STRONGLY ELLIPTIC PROBLEMS NEAR CUSPIDAL POINTS AND EDGES

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Abstract. *After an overview of the various geometrical situations occurring for two-dimensional piecewise smooth domains, we concentrate on the case of outgoing cusp points. We recall results by* P. Grisvard [4] *and* V.G. Mazya & B.A. Plamenevskii [10]*. Then, relying on a work by* J.-L. Steux [14]*, we state a result of regularity in the space of infinitely smooth functions: if the data are* C^∞ *, the solution is also* C^∞ *. We extend this result to the situation of cuspidal edges (for example the domain exterior to a cylinder lying on a plane, or two tangent tori).*

PROBLÈMES FORTEMENT ELLIPTIQUES PRÈS DE POINTS OU ARÊTES CUSPIDES

Résumé. *Après avoir passé en revue les différentes situations géométriques pouvant se produire pour un domaine à bord régulier par morceaux, nous nous concentrons sur le cas de point cuspides saillants. Nous rappelons des résultats de* P. Grisvard [4] *et* V.G. Mazya & B.A. Plamenevskii [10]*. Ensuite, nous basant sur un travail dû à* J.-L. Steux [14]*, nous établissons un résultats de régularité dans l'espace des fonctions infiniment différentiables : si les données sont* C^∞ *, la solution est aussi* C^∞ *. Enfin nous étendons ce résultat à la situation d'une arête cuspide (par exemple le domaine extérieur à un cylindre reposant sur un plan, ou encore à deux tores tangents).*

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1 PIECEWISE - SMOOTH PLANE DOMAINS

Let Ω be a piecewise-smooth plane domain. This means that the boundary $\partial\Omega$ of Ω is the union of finitely many arcs of $\mathcal{C}^\infty$ curves — which may be straight lines, of course. We call them the *sides* of Ω. A point belonging to the intersection of two sides is called a *vertex* of Ω.

The properties that we intend to investigate being *local*, we assume for simplicity that Ω has only *one* vertex, located at the origin $\mathcal{O}$ of the coordinate axes. Let $\vec{\tau}_1$ and $\vec{\tau}_2$ be the two tangents to $\partial\Omega$ at $\mathcal{O}$ and let ω be the measure of the angle between them. Five generic situations may occur:

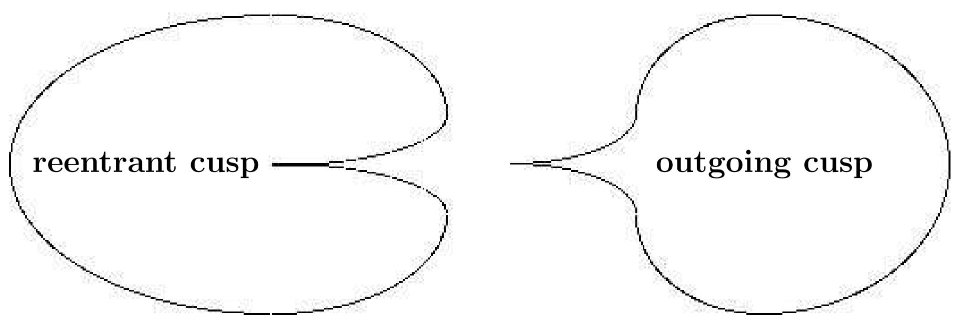
$\omega = 2\pi$: Ω has a crack (if the two arcs joining in $\mathcal{O}$ coincide), or a reentrant cusp point if not.

$\pi < \omega < 2\pi$: Ω has an ordinary non convex polygonal vertex.

$\pi = \omega$: Ω has a weak geometrical singularity (or is smooth if $\mathcal{O}$ is a dummy vertex!)

$0 < \omega < \pi$: Ω has an ordinary convex polygonal vertex.

$\omega = 0$: Ω has an outgoing cusp point.



With the help of a $\mathcal{C}^\infty$ diffeomorphism, it is always possible to flatten *one* of the sides of Ω in the neighborhood of $\mathcal{O}$, say the side tangent to $\vec{\tau}_1$. So, from now on, we assume that one side of Ω coincides with the horizontal axis in a neighborhood of $\mathcal{O}$. When $\pi < \omega < 2\pi$ or when $0 < \omega < \pi$, by a better choice of the diffeomorphism it is also possible to flatten the other side. But when Ω has a cusp, it is of course impossible.

Our object of consideration is the behavior of solutions of elliptic boundary value problems in Ω. Let L by a properly elliptic operator of order $2m$ with $\mathcal{C}^\infty$ coefficients in $\mathbb{R}^2$. Let us consider the Dirichlet boundary value problem for L on Ω:

$$\begin{cases} Lu = f \ \text{ in } \Omega, \\ u \in \mathring{H}^m(\Omega). \end{cases} \tag{1.1}$$

If f is more regular than $H^{-m}(\Omega)$, say $H^{s-m}(\Omega)$ with $s>0$, due to the presence of the corner in $\mathcal{O}$, we cannot expect that u belongs to $H^{s+m}(\Omega)$ for any s and any f.

When $\omega \neq 0$, i.e. in the four first situations, the structure of the solution u has similar properties: the function u has an asymptotics in $\mathcal{O}$, which, instead of being reduced to polynomials as in the case of the Taylor expansion of a smooth function, is made of special model functions which only depend on the geometry of Ω and the operator L. These model functions w are better described in *polar coordinates* (r,θ) centered in $\mathcal{O}$:

$$w(r,\theta) = r^{\mu} \sum_{q=0}^{Q} \psi_q(\theta)\, \log^q r \tag{1.2}$$

where μ is a complex (possibly real!) number, and the ψ_q are smooth functions of θ, "belonging" to w. In general, u admits a splitting:

$$u = u_{\text{sing}} + u_{\text{pol}} + u_{\text{flat}} \quad \text{with} \quad \begin{cases} u_{\text{sing}} = \sum_{k=1}^{K_s} c_k\, w_k, \\ u_{\text{pol}} \text{ a polynomial function,} \\ u_{\text{flat}} \in V^{s+m}(\Omega), \end{cases} \tag{1.3}$$

where the space $V^{s+m}(\Omega)$ is a space of flat functions contained in $H^{s+m}(\Omega)$. Note that for each fixed s, the number K_s of independent singular model functions is finite. Moreover the exponent μ_k belonging to w_k satisfies

$$m-1 < \operatorname{Re}\mu_1 \leq \ldots \leq \operatorname{Re}\mu_k \leq \ldots \leq \operatorname{Re}\mu_{K_s} < s+m-1. \tag{1.4}$$

In the case of the Laplacian Δ, the exponents μ_k are the $\frac{k\pi}{\omega}$ and $Q = Q(k)$ is equal to 1 if $\frac{k\pi}{\omega} \in \mathbb{N}$ and 0 if not. In general, the exponents μ_k are the eigenvalues of generalized Sturm-Liouville operators on the angular interval $]0,\omega[$. They are piecewise-smooth continuous functions of ω. For the opening π, all the μ_k are integers — if Ω is not smooth in $\mathcal{O}$, logarithmic terms occur in the asymptotics. For the opening 2π, all the μ_k are half-integers, i.e. belong to $\mathbb{N}/2$.

Under the form (1.2), the functions w does not depend smoothly on ω, even for $L = \Delta$. By mixing together the functions w and the polynomials, it is possible to construct *stable* linear combinations w_{stab}. Their radial behavior can be nicely described by contour integrals. The ordinary simple asymptotics can be written:

$$r^{\mu} \log^q r = \frac{q!}{2i\pi} \int_{\gamma} \frac{r^{\lambda}}{(\lambda-\mu)^{q+1}}\, d\lambda, \qquad q = 0,\ldots,Q, \tag{1.5}$$

where the contour γ surrounds μ. When the exponents μ depend smoothly on ω, stable behaviors are given by divided differences of the function $r \to r^{\lambda}$:

$$S[\mu_{(0)},\ldots,\mu_{(q)};r] = \frac{1}{2i\pi} \int_{\gamma} \frac{r^{\lambda}}{(\lambda-\mu_{(0)})\cdots(\lambda-\mu_{(q)})}\, d\lambda, \qquad q = 0,\ldots,Q, \tag{1.6}$$

where the $\mu_{(q)}$ occur in the exponents of the w_k or are integers — exponents of polynomials! In the general situation where the multiplicity of μ may change (for instance for $L = \Delta^2$ in the neighborhood of the angle $\omega_1 \simeq 0.813\pi$) stable behaviors are given by generalized divided differences of the function $r \to r^\lambda$:

$$S[\mu_{(0)}, \dots, \mu_{(Q)} | \, p_q; r] = \frac{1}{2i\pi} \int_\gamma \frac{r^\lambda \, p_q(\lambda)}{(\lambda - \mu_{(0)}) \cdots (\lambda - \mu_{(Q)})} \, d\lambda, \qquad q = 0, \dots, Q, \quad (1.7)$$

where the p_q for $q = 0, \dots, Q$ are a basis of $\mathbb{P}_Q$ the space of polynomials of 1 variable with degree $\leq Q$.

From a very abundant literature, we quote

- G.M. Verzbinskii & V.G. Maz'ya [15, 16, 17] concerning the Dirichlet problem for the Laplace operator in all the geometrical situations quoted above,
- P. Grisvard [5], V.A. Kondrat'ev [8] and V.G. Maz'ya & B.A. Plamenevskii [10] concerning the ordinary "conical" situation where the opening is neither 0 , nor π nor 2π ,
- [5] again and [1] for the cracks, V.G. Maz'ya, S.A. Nazarov & B.A. Plamenevskii [9] and A.B. Movchan & S.A. Nazarov [12] for reentrant cusps,
- V.G. Maz'ya & J. Rossmann [11] and our [2] for stable asymptotics in the full range $0 < \omega \leq 2\pi$ for the opening.

As a conclusion of this paragraph, we can say that in all the situations where the opening is > 0 , the asymptotics $u_{\text{asy}} := u_{\text{sing}} + u_{\text{pol}}$ of u in the neighborhood of $\mathcal{O}$ can be described in a unified and stable way, including even the case when the opening is equal to π and the domain smooth in $\mathcal{O}$ — the function u_{asy} is then the Taylor expansion of u .

Have we still a sort of stability when the opening tends to 0 ?

2 WHEN THE OPENING TENDS TO ZERO

We see that for $L = \Delta$, the first exponent occurring in the singular part u_{sing} of u is $\frac{\pi}{\omega}$ and it tends to infinity when $\omega \to 0$. The same phenomenon occurs for $L = \Delta^2$: the real part of the first exponent μ_1 tends to infinity when $\omega \to 0$. We have

Proposition 2.1 *Let L be a strongly elliptic operator. Let $\mu_1^{(\omega)}$ be the exponent with least real part occurring in* (1.4)*. Then*

$$\operatorname{Re} \mu_1^{(\omega)} \longrightarrow +\infty \quad \textit{when} \quad \omega \longrightarrow 0.$$

Proof. Let $\mathscr{L}$ be the principal part of L frozen in $\mathcal{O}$, written in the coordinates (t,θ) with $t = \log r$:

$$\mathscr{L}(\theta;\partial_t,\partial_\theta) = e^{2mt} L_{\mathrm{princ}}(\mathcal{O};\partial_x,\partial_y).$$

For any $\eta \in \mathbb{R}$, let $\mathscr{B}_\eta^{(\omega)}$ be the operator

$$\begin{aligned} \mathscr{B}_\eta^{(\omega)} : \{v \mid e^{-\eta t} v \in \mathring{H}^m(\mathbb{R}\times]0,\omega[)\} \quad &\longrightarrow \quad \{g \mid e^{-\eta t} g \in H^{-m}(\mathbb{R}\times]0,\omega[)\} \\ v \quad &\longmapsto \quad \mathscr{L}(\theta;\partial_t,\partial_\theta)\, v. \end{aligned}$$

From the general theory [8], we have for any $\eta > m-1$

$$\mathscr{B}_\eta^{(\omega)} \text{ isomorphism} \quad \iff \quad \forall k \geq 1,\ \operatorname{Re}\mu_k^{(\omega)} \neq \eta.$$

Thus, we are going to prove that $\forall \eta > m-1$, $\mathscr{B}_\eta^{(\omega)}$ is always an isomorphism if ω is small enough. Setting $\mathscr{A}_\eta^{(\omega)} = e^{-\eta t}\mathscr{B}_\eta^{(\omega)} e^{\eta t}$, acting from $\mathring{H}^m(\mathbb{R}\times]0,\omega[)$ into $H^{-m}(\mathbb{R}\times]0,\omega[)$ we have

$$\mathscr{B}_\eta^{(\omega)} \text{ isomorphism} \quad \iff \quad \mathscr{A}_\eta^{(\omega)} \text{ isomorphism}\,. \tag{2.1}$$

Let $\mathscr{A}^{(\omega)}$ be the principal part of $\mathscr{A}_\eta^{(\omega)}$. The operator $\mathscr{A}^{(\omega)}$ does not depend on η and we have the estimate

$$\begin{aligned} &\exists\, c > 0,\ \ \forall \omega \in]0,2\pi],\ \ \forall \eta \in \mathbb{R},\ \ \forall v \in \mathring{H}^m(\mathbb{R}\times]0,\omega[), \\ &\qquad \|(\mathscr{A}^{(\omega)} - \mathscr{A}_\eta^{(\omega)})\, v\|_{H^{-m}(\mathbb{R}\times]0,\omega[)} \leq c\,(1+|\eta|)^{2m}\|v\|_{H^{m-1}(\mathbb{R}\times]0,\omega[)}\,. \end{aligned} \tag{2.2}$$

Let $\mathscr{A}^{(\omega)}(0)$ be the operator $\mathscr{A}^{(\omega)}$ with its coefficients frozen in $\theta = 0$. Since

$$\partial_t = e^t\Big(\cos\theta\,\partial_x + \sin\theta\,\partial_y\Big) \quad \text{and} \quad \partial_\theta = e^t\Big(-\sin\theta\,\partial_x + \cos\theta\,\partial_y\Big)$$

we check that

$$\mathscr{A}(0)(\partial_t,\partial_\theta) = L_{\mathrm{princ}}(\mathcal{O};\partial_t,\partial_\theta). \tag{2.3}$$

Due to the strong ellipticity of L, $\mathscr{A}^{(\omega)}(0)$ is an isomorphism for all $\omega > 0$:

$$\begin{aligned} &\exists\, c > 0,\ \ \forall \omega > 0,\ \ \forall v \in \mathring{H}^m(\mathbb{R}\times]0,\omega[), \\ &\qquad |v|_{H^m(\mathbb{R}\times]0,\omega[)} \leq c\,|\mathscr{A}^{(\omega)}(0)\, v|_{H^{-m}(\mathbb{R}\times]0,\omega[)}\,. \end{aligned} \tag{2.4}$$

The Poincaré inequality on the strip reads

$$\begin{aligned} &\exists\, c > 0,\ \ \forall \omega > 0,\ \ \forall v \in \mathring{H}^m(\mathbb{R}\times]0,\omega[), \\ &\qquad \|v\|_{H^{m-1}(\mathbb{R}\times]0,\omega[)} \leq c\,\omega\,|v|_{H^m(\mathbb{R}\times]0,\omega[)}\,. \end{aligned} \tag{2.5}$$

The regularity of the coefficients of L yields

$$\begin{aligned}&\exists\, c>0,\quad \forall\omega\in]0,2\pi],\quad \forall v\in \mathring{H}^m(\mathbb{R}\times]0,\omega[),\\ &\qquad\qquad \|(\mathscr{A}^{(\omega)}(0)-\mathscr{A}^{(\omega)})\,v\|_{H^{-m}(\mathbb{R}\times]0,\omega[)}\le c\,\omega\,\|v\|_{H^m(\mathbb{R}\times]0,\omega[)}\,.\end{aligned}\tag{2.6}$$

From (2.4)-(2.6), we deduce that for ω small enough, $\mathscr{A}^{(\omega)}$ is an isomorphism satisfying

$$\begin{aligned}&\exists\, c>0,\quad \forall\omega,\ 0<\omega\le\omega_0,\quad \forall v\in \mathring{H}^m(\mathbb{R}\times]0,\omega[),\\ &\qquad\qquad \|v\|_{H^m(\mathbb{R}\times]0,\omega[)}\le c\,\|\mathscr{A}^{(\omega)}v\|_{H^{-m}(\mathbb{R}\times]0,\omega[)}\,.\end{aligned}\tag{2.7}$$

With (2.2) and (2.5), (2.7) yields that $\mathscr{A}^{(\omega)}_\eta$ is an isomorphism if $\omega\,(1+|\eta|)^{2m}$ is small enough. With (2.1), this gives the existence of a constant $c_0>0$ such that

$$\forall\eta\in\mathbb{R},\quad \forall\omega\le c_0(1+|\eta|)^{-2m},\quad \mathscr{B}^{(\omega)}_\eta \text{ isomorphism.}$$

Therefore $\mu_1^{(\omega)}$ satisfies

$$\omega>c_0(1+\operatorname{Re}\mu_1^{(\omega)})^{-2m}\quad\text{i.e.}\quad \operatorname{Re}\mu_1^{(\omega)}>\left(\frac{\omega}{c_0}\right)^{-\frac{1}{2m}}-1.$$

■

The strong ellipticity has served in only one place, to insure that $\mathscr{A}^{(\omega)}(0)$ is an isomorphism for all $\omega>0$ and satisfies the estimates (2.4). If we only assume that

$$L_{\text{princ}}(\mathcal{O};\partial_t,\partial_\theta):\mathring{H}^m(\mathbb{R}\times]0,1[)\longrightarrow H^{-m}(\mathbb{R}\times]0,1[)\text{ isomorphism,}\tag{2.8}$$

by a simple scaling argument we still obtain the estimates (2.4). By partial Fourier transform in the variable t, we obtain that (2.8) holds if

$$\forall\xi\in\mathbb{R},\quad L_{\text{princ}}(\mathcal{O};i\xi,\partial_\theta):\mathring{H}^m(]0,1[)\longrightarrow H^{-m}(]0,1[)\text{ isomorphism.}\tag{2.9}$$

Whence:

Proposition 2.2 *Let L be a properly elliptic operator satisfying* (2.9)*. Then*

$$\operatorname{Re}\mu_1^{(\omega)}\longrightarrow+\infty\quad\textit{when}\quad\omega\longrightarrow 0.$$

So, we can expect good regularity properties for the Dirichlet problem associated with operators L such as above in the neighborhood of outgoing cusp points.

3 OUTGOING CUSP POINTS : CASE OF FLAT FUNCTIONS

We assume that in a neighborhood $[-a, a] \times [-a, a]$ of $\mathcal{O}$, Ω is determined by the inequalities

$$(x, y) \in \Omega \cap [-a, a] \times [-a, a] \quad \Longleftrightarrow \quad 0 < x < a \quad \text{and} \quad 0 < y < \varphi(x), \tag{3.1}$$

where φ is a function $\mathcal{C}^\infty([-a, a])$, such that

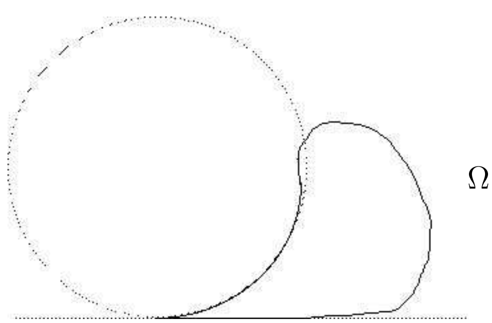
$$\varphi(0) = 0, \ \varphi'(0) = 0 \quad \text{and} \quad \varphi > 0 \text{ on }]0, a]. \tag{3.2}$$

We assume moreover that φ is not infinitely flat in 0 and let $p \in \mathbb{N}$ be the smallest integer such that

$$\varphi^{(p)}(0) \neq 0. \tag{3.3}$$

An example is given by the equation of a circle tangent to the x axis at $\mathcal{O}$: if the radius is equal to R

$$\varphi(x) = R\left(1 + \sqrt{1 - \frac{x^2}{R^2}}\right) = \frac{x^2}{2R^2} + \mathcal{O}(x^4).$$



We will see later (*cf* Remark 4.4) that our results can be applied to any domain $\Omega = \mathbb{R}^2 \setminus \overline{\mathcal{U}}$ exterior to the domain $\mathcal{U}$ formed by two tangent domains with analytic boundaries (for instance, $\mathcal{U}$ is the union of two tangent disks, or a disk tangent to a half plane.

As it has been proved in various frameworks by K. Ibuki [6], A. Khelif [7], V.G. Mazya & B.A. Plamenevskii [10], P. Grisvard [4] and J.-L. Steux [14], the operator of the Dirichlet problem (1.1) acts smoothly between spaces of flat functions: for any $j \in \mathbb{N}$, let

$$V^j(\Omega) = \{u \in L^2(\Omega) \mid \forall \alpha \in \mathbb{N}^2, \ |\alpha| \leq j, \quad \varphi^{|\alpha|-j} \, \partial^\alpha u \in L^2(\Omega)\}.$$

Moreover, the space $V^{-j}(\Omega)$ is defined as the dual space of $\mathring{V}^j(\Omega)$, where $\mathring{V}^j(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ in $V^j(\Omega)$.

Theorem 3.1 *Let L be a properly elliptic operator satisfying* (2.9). *In particular, L can be any strongly elliptic operator. Let $j \in \mathbb{Z}$, $j > -m$. Then any solution u of the Dirichlet problem* (1.1) *with right hand side $f \in V^j(\Omega)$ satisfies the optimal regularity property*

$$u \in V^{2m+j}(\Omega).$$

The proof of this theorem relies on the change of variables

$$(x,y) \longrightarrow (t,\theta) \quad \text{where} \quad \theta = \frac{y}{\varphi(x)} \quad \text{and} \quad t = -\int_x^a \frac{d\sigma}{\varphi(\sigma)}, \tag{3.4}$$

which transforms

$$\Omega_a := \Omega \cap \{(x,y) \mid 0 < x < a\} \quad \text{onto} \quad \Sigma := \{(t,\theta) \mid t < 0,\ \theta \in]0,1[\}.$$

The spaces $V^j(\Omega)$ are transformed in a simple way: we set

$$\tilde{\varphi}(t) = \varphi(x) \quad \text{and} \quad \text{for } \eta \in \mathbb{R},\ \ H^j_\eta(\Sigma) = \{v \mid \tilde{\varphi}^{-\eta} v \in H^j(\Sigma)\}.$$

Then the change of variables transforms

$$V^j(\Omega_a) \quad \text{onto} \quad H^j_{j-1}(\Sigma). \tag{3.5}$$

Note that

$$\tilde{\varphi}(t) = (p-1)^p\,|t|^{-1}\Big(|t|^{-\frac{1}{p-1}} + \mathcal{O}(|t|^{-\frac{2}{p-1}})\Big) \quad \text{when } t \to -\infty. \tag{3.6}$$

The transformation law of the operator L is

$$\varphi^{2m}(x) L(x,y;\partial_x,\partial_y) =: \widetilde{\mathscr{L}}(t,\theta;\partial_t,\partial_\theta) = L_{\text{princ}}(\mathcal{O};\partial_t,\partial_\theta) + M(t,\theta;\partial_t,\partial_\theta)$$

where the coefficients of M are smooth functions behaving like $\mathcal{O}(|t|^{-\frac{1}{p-1}})$ when $t \to -\infty$. For any $\eta \in \mathbb{R}$, let $\mathscr{B}_\eta$ be the operator

$$\begin{aligned} \mathscr{B}_\eta : \{v \mid \tilde{\varphi}^{-\eta} v \in \mathring{H}^m(\Sigma)\} \ &\longrightarrow\ \{g \mid \tilde{\varphi}^{-\eta} g \in H^{-m}(\Sigma)\} \\ v \ &\longmapsto\ \widetilde{\mathscr{L}}(t,\theta;\partial_t,\partial_\theta) v. \end{aligned}$$

Setting $\mathscr{A}_\eta = \tilde{\varphi}^{-\eta}\, \mathscr{B}_\eta\, \tilde{\varphi}^\eta$, acting from $\mathring{H}^m(\Sigma)$ into $H^{-m}(\Sigma)$ we have:

$$\begin{aligned} &\mathscr{A}_\eta = L_{\text{princ}}(\mathcal{O};\partial_t,\partial_\theta) + M_\eta(t,\theta;\partial_t,\partial_\theta) \\ &\qquad \text{with} \quad \|M_\eta\|_{\mathring{H}^m(]-\infty,-T[\times]0,1[) \to H^{-m}(]-\infty,-T[\times]0,1[)} = \mathcal{O}(T^{-\frac{1}{p-1}}). \end{aligned} \tag{3.7}$$

Since (2.9) allows for proving that $L_{\text{princ}}(\mathcal{O};\partial_t,\partial_\theta)$ induces an isomorphism from $\mathring{H}^m(\Sigma)$ onto $H^{-m}(\Sigma)$, the proof of the Theorem is a consequence of (3.5) and (3.7).

The fundamental difference between the present case of a cusp point and an acute plane sector where $\varphi(x)$ would be equal to γx and $\tilde{\varphi}(t)$ behave like e^t — compare with (3.6), is the decay property of the splitting (3.7) which does not hold for a plane sector.

4 OUTGOING CUSP POINTS : CASE OF SMOOTH FUNCTIONS

Following [14], we now intend to study the regularity of u solution of problem (1.1) when $f \in \mathcal{C}^\infty(\overline{\Omega})$. We can easily prove:

Lemma 4.1 *For $f \in \mathcal{C}^\infty(\overline{\Omega})$, we set for any $\ell \geq 1$:*

$$f_\ell(x,y) = \sum_{|\alpha|<\ell} \frac{x^{\alpha_1} y^{\alpha_2}}{\alpha_1!\, \alpha_2!}\, \partial^\alpha f(\mathcal{O}).$$

Then

$$\forall \ell \geq pj, \quad f - f_\ell \in V^j(\Omega).$$

In view of Theorem 3.1, it remains to investigate the *polynomial resolution.* For $\ell \geq 0$, let $\mathcal{C}^\infty_\ell(\overline{\Omega}_a)$ denote the space of functions:

$$\mathcal{C}^\infty_\ell(\overline{\Omega}_a) = \{f \in \mathcal{C}^\infty(\overline{\Omega}_a) \mid \forall \alpha,\ |\alpha| \leq \ell, \quad \partial^\alpha f(\mathcal{O}) = 0\}.$$

We note that

$$\forall \ell \geq pj, \quad \mathcal{C}^\infty_\ell(\overline{\Omega}_a) \subset V^j(\Omega_a).$$

Lemma 4.2 *We assume that L is elliptic. Let $\alpha \in \mathbb{N}^2$.*

There exists a function $U_\alpha \in \mathcal{C}^\infty(\overline{\Omega}_a)$ and constants $d_{\alpha,1}, \ldots, d_{\alpha,\alpha_1}$ such that

$$LU_\alpha - x^{\alpha_1} y^{\alpha_2} - \sum_{k=1}^{\alpha_1} d_{\alpha,k}\, x^{\alpha_1-k} y^{\alpha_2+k} \in \mathcal{C}^\infty_{|\alpha|}(\overline{\Omega}_a) \quad \textit{and} \quad \chi U_\alpha \in \mathring{H}^m(\Omega), \tag{4.1}$$

where χ is a smooth cut-off function $\equiv 1$ if $x \leq \frac{a}{2}$ and $\equiv 0$ if $x \geq a$.

In particular, if $\alpha_1 = 0$, the $d_{\alpha,k}$ are not there and the function U_α satisfies

$$LU_\alpha - x^{\alpha_1} y^{\alpha_2} \in \mathcal{C}^\infty_{|\alpha|}(\overline{\Omega}_a) \quad \textit{and} \quad \chi U_\alpha \in \mathring{H}^m(\Omega), \tag{4.2}$$

Proof. The method consists in solving the boundary value problem with respect to the variable y and considering x as a parameter.

The operator ∂_θ^{2m} is continuous between the spaces of polynomials:

$$\mathbb{P}_{2m+j} \cap \mathring{H}^m(]0,1[) \longrightarrow \mathbb{P}_j.$$

It is one to one and since the dimensions of the two spaces $\mathbb{P}_{2m+j} \cap \mathring{H}^m(]0,1[)$ and $\mathbb{P}_j$ are equal (to $j+1$), ∂_θ^{2m} is onto. Thus, there exists a unique polynomial $P_{\alpha_2}(\theta)$ such that

$$\partial_\theta^{2m} P_{\alpha_2} = \theta^{\alpha_2} \quad \text{and} \quad P_{\alpha_2} \in \mathring{H}^m(]0,1[).$$

With $a_{0,2m}(x,y)$ the coefficient of ∂_y^{2m} in L and $b_0 := 1/a_{0,2m}(\mathcal{O})$, we set

$$U_\alpha(x,y) = b_0\,\varphi(x)^{2m}\,x^{\alpha_1}\,\varphi(x)^{\alpha_2}\,P_{\alpha_2}(\frac{y}{\varphi(x)}).$$

Since P_{α_2} is a polynomial of degree $\le \alpha_2 + 2m$, we check that U_α has the form

$$U_\alpha(x,y) = x^{\alpha_1} \sum_{\substack{\alpha_2'\in\mathbb{N},\ \alpha_2''\in\mathbb{N}\\ \alpha_2'+\alpha_2''=\alpha_2+2m}} c_{\alpha_2',\alpha_2''}\,\varphi(x)^{\alpha_2'}\,y^{\alpha_2''}.$$

Thus U_α is smooth in a neighborhood of $\mathcal{O}$. By construction U_α satisfies the boundary conditions and

$$a_{0,2m}(\mathcal{O})\,\partial_y^{2m}\,U_\alpha(x,y) = x^{\alpha_1}y^{\alpha_2}.$$

A simple calculation proves (4.2) and (4.1). ■

The main result of this section is the regularity result [14]:

Theorem 4.3 *Let L be a properly elliptic operator satisfying* (2.9)*. In particular, L can be any strongly elliptic operator. Then any solution u of the Dirichlet problem* (1.1) *with right hand side $f \in \mathcal{C}^\infty(\overline{\Omega})$ satisfies the optimal regularity property*

$$u \in \mathcal{C}^\infty(\overline{\Omega}).$$

Proof. If suffices to prove that for any $j \in \mathbb{N}$, u can be written as the sum of a function u_j belonging to $\mathcal{C}^\infty(\overline{\Omega})$ and of a flat function $v_j \in V^{2m+j}(\Omega)$.

Let $j \in \mathbb{N}$. We begin with the following algorithm of polynomial resolution for the polynomial part f_{pj} of f given by Lemma 4.1. We start with $\alpha = (0,0)$, use Lemma 4.2 with $\alpha_1 = 0$ and put the remainder into the right hand side. Then we apply Lemma 4.2 for $(\alpha_1,\alpha_2) = (1,0)$, put the remainder into the right hand side and apply Lemma 4.2 for $(\alpha_1,\alpha_2) = (0,1)$, etc... The order in which the multi-indices α have to be treated is $|\alpha|$ increasing and α_2 increasing. In this way, we construct u_j in $\mathring{H}^m(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$ such that

$$Lu_j = f_{pj} + g_j \quad \text{with} \quad g_j \in \mathcal{C}^\infty_{pj}.$$

We conclude with Theorem 3.1 since $L(u-u_j) = (f - f_{pj}) - g_j \in V^j(\Omega)$. ■

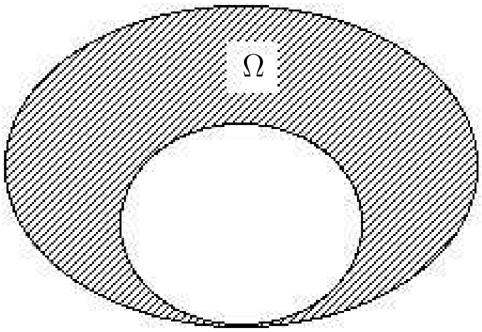
Remark 4.4 If in a neighborhood $[-a,a]\times[-a,a]$ of $\mathcal{O}$, Ω is determined by the inequalities

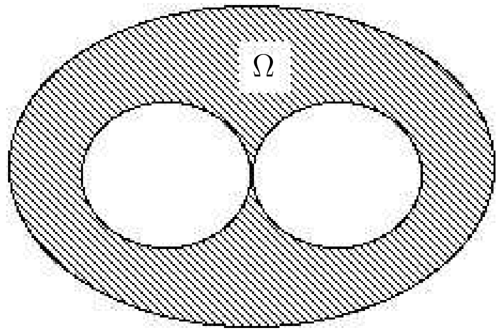
$$(x,y) \in \Omega \cap [-a,a]\times[-a,a] \iff -a < x < a \ \text{ and } \ \varphi_1(x) < y < \varphi_2(x), \tag{4.3}$$

where φ_1 and φ_2 are $\mathcal{C}^\infty([-a,a])$ such that

$$\varphi_1(0) = \varphi_2(0) = 0, \quad \varphi_1'(0) = \varphi_2'(0) = 0, \quad \varphi_2 - \varphi_1 > 0 \text{ on } [-a,0[\cup]0,a], \tag{4.4}$$

and such that $\varphi = \varphi_2 - \varphi_1$ is not infinitely flat in 0, then Theorem 4.3 still holds: the two Taylor expansions of u in $\mathcal{O}$ in the half-planes $x > 0$ and $x < 0$ are linear functions of the Taylor expansions of f, φ_1 and φ_2; therefore, one can prove that they fit together. ■





Remark 4.5 If φ has an asymptotics in non integer powers of x:

$$\varphi(x) = x^p + \gamma_1 x^{p_1} + \cdots + \gamma_N x^{p_N} + \mathcal{O}(x^{p_{N+1}}),$$

with p_n an increasing sequence tending to $+\infty$ (for example the profile of Joukowski has such a form with $p = \frac{3}{2}$ and $p_n = \frac{3}{2} + \frac{n}{2}$), Theorem 3.1 still holds [6, 7, 10, 4, 14], but the proofs of Lemma 4.2 and Theorem 4.3 yield the construction of a non smooth asymptotics for u, see [14]. ■

5 OTHER BOUNDARY CONDITIONS

While conditions corresponding to (2.9) are satisfied, we have the analogue of Theorems 3.1 and 4.3 for any other elliptic boundary problem. Namely, if $B_1(x, y; \partial_x, \partial_y)$ and $B_2(x, y; \partial_x, \partial_y)$ are two systems of boundary conditions on the sides $\theta = 0$ and $\theta = 1$ of Ω_a respectively, each of them covering L, then the behavior near $\mathcal{O}$ of the solutions of the boundary problem

$$\begin{cases} Lu = f \ \text{ in } \ \Omega, \\ B_1 u = g_1 \ \text{ if } \ y = 0, \\ B_2 u = g_2 \ \text{ if } \ y = \varphi(x), \\ \partial_x^k u = 0, \ \ k = 0, \dots, m-1 \ \text{ if } \ x = a, \end{cases} \tag{5.1}$$

depend on the problem $\Big(L_{\text{princ}}(\mathcal{O}; \partial_x, \partial_y), B_{1,\text{princ}}(\mathcal{O}; \partial_x, \partial_y), B_{2,\text{princ}}(\mathcal{O}; \partial_x, \partial_y)\Big)$ on the infinite strip $\mathbb{R}\times]0,1[$. The condition replacing (2.9) is now

$$\begin{aligned} \forall \xi \in \mathbb{R}, \quad & \Big(L_{\text{princ}}(\mathcal{O}; i\xi, \partial_\theta), B_{1,\text{princ}}(\mathcal{O}; i\xi, \partial_\theta), B_{2,\text{princ}}(\mathcal{O}; i\xi, \partial_\theta)\Big) : \\ & \qquad H^m(]0,1[) \longrightarrow H^{-m}(]0,1[) \times \mathbb{C}^{2m} \quad \text{isomorphism.} \end{aligned} \tag{5.2}$$

Under this assumption, the real part of the exponent $\mu_1^{(\omega)}$ tends to $+\infty$ when $\omega \to 0$, the regularity results in spaces of flat functions and in spaces of smooth functions still hold — note that, however, if B_1 and B_2 contain operators of the *same* order, the boundary data g_1 and g_2 have to satisfy a countable number of compatibility conditions at $\mathcal{O}$.

Examples are given by $B_1 = \boldsymbol{Id}$, $B_2 = \partial_n$ for $L = \Delta$, or for $B_1 = B_2 = (\boldsymbol{Id}, \Delta)$ for $L = \Delta^2$.

At the opposite, for Neumann problem, condition (5.2) is always violated in $\xi = 0$: all polynomials v in $\mathbb{P}_{m-1}$ satisfy

$$\Big(L_{\text{princ}}(\mathcal{O}; 0, \partial_\theta), B_{1,\text{princ}}(\mathcal{O}; 0, \partial_\theta), B_{2,\text{princ}}(\mathcal{O}; 0, \partial_\theta)\Big)v = (0, 0, 0).$$

That fact induces severe difficulties to handle flat right hand sides. Anyway for the Laplace operator for instance, it is still possible to construct an ansatz for the asymptotics of u from the Taylor expansion of the right hand side by alternating double integrations with respect to x of mean values of the type

$$x \longmapsto \frac{1}{\varphi(x)}\Big(g_1(x) - g_2(x) + \int_0^{\varphi(x)} f(x, y)\, dy\Big)$$

and double integrations in y like for Dirichlet, see S.A. NAZAROV & O.R. POLYAKOVA [13]. Such methods can be compared with what is done in elasticity for asymptotics in thin plates.

6 CUSPIDAL EDGES

Let now $\mathcal{W}$ be a three-dimensional domain with a cuspidal edge: this means that the boundary of $\mathcal{W}$ is smooth, except in the neighborhood of a smooth curve $\mathcal{E}$, the *edge* of $\mathcal{W}$, where $\mathcal{W}$ is locally diffeomorphic to $\mathbb{R} \times \Omega$, where Ω is a plane domain with an outgoing cusp in $\mathcal{O}$ as in §3-5.

Let M be a strongly elliptic operator with $\mathcal{C}^\infty$ coefficients in $\mathbb{R}^3$. We are interested in the regularity of the solutions of the Dirichlet problem:

$$\begin{cases} Mu = f \ \text{ in } \mathcal{W}, \\ u \in \mathring{H}^m(\mathcal{W}). \end{cases} \tag{6.1}$$

We study first the localized problem and prove that it is regular in spaces of flat and $\mathcal{C}^\infty$ functions respectively. Our method of proof is classical and relies on differential quotients.

Let (x,y) be the variables in Ω and z the variable in $\mathbb{R}$. Let φ be the function defining the boundary of Ω according to (3.1)-(3.3). The spaces $V^j(\mathbb{R}\times\Omega)$ are defined for $j\in\mathbb{N}$ by:

$$V^j(\mathbb{R}\times\Omega)=\{u\in L^2(\mathbb{R}\times\Omega)\mid \forall\alpha\in\mathbb{N}^3,\ |\alpha|\le j,\quad \varphi^{|\alpha|-j}\,\partial^\alpha u\in L^2(\mathbb{R}\times\Omega)\},$$

and by duality if $j<0$.

We have the tensorization properties for all $j\in\mathbb{N}$:

$$V^j(\mathbb{R}\times\Omega)=H^j(\mathbb{R},V^0(\Omega))\cap L^2(\mathbb{R},V^j(\Omega)) \tag{6.2}$$

and

$$V^{-j}(\mathbb{R}\times\Omega)=H^{-j}(\mathbb{R},V^0(\Omega))+L^2(\mathbb{R},V^{-j}(\Omega)). \tag{6.3}$$

Proposition 6.1 *Let M be a strongly elliptic operator of order 2. Let $j\in\mathbb{N}$. Then any solution u of the Dirichlet problem* (6.1) *with compact support and right hand side $f\in V^j(\mathbb{R}\times\Omega)$ satisfies the optimal regularity property*

$$u\in V^{2+j}(\mathbb{R}\times\Omega).$$

P. Grisvard [4] proved this result for $L=\Delta$ and $j=0$ in L^p Sobolev spaces by a completely different technique.

Proof. Thanks to the strong ellipticity of M, we have the a priori estimate

$$\|u\|_{H^1(\mathbb{R}\times\Omega)}\le c\,(\|Mu\|_{H^{-1}(\mathbb{R}\times\Omega)}+\|u\|_{L^2(\mathbb{R}\times\Omega)})$$

where c depends only on the support of u. This estimate can also be written as

$$\begin{aligned}\|u\|_{H^1(\mathbb{R},L^2(\Omega))}&+\|u\|_{L^2(\mathbb{R},H^1(\Omega))}\\ &\le c\left(\|Mu\|_{H^{-1}(\mathbb{R},L^2(\Omega))+L^2(\mathbb{R},H^{-1}(\Omega))}+\|u\|_{L^2(\mathbb{R}\times\Omega)}\right).\end{aligned} \tag{6.4}$$

Considering for $h>0$ small enough the function $(u(x,y,z+h)-u(x,y,z))h^{-1}$ and letting $h\to0$, we deduce from (6.4) by recurrence over $\ell\in\mathbb{N}$ that there holds:

$$\begin{aligned}\|u\|_{H^{\ell+1}(\mathbb{R},L^2(\Omega))}&+\|u\|_{H^{\ell}(\mathbb{R},H^1(\Omega))}\\ &\le c\left(\|Mu\|_{H^{\ell-1}(\mathbb{R},L^2(\Omega))+H^{\ell}(\mathbb{R},H^{-1}(\Omega))}+\|u\|_{L^2(\mathbb{R}\times\Omega)}\right).\end{aligned} \tag{6.5}$$

Integrating in y from the side $y=0$, we easily prove that

$$\mathring{H}^1(\Omega)=\mathring{V}^1(\Omega). \tag{6.6}$$

Thus, (6.5) writes

$$\begin{aligned}\|u\|_{H^{\ell+1}(\mathbb{R},V^0(\Omega))}&+\|u\|_{H^{\ell}(\mathbb{R},V^1(\Omega))}\\ &\le c\left(\|Mu\|_{H^{\ell-1}(\mathbb{R},V^0(\Omega))+H^{\ell}(\mathbb{R},V^{-1}(\Omega))}+\|u\|_{L^2(\mathbb{R}\times\Omega)}\right).\end{aligned} \tag{6.7}$$

Thus, for $f \in V^j(\mathbb{R} \times \Omega)$, the above estimate for $\ell = j+1$ yields that

$$u \in H^{2+j}(\mathbb{R}, V^0(\Omega)) \cap H^{1+j}(\mathbb{R}, V^1(\Omega)). \tag{6.8}$$

Let L be the operator

$$L(x, y, z; \partial_x, \partial_y) = M(x, y, z; \partial_x, \partial_y, 0), \tag{6.9}$$

so that there exists an operator P of order ≤ 1 such that

$$M(x, y, z; \partial_x, \partial_y, \partial_z) = L(x, y, z; \partial_x, \partial_y) + P(x, y, z; \partial_x, \partial_y, \partial_z)\, \partial_z. \tag{6.10}$$

Since $Mu = f$ belongs to $H^j(\mathbb{R}, V^0(\Omega))$, we deduce from (6.8) and (6.10) that

$$Lu \in H^j(\mathbb{R}, V^0(\Omega)). \tag{6.11}$$

Theorem 3.1 applied for each z combined with an argument of differential quotients yields that (6.11) implies

$$u \in H^j(\mathbb{R}, V^2(\Omega)).$$

In that way, we prove by induction over $\ell = 0, \ldots, j$ that

$$Lu \in H^{j-\ell}(\mathbb{R}, V^\ell(\Omega))$$

which implies

$$u \in H^{j-\ell}(\mathbb{R}, V^{\ell+2}(\Omega)). \tag{6.12}$$

(6.12) for $\ell = j$ combined with (6.8) gives the Proposition. ■

For general operators of order $2m$, one encounters a technical difficulty in handling the norms with negative exponents. We have

Lemma 6.2 *Let M be a strongly elliptic operator of order $2m$ with $m \geq 2$. Let $j \in \mathbb{Z}$, $j > -m$. There exists an integer $k = k(m) > 0$ such that any solution u of the Dirichlet problem* (6.1) *with compact support and right hand side*

$$f \in H^{j+k}(\mathbb{R}, V^0(\Omega)) \cap H^k(\mathbb{R}, V^j(\Omega)) \tag{6.13}$$

satisfies the regularity property

$$u \in V^{2m+j}(\mathbb{R} \times \Omega).$$

Proof. Thanks to the strong ellipticity of M, just like above we obtain the a priori estimate for any $\ell \in \mathbb{N}$ — compare with (6.7)

$$\|u\|_{H^{\ell+m}(\mathbb{R}, V^0(\Omega))} + \|u\|_{H^\ell(\mathbb{R}, V^m(\Omega))}$$
$$\leq c\left(\|Mu\|_{H^{\ell-m}(\mathbb{R}, V^0(\Omega)) + H^\ell(\mathbb{R}, V^{-m}(\Omega))} + \|u\|_{L^2(\mathbb{R} \times \Omega)}\right).$$

Thus, for f satisfying (6.13) the above estimate for $\ell = j+m+k$ yields that

$$u \in H^{2m+j+k}(\mathbb{R}, V^0(\Omega)) \cap H^{m+j+k}(\mathbb{R}, V^m(\Omega)).$$

The operator L is defined by (6.9) so that there exists an operator P of order $\leq 2m-1$ satisfying (6.10). Since $Mu = f$ belongs to $H^{j+k}(\mathbb{R}, V^0(\Omega))$, we deduce that

$$Lu \in H^{j+k}(\mathbb{R}, V^{1-m}(\Omega)).$$

Theorem 3.1 yields that

$$u \in H^{j+k}(\mathbb{R}, V^{1+m}(\Omega)).$$

In that way, we prove by induction over $\ell = 0, \ldots, j+m$ that

$$Lu \in H^{j+m-\ell+k_\ell}(\mathbb{R}, V^{\ell-m}(\Omega))$$

with $k_0 = k$, $k_1 = k_0 - (m-1)$, $k_2 = k_1 - (m-2)$, ... , $k_{m-1} = k_m = \ldots = k_{m+j}$, and

$$u \in H^{j+m-\ell+k_\ell}(\mathbb{R}, V^{\ell+m}(\Omega)).$$

It suffices to choose k_0 such that $k_{m-1} \geq 0$ to obtain finally for $\ell = m+j$ that u belongs to $L^2(\mathbb{R}, V^{2m+j}(\Omega))$. As u also belongs to $u \in H^{2m+j}(\mathbb{R}, V^0(\Omega))$, the Lemma is proved. ■

Theorem 6.3 *Let M be a strongly elliptic operator. Then any solution u of the Dirichlet problem* (6.1) *with compact support and right hand side $f \in \mathcal{C}^\infty(\mathbb{R} \times \overline{\Omega})$ satisfies the optimal regularity property*

$$u \in \mathcal{C}^\infty(\mathbb{R} \times \overline{\Omega}).$$

Proof. We denote by H^∞ the intersection of all spaces H^k for $k \in \mathbb{N}$. Since f belongs to $H^\infty(\mathbb{R}, V^0(\Omega))$, Proposition 6.1 and Lemma 6.2 yield that

$$u \in H^\infty(\mathbb{R}, V^{2m}(\Omega)). \tag{6.14}$$

The proof runs as above, using as spaces on Ω the spaces $V^j(\Omega)$ augmented by the spaces of polynomials:

$$\widetilde{V}^j(\Omega) = V^j(\Omega) + \mathbb{P}_{pj-1}(\Omega).$$

Indeed, Theorem 3.1 and Lemma 4.2 give that

$$\begin{cases} Lu \in \widetilde{V}^j(\Omega), \\ u \in \mathring{H}^m(\Omega). \end{cases} \implies u \in \widetilde{V}^{2m+j}(\Omega).$$

So, starting from (6.14), we have $Lu \in H^\infty(\mathbb{R}, \widetilde{V}^1(\Omega))$. Thus $u \in H^\infty(\mathbb{R}, \widetilde{V}^{2m+1}(\Omega))$. Going on, we prove by induction that $\forall \ell \in \mathbb{N}$,

$$u \in H^\infty(\mathbb{R}, \widetilde{V}^{2m+\ell}(\Omega)).$$

$\blacksquare$

Similarly to Remark 4.4, we deduce from all these statements a local regularity result in the space $\mathcal{C}^\infty(\overline{\mathcal{W}})$ for any domain $\mathcal{W} = \mathbb{R}^3 \setminus \overline{\mathcal{U}}$ where the domain $\mathcal{U}$ is the disjoint union of two (or more) domains with analytic boundaries $\mathcal{U}_1$ and $\mathcal{U}_2$ such that $\overline{\mathcal{U}}_1 \cap \overline{\mathcal{U}}_2$ is a curve $\mathcal{E}$. As examples, we can take for $\mathcal{U}_1$ and $\mathcal{U}_2$: a cylinder and a half-space, two cylinders, a torus and a cylinder, a torus and a ball, a torus and a half-space, two tori, etc...

Remark 6.4 Let us give a short description of the case where there is only one contact point: we take as $\mathcal{U}_1$ the half-space $y < 0$ and as $\mathcal{U}_2$ the ball of radius 1 and of center $(x_1, x_2, y) = (0, 0, 1)$ and we consider the Dirichlet problem for the Laplace operator. Then it is possible to prove that for a right hand side flat enough in $\mathcal{O} = (0, 0, 0)$, the solution is as flat as desired. The idea is to work in cylindrical coordinates (r, θ, y) with $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ and to combine arguments of differential quotients with respect to the variable θ and estimates on $\frac{u}{r}$ by considering $\Delta(\frac{u}{r})$. It is still possible to construct the asymptotics of u by the polynomial resolution, but the outcome of the construction gives not only polynomial terms but also non smooth terms. The most singular that we have found are

$$\frac{x_1\, y^4}{r^2} \qquad \text{and} \qquad \frac{x_2\, y^4}{r^2}.$$

$\blacksquare$

NOTE

During the Conference, V.G. Mazya pointed out to us a work by V.I. Feigin. We only found in the literature the short note [3], where are stated results very similar to ours.

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