NEUMANN AND MIXED PROBLEMS ON CURVILINEAR POLYHEDRA

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We prove regularity results in L^p Sobolev spaces. On one hand, we state some abstract results by L^p functional techniques : exponentially decreasing estimates in dyadic partitions of cones and dihedra, operator valued symbols and Marcinkievicz's theorem. On the other hand, we derive more concrete statements with the help of estimates about the first non-zero eigenvalue of some Laplace-Beltrami operators on spherical domains.

1. INTRODUCTION

It is well known [1] that elliptic boundary value problems in a smooth domain Ω are regular in the class of L^p Sobolev spaces. For instance, if u is a strong solution of the Neumann problem with $\Delta u \in W^{k,p}(\Omega)$, then u belongs to $W^{k+2,p}(\Omega)$. Here $k \in \mathbb{N}$. Such properties no longer hold in general in non-smooth domains : for instance for $p = 2$, $k = 0$, the regularity property fails for non-convex polygons. But non-smooth domains, and especially piecewise smooth domains, are very important in numerous applications to problems arising from physics. Moreover, one also may have to consider boundary conditions which are discontinuous on the boundary : this is what is called mixed conditions.

The L^p regularity of elliptic boundary value problems on non-smooth domains has been studied by several authors. Let us quote Fichera [16], Grisvard [17], [18], Maz'ya and Plamenevskii [25], [24], Maz'ya and Roßmann [26]. But all of them use weighted spaces techniques. In particular, this does not provide complete results for the Neumann problem associated with the Laplace operator. There exists more literature in Hilbert Sobolev spaces. Let us quote Kondrat'ev [20], Rempel and Schulze [27], Schulze [28] and the author [11]. The extension from $p = 2$ to $p \neq 2$ is in no way straightforward, because of the constant use of Mellin transform on cones and partial Fourier transform on edges. Such an extension gives rise to interesting and difficult techniques : study of operator kernels in [1] and [18], exponentially decreasing estimates in dyadic partitions of cones and dihedra in [25], [24].

Our method is inspired by [25], [24]. We have performed some simplifications and introduced improvements in order to obtain results in non-weighted spaces for a *variational* solution in $H^1(\Omega)$, which seems impossible by a strict application of the above references. This leads to an "abstract" result, as stated in the following Theorem 1.1. In this statement, the regularity is linked with the location of some spectra. Estimates from below for the first non-zero element of these spectra leads to explicit bounds for the couples (k, p) so that the regularity property

$$\Delta u \in W^{k,p}(\Omega) \Longrightarrow u \in W^{k+2,p}(\Omega)$$

holds for any solution of the mixed problem. The above mentioned spectra are only determined by the geometry of the domain and of the boundary conditions.

Here is what we obtain in a simple situation. Let Ω be a polyhedron in $\mathbb{R}^3$. We denote by $\mathfrak{E}$ the set of its edge points and by $\mathfrak{S}$ the set of its vertices. To each $x \in \mathfrak{S}$ and to each $x \in \mathfrak{E}$, we associate some spectrum $\sigma(x)$. This spectrum is determined (except possibly for its integer elements) by the set of the eigenvalues of some Laplace-Beltrami operator. Let $\lambda_1(x)$ be the least > 0 element of $\sigma(x)$.

Theorem 1.1 *Let $k \in \{-1, 0, 1, \ldots\}$. Let u be the variational solution in $H^1(\Omega)$ of*

$$\forall v \in H^1(\Omega) \quad \int_\Omega \nabla u \nabla \overline{v} \;= < f, v >$$

with $f \in W^{k,p}(\Omega)$. Then $u \in W^{k+2,p}(\Omega)$ if both following conditions are fulfilled
(i) $\forall x \in \mathfrak{E} \quad k + 2 - 2/p < \lambda_1(x)$,
(ii) $\forall x \in \mathfrak{S} \quad k + 2 - 3/p < \lambda_1(x)$.

When x belongs to an edge, $\lambda_1(x)$ is well known : it is equal to π/ω where ω is the opening of the edge. When x is a vertex, the exact value of $\lambda_1(x)$ is not known in general. For Dirichlet boundary conditions, minorizations using the monotonicity with respect to the domain are possible and yield information. For Neumann conditions, minorizations are more difficult to obtain since the monotonicity principle no longer holds : see [14] for instance. We have obtained that when Ω is convex

$$\lambda_1(x) \geq \frac{\sqrt{5} - 1}{2} \, .$$

Here is the outline of the paper.

In section 2, we introduce our class of domains : this is a class of piecewise C^ρ domains. We also introduce our class of operators : we are going to study second order operators defined by a symmetric integrodifferential form with real coefficients with limited regularity ($C^{\rho-1}$ for the principal part, where ρ is the same parameter as previously).

In section 3, we introduce the characteristic spectra at the vertices and edges of the domain and we state our main results, the abstract one (Theorem 3.2) and several more concrete ones using estimates about the "first eigenvalue" λ_1.

In section 4, we study this "first eigenvalue" and prove various minorizations which allow to deduce concrete statements from Theorem 3.2.

We give the proof of Theorem 3.2 in the remaining sections. By perturbation arguments we show in section 5 how to reduce to the case of constant coefficients. We prove regularity results for the Laplacian in cones (section 6) and wedges (sections 8 – 11). Following the idea of [25, 24] we prove exponentially decreasing estimates : see in section 7 Lemma 7.1 where we state such estimates in an "abstract" framework and Lemma 7.2 which gives the use of such estimates. For wedges, such estimates are applied to operator valued symbols on the edge.

A complete study of boundary value problems on polyhedral domains would require either the introduction of spaces with asymptotics like in [27, 28] or a more direct description of the variational solution with the help of a splitting into regular and singular parts near the edges and corners. If the faces of the polyhedron are flat and if the operator is the Laplacian, such a description is given in [11]. But when the edges are curved, the situation is much more complicated as can be seen in [26] and in the joint work with M. Costabel [8, 7].

Part of the results of this paper have been announced in the two notes [12, 13]. In [4], we apply these results to the problem of coerciveness of the **curl** form on divergence free vector functions with mixed conditions on the boundary and to the study of vector potentials for the three-dimensional Stokes problem. Discussions about this problem were the starting point of the present work.

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2. OPERATORS AND DOMAINS

The regularity of the coefficients of the operator and of the faces of the domain are limited by a parameter $\rho \geq 1$. The domain Ω is a polyhedral domain in the class $\mathcal{Q}_\rho$ we are going to define below. The boundary value problem we consider is determined by an integrodifferential form $\mathfrak{a}$ and a partition of the boundary $\partial\Omega$. Let $a_{i,j} \in C^{\rho-1}(\overline{\Omega})$ be real functions such that $a_{i,j} = a_{j,i}$ and satisfying the strong ellipticity condition

$$\exists c > 0,\ \forall x \in \overline{\Omega},\ \forall \xi \in \mathbb{R}^3, \quad \sum_{i,j} a_{i,j}(x)\, \xi_i \xi_j \geq c|\xi|^2. \tag{2.1}$$

Let $a_0 \in C^{(\rho-2)_+}(\overline{\Omega})$ be another real function, where

Notation 2.1 *For any* $\tau \in \mathbb{R}$, τ_+ *denotes* $\max(\tau, 0)$.

Our integrodifferential form is defined by

$$\mathfrak{a}(u, v) = \int_\Omega \sum_{i,j} a_{i,j}\, \partial_i u\, \partial_j \overline{v} + a_0\, u\, \overline{v}\, dx\,. \tag{2.2}$$

The form $\mathfrak{a}$ is coercive on $H^1(\Omega)$. In every point $x \in \overline{\Omega}$, the quadratic form $\sum_{i,j} a_{i,j}(x)\, \xi_i \xi_j$ is diagonalizable, i. e. there exists an invertible matrix $\mathfrak{M}_x = (\mathfrak{m}_{i,j}(x))$ such that

$$\sum_{i,j} a_{i,j}(x)\, \xi_i \xi_j = \sum_i \left(\sum_j \mathfrak{m}_{i,j}(x) \xi_j \right)^2 . \tag{2.3}$$

For each $x \in \partial\Omega$, this matrix $\mathfrak{M}_x$ allows to introduce some geometric object attached to x. This object is a polyhedral cone Ξ_x in $\mathbb{R}^3$, i. e. a positive homogeneous cone with plane faces.

Definition 2.2 *We say that $\Omega \in \mathcal{Q}_\rho$ if Ω is a bounded domain in $\mathbb{R}^3$ satisfying that for any $x \in \partial\Omega$ there exists a C^ρ local map χ_x in a neighborhood of x and a polyhedral cone Ξ such that χ_x maps a neighborhood of x in $\overline{\Omega}$ onto a neighborhood of 0 in $\overline{\Xi}$. Let us suppose, which is not a restriction, that $D\chi_x(x)$ is the identity map in $\mathbb{R}^3$. We set*

$$\Xi_x = {}^t\mathfrak{M}_x^{-1}\Xi\,.$$

The reason for the introduction of Ξ_x is the following : the change of variables $X \mapsto {}^t\mathfrak{M}_x^{-1}X$ transforms Ξ into Ξ_x and the form $\mathfrak{a}$ into another form $\mathfrak{a}_x$ with principal coefficients $a^x_{i,j}$ satisfying

$$a^x_{i,j}(0) = \delta_{i,j}.$$

So, in x, the principal part of $\mathfrak{a}$ becomes the standard gradient form.

Definition 2.3 *Let $\Omega \in \mathcal{Q}_\rho$ and let $x \in \partial\Omega$.*
(i) If Ξ_x is a half-space, we say that x belongs to a "face".
(ii) If Ξ_x is not a half-space but can be written in the form of a product $\mathbb{R} \times \Gamma_x$ where Γ_x is a plane sector, we say that x belongs to an "edge" ; ω_x denotes the opening of Γ_x and $G_x :=]0, \omega_x[$.
(iii) If Ξ_x cannot be written in the form of a product $\mathbb{R} \times \Gamma$, we say that x is a "vertex" ; we also denote $\Gamma_x := \Xi_x$ and $G_x := \Xi_x \cap S^2$.
We denote by $\mathfrak{S}$ the set of the vertices and by $\mathfrak{E}$ the closure of the set of the edge points.

We now introduce a partition of the boundary of Ω : we fix two open sets $\partial_D\Omega$ and $\partial_N\Omega$ in $\partial\Omega$ such that

$$\partial\Omega = \overline{\partial_D\Omega} \cup \overline{\partial_N\Omega} \quad \text{and} \quad \partial_D\Omega \cap \partial_N\Omega = \emptyset\,. \tag{2.4}$$

Such a partition is usually a partition of the faces of Ω. But it is also possible to divide each face itself. We then introduce new "edges" or "vertices" on the border lines. Generally speaking, all points in $\overline{\partial_D\Omega} \cap \overline{\partial_N\Omega}$ have to be considered as singular points.

We define our boundary value problem in variational form. We introduce the variational space

$$V = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial_D\Omega\}.$$

The boundary value problem we consider is defined by

$$\begin{array}{ccl} V & \rightarrow & V' \\ u & \mapsto & f = \{v \mapsto \mathfrak{a}(u, v)\}. \end{array} \tag{2.5}$$

If $f \in L^p(\Omega)$ and is identified with $v \mapsto \int_\Omega f\overline{v}$, then problem (2.5) can be interpreted in a classical way (ν_j is the j-th component of the outward normal to $\partial\Omega$)

$$\left\{ \begin{array}{lll} \sum\limits_{i,j} \partial_j a_{i,j}\, \partial_i u = f & \text{on } \Omega, & \\ u = 0 & \text{on } \partial_D\Omega & \text{(Dirichlet condition)}, \\ \sum\limits_{i,j} \nu_j a_{i,j}\, \partial_i u = 0 & \text{on } \partial_N\Omega & \text{(Neumann condition)}. \end{array} \right.$$

3. MAIN RESULTS

We assume that $\Omega \in \mathcal{Q}_\rho$. We are interested in the regularity of a solution u of (2.5) when f belongs to some L^p Sobolev space. For any $k \in \mathbb{N}$ and any $p \in]1, +\infty[$, $W^{k,p}(\Omega)$ denotes the usual space of functions f such that

$$\forall \alpha,\ |\alpha| \le k, \quad D^\alpha f \in L^p(\Omega).$$

We also introduce the space $W^{-1,p}(\Omega)$ as the dual space of

$$\{v \in W^{1,q}(\Omega) \mid v = 0 \text{ on } \partial_D \Omega\},$$

where q denotes the conjugate of p : $\frac{1}{p} + \frac{1}{q} = 1$.

We assume conditions on (k,p) so that $W^{k,p}(\Omega) \subset V'$. We also want to avoid some limit cases in Sobolev imbedding theorems. Thus, we suppose

$$\begin{cases} p > 2, & p \neq 3 & \text{if } k = -1 \\ p \ge \frac{6}{5}, & p \neq 2 & \text{if } k = 0 \\ p > 1, & p \notin \{2,3\} & \text{if } k \ge 1. \end{cases} \tag{3.1}$$

The abstract form of our statement relies on the notion of "injectivity modulo polynomials" we introduced in §3.C and §23.B of [11] and which we recall hereafter. Let x be a singular point of the boundary of Ω. To x we have associated a cone Γ_x. We denote by $\partial_D \Gamma_x$ the faces of Γ_x which correspond to $\partial_D \Omega$ via the diffeomorphism χ_x and the linear transformation ${}^t\mathfrak{M}_x^{-1}$.

Definition 3.1 *The "spectrum" $\sigma(x)$ is the complement in $\mathbb{C}$ of the set of the λ such that the following condition $\mathcal{C}^\lambda(\Gamma_x, \partial_D \Gamma_x)$ of injectivity modulo polynomials holds. Denoting by (r, θ) the polar coordinates, we set*

$$S^\lambda(\Gamma, \partial_D \Gamma) = \{u = \sum_{\substack{q \ge 0 \\ \text{finite}}} r^\lambda \log^q r\, u_q(\theta) \mid u_q \in H^1(G) \text{ and } u_q = 0 \text{ on } \partial_D G\}.$$

Then the condition is

$$\mathcal{C}^\lambda(\Gamma_x, \partial_D \Gamma_x) \begin{cases} u \in S^\lambda(\Gamma_x, \partial_D \Gamma_x) \text{ and } \Delta u = f \text{ with } f \text{ polynomial in cartesian variables} \\ \text{implies} \\ u \text{ polynomial in cartesian variables.} \end{cases}$$

Here $\Delta u = f$ means that $\forall v \in H^1(\Gamma)$ such that $v = 0$ on $\partial_D \Gamma$ and with compact support, we have $\int_\Omega \nabla u \nabla \overline{v} = < f, v >$.

Our basic result is the following

Theorem 3.2 *We assume that $\rho \ge k+2$. Let u be a solution of problem (2.5) with $f \in W^{k,p}(\Omega)$. Then $u \in W^{k+2,p}(\Omega)$ if both following conditions are fulfilled :*
(i) $\forall x \in \mathfrak{E} \quad \forall \lambda,\ 0 \le \operatorname{Re} \lambda \le k + 2 - 2/p, \quad \lambda \notin \sigma(x)$
(ii) $\forall x \in \mathfrak{S} \quad \forall \lambda,\ -1/2 \le \operatorname{Re} \lambda \le k + 2 - 3/p, \quad \lambda \notin \sigma(x)$.

The conjunction of conditions *(i)* and *(ii)* is indeed a necessary and sufficient condition to have the regularity of solutions. Though such a statement is optimal in a certain sense, it is not informative enough. That is why we investigate now conditions *(i)* and *(ii)*.

With the help of [17], [11], it is possible to determine completely the spectrum $\sigma(x)$ for any point in an edge. For Dirichlet or Neumann problems

$$\sigma(x) = \{\frac{l\pi}{\omega_x} \mid l \in \mathbb{Z} \setminus \{0\}\}$$

except when $\omega_x = 2\pi$, where $\sigma(x)$ coincides with the previous set from which one removes the positive integers. For the mixed Dirichlet-Neumann problem

$$\sigma(x) = \{\frac{(l+1/2)\pi}{\omega_x} \mid l \in \mathbb{Z}\}.$$

Notation 3.3

$$\lambda_1(x) = \begin{cases} \pi/\omega_x & \textit{for Dirichlet or Neumann conditions on } \Gamma_x \\ \pi/2\omega_x & \textit{for mixed conditions on } \Gamma_x\,. \end{cases}$$

Then condition *(i)* holds if $k + 2 - 2/p < \lambda_1(x)$ for any $x \in \mathfrak{E}$.

Let us now study the case of the vertices.

Notation 3.4 *For a domain G in the sphere S^2 and a subset $\partial_D G$ of its boundary, we denote by $\mu_1(G, \partial_D G)$ the first non-zero eigenvalue of the positive Laplace-Beltrami operator L on the space $H^1(G, \partial_D G) := \{v \in H^1(G) \mid v = 0$ on $\partial_D G\}$.*

We can prove

Lemma 3.5 *Let $x \in \mathfrak{S}$. If λ has its real part < 2 and belongs to $\sigma(x)$ then $\lambda(\lambda+1)$ is an eigenvalue of the Laplace-Beltrami operator L on the space $H^1(G_x, \partial_D G_x)$.*

This lemma is a consequence of the more general results we proved in [11], §4. We only give here two arguments which help to understand the link between the injectivity modulo polynomials and the eigenvalues of L : let us consider $u \in S^\lambda(\Gamma_x, \partial_D\Gamma_x)$ such that $\Delta u = f$ with f polynomial ; firstly, if $\operatorname{Re}\lambda < 2$ then $f = 0$; secondly,

$$\Delta(r^\lambda u_0(\theta)) = r^{\lambda-2}\Big(\lambda(\lambda+1) - L\Big)u_0(\theta)$$

and u_0 belongs to the domain $H^1(G_x, \partial_D G_x)$ of L. Indeed there holds :

$$\textit{If } \lambda \notin \mathbb{N}, \quad \mathcal{C}^\lambda(\Gamma, \partial_D\Gamma) \textit{ does not hold } \iff \lambda(\lambda+1) \textit{ is an eigenvalue of } L \textit{ on } H^1(G, \partial_D G). \tag{3.2}$$

If $\lambda(\lambda+1)$ is an eigenvalue μ of L, since we are only interested in the λ such that $\operatorname{Re}\lambda > -1/2$, we have

$$\lambda = -\frac{1}{2} + \sqrt{\mu + \frac{1}{4}}\,.$$

Then the least value of λ corresponds to the least value of μ. For Dirichlet or mixed problems, the least value of μ is > 0 and it is equal to $\mu_1(G_x, \partial_D G_x)$ according to Notation 3.4. For the Neumann problem, the least possible value of μ is 0. But we have proved in [11], §23.C that the condition of injectivity modulo polynomials always holds in $\lambda = 0$. Thus, the important value is again $\mu_1(G_x, \partial_D G_x)$.

Notation 3.6 *We set for $x \in \mathfrak{S}$*

$$\lambda_1(x) = -\frac{1}{2} + \sqrt{\mu_1(G_x, \partial_D G_x) + \frac{1}{4}} \quad \text{and} \quad \tilde{\lambda}_1(x) = \min(\lambda_1, 2)\,.$$

As a consequence of Theorem 3.2 and using Notations 3.3 and 3.6 we get

Corollary 3.7 *Let u be a solution of problem (2.5) with $f \in W^{k,p}(\Omega)$. Then $u \in W^{k+2,p}(\Omega)$ if both following conditions are fulfilled :*
(i) $\forall x \in \mathfrak{E}, \quad k+2-2/p < \lambda_1(x)$,
(ii) $\forall x \in \mathfrak{S}, \quad k+2-3/p < \tilde{\lambda}_1(x)$.

Remark 3.8 It is possible to prove [9], that for the Neumann problem one can take $\min(\lambda_1, 3)$ as $\tilde{\lambda}_1$ instead of $\min(\lambda_1, 2)$.

Introducing for each fixed k the greatest possible value of p such that the conditions of Theorem 3.2 hold, we write the previous corollary in another way :

Corollary 3.9 *Let $k \in \{-1, 0, 1, \ldots\}$. Let $p_k(\Omega, \partial_D\Omega)$ be the greatest real $\tilde{p}$ such that conditions (i) and (ii) of Theorem 3.2 hold $\forall p < \tilde{p}$. Let $\nu(\mathfrak{E})$ be the minimum of $\lambda_1(x)$ when $x \in \mathfrak{E}$. Let $\nu(\mathfrak{S})$ be the minimum of $\tilde{\lambda}_1(x)$ when $x \in \mathfrak{S}$. Then*

$$p_k(\Omega, \partial_D\Omega) \geq \min\left\{ \frac{2}{(k+2-\nu(\mathfrak{E}))_+}\,,\, \frac{3}{(k+2-\nu(\mathfrak{S}))_+} \right\}.$$

Minorizations of $\nu(\mathfrak{E})$ and $\nu(\mathfrak{S})$ yield minorizations of $p_k(\Omega, \partial_D\Omega)$. Since $\nu(\mathfrak{E})$ is greater than $\frac{1}{2}$ for Dirichlet or Neumann problems, and greater than $\frac{1}{4}$ for mixed problems and since $\nu(\mathfrak{S}) > 0$ we obtain

Corollary 3.10 *For Dirichlet or Neumann problems, $p_{-1} > 3$ and $p_0 \geq \frac{4}{3}$. For mixed problems, $p_{-1} \geq \frac{8}{3}$.*

Remark 3.11 By a duality argument it is easy to deduce from the previous statement that the Laplace operator

$$\{u \in W^{1,p}(\Omega) \mid u = 0 \text{ on } \partial_D\Omega\} \to W^{-1,p}(\Omega)$$

is an isomorphism for Dirichlet or Neumann conditions when $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$ for a $\varepsilon > 0$. Jerison and Kenig in [19] obtain the same result for Lipschitz domains.

The estimates we are going to prove in the next section (see §4.c and Proposition 4.5) allow to obtain

Corollary 3.12 *Let us suppose Ω convex. Then*
- *For Dirichlet problems,* $p_{-1} = +\infty$ *and* $p_0 > 2$.
- *For Neumann problems,* $p_{-1} \geq \frac{6}{3-\sqrt{5}} \sim 7.854$ *and* $p_0 > 2$.
- *For mixed problems,* $p_{-1} > 3$ *and* $p_0 > \frac{4}{3}$.

Lemma 4.9 gives an improvement of the above Corollary for mixed problems when all Neumann faces are isolated from each other.

The results in §4.c and Proposition 4.8 yield :

Corollary 3.13 *Let us suppose that Ω has all its edge openings $\omega_x \leq \frac{\pi}{2}$ and that the form $\mathfrak{a}$ is the gradient form. Then for the Dirichlet problem*

$$p_k(\Omega, \partial\Omega) \geq \frac{2}{k_+}$$

and for the Neumann problem

$$p_{-1}(\Omega, \emptyset) = +\infty, \quad p_0(\Omega, \emptyset) \geq 3, \quad p_1(\Omega, \emptyset) \geq \frac{3}{2}.$$

Corollary 3.7 joined with Lemma 4.1 allows to obtain precise results in the case when the domain Ω is a polygonal cylinder, i. e. of the form $P \times I$ where P is a curvilinear polygon and I is a bounded interval. In particular :

Corollary 3.14 *Let us suppose that Ω is a rectangular parallelepiped and that the form $\mathfrak{a}$ is the gradient form. Then for Dirichlet and Neumann problems*

$$p_{-1}, p_0 = +\infty, \quad p_1 = 2.$$

This statement is an extension of Faierman's result in [15] which can be expressed as $p_0 = +\infty$.

4. ESTIMATES FOR THE FIRST EIGENVALUE

4.a Explicit formulas.

In two special geometric situations : the dihedra Γ_ω and the half-dihedra Γ_ω^+, it is possible to determine explicitely the value of λ_1. Let ω be an opening angle. We denote by (y, ρ, ϑ) with $y \in \mathbb{R}$, $\rho \in \mathbb{R}^+$, $\vartheta \in [0, 2\pi[$, cylindrical coordinates in $\mathbb{R}^3$. We set

$$\Gamma_\omega := \{x \in \mathbb{R}^3 \mid 0 < \vartheta < \omega\}, \quad G_\omega := \Gamma_\omega \cap S^2$$

and

$$\Gamma_\omega^+ := \{x \in \mathbb{R}^3 \mid 0 < \vartheta < \omega, \ y > 0\}, \quad G_\omega^+ := \Gamma_\omega^+ \cap S^2.$$

We denote by T_0 the side $\vartheta = 0$ of G_ω or G_ω^+ and T^+ the side $y = 0$ of G_ω^+.

We denote by $\Lambda(G, \partial_D G)$ the set of the characteristic values

$$\lambda_j := -\frac{1}{2} + \sqrt{\mu_j(G, \partial_D G) + \frac{1}{4}}$$

where $\mu_j(G, \partial_D G)$ are the eigenvalues of the Laplace-Beltrami operator on $H^1(G, \partial_D G)$. We recall that $\lambda_1(G, \partial_D G)$ is the least positive element of this set.

Lemma 4.1

$$
\begin{array}{llll}
\Lambda(G_\omega,\partial G_\omega) & = \{\frac{l\pi}{\omega}+d \mid l\in\mathbb{N}^*,\ d\in\mathbb{N}\}, & \lambda_1(G_\omega,\partial G_\omega) & = \dfrac{\pi}{\omega}\\
\Lambda(G_\omega,\emptyset) & = \{\frac{l\pi}{\omega}+d \mid l,d\in\mathbb{N}\}, & \lambda_1(G_\omega,\emptyset) & = \min(1,\dfrac{\pi}{\omega})\\
\Lambda(G_\omega,T_0) & = \{\frac{l\pi}{\omega}+\frac{\pi}{2\omega}+d \mid l,d\in\mathbb{N}\}, & \lambda_1(G_\omega,T_0) & = \dfrac{\pi}{2\omega}\\
\Lambda(G^+_\omega,\partial G^+_\omega) & = \{\frac{l\pi}{\omega}+1+2d \mid l\in\mathbb{N}^*,\ d\in\mathbb{N}\}, & \lambda_1(G^+_\omega,\partial G^+_\omega) & = 1+\dfrac{\pi}{\omega}\\
\Lambda(G^+_\omega,\emptyset) & = \{\frac{l\pi}{\omega}+2d \mid l,d\in\mathbb{N}\}, & \lambda_1(G^+_\omega,\emptyset) & = \min(2,\dfrac{\pi}{\omega})\\
\Lambda(G^+_\omega,T_0) & = \{\frac{l\pi}{\omega}+\frac{\pi}{2\omega}+2d \mid l,d\in\mathbb{N}\}, & \lambda_1(G^+_\omega,T_0) & = \dfrac{\pi}{2\omega}\\
\Lambda(G^+_\omega,T^+) & = \{\frac{l\pi}{\omega}+1+2d \mid l,d\in\mathbb{N}\}, & \lambda_1(G^+_\omega,T^+) & = 1\\
\Lambda(G^+_\omega,T_0\cup T^+) & = \{\frac{l\pi}{\omega}+\frac{\pi}{2\omega}+1+2d \mid l,d\in\mathbb{N}\}, & \lambda_1(G^+_\omega,T_0\cup T^+) & = \dfrac{\pi}{2\omega}+1
\end{array}
$$

The proof of this lemma is based upon the knowledge of the structure of the eigenfunctions of the Laplace-Beltrami operator. Following [11], §18.C, one can show that they are the trace on the sphere S^2 of functions of the type

$$\sum_{q\in\mathbb{N}} \Delta_y^q Q(y)\,\gamma_q \rho^{2q} v(\rho,\vartheta)$$

where Q is an homogeneous polynomial (here, this is y^d), γ_q are constants which only depend on v and v is an homogeneous harmonic function on the plane sector $\{0<\vartheta<\omega\}$ and satisfies the boundary conditions induced by the initial problem on $(G,\partial_D G)$. The powers $\frac{l\pi}{\omega}$ come from v and the integer powers come from Q. Depending on the boundary conditions on T^+, the degree of Q is odd or even.

Remark 4.2 When λ_1 is equal to 1, the corresponding eigenfunction on Γ is a polynomial. Indeed the condition of injectivity modulo polynomials is satisfied in $\lambda=1$. A similar phenomenon happens on the plane sector with opening $\frac{\pi}{2}$ with the mixed conditions : the characteristic value $\lambda=1$ does not give rise to any singularity. Anyway, in such cases, we can consider the next characteristic value for the computation of the parameters $\nu(\mathfrak{E})$ and $\nu(\mathfrak{S})$ in Corollary 3.9 which give the limits for the regularity of the problem (2.5). This is due to the fact that we take zero Neumann boundary conditions. On the contrary, if the Neumann conditions are non-zero, there appears a logarithmic singularity in $\lambda=1$.

4.b A monotonicity result.

Here is an extension to mixed problems of the well-known monotonicity result for the eigenvalues of the Dirichlet problem.

Proposition 4.3 *Let G and G' be two connected curvilinear polygons in the sphere S^2. We consider partitions (2.4) of the boundary of G, resp. G', into $\partial_D G$ and $\partial_N G$, resp. $\partial_D G'$ and $\partial_N G'$. We assume that*

$$G\subset G' \quad \textit{and} \quad \partial_N G\subset\partial_N G'. \tag{4.1}$$

Then

$$\mu_1(G, \partial_D G) \geq \mu_1(G', \partial_D G') .$$

Proof. We can assume that $\partial_D G \neq \emptyset$, since if $\partial_D G$ were empty, we would have $G \subset G'$ and $\partial G \subset \partial G'$, thus $G = G'$. Since

$$\mu_1(G, \partial_D G) = \min_{\substack{u \in H^1(G) \\ u=0 \text{ on } \partial_D G}} \frac{||\nabla u||^2}{||u||^2}$$

and a similar formula for G', it is enough to prove that if $u \in H^1(G)$ is such that $u = 0$ on $\partial_D G$, then $\tilde{u} \in H^1(G')$ and $\tilde{u} = 0$ on $\partial_D G'$, where $\tilde{u}$ denotes the extension by 0 of u.

As $\partial_D G' \subset \overline{G'} \setminus (G \cup \partial_N G)$, we have indeed $\tilde{u} = 0$ on $\partial_D G'$. To end the proof, it suffices to prove that $\forall x \in \overline{G'}$, there exists a neighborhood $\mathcal{V}_x$ of x such that $\tilde{u} \in H^1(G' \cap \mathcal{V}_x)$.

Since for $x \in G$ or $x \in \overline{G'} \setminus G$, this is obvious, it remains to consider the case when $x \in \partial G$. But

$$\partial G = \partial_D G \cup \partial_N G \cup \mathcal{J},$$

where $\mathcal{J}$ is the finite set of junction points $\mathcal{J} = \overline{\partial_D G} \cap \overline{\partial_N G}$.

• If $x \in \partial_N G$, there is no extension in the neighborhood of x. So, $\tilde{u} = u$ on a suitable $\mathcal{V}_x$.

• If $x \in \partial_D G$, the usual result about extension by zero of functions with null traces yields the wanted result.

• If $x \in \mathcal{J}$, with the help of local map, we reduce to the case when $x = 0$ and $G \cap \mathcal{V}_x$ is a plane sector in $\mathbb{R}^2$ with opening ω. Let us denote by ϑ the angular variable in $\mathbb{R}^2$. We can assume that

$$\begin{array}{rcl} \partial_D G \cap \mathcal{V}_x & = & \{z \in \mathbb{R}^2 \mid \vartheta = 0,\ |z| < \varepsilon\}, \\ \partial_N G \cap \mathcal{V}_x & = & \{z \in \mathbb{R}^2 \mid \vartheta = \omega,\ |z| < \varepsilon\}, \\ G' \cap \mathcal{V}_x & \subset & \{z \in \mathbb{R}^2 \mid \omega' < \vartheta < \omega,\ |z| < \varepsilon\}, \end{array}$$

with $\omega - 2\pi \leq \omega' \leq 0$. Since $u \in H^1(G)$ and $u = 0$ on $\partial_D G$, then $|z|^{-1} u \in L^2$. So, in the coordinates (t, ϑ) with $t = \log |z|$, we have $\mathfrak{u} \in H^1(I \times]0, \omega[)$ with $I =]-\infty, \log \varepsilon[$. Since $\mathfrak{u}(t, 0) = 0\ \forall t \in I$, the extension by 0 yields that $\tilde{\mathfrak{u}} \in H^1(I \times]\omega', \omega[)$. Hence $\tilde{u} \in H^1(G' \cap \mathcal{V}_x)$. ∎

4.c Dirichlet problem.

In numerous situations the conjunction of Lemma 4.1 and Proposition 4.3 yields estimates on the values $\lambda_1(x)$ for any vertex x. For instance, if Ω is convex, and if ω denotes the greatest edge opening of Ω, then for any vertex x, the cone Γ_x is contained in Γ_ω. Thus

$$\lambda_1(x) \geq \frac{\pi}{\omega} .$$

In this case, $\nu(\mathfrak{E}) = \frac{\pi}{\omega}$ and $\nu(\mathfrak{S}) \geq \min(2, \frac{\pi}{\omega})$.

4.d Neumann problem.

We begin with a general result.

Lemma 4.4 *Let M be a convex domain on the sphere S^n with C^∞ boundary. Then $\mu_1(G,\emptyset) \geq n-1$.*

Proof. We follow the proof of [21], Theorem 7. As the Ricci curvature of S^n is $(n-1)I$, with the help of Theorem 3 in [21] we arrive at the minorization

$$\frac{\alpha-1}{\alpha}\left\{\frac{1}{4(n-1)d^2}\left(\log\frac{\alpha}{\alpha-1}\right)^2+(n-1)\right\} \leq \mu_1(G,\emptyset)\,,$$

where $\alpha > 1$ is arbitrary and d is the diameter of G, compare [21], (3.8). Therefore, $\forall \alpha > 1$

$$(1-\frac{1}{\alpha})(n-1) \leq \mu_1(G,\emptyset)\,.$$

Hence the lemma. ∎

We derive the statement which is useful for us.

Proposition 4.5 *Let G be a convex curvilinear polygon on the sphere S^2 with C^∞ sides. Then $\mu_1(G,\emptyset) \geq 1$.*

Hence $\lambda_1(G,\emptyset) \geq \frac{\sqrt{5}-1}{2}$ as stated in the introduction.

Proof. Regularizing ∂G in the neighborhood of each vertex (with preservation of the convexity), we construct a sequence of convex domains $G_n \subset G$ with smooth boundaries such that $\text{mes}(G \setminus G_n) \to 0$ when $n \to \infty$.

Let u be an eigenvector on G associated with $\mu_1(G,\emptyset)$. Let u_n be defined as

$$u_n := u\Big|_{G_n} - (\text{mes}\, G_n)^{-1}\int_{G_n} u(x)\,dx.$$

Then $u_n \in H^1(G_n)$ and $\int_{G_n} u_n(x)\,dx = 0$. Therefore Lemma 4.4 yields

$$1 \leq \frac{||\nabla u_n||^2}{||u_n||^2}\,.$$

Since $\text{mes}(G \setminus G_n) \to 0$, it is easy to see that

$$||u_n||_{L^2(G_n)} \to ||u||_{L^2(G)} \quad \text{and} \quad ||\nabla u_n||_{L^2(G_n)} \to ||\nabla u||_{L^2(G)}.$$

Hence

$$1 \leq \frac{||\nabla u||^2}{||u||^2}\,.$$

∎

We are going to prove a sharper result when G is a spherical triangle.

Proposition 4.6 *Let G be a spherical geodesic triangle (i. e. the cone Γ such that $G = \Gamma \cap S^2$ has three flat faces). Then there exists a side T of G such that*

$$\mu_1(G,\emptyset) \geq \mu_1(G,T).$$

Proof. Let u be a real eigenfunction associated with $\mu_1(G, \emptyset)$. Let $G_1, \dots, G_k, \dots$ be the nodal domains of u, i. e. the connected components of $u^{-1}(\mathbb{R} \setminus \{0\})$. Let u_k be the restriction of u to G_k and $\tilde{u}_k$ be the extension of u_k by zero outside G_k. We rely on the following lemma which is stated in [3] and proved in the appendix of [9].

Lemma 4.7 *Let D be a bounded domain. Let u be a function on D such that*

$$u \in H^1(D), \quad u \in C^0(\overline{D}), \quad u \in \mathrm{Lip}_{\mathrm{loc}}(D).$$

If moreover $u\Big|_{\partial D} = 0$ then $u \in H^1_0(D)$, i. e. u is the limit in $H^1(D)$ of a sequence of $C^\infty_0(D)$ functions.

We deduce from this lemma that u_k is the limit in $H^1(G_k)$ of a sequence of $C^\infty(\overline{G}_k)$ functions whose supports do not meet T_k, where T_k denotes the complement of ∂G in ∂G_k.

Therefore, u_k is an eigenvector of the positive Laplace-Beltrami operator with the Dirichlet conditions on T_k associated with the eigenvalue $\mu_1(G, \emptyset)$. Since u_k has a constant sign on G_k, we infer that

$$\mu_1(G, \emptyset) = \mu_1(G_k, T_k). \tag{4.2}$$

As another consequence of the above property of u_k, we obtain that $\tilde{u}_k \in H^1(G)$. Therefore, the Max–Min principle yields :

$$\mu_1(G_k, T_k) \geq \mu_1(G, \partial_k G), \tag{4.3}$$

where $\partial_k G$ denotes the complement of ∂G_k in ∂G.

Since there exists at least one nodal domain G_k such that $\partial_k G$ contains a full side T of G, (4.2), (4.3) and Proposition 4.3 yield Proposition 4.6. ∎

Proposition 4.8 *Let G be a spherical geodesic triangle such that all its openings are $\leq \frac{\pi}{2}$. Then*

$$\mu_1(G, \emptyset) \geq 2.$$

Proof. Applying Proposition 4.6, we obtain that there exists a side T of G such that $\mu_1(G, \emptyset) \geq \mu_1(G, T)$. Let N be the vertex of G such that $N \notin \overline{T}$. Let us choose N as north pole and $(\varphi, \theta) \in [-\pi, \pi[\times [0, \pi]$ as spherical coordinates such that in the neighborhood of N, G coincides with the spherical sector G_ω

$$\{\psi \in S^2 \mid 0 < \varphi < \omega,\ 0 < \theta < \pi\}.$$

We introduce

$$G^+_\omega := \{\psi \in S^2 \mid 0 < \varphi < \omega,\ 0 < \theta < \frac{\pi}{2}\} \quad T^+ := \{\psi \in S^2 \mid 0 < \varphi < \omega,\ \theta = \frac{\pi}{2}\}.$$

It is possible to prove [9] Lemma 8.11, that G is included in G^+_ω. Then Proposition 4.3 implies that

$$\mu_1(G, T) \geq \mu_1(G^+_\omega, T^+).$$

But Lemma 4.1 yields that $\lambda_1(G^+_\omega, T^+) = 1$. Therefore $\mu_1(G^+_\omega, T^+) = 2$. Hence $\mu_1(G, T) \geq 2$. ∎

4.e Mixed problems.

We only give two results which can help to obtain minorizations of $\nu(\mathfrak{S})$ in Corollary 3.9.

Lemma 4.9 *Let G be a convex geodesic polygon on the sphere and T be one of its sides. If $\partial_D G$ is the complement of T in ∂G, i. e. if the Neumann condition is prescribed only on T, then*

$$\lambda_1(G, \partial_D G) \geq \frac{\pi}{2\omega}$$

where ω is the smallest of the opening angles of G at the vertices which belong to the side T.

Proof. It is possible to include G in a spherical sector G_ω so that one of the sides of G_ω contains T. Then according to Proposition 4.3

$$\lambda_1(G, \partial_D G) \geq \lambda_1(G_\omega, T_0)$$

with the notation of Lemma 4.1. This Lemma yields the result. ∎

Lemma 4.10 *Let G be a spherical geodesic triangle such that all its openings are $\leq \frac{\pi}{2}$. Then*

$$\mu_1(G, \partial_D G) \geq 2.$$

Proof. Owing to Proposition 4.3

$$\mu_1(G, \partial_D G) \geq \mu_1(G, \emptyset)$$

Proposition 4.8 yields the result. ∎

4.f Last comments about eigenvalues.

If one knows the numerical value of the first non-zero eigenvalue μ_1 at each vertex, one can deduce regularity results on the domain Ω. Till recently, the existing eigenvalues approximations were only done for the Dirichlet problem. But in this case there are also theoretical ways to get minorizations :
(i) Monotonicity with respect to the domain,
(ii) Faber-Krahn principle, which holds on all spheres S^n, see [5] for example.

For the Neumann problem, theoretical questions remain open : for example, is the minorization in Proposition 4.5 optimal ? When explicit theoretical computation is possible, one gets $\mu_1(G, \emptyset) \geq 2$ (for convex domains of the following form : spherical sectors, half spherical sectors, spherical disks, ...). But the theoretical general lower bound is 1 and not 2 ... Do correct numerical computations exist for the Neumann problem ? There is no monotonicity with respect to the domain, even for convex domains. But it seems that, in a neighbourhood of the hemisphere H (for which we have exactly $\mu_1(G, \emptyset) = 2$), the eigenvalues are decreasing when the domain increases, i. e. if we consider an analytic decreasing one parameter family of spherical domains $\tau \to G_\tau$, $\tau \in [0, 1]$ such that $G_0 = H$, then $\frac{d\lambda_1}{d\tau}$ would be > 0 in $\tau = 0$. To see that, one may use [14]'s method without returning to the equation and compute the sign of the derivative "on the hemisphere". But what is the situation outside such a neighborhood ?

Now, let us give a little list of works about numerical approximation of Dirichlet eigenvalues :

• Fichera [16] for the complement of an octant (which gives upper and lower bounds (this last one by Faber-Krahn) : $0.4335 < \lambda_1 < 0.4645$)

• Beagles and Whiteman [2] for theoretical formulas and computations : in particular when separation of variables is possible, formulas involving Legendre functions are given, and methods for calculating them – compare also [14] relating to that topic. [2] also indicates a finite elements method combined with stereographic projection. Results of computations are given for the complement of an octant (see above).

• Walden and Kellogg [30] use finite difference approximation for the computing of the first Dirichlet eigenvalue on various spherical geodesic triangles. Tests are made for theoretically known situations.

• A programm by Costabel [6] uses a first kind integral equation and approximation by boundary elements. The problem is reduced to the detection of a zero eigenvalue for a preconditioned matrix. The numerical results are sharp with a short computing time.

5. PROOF : REDUCTION TO CONSTANT COEFFICIENTS

In the next sections, we will prove Theorem 3.2 in the case when the integrodifferential form $\mathfrak{a}$ is the gradient form and Ω is a polyhedron with flat faces : Theorems 6.4 and 8.1. We explain here how to perform the reduction from the variable coefficients case to the constant coefficients case.

We assume that conditions *(i)* and *(ii)* of Theorem 3.2 hold. We take u and f as in Theorem 3.2. Let $x \in \overline{\Omega}$. We want to prove that u belongs to $W^{k+2,p}$ in a neighborhood of x. With the help of the local map χ_x and the matrix ${}^t\mathfrak{M}_x^{-1}$ we transform locally the domain into the straight cone Ξ_x and the integrodifferential form into a form $\mathfrak{b}$ with coefficients $b_{i,j}$ and b_0 satisfying

$$b_{i,j} \in C^{k+1} \quad \text{and} \quad b_0 \in C^{k_+} \tag{5.1}$$

and

$$b_{i,j}(0) = \delta_{i,j}. \tag{5.2}$$

Now we are going to use a perturbation argument, like in §10.C and §10.D of [11]. Let ψ be a smooth cut-off function which is equal to 1 in a neighborhood of 0. For $\sigma \geq 0$ we introduce

$$b^\sigma_{i,j}(z) := \delta_{i,j} + \psi(z)[b_{i,j}(\sigma z) - \delta_{i,j}] \quad \text{and} \quad b^\sigma_0(z) := \sigma^2 b_0(\sigma z). \tag{5.3}$$

These coefficients define an integrodifferential form $\mathfrak{b}_\sigma$ on the cone Ξ_x. We note that for $\sigma = 0$, $\mathfrak{b}_\sigma$ coincides with the gradient form and that for $\sigma \neq 0$, the change of variables $z \to \sigma z$ transforms $\sigma^{-2}\mathfrak{b}_\sigma$ into a form whose coefficients coincide with those of $\mathfrak{b}$ in a neighborhood of 0. We will prove in the next sections that *(i)* and *(ii)* of Theorem 3.2 imply the regularity result on the cone Ξ_x for $\mathfrak{b}_0$. In order to deduce the corresponding result for $\mathfrak{b}$ it suffices to prove that

$$|||\, \mathfrak{b}_\sigma - \mathfrak{b}_0 \,|||k,p \to 0 \text{ when } \sigma \to 0, \tag{5.4}$$

where $|||\ .\ |||k,p$ denotes the norm of the induced operator from $W^{k+2,p}(\Xi_x)$ into $W^{k,p}(\Xi_x)$.

Let K be the support of ψ. We have

$$\|b_0(\sigma z)u(z)\|_{W^{k,p}(K)} \leq C\|b_0(\sigma z)\|_{C^{k_+}(\overline{K})} \ \|u\|_{W^{k+2,p}(K)}$$

and

$$\|\partial_j[b_{i,j}(\sigma z)-\delta_{i,j}]\partial_i u(z)\|_{W^{k,p}(K)} \leq C\|b_{i,j}(\sigma z)-\delta_{i,j}\|_{C^{k+1}(\overline{K})} \ \|u\|_{W^{k+2,p}(K)}\,.$$

As a consequence of (5.1), $\|b_0(\sigma z)\|_{C^{k_+}(\overline{K})}$ is uniformly bounded when $\sigma \in [0,1]$ and with (5.2)

$$\|b_{i,j}(\sigma z)-\delta_{i,j}\|_{C^{k+1}(\overline{K})} \to 0 \text{ when } \sigma \to 0.$$

All that yields (5.4).

We will now show by two examples that the assumptions about the regularity of the coefficients $a_{i,j}$ are not far from optimality : it is impossible to replace "continuous" by "bounded" without destroying the regularity result.

Let us take $k=-1$ and Ω a convex plane polygon. Let $a_{i,j}$ be real continuous functions on $\overline{\Omega}$ satisfying the ellipticity condition. Let A be the operator

$$\sum_{i,j} \partial_j a_{i,j} \partial_i.$$

Our statement, adapted for 2-dimensional domains, gives immediately that any function u such that

$$u \in \overset{\circ}{H}{}^1(\Omega) \quad \text{and} \quad Au \in W^{-1,p}(\Omega) \tag{5.5}$$

satisfies

$$u \in W^{1,p}(\Omega)$$

for any $p \geq 2$. Such a result no longer holds in general when the coefficients are only bounded. We are going to study two examples.

Example 5.1 Let a be a positive parameter. We write (x,y) the coordinates in $\mathbb{R}^2$, (r,θ) the polar coordinates and we suppose that $0 \in \Omega$. We define A as

$$\Delta + (a-1)\left(\partial_x \frac{y^2}{r^2}\partial_x + \partial_y \frac{x^2}{r^2}\partial_y - \partial_x \frac{xy}{r^2}\partial_y - \partial_y \frac{xy}{r^2}\partial_x\right).$$

In polar coordinates we have

$$r^2 A = (r\partial_r)^2 + a\partial_\theta^2.$$

A is an elliptic operator. Let ψ be a cut-off function with support in Ω and equal to 1 in a neighborhood of 0. For any $p \geq 2$, the function $u := \psi\, r^{\sqrt{a}} \cos\theta$ satisfies (5.5) but if $a \in]0,1[$:

$$u \in W^{1,p}(\Omega) \quad \Longleftrightarrow \quad 1-\frac{2}{p} < \sqrt{a}.$$

Indeed we can show that for this example, with the notation of Corollary 3.9,

$$p_{-1} = \frac{2}{1-\sqrt{a}}\,.$$

∎

Example 5.2 We consider a case when the coefficients $a_{i,j}$ are piecewise constant. This is a framework for transmission problems. The polygon Ω is covered by a finite number of disjoint polygons Ω_k and there are given $\alpha_1, \ldots, \alpha_K > 0$ and

$$a_{i,j}\Big|_{\Omega_k} = \alpha_k\,\delta_{i,j}.$$

Let us consider in the neighborhood of 0 the covering of the half-space $x > 0$ by the 3 sectors $-\frac{\pi}{2} < \theta < -\frac{\pi}{4}$, $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$ and $\frac{\pi}{4} < \theta < \frac{\pi}{2}$. We set

$$\alpha_1,\ \alpha_3 = 1, \quad \alpha_2 = (\tan\frac{4}{\pi}\lambda)^{-2},$$

where λ is a positive parameter. Then $p_{-1} \leq \frac{2}{1-\lambda}$ as shown by the consideration of the function $r^\lambda v(\theta)$ where $v = (v_1, v_2, v_3)$ defined by

$$\begin{cases} v_1(\theta) = \kappa\sin\lambda(\theta + \frac{\pi}{2}) & -\frac{\pi}{2} < \theta < -\frac{\pi}{4} \\ v_2(\theta) = \cos\lambda\theta & -\frac{\pi}{4} < \theta < \frac{\pi}{4} \\ v_3(\theta) = \kappa\sin\lambda(\frac{\pi}{2} - \theta) & \frac{\pi}{4} < \theta < \frac{\pi}{2} \end{cases}$$

with $\kappa = (\tan\frac{4}{\pi}\lambda)^{-1}$. ∎

6. PROOF : REGULARITY ON CONES

Let Γ be a cone in $\mathbb{R}^n$ with vertex 0. We set $G := \Gamma \cap S^{n-1}$. We assume that G has a Lipschitz boundary. Let $\partial_D G$ be an open set in ∂G. We denote

$$\partial_D\Gamma := \{z \in \mathbb{R}^n \mid \frac{z}{|z|} \in \partial_D G\}.$$

We denote by $H^1(\Gamma, \partial_D\Gamma)$ the space of the functions $u \in H^1(\Gamma)$ which are zero on $\partial_D\Gamma$. Let us recall that we define the space $W^{-1,p}(\Gamma)$ as the dual space of

$$\{v \in W^{1,q}(\Gamma) \mid v = 0 \text{ on } \partial_D\Gamma\},$$

where q denotes the conjugate of p : $\frac{1}{p} + \frac{1}{q} = 1$.

We take $k \in \{-1, 0, 1\ldots\}$ and $p \in]1, \infty[$ such that $k + 2 - \frac{n}{p} \geq 1 - \frac{n}{2}$. Thus we have the imbeddings

$$W^{k+2,p}(\Gamma) \subset H^1_{\mathrm{loc}}(\overline{\Gamma}) \quad \text{and} \quad W^{k,p}(\Gamma) \subset W^{-1,2}_{\mathrm{loc}}(\overline{\Gamma}).$$

We also assume that $k + 2 - \frac{n}{p} \notin \mathbb{N}$. Thus $W^{k,p}(\Gamma)$ is imbedded in $C^{[k-n/p]}(\overline{\Gamma})$. Moreover, setting

$$V^{k,p}_\beta(\Gamma) = \{u \in \mathcal{D}'(\Gamma) \mid r^{\beta+|\alpha|-k} D^\alpha u \in L^p(\Gamma)\ \forall\alpha,\ |\alpha| \leq k\},$$

we obtain as a standard consequence of Hardy's inequality

Lemma 6.1 *We assume that $k - \frac{n}{p} \notin \mathbb{N}$. For any $f \in W^{k,p}(\Gamma)$ we have*

$$\forall\alpha,\ |\alpha| \leq k - \frac{n}{p}, \quad D^\alpha f(0) = 0 \quad \Longrightarrow \quad f \in V^{k,p}_0(\Gamma).$$

We are going to deduce a regularity result in L^p Sobolev spaces on Γ from two hypotheses : the first one about the regularity "far from 0" and the second one about the absence of singular functions at the vertex.

Hypothesis 6.2 *For $j = 1, 2$, let Γ^j be the subsets $\{x \in \Gamma \mid 2^{-j} < |x| < 2^j\}$. For any $u \in H^1(\Gamma, \partial_D\Omega)$ such that $\Delta u \in W^{k,p}(\Gamma)$ we have $u \in W^{k+2,p}(\Gamma)$ with the a priori estimate*

$$\|u\|_{W^{k+2,p}(\Gamma^1)} \le C(\|f\|_{W^{k,p}(\Gamma^2)} + \|u\|_{H^1(\Gamma^2)}).$$

This hypothesis always holds when the section G of Γ on the sphere is smooth (Γ is a "regular" cone). But we are interested here in the polyhedral cones Γ_x for $x \in \mathfrak{S}$. The above hypothesis will be satisfied if the regularity property holds in the neighborhood of *each edge* of Γ_x in $\{x \mid 1/2 < |x| < 2\}$. Such a regularity is a consequence of condition *(i)* of Theorem 3.2 as will be shown in Theorem 8.1.

Hypothesis 6.3 *For any λ, $\operatorname{Re}\lambda \in [1 - \frac{n}{2}, k + 2 - \frac{n}{p}]$, the condition $\mathcal{C}^\lambda(\Gamma, \partial_D\Gamma)$ of injectivity modulo polynomials holds. Let us recall from Definition* 3.1 *that*

$$\mathcal{C}^\lambda(\Gamma, \partial_D\Gamma) \begin{cases} u \in S^\lambda(\Gamma, \partial_D\Gamma) \text{ and } \Delta u = f \text{ with } f \text{ polynomial in cartesian variables} \\ \text{implies} \\ u \text{ polynomial in cartesian variables.} \end{cases}$$

When $x \in \mathfrak{S}$, this hypothesis corresponds exactly to condition *(ii)* of Theorem 3.2.

The main result of this section is

Theorem 6.4 *We suppose that $k + 2 - \frac{n}{p} \notin \mathbb{N}$ is greater than $1 - \frac{n}{2}$. We assume the above hypotheses* 6.2 *and* 6.3. *Let $u \in H^1(\Gamma, \partial_D\Gamma)$ with compact support such that*

$$\forall v \in H^1(\Gamma, \partial_D\Gamma) \quad < \nabla u, \nabla v > = < f, v > \quad \text{with} \quad f \in W^{k,p}(\Gamma).$$

Then $u \in W^{k+2,p}(\Gamma)$.

Note that if $k \ge 0$, u is solution of the following boundary value problem

$$\begin{cases} \Delta u = f & \text{on } \Gamma, \\ u = 0 & \text{on } \partial_D\Gamma \qquad \text{(Dirichlet condition)}, \\ \partial_n u = 0 & \text{on } \partial_N\Gamma \qquad \text{(Neumann condition)}. \end{cases}$$

Proof.

First step : reduction to weighted spaces on Γ. With the help of Lemma 6.1, we write f as the sum of a polynomial Q of degree $[k - \frac{n}{p}]$ and a function in $V_0^{k,p}(\Gamma)$. With the results of §4.C in [11], we obtain that for each homogeneous component Q_λ of degree $\lambda - 2$ of Q, with $\lambda = 2, 3, \ldots, [k + 2 - \frac{n}{p}]$ there exists an element $P_\lambda \in S^\lambda(\Gamma, \partial_D\Gamma)$ satisfying $\Delta P_\lambda = Q_\lambda$. Hypothesis 6.3 yields that P_λ is a polynomial.

Second step : problem on the strip $\mathbb{R} \times G$. Now we can assume that $f \in V_0^{k,p}(\Gamma)$ with compact support. We introduce the new coordinates

$$t := \log|x| \quad \text{and} \quad \theta := \frac{x}{|x|} \in S^{n-1}$$

and the following functions on $\mathbb{R}\times G$

$$v(t,\theta)=u(x) \quad \text{and} \quad g(t,\theta)=e^{2t}f(x).$$

We need some spaces on $\mathbb{R}\times G$. It is well known [20, 25] that the previous change of variables transforms the weighted spaces of the type of $V^{k,p}_\beta(\Gamma)$ into spaces with an exponential weight on $\mathbb{R}\times G$. We set for any real ζ

$$W^{k,p}_\zeta(\mathbb{R}\times G)=\{h\in\mathcal{D}'(\mathbb{R}\times G)\mid e^{-\zeta t}h\in W^{k,p}(\mathbb{R}\times G)\},$$

where, as usual, $W^{-1,p}(\mathbb{R}\times G)$ denotes the dual space of

$$\{w\in W^{1,q}(\mathbb{R}\times G)\mid w=0 \text{ on } \mathbb{R}\times\partial_D G\}.$$

We have

Lemma 6.5 *The change of variables $z\mapsto(t,\theta)$ induces an isomorphism*

$$V^{k,p}_\beta(\Gamma)\longrightarrow W^{k,p}_\zeta(\mathbb{R}\times G) \quad \textit{with} \quad \zeta=k-\beta-n/p.$$

We will also use some kind of "variational" spaces :

$$\mathcal{H}^1_\zeta=\{w\in W^{1,2}_\zeta(\mathbb{R}\times G)\mid w=0 \text{ on } \mathbb{R}\times\partial_D G\}$$

and, with the above notation

$$\mathcal{H}^{-1}_\zeta=W^{-1,2}_\zeta(\mathbb{R}\times G).$$

Indeed, $\mathcal{H}^{-1}_\zeta$ is the dual space of $\mathcal{H}^1_{-\zeta}$. $\mathcal{H}^*$ denotes $\mathcal{H}^*_0$.

Functions g and h satisfy the following properties

Lemma 6.6 *We set*

$$\eta=k+2-\frac{n}{p}.$$

Then

$$\forall\zeta\le\eta,\qquad g\in W^{k,p}_\zeta(\mathbb{R}\times G) \tag{6.1}$$

$$\forall\zeta<\eta,\qquad g\in\mathcal{H}^{-1}_\zeta \tag{6.2}$$

and

$$\forall\zeta<1-\frac{n}{2},\qquad v\in\mathcal{H}^1_\zeta. \tag{6.3}$$

Moreover

$$(\partial_t^2+(n-2)\partial_t-L)v=g \tag{6.4}$$

where L is the positive Laplace-Beltrami operator on $H^1(G,\partial_D G)$.

The assertions (6.1), (6.3), (6.4) are not difficult to obtain. We are going to prove (6.2), which uses the basic tool of translation invariant partition of unity on $\mathbb{R}\times G$.

Lemma 6.7 *Let $\chi_0\in\mathcal{D}(]-1,1[)$ such that*

$$\chi_\nu(t):=\chi_0(t-\nu),\quad \nu\in\mathbb{Z}$$

forms a partition of unity on $\mathbb{R}$. Then

$$\|h\|_{W^{k,p}(\mathbb{R}\times G)}\simeq\Big(\sum_{\nu\in\mathbb{Z}}\|\chi_\nu h\|^p_{W^{k,p}(\mathbb{R}\times G)}\Big)^{1/p}.$$

Let us deduce (6.2) from (6.1). We set $g_\nu := \chi_\nu g$. Let $\zeta < \eta$. We have the uniform estimate

$$\|e^{-\zeta t} g_\nu\|_{\mathcal{H}^{-1}} \leq C \|e^{-\zeta t} g_\nu\|_{W^{k,p}}$$

which implies :

$$\Big(\sum_{\nu\in\mathbb{Z}} \|e^{-\zeta t} g_\nu\|^2_{\mathcal{H}^{-1}}\Big)^{1/2} \leq C\Big(\sum_{\nu\in\mathbb{Z}} \|e^{-\zeta t} g_\nu\|^2_{W^{k,p}}\Big)^{1/2}.$$

• If $p \in]1,2[$, as $l^p \subset l^2$ the right hand side is smaller than

$$C\Big(\sum_{\nu\in\mathbb{Z}} \|e^{-\zeta t} g_\nu\|^p_{W^{k,p}}\Big)^{1/p}.$$

Lemma 6.7 then imply that $\|e^{-\zeta t} g\|_{\mathcal{H}^{-1}} \leq \|e^{-\zeta t} g\|_{W^{k,p}}$.

• If $p > 2$, we fix $\varepsilon > 0$ such that $\zeta + \varepsilon \leq \eta$ and we write :

$$\|e^{-\zeta t} g_\nu\|_{W^{k,p}} \simeq e^{\varepsilon\nu} \|e^{-(\zeta+\varepsilon) t} g_\nu\|_{W^{k,p}}.$$

As the support of g is bounded from above, and as the sequence $(e^{\varepsilon\nu})_{\nu\in\mathbb{Z}^-}$ belongs to l^1, the Hölder inequality yields that

$$\Big(\sum_{\nu\in\mathbb{Z}} \|e^{-\zeta t} g_\nu\|^2_{W^{k,p}}\Big)^{1/2} \leq C\Big(\sum_{\nu\in\mathbb{Z}} \|e^{-(\zeta+\varepsilon) t} g_\nu\|^p_{W^{k,p}}\Big)^{1/p}.$$

We conclude as previously.

Third step : regularity on the strip $\mathbb{R} \times G$ in Hilbert spaces. Among other things, we use classical facts of the theory of such problems in Hilbert Sobolev spaces. We recall these facts without proof in the two following lemmas. The techniques of proof are essentially based on the partial Fourier-Laplace transform in the variable t and the Cauchy residue formula joined with some *a priori* estimates : see [20, 11] for instance.

Lemma 6.8 *If $\zeta \in \mathbb{R}$ satisfies*

$$\zeta(\zeta + n - 2) \text{ is not an eigenvalue of } L$$

then the operator $\partial_t^2 + (n-2)\partial_t - L$ induces an isomorphism :

$$A^\zeta : \mathcal{H}^1_\zeta \longrightarrow \mathcal{H}^{-1}_\zeta.$$

Lemma 6.9 *Let $\zeta_1, \zeta_2 \in \mathbb{R}$, $\zeta_1 < \zeta_2$ such that*

$$\zeta_j(\zeta_j + n - 2) \text{ is not an eigenvalue of } L \quad \text{for} \quad j = 1, 2.$$

Let $h \in \mathcal{H}^{-1}_{\zeta_1} \cap \mathcal{H}^{-1}_{\zeta_2}$. Lemma 6.8 gives the existence of $w_j := (A^{\zeta_j})^{-1} h$ for $j = 1, 2$. We denote by u_j the function on Γ defined by $u_j(z) := w_j(t, \theta)$. Then, if $\lambda_1, \ldots, \lambda_K$ denotes the set of the $\lambda \in]\zeta_1, \zeta_2[$ such that $\lambda(\lambda + n - 2)$ is an eigenvalue of L, we have

$$u_1 - u_2 = \sum_{k=1}^{K} P_k \quad \text{where} \quad P_k \in S^{\lambda_k}(\Gamma, \partial_D \Gamma) \text{ and } \Delta P_k = 0.$$

In particular, if $\forall \lambda \in]\zeta_1, \zeta_2[$, $\lambda(\lambda + n - 2)$ is not an eigenvalue of L, $w_1 = w_2$.

We come back now to the functions g and v which were previously defined and satisfy the properties in Lemma 6.6. Since, according to our hypotheses $\eta \notin \mathbb{N}$ and the condition

$\mathcal{C}^\lambda(\Gamma, \partial_D\Gamma)$ holds for $\lambda = \eta$, then $\eta(\eta+n-2)$ is not an eigenvalue of L (see (3.2)). Let $\zeta_0 < \eta$ be such that

$$\forall\zeta \in [\zeta_0, \eta], \quad \zeta(\zeta + n - 2) \text{ is not an eigenvalue of } L\,.$$

The conjunction of (6.2) and Lemma 6.9 yields that $\forall\zeta \in [\zeta_0, \eta[$, $(A^\zeta)^{-1}g$ gives the same function w. Let us set $u_0(z) = w(t, \theta)$.

From (6.3) and (6.4), we deduce that $\forall\zeta < 1 - \frac{n}{2}$, $(A^\zeta)^{-1}g = v$. As a consequence of Lemma 6.9, we obtain that

$$u - u_0 = \sum_{k=1}^{K} P_k$$

and Hypothesis 6.3 yields that the P_k are polynomial. Theorem 6.4 will be proved if we show that $u_0 \in V_0^{k+2,p}(\Gamma)$, i. e.

$$e^{-\eta t} w \in W^{k+2,p}(\mathbb{R} \times G). \tag{6.5}$$

Fourth step : regularity on the strip $\mathbb{R} \times G$ in L^p spaces. We are going to prove (6.5). This is the basic step, where we use the idea [25] of pseudo-localization of the operators $(A^\zeta)^{-1}$. We introduce the sequence (v_ν) defined for $\nu \in \mathbb{Z}$, by :

$$v_\nu = (A^\eta)^{-1} g_\nu \quad \text{with} \quad g_\nu = \chi_\nu g,$$

as previously (cf Lemma 6.7). The properties of ζ_0 imply that :

$$\forall\zeta \in [\zeta_0, \eta], \quad v_\nu = (A^\zeta)^{-1} g_\nu.$$

As $\sum g_\nu = g$ with convergence in $\mathcal{H}^{-1}_\zeta$ for all $\zeta \in [\zeta_0, \eta[$ (compare (6.2)), we have

$$\sum v_\nu = w \text{ with convergence in } \mathcal{H}^1_\zeta, \quad \forall\zeta \in [\zeta_0, \eta[.$$

Thus, to obtain (6.5) it is enough to prove that :

$$e^{-\eta t} \sum v_\nu \in W^{k+2,p}(\mathbb{R} \times G). \tag{6.6}$$

For doing that, we first prove :

$$\text{the sequence } (\chi_\mu \sum e^{-\eta t} v_\nu)_{\mu\in\mathbb{Z}} \text{ belongs to } l^p(H^1(\mathbb{R} \times G)). \tag{6.7}$$

Next, we will complete the proof of (6.6) by a priori estimates.

In order to prove (6.7), we start from the isomorphisms A^ζ for $\zeta = \eta \pm \varepsilon$, with a suitable small enough $\varepsilon > 0$, such that the property

$$\zeta(\zeta + n - 2) \text{ is not an eigenvalue of } L$$

still holds for any ζ in the interval $[\eta - \varepsilon, \eta + \varepsilon]$. So

$$\forall\zeta \in [\eta - \varepsilon, \eta + \varepsilon] \quad v_\nu = (A^\zeta)^{-1} g_\nu.$$

We are going to estimate $\chi_\mu v_\nu$ where μ spans $\mathbb{Z}$.

As, we have, for $\zeta = \eta \pm \varepsilon$:

$$\|e^{-\zeta t} \chi_\mu v_\nu\|_{\mathcal{H}^1} \le C \|e^{-\zeta t} g_\nu\|_{\mathcal{H}^{-1}}$$

we deduce that :

$$e^{\pm\varepsilon\mu} \|\chi_\mu v_\nu\|_{\mathcal{H}^1_\eta} \le C e^{\pm\varepsilon\nu} \|g_\nu\|_{\mathcal{H}^{-1}_\eta}\,.$$

When $\nu \geq \mu$, we choose $-\varepsilon$; when $\nu < \mu$, we choose ε. So, we obtain :

$$\|e^{-\eta t}\chi_\mu v_\nu\|_{\mathcal{H}^1} \leq Ce^{-\varepsilon|\mu-\nu|}\|e^{-\eta t}g_\nu\|_{\mathcal{H}^{-1}}. \tag{6.8}$$

But $\exists C > 0$, $\forall \nu$:

$$\|e^{-\eta t}g_\nu\|_{\mathcal{H}^{-1}} \leq C\|e^{-\eta t}g_\nu\|_{W^{k,p}}.$$

That estimate yields :

$$\|e^{-\eta t}\chi_\mu v_\nu\|_{\mathcal{H}^1} \leq Ce^{-\varepsilon|\mu-\nu|}\|e^{-\eta t}g_\nu\|_{W^{k,p}}. \tag{6.9}$$

Now, we define the four following sequences :

$$\begin{aligned} a_\nu &= \|e^{-\eta t}g_\nu\|_{W^{k,p}} \\ b_\mu &= \|\chi_\mu \sum_\nu e^{-\eta t}v_\nu\|_{\mathcal{H}^1} \\ c_\mu &= \sum_\nu e^{-\varepsilon|\mu-\nu|}a_\nu \\ e_\mu &= e^{-\varepsilon|\mu|}. \end{aligned}$$

The sequence c is equal to the convolution $(e_\mu) * (a_\nu)$. As $e^{-\eta t}g \in W^{k,p}$, Lemma 6.7 give us

$$a \in l^p.$$

As (e_μ) belongs to l^1, the convolution by a belongs to l^p . But (6.9) yields :

$$|b_\mu| \leq C|c_\mu|.$$

So $b \in l^p$. Thus we have just gotten (6.7) :

$$\sum_\mu \|\chi_\mu e^{-\eta t}\sum_\nu v_\nu\|^p_{H^1(\mathbb{R}\times G)} \leq C\|e^{-\eta t}g\|^p_{W^{k,p}(\mathbb{R}\times G)}.$$

Let us recall that w is equal to the sum $\sum_\nu v_\nu$. We easily deduce from the latter estimate that :

$$\sum_\mu \|e^{-\eta t}w\|^p_{H^1(]\mu-1,\mu+1[\times G)} \leq C\|e^{-\eta t}g\|^p_{W^{k,p}}. \tag{6.10}$$

Now, Hypothesis 6.2 yields the uniform estimate (as the operator $\partial_t^2 + (n-2)\partial_t - L$ is translation invariant, the constant C does not depend on μ)

$$\|\chi_\mu w\|_{W^{k+2,p}} \leq C\Big(\|g\|_{W^{k,p}(]\mu-1,\mu+1[\times G)} + \|w\|_{H^1(]\mu-1,\mu+1[\times G)}\Big).$$

Since the weight $e^{-\eta t}$ is equivalent to $e^{-\eta\mu}$ on $]\mu-1,\mu+1[$ we deduce the estimate

$$\|e^{-\eta t}\chi_\mu w\|^p_{W^{k+2,p}} \leq C\Big(\|e^{-\eta t}g\|^p_{W^{k,p}(]\mu-1,\mu+1[\times G)} + \|e^{-\eta t}w\|^p_{H^1(]\mu-1,\mu+1[\times G)}\Big). \tag{6.11}$$

The two inequalities (6.10) and (6.11) yield :

$$\sum_\mu \|e^{-\eta t}\chi_\mu w\|^p_{W^{k+2,p}} \leq C\|e^{-\eta t}g\|^p_{W^{k,p}}.$$

So, we obtain that $e^{-\eta t}w \in W^{k+2,p}(\mathbb{R}\times G)$; thus $u_0 \in V_0^{k+2,p}(\Gamma)$ hence $u \in W^{k+2,p}(\Gamma)$. ∎

7. FUNCTIONAL LEMMAS

The last steps of the proof of Theorem 6.4 state results (estimates (6.8) and (6.10)) which can be formulated under more general assumptions. We state them here, because we will be lead to use them anew in the proofs below.

The domain has the following form : $\mathbb{R}\times\mathfrak{U}$. All the spaces B we introduce are Banach spaces of distributions over $\mathbb{R}\times\mathfrak{U}$ such that $\forall\psi \in C_0^\infty(\mathbb{R})$, the mapping $g(t,\theta) \to \psi(t)g(t,\theta)$ defines a bounded operator from B into B.

For $\nu \in \mathbb{Z}$, T_ν denotes the translation : $T_\nu g(t,\theta) = g(t-\nu,\theta)$. We say that B is translation invariant if :

$$\forall\nu \in \mathbb{Z},\ \forall g \in B, \quad \|T_\nu g\|_B = \|g\|_B .$$

Lemma 7.1 *Suppose that B_1 and B_2 are Banach spaces as above, and translation invariant. For $\zeta \in \mathbb{R}$ and $j = 1, 2$ we set :*

$$B_{j,\zeta} := \{g \in \mathcal{D}'(\mathbb{R}\times\mathfrak{U}) \mid e^{-\zeta t}g \in B_j\}.$$

Let $\eta \in \mathbb{R}$ and $\varepsilon > 0$. For $\zeta \in \{\eta-\varepsilon, \eta, \eta+\varepsilon\}$, let $\mathcal{A}^\zeta$ be such that :

(i) $\mathcal{A}^\zeta : B_{1,\zeta} \to B_{2,\zeta}$ is bounded,

(ii) $g \in B_{1,\eta-\varepsilon} \cap B_{1,\eta+\varepsilon} \Longrightarrow \mathcal{A}^{\eta-\varepsilon}g = \mathcal{A}^\eta g = \mathcal{A}^{\eta+\varepsilon}g$.

Then $\exists C > 0$, $\forall g \in B_{1,\eta}$, $\forall\mu,\nu \in \mathbb{Z}$:

$$\|\chi_\mu \mathcal{A}^\eta(\chi_\nu g)\|_{B_{2,\eta}} \leq Ce^{-\varepsilon|\mu-\nu|}\|\chi_\nu g\|_{B_{1,\eta}} .$$

We say that a Banach space B as above is of l^p-type if we have the equivalence (compare Lemma 6.7) :

$$\|g\|_B \simeq \Big(\sum_{\nu\in\mathbb{Z}} \|\chi_\nu g\|_B^p\Big)^{1/p}.$$

Lemma 7.2 *Suppose that B_1 and B_2 are of l^p-type. We set*

$$B_1^\infty := \{g \in B_1 \mid g \text{ compactly supported in the } t\text{-variable}\}.$$

Let $\mathcal{A}$ be an operator from B_1^∞ into B_2 such that :
$\exists C > 0$, $\exists\varepsilon > 0$, $\forall g \in B_1$, $\forall\mu,\nu \in \mathbb{Z}$

$$\|\chi_\mu \mathcal{A}(\chi_\nu g)\|_{B_2} \leq Ce^{-\varepsilon|\mu-\nu|}\|\chi_\nu g\|_{B_1} .$$

Then $\mathcal{A}$ extends to a bounded operator $B_1 \to B_2$ defined by $\mathcal{A}g = \sum_\nu \mathcal{A}(\chi_\nu g)$.

8. PROOF : REGULARITY ON WEDGES – REDUCTION TO FLAT DATA

Let Γ be a plane sector ; we denote by ω its opening angle. Let $\mathcal{D}$ be the wedge $\mathbb{R}\times\Gamma$. The variables are (y,z) with $y\in\mathbb{R}$ and $z\in\Gamma$. We denote by $\partial_j\Gamma$ for $j=1,2$ the two sides of Γ and $\partial_j\mathcal{D}$ the two corresponding sides of $\mathcal{D}$. On each of these sides we prescribe the Dirichlet or the Neumann condition. For doing that we choose $\partial_D\Gamma$ equal to

$$\emptyset,\quad \text{or}\quad \partial\Gamma,\quad \text{or}\quad \partial_1\Gamma,\quad \text{or}\quad \partial_2\Gamma$$

and we define $\partial_D\mathcal{D}$ as the set $\mathbb{R}\times\partial_D\Gamma$. The first case corresponds to the Neumann problem, the second to the Dirichlet problem and the last ones to mixed problems.

Following Notation 3.3, we set

$$\lambda_1=\begin{cases}\pi/\omega & \text{for Dirichlet or Neumann problems}\\ \pi/2\omega & \text{for mixed problems}\,.\end{cases}$$

The last sections of this work are devoted to the proof of

Theorem 8.1 *We assume that $0<k+2-2/p<\lambda_1$. Let $u\in H^1(\mathcal{D},\partial_D\mathcal{D})$ with compact support such that*

$$\forall v\in H^1(\mathcal{D},\partial_D\mathcal{D})\quad <\nabla u,\nabla v> \,=\, <f,v>\quad \textit{with}\quad f\in W^{k,p}(\mathcal{D}).$$

Then $u\in W^{k+2,p}(\mathcal{D})$.

The first stage of the proof is to reduce to the case when f has null traces on the edge $E:=\mathbb{R}\times\{0\}$ of $\mathcal{D}$. As in section 6, we use weighted spaces on $\mathcal{D}$:

$$V^{k,p}_\beta(\mathcal{D})=\{u\in\mathcal{D}'(\mathcal{D})\mid r^{\beta+|\alpha|-k}D^\alpha u\in L^p(\mathcal{D})\ \forall\alpha,\ |\alpha|\le k\}.$$

We obtain as a standard consequence of Hardy's inequality (compare Lemma 6.1)

Lemma 8.2 *We assume that $p\neq 2$. For any $f\in W^{k,p}(\mathcal{D})$ we have*

$$\forall\alpha\in\mathbb{N}^2,\ |\alpha|\le k-\frac{2}{p},\quad \partial_z^\alpha f\Big|_E=0\quad\Longrightarrow\quad f\in V^{k,p}_0(\mathcal{D}).$$

We also need the corresponding result about trace spaces. We denote by τ_j, resp. n_j, the tangent, resp. normal, unitary vectors to the side $\partial_j\Gamma$ for $j=1,2$.

Lemma 8.3 *We assume that $p\neq 2$. Let $m_j\in\{0,1\}$ for $j=1,2$. If $\psi_j\in W^{k+2-m_j-1/p,p}(\partial_j\mathcal{D})$ satisfy for $j=1,2$:*

$$\forall\alpha\in\mathbb{N},\ \alpha<k+2-m_j-2/p,\quad \partial^\alpha_{\tau_j}\psi_j\Big|_E=0,$$

then there exists $u\in V^{k+2,p}_0(\mathcal{D})$ such that :

$$\partial^{m_j}_{n_j}u=\psi_j\quad \textit{on}\quad \partial_j\mathcal{D},\ j=1,2.$$

We will use both these lemmas to prove the reduction to flat data.

Proposition 8.4 *Let $f \in W^{k,p}(\mathcal{D})$, with $k-2/p>0$ and $p \neq 2$. If $k+2-2/p<\lambda_1$, then there exists $u \in W^{k+2,p}(\mathcal{D})$ such that :*

$$u\Big|_E = 0,\ u=0 \text{ on } \partial_D\mathcal{D},\ \partial_n u = 0 \text{ on } \partial_N\mathcal{D} \quad \text{and} \quad \Delta u - f \in V_0^{k,p}(\mathcal{D}).$$

Proof. To fix the notations, we prove this statement for the Neumann conditions ($\partial_D\mathcal{D}=\emptyset$). Using Lemmas 8.2 and 8.3 we are reduced to find $u \in W^{k+2,p}(\mathcal{D})$ such that :

$$\left.\begin{array}{ll} \forall\beta, |\beta|<k-2/p & \partial_z^\beta(\Delta u - f)\Big|_E = 0 \\ \forall\beta, |\beta|<k+1-2/p & \partial_{\tau_j}^\beta \partial_{n_j} u\Big|_E = 0, \quad j=1,2 \\ & u\Big|_E = 0 \end{array}\right\} \tag{8.1}$$

We can be write (8.1) in the following form, with suitable coefficients $c_\alpha^{K,\beta}$ and $c_\alpha^{K,j}$:

$$\left.\begin{array}{ll} \forall K=2,\dots,[k+2-2/p],\ \forall\beta, |\beta|=K-2 & \Big(\sum\limits_{|\alpha|=K} c_\alpha^{K,\beta}\partial_z^\alpha u + \partial_y^2\partial_z^\beta u - \partial_z^\beta f\Big)\Big|_E = 0 \\ \forall K=1,\dots,[k+2-2/p],\ \forall j=1,2 & \Big(\sum\limits_{|\alpha|=K} c_\alpha^{K,j}\partial_z^\alpha u\Big)\Big|_E = 0 \\ & u\Big|_E = 0. \end{array}\right\} \tag{8.2}$$

If we find for each $\alpha \in \mathbb{N}^2$, $|\alpha|<k+2-2/p$,

$$u_\alpha \in W^{k+2-|\alpha|-2/p,p}(E)$$

such that :

$$\left.\begin{array}{ll} \forall K=2,\dots,[k+2-2/p],\ \forall\beta, |\beta|=K-2 & \Big(\sum\limits_{|\alpha|=K} c_\alpha^{K,\beta} u_\alpha + \partial_y^2 u_\beta - \partial_z^\beta f\Big)\Big|_E = 0 \\ \forall K=1,\dots,[k+2-2/p],\ \forall j=1,2 & \Big(\sum\limits_{|\alpha|=K} c_\alpha^{K,j} u_\alpha\Big)\Big|_E = 0 \\ & u_0\Big|_E = 0, \end{array}\right\} \tag{8.3}$$

then problem (8.2) will be solved by a lifting

$$u \in W^{k+2,p}(\mathcal{D}) \quad \text{such that} \quad \forall\alpha : \partial_z^\alpha u\Big|_E = u_\alpha.$$

Such a lifting of traces does exist ([29], p.223). As $\partial_z^\beta f \in W^{k-|\beta|-2/p,p}(E)$, it is easy to prove by induction over K that (8.3) will be solved if

$$\forall g_\beta \in W^{k-|\beta|-2/p,p}(E)$$

there exists u_α such that :

$$\left.\begin{array}{ll} \forall K=2,\dots,[k+2-2/p],\ \ \forall\beta, |\beta|=K-2 & \Big(\sum\limits_{|\alpha|=K} c_\alpha^{K,\beta} u_\alpha\Big)\Big|_E = g_\beta \\ \forall K=1,\dots,[k+2-2/p],\ \ \forall j=1,2 & \Big(\sum\limits_{|\alpha|=K} c_\alpha^{K,j} u_\alpha\Big)\Big|_E = 0 \\ & u_0\Big|_E = 0. \end{array}\right\} \tag{8.4}$$

To solve that system we introduce for $K = 0, \dots, [k+2-2/p]$ the square $(K+1)\times(K+1)$ matrix $M_K = (\mathfrak{m}^K_{\alpha,\gamma})$ defined by :

$$\begin{array}{llll} \text{for } K=0, & M_0 = 1 & & \\ \text{for } K \geq 1, & \mathfrak{m}^K_{\alpha,\gamma} = c^{K,j}_\alpha & \text{with} & \gamma = (K,0) \text{ for } j=1 \\ & & & \gamma = (0,K) \text{ for } j=2 \\ & \mathfrak{m}^K_{\alpha,\gamma} = c^{K,\beta}_\alpha & \text{with} & \gamma = (\beta_1+1, \beta_2+1). \end{array}$$

It is enough to prove that each M_K is injective, thus invertible. We will indeed have :

$$(u_\alpha)_{|\alpha|=K} = M_K^{-1}\Big(0, 0, (g_\beta)_{|\beta|=K-2}\Big).$$

As $g_\beta \in W^{k-K+2-2/p,p}(E)$, we will obtain $u_\alpha \in W^{k-K+2-2/p,p}(E)$.

Let us prove that M_K is injective. Let us consider $v_\alpha \in \mathbb{R}$ such that $M_K(v_\alpha)_{|\alpha|=K} = 0$. If we set

$$P := \sum_{|\alpha|=K} \frac{1}{\alpha!}\, v_\alpha\, z^\alpha,$$

we have $\partial^\alpha P(0) = v_\alpha$ and (8.3) yields :

$$\begin{array}{ll} \forall\beta,\ |\beta| = K-2 & \displaystyle\sum_{|\alpha|=K} c^{K,\beta}_\alpha\, \partial^\alpha P(0) = 0 \\ j = 1,2 & \displaystyle\sum_{|\alpha|=K} c^{K,j}_\alpha\, \partial^\alpha P(0) = 0. \end{array}$$

Coming back to (8.1), that means :

$$\begin{array}{ll} \forall\beta,\ |\beta| = K-2 & \partial^\beta \Delta P = 0 \\ j = 1,2 & \partial^{K-1}_{\tau_j}\partial_{n_j} P = 0. \end{array}$$

As P is homogeneous of degree K, we obtain :

$$\Delta P = 0 \quad \text{and} \quad \partial_{n_j} P = 0 \text{ on } \partial_j\Gamma.$$

Since $K < \lambda_1$, no such polynomial exists. ∎

9. PROOF : REGULARITY ON WEDGES – SYMBOLIC CALCULUS

With the help of Proposition 8.4, the proof of Theorem 8.1 is reduced to prove :

Theorem 9.1 *We assume that* $0 < k+2-2/p < \lambda_1$. *Let* $u \in H^1(\mathcal{D}, \partial_D\mathcal{D})$ *with compact support such that*

$$\forall v \in H^1(\mathcal{D}, \partial_D\mathcal{D}) \quad <\nabla u, \nabla v> = <f, v> \quad \textit{with} \quad f \in V^{k,p}_0(\mathcal{D}).$$

If, moreover $u\Big|_E = 0$*, then* $u \in V^{k+2,p}_0(\mathcal{D})$.

Remark 9.2 If $\partial_D\mathcal{D} \neq \emptyset$, i. e. if this is not the Neumann problem, the condition $u\Big|_E = 0$ is a consequence of $u \in H^1(\mathcal{D}, \partial_D\mathcal{D})$. We prove in the appendix how we can reduce to the case when $u\Big|_E = 0$ for the Neumann problem (see Proposition 11.3).

In this section, we are going to show how Theorem 9.1 can be proved by partial Fourier transform along the edge with the help of Marcinkiewicz's theorem. We will prove suitable estimates for the operator valued symbol in the next section.

Relying upon local a priori estimates on domains in the form

$$\{(y,z) \mid |y-y_0| < e^{-\nu}, e^{-\nu-1} < r < e^{-\nu+1}\}$$

for $\nu \in \mathbb{Z}$ and $y_0 \in e^{-\nu}\mathbb{Z}$, we can show that (compare [24], §4) :

$$\|u\|_{V_0^{k+2,p}(\mathcal{D})} \le C(\|f\|_{V_0^{k,p}(\mathcal{D})} + \|u\|_{V_{-k-2}^{0,p}(\mathcal{D})}).$$

So, we are reduced to prove :

$$u \in V_{-k-2}^{0,p}(\mathcal{D}). \tag{9.1}$$

We set :

$$B_1 = V_{-k}^{0,p}(\Gamma) \quad \text{and} \quad B_2 = V_{-k-2}^{0,p}(\Gamma). \tag{9.2}$$

When $k \ge 0$, we have : $f \in L^p(\mathbb{R}, B_1)$. When $k = -1$, similar arguments can be used [10].

So, to obtain (9.1), it suffices to state the following estimate :

$$\|u\|_{L^p(\mathbb{R},B_2)} \le C\|f\|_{L^p(\mathbb{R},B_1)}. \tag{9.3}$$

To state that, we use Lemma 7.2. We are going to construct a bounded operator $\mathfrak{R}$: such that :

$$\mathfrak{R} : L^p(\mathbb{R}, B_1) \to L^p(\mathbb{R}, B_2) \quad \text{such that} \quad \mathfrak{R}u = f. \tag{9.4}$$

Since $L^p(\mathbb{R}, B_j)$ for $j = 1, 2$ has l^p-type, Lemma 7.2 yields that it suffices to construct $\mathfrak{R}$ on compactly supported functions and satisfying :

$$\exists C > 0,\ \forall g \in L^p(\mathbb{R}, B_1),\ \forall \mu, \nu \in \mathbb{Z} \quad \|\chi_\mu \mathfrak{R} \chi_\nu g\|_{L^p(\mathbb{R},B_2)} \le C e^{-\varepsilon|\mu-\nu|} \|\chi_\nu g\|_{L^p(\mathbb{R},B_1)}. \tag{9.5}$$

That will give estimate (9.3).

In order to apply Marcinkiewicz's theorem, we are going to replace B_1 and B_2 by suitable Hilbert spaces H_1 and H_2. We want the estimates :

$\exists C > 0, \forall w, \forall \nu \in \mathbb{Z}$

$$\|\chi_\nu w\|_{L^p(\mathbb{R},H_1)} \le C\|\chi_\nu w\|_{L^p(\mathbb{R},B_1)} \quad \text{and} \quad \|\chi_\nu w\|_{L^p(\mathbb{R},B_2)} \le C\|\chi_\nu w\|_{L^p(\mathbb{R},H_2)} \tag{9.6}$$

We also want to construct an operator $\widetilde{\mathfrak{R}}$ such that :

$$\widetilde{\mathfrak{R}}(\chi_\nu f) = \mathfrak{R}(\chi_\nu f) \quad \forall \nu$$

and

$$\|\chi_\mu \widetilde{\mathfrak{R}} \chi_\nu g\|_{L^p(\mathbb{R},H_2)} \le C e^{-\varepsilon|\mu-\nu|} \|\chi_\nu g\|_{L^p(\mathbb{R},H_1)}. \tag{9.7}$$

It is clear that (9.6) and (9.7) imply (9.5). We set

$$\beta = -k - 2 + 2/p$$

and we choose :

$$\left.\begin{array}{c} H_2 = \{w \in V_\beta^{1,2}(\Gamma) \mid w = 0 \text{ on } \partial_D\Gamma\} \\ H_1 \text{ is the dual space of } \{w \in V_{-\beta}^{1,2}(\Gamma) \mid w = 0 \text{ on } \partial_D\Gamma\}. \end{array}\right\} \tag{9.8}$$

The estimates (9.6) are a consequence of the isomorphism in Lemma 6.5 and of the assumptions on (k, p). As another consequence of these assumptions, we also have

$$-\lambda_1 < \beta < 0.$$

To end the proof of theorem 9.1, we only have to construct $\widetilde{\mathfrak{R}}$ satisfying (9.7) and such that :

$$\sum_\nu \widetilde{\mathfrak{R}}(\chi_\nu f) = u. \tag{9.9}$$

We search for $\widetilde{\mathfrak{R}}$ with the following form :

$$\widetilde{\mathfrak{R}} = \mathcal{F}^{-1}\Pi(\xi)\mathcal{F} \tag{9.10}$$

where $\xi \to \Pi(\xi)$ is a suitable symbol with values in $\mathcal{L}(H_1, H_2)$ – the bounded operators from H_1 into H_2. Here, $\mathcal{F}$ denotes the partial Fourier transform on $\mathbb{R} \times \Gamma$. Roughly speaking, $\Pi(\xi)$ will be $(-\Delta + \xi^2)^{-1}$. In the situation of the Neumann problem, a correction is needed : it will consist of the elimination of the first trace along the edge. As we have supposed that $u\Big|_E = 0$, we will still obtain the convergence (9.9) towards u (see the appendix). Now, we rely on the following Lemma 9.4 (compare Proposition 11.1 in the appendix).

Notation 9.3 *For $\gamma \in \mathbb{R}$, we introduce the following spaces*

$$E_\gamma(\Gamma, \partial_D\Gamma) := \{w \in V_\gamma^{1,2}(\Gamma) \mid r^\gamma w \in L^2(\Gamma) \text{ and } w = 0 \text{ on } \partial_D\Gamma\}.$$

and the operator

$$\begin{array}{rcc} \mathfrak{E}_\gamma : E_\gamma(\Gamma, \partial_D\Gamma) & \longrightarrow & E_{-\gamma}(\Gamma, \partial_D\Gamma)' \\ \mathfrak{u} & \longrightarrow & (\mathfrak{v} \to < \nabla\mathfrak{u}, \nabla\mathfrak{v} > + < \mathfrak{u}, \mathfrak{v} >). \end{array}$$

We have

Lemma 9.4 *We assume that $\partial_D\Gamma \neq \emptyset$. Let $\gamma \in]-\lambda_1, \lambda_1[$. Then $\mathfrak{E}_\gamma$ is one to one.*

The important fact is the equality

$$E_0(\Gamma, \partial_D\Gamma) = H^1(\Gamma, \partial_D\Gamma)$$

which is no longer true when $\partial_D\Gamma = \emptyset$. So, as a consequence of the Lax–Milgram lemma, $\mathfrak{E}_0$ is one to one ; then we obtain the Lemma by classical "corner" analysis on the plane sector Γ ([24], see also Proposition 11.1).

Let us introduce the homotheties $\mathcal{H}_\rho$

$$\mathcal{H}_\rho w(z) = w(\rho z). \tag{9.11}$$

We note that :

$$-\Delta + |\xi|^2 = \xi^2\, \mathcal{H}_{1/|\xi|}\, (-\Delta + 1)\, \mathcal{H}_{|\xi|}. \tag{9.12}$$

Since $(\mathfrak{E}_\beta)^{-1}$ exists and solves $-\Delta + 1$ in the convenient spaces, we set :

$$\Pi(\xi) = \xi^{-2}\, \mathcal{H}_{|\xi|}\, (\mathfrak{E}_\beta)^{-1}\, \mathcal{H}_{1/|\xi|}. \tag{9.13}$$

Then we note that $\Pi(\xi)\mathfrak{f}$ solves the Dirichlet or mixed problem in Γ for the operator $-\Delta + |\xi|^2$ with right hand side $\mathfrak{f}$. From that, we easily deduce the operator $\widetilde{\mathfrak{R}}$ defined by (9.10) and (9.13) satisfies (9.9).

It remains to prove (9.7). We have :

$$\chi_\mu\, \widetilde{\mathfrak{R}}\, \chi_\nu = \mathcal{F}^{-1} \Pi_{\mu,\nu}(\xi) \mathcal{F}$$

where $\Pi_{\mu,\nu}(\xi) := \chi_\mu \Pi(\xi) \chi_\nu$. We are going to prove in the next paragraph the following estimate, where $|||\quad|||$ denotes the norm of bounded operators $H_1 \to H_2$:

$\exists \varepsilon > 0$, $\exists C > 0$, $\forall \mu, \nu \in \mathbb{Z}$, $\forall \xi \neq 0$:

$$|||\, \Pi_{\mu,\nu}(\xi)\, ||| + |||\, \xi \frac{d}{d\xi} \Pi_{\mu,\nu}(\xi)\, ||| \leq C e^{-\varepsilon|\mu - \nu|}. \tag{9.14}$$

This estimate will imply (9.7) with the application of Marcinkiewicz's theorem.

The idea of applying Marcinkiewicz's theorem with the help of the estimate (9.14) comes from [24]. Our method to obtain (9.14) is different from [24].

10. PROOF : REGULARITY ON WEDGES – ESTIMATES OF THE SYMBOLS

The mapping $\xi \to \Pi(\xi)$ is C^∞ from $\mathbb{R} \setminus \{0\}$ into $\mathcal{L}(H_1, H_2)$. For $\varepsilon > 0$ such that

$$[\beta - \varepsilon, \beta + \varepsilon] \subset] - \lambda_1, 0],$$

Π is still C^∞ from $\mathbb{R} \setminus \{0\}$ into $\mathcal{L}\Big(V^{1,2}_{-\zeta}(\Gamma, \partial_D \Gamma)', V^{1,2}_{\zeta}(\Gamma, \partial_D \Gamma)\Big)$, for all $\zeta \in [\beta - \varepsilon, \beta + \varepsilon]$. Thus, such is also the case for $\Pi'(\xi) := (\frac{d}{d\xi}\Pi)(\xi)$. Therefore, with the help of Lemma 7.1, we deduce that there exists $C > 0$, such that $\forall \mu, \nu \in \mathbb{Z}$, $\forall \xi \in [1, 2]$:

$$|||\, \chi_\mu\, \Pi(\xi)\, \chi_\nu\, ||| \leq C e^{-\varepsilon|\mu - \nu|}, \quad \text{and} \quad |||\, \chi_\mu\, \xi \Pi'(\xi)\, \chi_\nu\, ||| \leq C e^{-\varepsilon|\mu - \nu|}. \tag{10.1}$$

We deduce from (9.13) the formula

$$\Pi(\xi) = \xi^{-2}\, \mathcal{H}_{|\xi|}\, \Pi(1)\, \mathcal{H}_{1/|\xi|}. \tag{10.2}$$

We are now going to prove that we have for $\xi\Pi'(\xi)$ the following similar formula :

$$\xi\Pi'(\xi) = \xi^{-2}\,\mathcal{H}_{|\xi|}\,\Pi'(1)\,\mathcal{H}_{1/|\xi|}. \tag{10.3}$$

We differentiate formula (10.2) and we use that for $\xi > 0$:

$$\frac{d}{d\xi}\mathcal{H}_\xi = \frac{1}{\xi}\mathcal{H}_\xi \circ (r\partial_r) \quad \text{and} \quad \frac{d}{d\xi}\mathcal{H}_{1/\xi} = -\frac{1}{\xi}(r\partial_r) \circ \mathcal{H}_{1/\xi}.$$

Thus, we easily obtain that :

$$\xi\Pi'(\xi) = \xi^{-2}\,\mathcal{H}_{|\xi|}\,\{[r\partial_r, \Pi(1)] - 2\Pi(1)\}\,\mathcal{H}_{1/|\xi|}.$$

For $\xi = 1$, that yields :

$$\Pi'(1) = [r\partial_r, \Pi(1)] - 2\Pi(1).$$

The two previous formulas give us (10.3).

Now, we want to prove (10.1) for all $\xi \neq 0$. To simplify the notations, let $\xi > 0$. There exists $l \in \mathbb{Z}$ such that $\xi \in [2^l, 2^{l+1}[$. We set $\rho := 2^l$. (10.2) gives that :

$$\Pi(\xi) = \rho^{-2}\,\mathcal{H}_\rho\,\Pi(\frac{\xi}{\rho})\,\mathcal{H}_{1/\rho}.$$

So :

$$\begin{aligned} \chi_\mu\,\Pi(\xi)\,\chi_\nu &= \rho^{-2}\,\chi_\mu\,\mathcal{H}_\rho\,\Pi(\frac{\xi}{\rho})\,\mathcal{H}_{1/\rho}\,\chi_\nu \\ &= \rho^{-2}\,\mathcal{H}_\rho\,\chi_{\mu-l}\,\Pi(\frac{\xi}{\rho})\,\chi_{\nu-l}\,\mathcal{H}_{1/\rho}. \end{aligned}$$

Therefore :

$$\|\chi_\mu\,\Pi(\xi)\,\chi_\nu\|_{L(H_1,H_2)} \leq \rho^{-2}\|\mathcal{H}_\rho\|_{\mathcal{L}(H_2,H_2)} \times \|\chi_{\mu-l}\,\Pi(\frac{\xi}{\rho})\,\chi_{\nu-l}\|_{\mathcal{L}(H_1,H_2)} \times \|\mathcal{H}_{1/\rho}\|_{L(H_1,H_1)}. \tag{10.4}$$

But it is easy to prove that :

$$\|\mathcal{H}_\rho\|_{\mathcal{L}(H_2,H_2)} = \rho^{-\beta} \quad \text{and} \quad \|\mathcal{H}_{1/\rho}\|_{\mathcal{L}(H_1,H_1)} = \rho^{2+\beta}. \tag{10.5}$$

On the other hand, as $\xi/\rho \in [1,2[$, the first estimate in (10.1) yields :

$$\|\chi_{\mu-l}\,\Pi(\frac{\xi}{\rho})\,\chi_{\nu-l}\|_{\mathcal{L}(H_1,H_2)} \leq Ce^{-\varepsilon|\mu-l-\nu+l|}. \tag{10.6}$$

Inequalities (10.4), (10.5) and (10.6) give the uniform estimate :

$$|||\,\chi_\mu\,\Pi(\xi)\,\chi_\nu\,||| \leq Ce^{-\varepsilon|\mu-\nu|},$$

for $\xi \in \mathbb{R}\setminus\{0\}$. We obtain the corresponding estimate for $\xi\Pi'(\xi)$ by starting from the second estimate in (10.1) : (10.3) yields an estimate as (10.4) and we deduce that :

$$|||\,\chi_\mu\,\xi\Pi'(\xi)\,\chi_\nu\,||| \leq Ce^{-\varepsilon|\mu-\nu|}.$$

11. APPENDIX : NEUMANN PROBLEM IN A WEDGE

For the Neumann problem, it is impossible to define the solving symbol $\Pi(\xi)$ by formula (9.13) as in section 9, because the operators $\mathfrak{E}_\gamma$ are *never* invertible. We begin this section by a study of the properties of these operators. They are still defined according to Notation 9.3 as operators

$$E_\gamma(\Gamma) \longrightarrow E_{-\gamma}(\Gamma)'$$

where $E_\gamma(\Gamma) := E_\gamma(\Gamma, \emptyset)$.

Let us denote by $\mathfrak{A}$ the operator

$$\mathfrak{u} \longrightarrow (\mathfrak{v} \rightarrow < \nabla\mathfrak{u}, \nabla\mathfrak{v} > + < \mathfrak{u}, \mathfrak{v} >)$$

acting from $H^1(\Gamma)$ into its dual space. $\mathfrak{A}$ is one to one.
Now $\lambda_1 = \pi/\omega$.

Proposition 11.1
(i) If $-\pi/\omega < \gamma < 0$, $\mathfrak{E}_\gamma$ is injective and Fredhom ; the codimension of its range is equal to one ; there exists a compactly supported function $\sigma \in C^\infty(\overline{\Gamma})$ such that :

$$\sigma(0) = 1 \quad (\textit{thus } \sigma \notin E_\gamma(\Gamma)) \quad \textit{and} \quad \mathrm{Rg}\, \mathfrak{E}_\gamma \oplus (\mathfrak{A}\sigma) = E_{-\gamma}(\Gamma)'.$$

(ii) If $0 < \gamma < \pi/\omega$, $\mathfrak{E}_\gamma$ is onto, and its kernel is one dimensional ; there exists a generator $\mathfrak{K}$ of Ker $\mathfrak{E}_\gamma$ *(which does not depend on γ) such that $< \mathfrak{K}, \mathfrak{A}\sigma >= 1$.*

Proof.
(i) As a consequence of the general theory of such problems [24], $\mathfrak{E}_\gamma$ is Fredholm if and only if γ^2 is not an eigenvalue of the Neumann problem for the opeator $-\partial_\theta^2$ on $]0, \omega[$. The first eigenvalue being 0 and the next one being $(\frac{\pi}{\omega})^2$, we obtain that $\mathfrak{E}_\gamma$ is Fredholm when $-\pi/\omega < \gamma < 0$. On the other hand, the following property holds if $\gamma < \gamma'$ (cf [24])

$$\mathfrak{u} \in E_\gamma(\Gamma) \text{ and } \mathfrak{E}_\gamma\mathfrak{u} \in E_{-\gamma'}(\Gamma)' \quad \Longrightarrow \quad \mathfrak{u} \in E_{\gamma'}(\Gamma)$$

Then any element of Ker $\mathfrak{E}_\gamma$ belongs to $H^1(\Gamma)$ thus is equal to 0.

When $-\pi/\omega < \gamma < 0$, a classical proof based on Mellin transform yields that there is only one singular function σ which is generated by the constant function. A convenient choice for σ is :

$$\sigma(z) = \varphi(r) \sum_{0 \le 2j \le -\gamma} (-1)^j \, 4^{-j} \, \frac{1}{(j!)^2} \, r^{2j} \tag{11.1}$$

where φ is a smooth cut-off function which is equal to 1 in a neighborhood of 0.

(ii) The dual of $\mathfrak{E}_\gamma$ is $\mathfrak{E}_{-\gamma}$. Thus

$$\mathrm{Rg}\, \mathfrak{E}_{-\gamma} = (\mathrm{Ker}\, \mathfrak{E}_\gamma)^\perp$$

and the statement is a consequence of the Fredholm alternative. ∎

Corollary 11.2 *Let* $\mathfrak{f} \in E_{-\gamma}(\Gamma)'$ *with* $-\pi/\omega < \gamma < 0$. *Then there exists a unique* $\mathfrak{v} \in E_\gamma(\Gamma)$ *such that*

$$\mathfrak{E}_\gamma \mathfrak{v} = \mathfrak{f} - < \mathfrak{K}, \mathfrak{f} > \mathfrak{A}\sigma.$$

If moreover $\mathfrak{f} \in H^1(\Gamma)'$ *and* $\mathfrak{u} := \mathfrak{A}^{-1}\mathfrak{f}$ *then*

$$\mathfrak{u} = \mathfrak{v} + < \mathfrak{K}, \mathfrak{f} > \sigma.$$

If moreover $\mathfrak{f} \in V_0^{k,p}(\Gamma)$ *with* $0 < k + 2 - 2/p$, *then*

$$< \mathfrak{K}, \mathfrak{f} > = \mathfrak{u}(0) \quad \textit{and} \quad \mathfrak{u} = \mathfrak{v} + \mathfrak{u}(0)\sigma.$$

We are going to show how to modify the proof of Theorem 9.1 to reach the case of the Neumann problem. Then, we will prove how to reduce to the case when u is zero on the edge of $\mathcal{D}$.

Proof of Theorem 9.1 for Neumann. We recall that $-\pi/\omega < \beta < 0$. Applying the above corollary, we define $\Pi(1)$ as

$$\Pi(1)\mathfrak{f} := (\mathfrak{E}_\beta)^{-1}(\mathfrak{f} - < \mathfrak{K}, \mathfrak{f} > \mathfrak{A}\sigma)$$

and $\Pi(\xi)$ by the formula (cf (9.13) and (10.2))

$$\Pi(\xi) = \xi^{-2}\, \mathcal{H}_{|\xi|}\, \Pi(1)\, \mathcal{H}_{1/|\xi|}.$$

Since this formula allows, exactly as in section 10, to prove the estimate (9.14) on the symbol, it remains to prove that equality (9.9) holds, i. e. setting ($\hat{}$ denotes the Fourier transform $\mathcal{F}$)

$$\hat{v}_\nu(\xi) := \Pi(\xi)(\chi_\nu \hat{f})$$

that we have

$$\sum_\nu \hat{v}_\nu = \hat{u}. \tag{11.2}$$

Setting

$$\mathfrak{A}(\xi) = \xi^{-2}\, \mathcal{H}_{|\xi|}\, \mathfrak{A}^{-1}\, \mathcal{H}_{1/|\xi|} \quad \text{and} \quad \hat{u}_\nu(\xi) := \mathfrak{A}(\xi)(\chi_\nu \hat{f})$$

it is clear that $\sum_\nu \hat{u}_\nu = \hat{u}$. Then using corollary 11.2 we obtain that

$$u_\nu(\xi, z) = v_\nu(\xi, z) + u_\nu(\xi, 0)\, \sigma(|\xi| z).$$

Since $\sum_\nu \hat{u}_\nu(\xi, 0) = \hat{u}(\xi, 0)$ and $\hat{u}(\xi, 0) = 0$ we deduce (11.2). ∎

Our last task is to prove

Proposition 11.3 *We assume that* $0 < k + 2 - 2/p < \pi/\omega$. *Let* $u \in H^1(\mathcal{D})$ *with compact support such that*

$$\forall v \in H^1(\mathcal{D}) \quad < \nabla u, \nabla v > = < f, v > \quad \textit{with} \quad f \in V_0^{k,p}(\mathcal{D}).$$

Then there exists $U \in W^{k+2,p}(\mathcal{D})$ *such that* $u - U\Big|_E = 0$. *If* $k \geq 0$, *we have moreover* $\partial_n U = 0$ *on* $\partial\mathcal{D}$.

The main point is to prove that u has a trace u_0 on the edge E and that this trace belongs to $W^{k+2-2/p,p}(E)$. When we have that, the construction of U will be done with the help of lifting of traces (compare Proposition 8.4 and Lemma 8.3).

Let us prove that u_0 makes sense in $\mathcal{S}'(\mathbb{R})$. The partial Fourier transform yields that $\hat{u}(\xi) \in H^1(\Gamma)$ and is solution of the Neumann problem associated to the operator $-\Delta + |\xi|^2$ with right hand side $\hat{f}(\xi)$. But $f \in \mathcal{S}'(\mathbb{R}, W^{k,p}(\Gamma))$; so $\hat{f} \in \mathcal{S}'(\mathbb{R}, W^{k,p}(\Gamma))$. The result of Theorem 6.4 still holds for $-\Delta + |\xi|^2$ instead of $-\Delta$, and it is easy to obtain polynomial estimates with respect to ξ. That gives $u \in \mathcal{S}'(\mathbb{R}, W^{k+2,p}(\Gamma))$ hence $u_0 \in \mathcal{S}'(\mathbb{R})$.

To obtain the regularity of u_0, we are going to prove :

Lemma 11.4 *There exist functions $K_\xi(r)$ independent from ω such that*

$$\forall \xi \neq 0 : \quad \hat{u}_0(\xi) = \frac{1}{\omega} \int_\Gamma \hat{f}(\xi, z)\, K_\xi(r)\, dz.$$

That lemma will allow to state that u_0 is also the trace of the solution of a Neumann problem on the half-space with a right hand side in $W^{k,p}$.

Proof. We deduce from Corollary 11.2 that

$$\hat{u}_0(\xi) = \int_\Gamma \hat{f}(\xi, z)\, \mathfrak{K}(|\xi| z)\, dz.$$

We are going to prove that there exists a function $K(r)$ such that

$$\mathfrak{K}(z) = \frac{1}{\omega}\, K(r). \tag{11.3}$$

So, setting $K_\xi(r) := K(r|\xi|)$, we will obtain the Lemma. To prove (11.3) we use the separation of variables in polar coordinates. Similarly to the operators $\mathfrak{C}_\gamma$, we introduce the following operators $\mathfrak{B}_\gamma$ which come from the radial part of $-\Delta$ (compare Notation 9.3) :

$$\begin{array}{rcc} \mathfrak{B}_\gamma : E_\gamma(\mathbb{R}_+) & \longrightarrow & E_{-\gamma}(\mathbb{R}_+)' \\ \mathfrak{u} & \longrightarrow & (\mathfrak{v} \to < d_r\mathfrak{u}, d_r\mathfrak{v} > + < \mathfrak{u}, \mathfrak{v} >) \end{array}$$

with $r\,dr$ as mesure on $\mathbb{R}_+$ in relation with the polar coordinates in Γ : $dz = r\,dr\,d\theta$. $\mathfrak{B}$ denotes the same operator acting from $H^1(\mathbb{R}_+)$ onto its dual space.

Now, all the results (Proposition 11.1 and Corollary 11.2) we proved for $\mathfrak{C}_\gamma$ and $\mathfrak{A}$ remain true for $\mathfrak{B}_\gamma$ and $\mathfrak{B}$ (moreover *(i)* holds for any $\gamma < 0$ and *(ii)* holds for any $\gamma > 0$) : we may take as singular function $\tilde{\sigma}(r)$ such that $\tilde{\sigma}(r) = \sigma(z)$, cf (11.1) ; there is a function $K \in \cap_{\gamma>0} \operatorname{Ker} \mathfrak{B}_\gamma$ which satisfies :

$$< K, \mathfrak{B}\tilde{\sigma} >= 1.$$

But $\mathfrak{B}\tilde{\sigma} = \mathfrak{A}\sigma$. So :

$$\begin{aligned} \int_\Gamma K(r)\, \mathfrak{A}\sigma\, dz &= \int_0^\omega \int_0^{+\infty} K(r)\, \mathfrak{B}\tilde{\sigma}\, r\, dr\, d\theta \\ &= \omega < K, \mathfrak{B}\tilde{\sigma} >= \omega. \end{aligned}$$

So, we have proved that (11.3) holds. ∎

Proof of Proposition 11.3. Lemma 11.4 still holds when $\omega = \pi$ (the domain is then the half plane P). f satisfying the assumptions of Proposition 11.3, we introduce $\tilde{f}$

defined by :

$$\tilde{f}(y,\tilde{z}) = f(y,z), \quad y \in \mathbb{R}, \quad \tilde{z} \in P$$

where, for $\tilde{z} \in P$:

$$z \in \Gamma \quad \text{is such that} \quad \frac{\tilde{z}}{|\tilde{z}|} = \frac{\pi}{\omega}\,\frac{z}{|z|}.$$

Since $f \in V_0^{k,p}(\mathbb{R}\times\Gamma)$, we obtain that $\tilde{f} \in V_0^{k,p}(\mathbb{R}\times P)$; as the support of f is compact, we deduce that $\tilde{f}$ belongs to $W^{k,p}(\mathbb{R}\times P) \cap H^1(\mathbb{R}\times P)'$. So, we can consider the solution

$$w \in W := \{v \in L^2_{\rm loc}(\mathbb{R}\times P) \mid \nabla v \in L^2(\mathbb{R}\times P)\}$$

of the Neumann problem on $\mathbb{R}\times P$:

$$< \nabla w, \nabla v > = < \tilde{f}, v > \quad \forall v \in W.$$

As $\mathbb{R}\times P$ has a smooth boundary, $w \in W^{k+2,p}_{\rm loc}(\mathbb{R}\times P)$. So, the first trace w_0 of w on the "edge" E $(r=0)$ belongs to $W^{k+2-2/p,p}_{\rm loc}(E)$. Finally, Lemma 11.4 yields

$$\begin{aligned}
\hat{w}_0(\xi) &= \frac{1}{\pi}\int_P \hat{\tilde{f}}(\xi,\tilde{z})\,K_\xi(r)\,d\tilde{z} \\
&= \frac{1}{\pi}\int_0^\pi \int_0^{+\infty} \hat{\tilde{f}}(\xi,\tilde{z})\,K_\xi(r)\,r dr\,d\tilde{\theta} \\
&= \frac{1}{\omega}\int_0^\omega \int_0^{+\infty} \hat{f}(\xi,z)\,K_\xi(r)\,r dr\,d\theta \\
&= \hat{u}_0(\xi).
\end{aligned}$$

∎

REFERENCES

[1] S. Agmon, A. Douglis, L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I. *Comm. Pure Appl. Math.* **12** (1959) 623–727.

[2] A. E. Beagles, J. R. Whiteman. General conical singularities in three-dimensional poisson problems. *Math. Meth. in the Appl. Sci.* **11** (1989) 215–235.

[3] P. Berard, D. Meyer. Inégalités isopérimétriques et applications. *Ec. Norm. Sup. Série 4* **15** (1982) 513–542.

[4] C. Bernardi, C. Amrouche, M. Dauge, V. Girault. Vector potentials for the three-dimensional Stokes problem. In preparation, 1991.

[5] I. Chavel. *Eigenvalues in Riemannian geometry.* Academic Press 1984.

[6] M. Costabel. Personal communication. Darmstadt, 1989.

[7] M. Costabel, M. Dauge. General edge asymptotics of solutions of second order elliptic boundary value problems. Preprint. Paris VI, Nantes. 1991.

[8] M. Costabel, M. Dauge. Développement asymptotique le long d'une arête pour des équations elliptiques d'ordre 2 dans $\mathbb{R}^3$. *C. R. Acad. Sc. Paris, Série I* **312** (1991) 227–232.

[9] M. Dauge. Neumann and mixed problems on curvilinear polyhedra in L^p Sobolev spaces. Séminaire Equations aux Dérivées Partielles, Nantes. 1989.

[10] M. Dauge. Neumann problems on polyhedra in L^p Sobolev spaces. Séminaire Equations aux Dérivées Partielles, Nantes. 1988.

[11] M. Dauge. *Elliptic Boundary Value Problems in Corner Domains — Smoothness and Asymptotics of Solutions.* Lecture Notes in Mathematics, Vol. 1341. Springer-Verlag, Berlin 1988.

[12] M. Dauge. Problèmes de Neumann et de Dirichlet sur un polyèdre dans $\mathbb{R}^3$: régularité dans des espaces de Sobolev L^p. *C. R. Acad. Sc. Paris, Série I* **307** (1988) 27–32.

[13] M. Dauge. Problèmes mixtes pour le laplacien dans des domaines polyédraux courbes. *C. R. Acad. Sc. Paris, Série I* **309** (1989) 553–558.

[14] M. Dauge, B. Helffer. Eigenvalues variation II. Mutidimensional problems. Séminaire Equations aux Dérivées Partielles, Nantes. 1989.

[15] M. Faierman. Regularity of solutions of an elliptic boundary value problem in a rectangle. *Comm. in Partial Diff. Equat.* **12 (3)** (1987) 285–305.

[16] G. Fichera. Comportamento asintotico del campo elettrico e della densità elettrica in prossimità dei punti singolari della superficie conduttore. *Rend. Sem. Univers. Politecn. Torino* **32** (1973–74) 111–143.

[17] P. Grisvard. Behavior of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain. *SYNSPADE III, B.Hubbard Ed. Academic Press* (1975) 207–274.

[18] P. Grisvard. *Boundary Value Problems in Non-Smooth Domains.* Pitman, London 1985.

[19] D. Jerison, C. E. Kenig. The functional calculus for the Laplacian on Lipschitz domains. Journées Equations aux Dérivées Partielles, Saint-Jean-de-Monts. 1989.

[20] V. A. Kondrat'ev. Boundary-value problems for elliptic equations in domains with conical or angular points. *Trans. Moscow Math. Soc.* **16** (1967) 227–313.

[21] P. Li, S. T. Yau. Estimates of eigenvalues of a compact riemannian manifold. *Proceedings of Symposia in Pure Mathematics* **36** (1980) 205–239.

[22] V. Maz'ya, T. Shaposhnikova. Requirements on the boundary in the L^p theory of elliptic boundary value problems. *Soviet Math. Dokl.* **21, 2** (1980) 576–580.

[23] V. Maz'ya, T. Shaposhnikova. Multipliers of Sobolev spaces in a domain. *Amer. Math. Soc. Transl. (2)* **124** (1984) 25–43.

[24] V. G. Maz'ya, B. A. Plamenevskii. L^p estimates of solutions of elliptic boundary value problems in a domain with edges. *Trans. Moscow Math. Soc.* **1** (1980) 49–97.

[25] V. G. Maz'ya, B. A. Plamenevskii. Estimates in L^p and in Hölder classes and the Miranda-Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary. *Amer. Math. Soc. Transl. (2)* **123** (1984) 1–56.

[26] V. G. Maz'ya, J. Rossmann. Über die Asymptotik der Lösungen elliptischer Randwertaufgaben in der Umgebung von Kanten. *Math. Nachr.* **138** (1988) 27–53.

[27] S. Rempel, B. W. Schulze. *Asymptotics for Elliptic Mixed Boundary Problems.* Akademie-Verlag, Berlin 1989.

[28] B. W. Schulze. Regularity with continuous and branching asymptotics for elliptic operators on manifold with edges. *Integral Equations and Operator Theory* **11, 4** (1988) 557–602.

[29] H. Triebel. *Interpolation theory. Function spaces. Differential operators.* North-Holland Mathematical Library. North-Holland, Amsterdam 1978.

[30] H. Walden, R. B. Kellogg. Numerical determination of the fundamental eigenvalue for the laplace operator on a spherical domain. *Journal of Engineering Mathematics* **11, 4** (1977) 299–318.

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