dir}, R_*} \geq \mathrm{S}_0^* ,$$

where $\mathscr{L}_\varepsilon^{\mathsf{Dir}, R_*}$ is the Dirichlet realization of the operator $\mathscr{L}_\varepsilon$ outside the ball of center 0 and radius R_*.

Proof. To deduce the second assertion, it suffices to note that $\lim\limits_{|s|+|t| \to +\infty} |s^2 - t^2| + 2|t| = +\infty$. $\qquad\square$

It is classical to deduce from Proposition 4.9 that the eigenfunctions associated with the low lying eigenvalues satisfy Agmon estimates with respect to (s, t) (see, for instance, [9, Section 2.2]). By taking derivatives of the eigenvalue equation, we finally get the following corollary.

Corollary 4.10. *Let C_0 and $k, l, d \in \mathbb{N}$. There exist $\varepsilon_0 > 0, C > 0$, and $c_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all eigenpairs $(\varkappa, \psi)$ of the operator $\mathscr{L}_\varepsilon$ with $\varkappa \leq \mathrm{S}_0 + C_0 \varepsilon$, we have*

$$\|t^k s^l \psi\| \leq C \|\psi\|, \quad \mathcal{Q}_\varepsilon(t^k s^l \psi) \leq C \|\psi\|^2,$$

$$\|(D_t)^d s^l t^k \psi\| \leq C \|\psi\|, \quad \|(\varepsilon D_s)^d s^l t^k \psi)\| \leq C \|\psi\| .$$

4.2.3. *Coherent states.* Following a classical formalism, see [4, 10] for instance, we introduce the annihilation operator a and the creation operator a^*

$$a = \frac{1}{\sqrt{2}} \left(\mathfrak{s} + \partial_{\mathfrak{s}} \right), \quad a^* = \frac{1}{\sqrt{2}} \left(\mathfrak{s} - \partial_{\mathfrak{s}} \right) .$$

We have the following identities

$$(4.28) \qquad \mathfrak{s} = \frac{a + a^*}{\sqrt{2}}, \quad \partial_{\mathfrak{s}} = \frac{a - a^*}{\sqrt{2}}, \quad \text{and} \quad [a, a^*] = 1.$$

Setting $g_0(\mathfrak{s}) = \pi^{-1/4} e^{-\mathfrak{s}^2/2}$, we introduce the coherent states for any $u \in \mathbb{R}$ and $p \in \mathbb{R}$,

$$f_{u,p}(\mathfrak{s}) = e^{ip\mathfrak{s}} g_0(\mathfrak{s} - u) ,$$

and for all $\psi \in L^2(\mathbb{R})$, the associated projection defined by

$$\Pi_{u,p} \psi = \langle \psi, f_{u,p} \rangle_{L^2(\mathbb{R}, d\mathfrak{s})} f_{u,p} .$$

We have the resolution of the identity

$$\psi = \int_{\mathbb{R}^2} \Pi_{u,p} \psi \, \mathrm{d}u \mathrm{d}p ,$$

and the Parseval formula

$$\|\psi\|^2 = \int_{\mathbb{R}} \int_{\mathbb{R}^2} |\Pi_{u,p} \psi|^2 \, \mathrm{d}u \mathrm{d}p \, \mathrm{d}\mathfrak{s} .$$

Lemma 4.11 ([4])**.** *For all $n, m \in \mathbb{N}$, we have*

$$(a)^m (a^*)^n = \int_{\mathbb{R}^2} \left(\frac{u + ip}{\sqrt{2}} \right)^m \left(\frac{u - ip}{\sqrt{2}} \right)^n \Pi_{u,p} \, \mathrm{d}u \mathrm{d}p.$$

Now we have all tools at hand to end the proof of Theorem 1.10. Taking Proposition 4.7 into account, it remains to prove the following lemma:

Lemma 4.12. *Consider an eigenfunction Ψ_ε of $\mathscr{X}_\varepsilon$ associated with an eigenvalue $\varkappa(\varepsilon) \leq \mathrm{S}_0 + C_0 \varepsilon$. Then there exists C only depending on C_0 such that*

$$(4.29) \qquad \varkappa(\varepsilon) \|\Psi_\varepsilon\|^2 = \langle \mathscr{X}_\varepsilon \Psi_\varepsilon, \Psi_\varepsilon \rangle \geq (\mathrm{S}_0 - C\varepsilon) \|\Psi_\varepsilon\|^2.$$

Proof. In view of (4.19), it suffices to prove (4.29) for an eigenpair $(\varkappa(\varepsilon), \psi_\varepsilon)$ of the operator $\mathscr{L}_\varepsilon$. By the scaling $s = \varepsilon^{\frac{1}{2}} \mathfrak{s}$, $\mathscr{L}_\varepsilon$ becomes the unitarily equivalent operator

$$\widetilde{\mathfrak{L}}_\varepsilon = D_{\mathfrak{t}}^2 + \left(\varepsilon^{1/2} D_{\mathfrak{s}} + \varepsilon \mathfrak{s}^2 \mathfrak{t} - \frac{\mathfrak{t}^3}{3} \right)^2 ,$$

which we expand as

$$\widetilde{\mathfrak{L}}_\varepsilon = \widetilde{\mathfrak{L}}_0 + \varepsilon^{1/2} \widetilde{\mathfrak{L}}_1 + \varepsilon \widetilde{\mathfrak{L}}_2 + \varepsilon^{3/2} \widetilde{\mathfrak{L}}_3 + \varepsilon^2 \widetilde{\mathfrak{L}}_4,$$

where

$$\widetilde{\mathfrak{L}}_0 = D_{\mathfrak{t}}^2 + \frac{\mathfrak{t}^6}{9}, \quad \widetilde{\mathfrak{L}}_1 = -\frac{2\mathfrak{t}^3}{3} D_{\mathfrak{s}}, \quad \widetilde{\mathfrak{L}}_2 = -\frac{2}{3} \mathfrak{t}^4 \mathfrak{s}^2 + D_{\mathfrak{s}}^2, \quad \widetilde{\mathfrak{L}}_3 = \mathfrak{t} \left(\mathfrak{s}^2 D_{\mathfrak{s}} + D_{\mathfrak{s}} \mathfrak{s}^2 \right), \quad \widetilde{\mathfrak{L}}_4 = \mathfrak{s}^4 \mathfrak{t}^2.$$

We use (4.28) and we commute a and a^* to put all the a^* on the right. With Lemma 4.11, we get that there exist $(\gamma, \delta) \in \mathbb{R}^2$, a homogeneous polynomial of order 1, $L(X, Y)$, and a non-commutative homogeneous polynomial of order 2, $P(X, Y)$, such that

$$(4.30) \qquad \widetilde{\mathfrak{L}}_\varepsilon = \mathfrak{L}_\varepsilon^{\mathsf{W}} + \varepsilon \mathfrak{L}_{2,\mathsf{rem}} + \varepsilon^{3/2} \mathfrak{L}_{3,\mathsf{rem}} + \varepsilon^2 \mathfrak{L}_{4,\mathsf{rem}} ,$$

where

$$\mathfrak{L}_\varepsilon^{\mathsf{W}} = \int_{\mathbb{R}^2} \mathsf{X}_{u\sqrt{\varepsilon},\, p\sqrt{\varepsilon}} \, \Pi_{u,p} \, \mathrm{d}u \mathrm{d}p ,$$

and

$$\mathfrak{L}_{2,\mathsf{rem}} = \gamma \mathfrak{t}^4 + \delta, \quad \mathfrak{L}_{3,\mathsf{rem}} = \mathfrak{t} L(a, a^*), \quad \mathfrak{L}_{4,\mathsf{rem}} = \mathfrak{t}^2 P(a, a^*) .$$

Consider an eigenfunction $\widetilde{\psi}_\varepsilon$ of $\widetilde{\mathfrak{L}}_\varepsilon$ associated with an eigenvalue $\varkappa(\varepsilon) \leq \mathrm{S}_0 + C_0 \varepsilon$. From (4.30) and using the quadratic forms $\widetilde{\mathfrak{Q}}_\varepsilon$ of $\widetilde{\mathfrak{L}}_\varepsilon$ and $\mathsf{Q}_{\alpha,\xi}$ of $\mathsf{X}_{\alpha,\xi}$, we get

$$(4.31) \quad \widetilde{\mathfrak{Q}}_\varepsilon(\widetilde{\psi}_\varepsilon) \geq \int_{\mathbb{R}} \int_{\mathbb{R}^2} \mathsf{Q}_{u\sqrt{\varepsilon},\, p\sqrt{\varepsilon}}(\Pi_{u,p} \widetilde{\psi}_\varepsilon) \, \mathrm{d}u \mathrm{d}p \, \mathrm{d}\mathfrak{s}$$
$$- \varepsilon \|(\gamma \mathfrak{t}^4 + \delta) \widetilde{\psi}_\varepsilon\| \, \|\widetilde{\psi}_\varepsilon\| - \varepsilon^{3/2} \|L(a, a^*) \widetilde{\psi}_\varepsilon\| \, \|\mathfrak{t}\widetilde{\psi}_\varepsilon\| - \varepsilon^2 \|P(a, a^*) \widetilde{\psi}_\varepsilon\| \, \|\mathfrak{t}^2 \widetilde{\psi}_\varepsilon\|.$$

Since S_0 is a minimum for the band function of $\mathsf{X}_{\alpha,\xi}$, we have, for each $(u, p) \in \mathbb{R}^2$ and $\mathfrak{s} \in \mathbb{R}$, the bound from below

$$\mathsf{Q}_{u\sqrt{\varepsilon},\, p\sqrt{\varepsilon}}(\Pi_{u,p} \widetilde{\psi}_\varepsilon)(\mathfrak{s}, \cdot) \geq \mathrm{S}_0 \|\Pi_{u,p} \widetilde{\psi}_\varepsilon(\mathfrak{s}, \cdot)\|^2_{L^2(\mathbb{R}, \mathrm{d}\mathfrak{t})} .$$

Thus by Parseval formula

$$(4.32) \qquad \int_{\mathbb{R}} \int_{\mathbb{R}^2} \mathsf{Q}_{u\sqrt{\varepsilon},\, p\sqrt{\varepsilon}}(\Pi_{u,p} \widetilde{\psi}_\varepsilon) \, \mathrm{d}u \mathrm{d}p \, \mathrm{d}\mathfrak{s} \geq \mathrm{S}_0 \|\widetilde{\psi}_\varepsilon\|^2_{L^2(\mathbb{R}^2, \mathrm{d}\mathfrak{s}\mathrm{d}\mathfrak{t})}.$$

Let us estimate the remainder terms in (4.31). The homogeneity of the polynomials P and L yields

$$\varepsilon^{3/2} L(a, a^*) \widetilde{\psi}_\varepsilon = \varepsilon L(\varepsilon^{1/2} a, \varepsilon^{1/2} a^*) \widetilde{\psi}_\varepsilon \quad \text{and} \quad \varepsilon^2 P(a, a^*) \widetilde{\psi}_\varepsilon = \varepsilon P(\varepsilon^{1/2} a, \varepsilon^{1/2} a^*) \widetilde{\psi}_\varepsilon.$$

Thus, using the scaling $s = \varepsilon^{1/2}\mathfrak{s}$ and Corollary 4.10, we find

$$\varepsilon\|(\gamma\mathfrak{t}^4 + \delta)\widetilde{\psi}_\varepsilon\|\,\|\widetilde{\psi}_\varepsilon\| + \varepsilon^{3/2}\|L(a, a^*)\widetilde{\psi}_\varepsilon\|\,\|\mathfrak{t}\widetilde{\psi}_\varepsilon\| + \varepsilon^2\|P(a, a^*)\widetilde{\psi}_\varepsilon\|\,\|\mathfrak{t}^2\widetilde{\psi}_\varepsilon\| \leq C\varepsilon\|\widetilde{\psi}_\varepsilon\|^2.$$

From this, combined with (4.31) and (4.32), follows that

$$\varkappa(\varepsilon)\|\widetilde{\psi}_\varepsilon\|^2 = \widetilde{\mathfrak{Q}}_\varepsilon(\widetilde{\psi}_\varepsilon) \geq (\mathrm{S}_0 - C\varepsilon)\|\widetilde{\psi}_\varepsilon\|^2.$$

Coming back to the operator $\mathscr{L}_\varepsilon$ and finally to the operator $\mathscr{X}_\varepsilon$, this ends the proof of Lemma 4.12, and thus of Theorem 1.10. □

4.2.4. *Computation of the ground states of $\mathscr{X}_\varepsilon$.* In the last section of this paper, we present computations of the first eigenpair of the operator $\mathscr{X}_\varepsilon$ for a decreasing sequence of values of ε. Let us agree that

$$\varepsilon_\ell = 2^{-1-\ell/2}, \quad \ell \geq 0$$

so that $\varepsilon_0 = 1/2$, $\varepsilon_1 = \sqrt{2}/4$, $\varepsilon_2 = 1/4$, etc... We have computed the first eigenpair of $\mathscr{X}_\varepsilon$ for $\varepsilon = \varepsilon_\ell$, $\ell \in \{0, 1, \ldots, 12\}$ with natural boundary conditions on the domain $[-a_\ell, a_\ell] \times [-8, 8]$ (with $a_\ell = 4/\varepsilon_\ell = 8 \cdot 2^{\ell/2}$) by a finite element discretization on a uniform rectangular grid of 48×6 elements of partial degree 10 in each variable.

Theorem 1.10 yields the convergence of $\varkappa_1(\varepsilon)$ to S_0 at a rate of $\mathcal{O}(\varepsilon)$. Our computations (see Figures 1 and 2) suggest that $\mathrm{S}_0 \simeq 0.4941$. In Figure 3 we plot the difference $\varkappa_1(\varepsilon_\ell) - 0.4941$ versus ε_ℓ (here we use a $\log - \log$ scale in base 2).
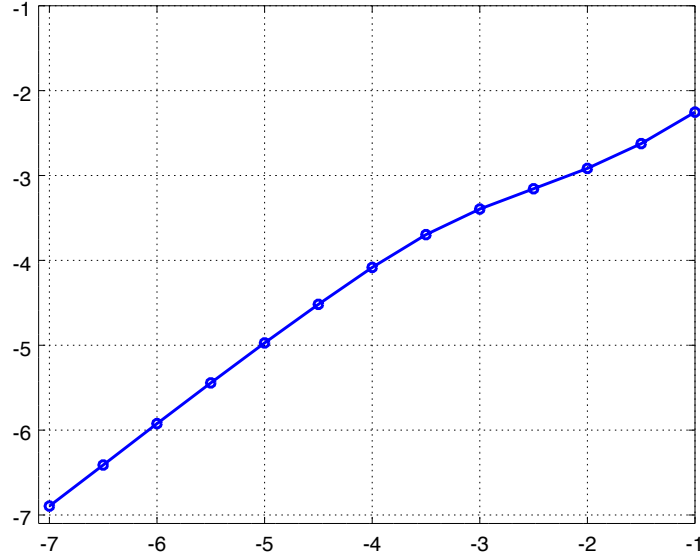

FIGURE 3. Plot of $\log_2(\varkappa_1(\varepsilon_\ell) - 0.495)$ versus $\log_2(\varepsilon_\ell)$.

Figure 4 gives the modulus of the first eigenvector of the operator $\mathscr{X}_\varepsilon$ and a numerical value of the eigenvalue $\varkappa_1(\varepsilon)$, for $\varepsilon = \varepsilon_\ell$ with $\ell = 0, \ldots, 5$. For $\varepsilon = \varepsilon_\ell$, the horizontal scale (variable σ) is $[-a'_\ell, a'_\ell]$ with $a'_\ell = 5 \cdot 2^{\ell/2}$ and the vertical scale (variable τ) is always $[-5, 5]$. The proportion between the two scales is kept, which makes the plot of the vertical scale shrink as ℓ increases.

In Figure 5, we zoom some region around the point α_0/ε of the plot of the first eigenvector of $\mathscr{X}_\varepsilon$ when $\varepsilon = \varepsilon_\ell$ with $\ell = 3, \ldots, 10$. When $\ell = 6, \ldots, 10$, we have computed the eigenvector

$\varkappa_1 = 0.7039 \ (\ell = 0)$ $\varkappa_1 = 0.6563 \ (\ell = 1)$ $\varkappa_1 = 0.6266 \ (\ell = 2)$

$\varkappa_1 = 0.6063 \ (\ell = 3)$ $\varkappa_1 = 0.5892 \ (\ell = 4)$ $\varkappa_1 = 0.5712 \ (\ell = 5)$
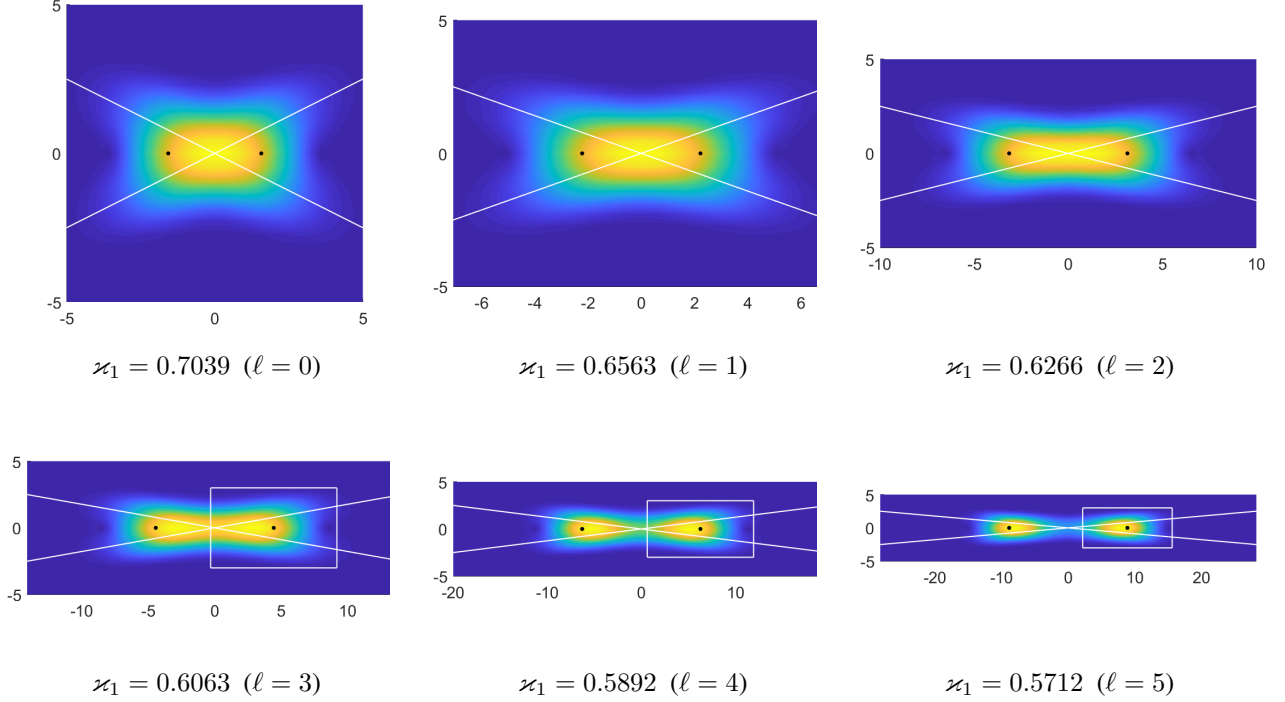
FIGURE 4. Modulus of the first eigenvector of $\mathscr{X}_{\varepsilon_\ell}$ for $\ell = 0, \ldots, 5$. The white cross represents the two lines of cancellation of the magnetic field and the black dots mark the two points $(\pm\alpha_0/\varepsilon_\ell, 0)$. The rectangular boxes in the second row indicate the zoom region displayed in Figure 5.

on the smaller region $[-\frac{a_\ell}{2}, \frac{a_\ell}{2}] \times [-4, 4]$ on a uniform rectangular grid of 48×6 elements of partial degree 10 in each variable. We represent the modulus of the eigenvector on the region

$$\left[\frac{\alpha_0}{\varepsilon_\ell} - \frac{2}{\sqrt{\varepsilon_\ell}}, \ \frac{\alpha_0}{\varepsilon_\ell} + \frac{2}{\sqrt{\varepsilon_\ell}} \right] \times [-3, 3].$$

This choice is driven by the structure of the first term $(\mathfrak{s}, \mathfrak{t}) \mapsto \psi_0(\mathfrak{s}, \mathfrak{t}) = f_0(\mathfrak{s})u_0(\mathfrak{t})$ (4.22) of a possible eigenvector asymptotics for the operator $\mathfrak{L}_\varepsilon$ (4.21). We note that with $\xi_0 = 0$, we do not need any gauge transform to go from $\mathscr{L}_\varepsilon$ to $\mathfrak{L}_\varepsilon$. Thus, using the change of variables (4.20), we find that

$$\psi_0(\mathfrak{s}, \mathfrak{t}) = f_0\left(\frac{s - \alpha_0}{\sqrt{\varepsilon}} \right) u_0(\mathfrak{t}).$$

Going back to the "physical variables" σ, τ with (4.18), we find

$$\psi_0(\mathfrak{s}, \mathfrak{t}) = f_0\left(\frac{\varepsilon\sigma - \alpha_0}{\sqrt{\varepsilon}} \right) u_0(\tau).$$

Taking $\sigma \in \left[\frac{\alpha_0}{\varepsilon} - \frac{2}{\sqrt{\varepsilon}}, \ \frac{\alpha_0}{\varepsilon} + \frac{2}{\sqrt{\varepsilon}} \right]$, we see that $\frac{\varepsilon\sigma - \alpha_0}{\sqrt{\varepsilon}}$ spans the fixed interval $[-2, 2]$. Hence we expect that the zoom can yield the image of a convergence as $\varepsilon \to 0$. Indeed, this is exactly what we can detect from Figure 5.

Thus, in the small angle limit $\varepsilon \to 0$, there are two centers of localization that are spread at the scale $\varepsilon^{-1/2}$ and go away at the order $\frac{1}{\varepsilon}$. The "area of localization" of the first eigenfunctions goes to infinity. This fact is quite understandable. Considering that the limit $\varepsilon = 0$ is singular (the two lines of cancellation become identical and we recover the Montgomery operator $\mathscr{M}^{[2]}$ whose
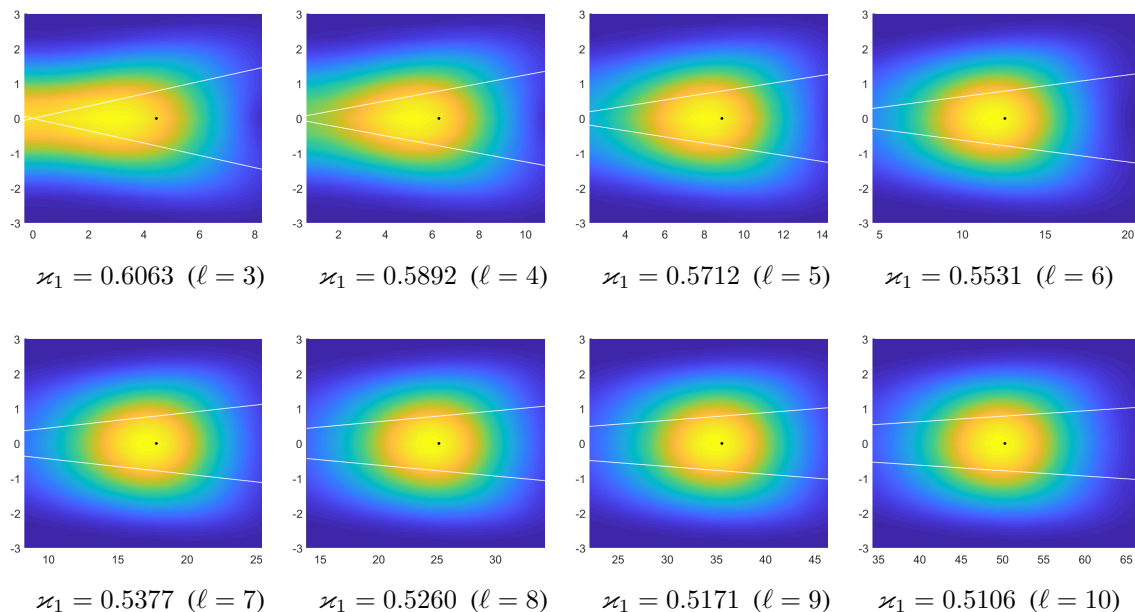
$\varkappa_1 = 0.6063 \ (\ell = 3)$ $\varkappa_1 = 0.5892 \ (\ell = 4)$ $\varkappa_1 = 0.5712 \ (\ell = 5)$ $\varkappa_1 = 0.5531 \ (\ell = 6)$

$\varkappa_1 = 0.5377 \ (\ell = 7)$ $\varkappa_1 = 0.5260 \ (\ell = 8)$ $\varkappa_1 = 0.5171 \ (\ell = 9)$ $\varkappa_1 = 0.5106 \ (\ell = 10)$

FIGURE 5. Modulus of the first eigenvector of $\mathscr{X}_{\varepsilon_\ell}$ for $\ell = 3, \ldots, 10$. The white cross represents the two lines of cancellation of the magnetic field and the black dot marks the point $(\alpha_0/\varepsilon_\ell, 0)$.

essential spectrum is non-empty); it is rather natural to observe a "loss of mass at infinity", characteristic of Weyl sequences.

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