STRONGLY ELLIPTIC PROBLEMS
NEAR CUSPIDAL POINTS AND EDGES

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Abstract. After an overview of the various geometrical situations occurring for two-dimensional piecewise smooth domains, we concentrate on the case of outgoing cusp points. We recall results by P. Grisvard [4] and V.G. Mazya & B.A. Plamenevskii [10]. Then, relying on a work by J.-L. Steux [14], we state a result of regularity in the space of infinitely smooth functions: if the data are $C^\infty$, the solution is also $C^\infty$. We extend this result to the situation of cuspidal edges (for example the domain exterior to a cylinder lying on a plane, or two tangent tori).

PROBLÈMES FORTEMENT ELLIPTIQUES PRÈS DE POINTS OU ARÊTES CUSPIDES


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Let $\Omega$ be a piecewise-smooth plane domain. This means that the boundary $\partial \Omega$ of $\Omega$ is the union of finitely many arcs of $C^\infty$ curves — which may be straight lines, of course. We call them the sides of $\Omega$. A point belonging to the intersection of two sides is called a vertex of $\Omega$.

The properties that we intend to investigate being local, we assume for simplicity that $\Omega$ has only one vertex, located at the origin $O$ of the coordinate axes. Let $\vec{\tau}_1$ and $\vec{\tau}_2$ be the two tangents to $\partial \Omega$ at $O$ and let $\omega$ be the measure of the angle between them. Five generic situations may occur:

$\omega = 2\pi$: $\Omega$ has a crack (if the two arcs joining in $O$ coincide), or a reentrant cusp point if not.

$\pi < \omega < 2\pi$: $\Omega$ has an ordinary non convex polygonal vertex.

$\pi = \omega$: $\Omega$ has a weak geometrical singularity (or is smooth if $O$ is a dummy vertex!)

$0 < \omega < \pi$: $\Omega$ has an ordinary convex polygonal vertex.

$\omega = 0$: $\Omega$ has an outgoing cusp point.

With the help of a $C^\infty$ diffeomorphism, it is always possible to flatten one of the sides of $\Omega$ in the neighborhood of $O$, say the side tangent to $\vec{\tau}_1$. So, from now on, we assume that one side of $\Omega$ coincides with the horizontal axis in a neighborhood of $O$. When $\pi < \omega < 2\pi$ or when $0 < \omega < \pi$, by a better choice of the diffeomorphism it is also possible to flatten the other side. But when $\Omega$ has a cusp, it is of course impossible.

Our object of consideration is the behavior of solutions of elliptic boundary value problems in $\Omega$. Let $L$ by a properly elliptic operator of order $2m$ with $C^\infty$ coefficients in $\mathbb{R}^2$. Let us consider the Dirichlet boundary value problem for $L$ on $\Omega$:

\[
\begin{align*}
Lu &= f \quad \text{in } \Omega, \\
u &\in \dot{H}^m(\Omega).
\end{align*}
\]
If \( f \) is more regular than \( H^{-m}(\Omega) \), say \( H^{s-m}(\Omega) \) with \( s > 0 \), due to the presence of the corner in \( \Omega \), we cannot expect that \( u \) belongs to \( H^{s+m}(\Omega) \) for any \( s \) and any \( f \).

When \( \omega \neq 0 \), i.e. in the four first situations, the structure of the solution \( u \) has similar properties: the function \( u \) has an asymptotics in \( \Omega \), which, instead of being reduced to polynomials as in the case of the Taylor expansion of a smooth function, is made of special model functions which only depend on the geometry of \( \Omega \) and the operator \( L \). These model functions \( w \) are better described in polar coordinates \((r, \theta)\) centered in \( \Omega \):

\[
w(r, \theta) = r^{\mu} \sum_{q=0}^{Q} \psi_q(\theta) \log^q r \tag{1.2}
\]

where \( \mu \) is a complex (possibly real!) number, and the \( \psi_q \) are smooth functions of \( \theta \), “belonging” to \( w \). In general, \( u \) admits a splitting:

\[
u = u_{\text{sing}} + u_{\text{pol}} + u_{\text{flat}} \quad \text{with} \quad \left\{
\begin{array}{l}
u_{\text{sing}} = \sum_{k=1}^{K^*} c_k w_k, \\
u_{\text{pol}} \text{ a polynomial function,} \\
u_{\text{flat}} \in V^{s+m}(\Omega),
\end{array}
\right. \tag{1.3}
\]

where the space \( V^{s+m}(\Omega) \) is a space of flat functions contained in \( H^{s+m}(\Omega) \). Note that for each fixed \( s \), the number \( K^*_s \) of independent singular model functions is finite. Moreover the exponent \( \mu_k \) belonging to \( w_k \) satisfies

\[
m - 1 < \Re \mu_1 \leq \ldots \leq \Re \mu_k \leq \ldots \leq \Re \mu_{K^*_s} < s + m - 1. \tag{1.4}
\]

In the case of the Laplacian \( \Delta \), the exponents \( \mu_k \) are the \( \frac{k\pi}{\omega} \) and \( Q = Q(k) \) is equal to 1 if \( \frac{k\pi}{\omega} \in \mathbb{N} \) and 0 if not. In general, the exponents \( \mu_k \) are the eigenvalues of generalized Sturm-Liouville operators on the angular interval \( [0, \omega[ \). They are piecewise-smooth continuous functions of \( \omega \). For the opening \( \pi \), all the \( \mu_k \) are integers — if \( \Omega \) is not smooth in \( \Omega \), logarithmic terms occur in the asymptotics. For the opening \( 2\pi \), all the \( \mu_k \) are half-integers, i.e. belong to \( \mathbb{N}/2 \).

Under the form (1.2), the functions \( w \) does not depend smoothly on \( \omega \), even for \( L = \Delta \). By mixing together the functions \( w \) and the polynomials, it is possible to construct stable linear combinations \( u_{\text{stab}} \). Their radial behavior can be nicely described by contour integrals. The ordinary simple asymptotics can be written:

\[
r^{\mu} \log^q r = \frac{q!}{2i\pi} \int_{\gamma} \frac{r^{\lambda}}{(\lambda - \mu)^{q+1}} d\lambda, \quad q = 0, \ldots, Q, \tag{1.5}
\]

where the contour \( \gamma \) surrounds \( \mu \). When the exponents \( \mu \) depend smoothly on \( \omega \), stable behaviors are given by divided differences of the function \( r \to r^{\lambda} \):

\[
S[\mu(0), \ldots, \mu(q); r] = \frac{1}{2i\pi} \int_{\gamma} \frac{r^{\lambda}}{(\lambda - \mu(0)) \cdots (\lambda - \mu(q))} d\lambda, \quad q = 0, \ldots, Q. \tag{1.6}
\]
where the $\mu(q)$ occur in the exponents of the $w_k$ or are integers — exponents of polynomials! In the general situation where the multiplicity of $\mu$ may change (for instance for $L = \Delta^2$ in the neighborhood of the angle $\omega_1 \simeq 0.813\pi$) stable behaviors are given by generalized divided differences of the function $r \to r^\lambda$:

$$S[\mu(0), \ldots, \mu(Q)] p_q r = \frac{1}{2i\pi} \int_\gamma \frac{r^\lambda p_q(\lambda)}{(\lambda - \mu(0)) \cdots (\lambda - \mu(Q))} d\lambda, \quad q = 0, \ldots, Q, \quad (1.7)$$

where the $p_q$ for $q = 0, \ldots, Q$ are a basis of $\mathbb{P}_Q$ the space of polynomials of 1 variable with degree $\leq Q$.

From a very abundant literature, we quote
- G.M. Verzbinskii & V.G. Maz’ya [15, 16, 17] concerning the Dirichlet problem for the Laplace operator in all the geometrical situations quoted above,
- P. Grisvard [5], V.A. Kondrat’ev [8] and V.G. Maz’ya & B.A. Plame- nevskii [10] concerning the ordinary “conical” situation where the opening is neither 0, nor $\pi$ nor $2\pi$,

As a conclusion of this paragraph, we can say that in all the situations where the opening is $> 0$, the asymptotics $u_{asy} := u_{sing} + u_{pol}$ of $u$ in the neighborhood of $O$ can be described in a unified and stable way, including even the case when the opening is equal to $\pi$ and the domain smooth in $O$ — the function $u_{asy}$ is then the Taylor expansion of $u$.

Have we still a sort of stability when the opening tends to 0?

## 2 WHEN THE OPENING TENDS TO ZERO

We see that for $L = \Delta$, the first exponent occurring in the singular part $u_{sing}$ of $u$ is $\frac{\pi}{\omega}$ and it tends to infinity when $\omega \to 0$. The same phenomenon occurs for $L = \Delta^2$: the real part of the first exponent $\mu_1$ tends to infinity when $\omega \to 0$. We have

**Proposition 2.1** Let $L$ be a strongly elliptic operator. Let $\mu_1^{(\omega)}$ be the exponent with least real part occurring in (1.4). Then

$$\text{Re} \mu_1^{(\omega)} \longrightarrow +\infty \quad \text{when} \quad \omega \longrightarrow 0.$$
Proof. Let $\mathcal{L}$ be the principal part of $L$ frozen in $\mathcal{O}$, written in the coordinates $(t, \theta)$ with $t = \log r$:

$$\mathcal{L}(\theta; \partial_t, \partial_\theta) = e^{2mt} L_{\text{princ}}(\mathcal{O}; \partial_x, \partial_y).$$

For any $\eta \in \mathbb{R}$, let $\mathcal{B}_{\eta}^{(\omega)}$ be the operator

$$\mathcal{B}_{\eta}^{(\omega)} : \{ v \mid e^{-\eta t} v \in \dot{H}^m(\mathbb{R} \times [0, \omega[) \} \rightarrow \{ g \mid e^{-\eta t} g \in H^{-m}(\mathbb{R} \times [0, \omega[) \}$$

$$v \mapsto L(\theta; \partial_t, \partial_\theta) v.$$

From the general theory [8], we have for any $\eta > m$

$$\mathcal{B}_{\eta}^{(\omega)} \text{ isomorphism} \iff \forall k \geq 1, \Re \mu_k^{(\omega)} \neq \eta.$$

Thus, we are going to prove that $\forall \eta > m - 1$, $\mathcal{B}_{\eta}^{(\omega)}$ is always an isomorphism if $\omega$ is small enough. Setting $A_{\eta}^{(\omega)} = e^{-\eta t} \mathcal{B}_{\eta}^{(\omega)} e^{\eta t}$, acting from $\dot{H}^m(\mathbb{R} \times [0, \omega[)$ into $H^{-m}(\mathbb{R} \times [0, \omega[)$ we have

$$A_{\eta}^{(\omega)} \text{ isomorphism} \iff A_{\eta}^{(\omega)} \text{ isomorphism}.$$

Let $A_{\eta}^{(\omega)}$ be the principal part of $A_{\eta}^{(\omega)}$. The operator $A_{\eta}^{(\omega)}$ does not depend on $\eta$ and we have the estimate

$$\exists c > 0, \forall \omega \in [0, 2\pi], \forall \eta \in \mathbb{R}, \forall v \in \dot{H}^m(\mathbb{R} \times [0, \omega[),$$

$$\| (A_{\eta}^{(\omega)} - A_{\eta}^{(\omega)}) v \|_{H^{-m}(\mathbb{R} \times [0, \omega[)} \leq c (1 + |\eta|)^2 \| \omega \|_{H^{-1}(\mathbb{R} \times [0, \omega[)}.$$  (2.2)

Let $A_{\eta}^{(0)}$ be the operator $A_{\eta}^{(\omega)}$ with its coefficients frozen in $\theta = 0$. Since

$$\partial_t = e^t \left( \cos \theta \partial_x + \sin \theta \partial_y \right) \quad \text{and} \quad \partial_\theta = e^t \left( -\sin \theta \partial_x + \cos \theta \partial_y \right)$$

we check that

$$A_{\eta}^{(0)}(\partial_t, \partial_\theta) = L_{\text{princ}}(\mathcal{O}; \partial_x, \partial_y).$$  (2.3)

Due to the strong ellipticity of $L$, $A_{\eta}^{(0)}$ is an isomorphism for all $\omega > 0$:

$$\exists c > 0, \forall \omega > 0, \forall v \in \dot{H}^m(\mathbb{R} \times [0, \omega[),$$

$$|v|_{H^m(\mathbb{R} \times [0, \omega[)} \leq c |A_{\eta}^{(0)} v|_{H^{-m}(\mathbb{R} \times [0, \omega[)}.$$  (2.4)

The Poincaré inequality on the strip reads

$$\exists c > 0, \forall \omega > 0, \forall v \in \dot{H}^m(\mathbb{R} \times [0, \omega[),$$

$$\| v \|_{H^{m-1}(\mathbb{R} \times [0, \omega[)} \leq c \omega |v|_{H^m(\mathbb{R} \times [0, \omega[)}.$$  (2.5)
The regularity of the coefficients of $L$ yields

$$\exists c > 0, \ \forall \omega \in [0, 2\pi], \ \forall v \in \tilde{H}^m(\mathbb{R} \times [0, \omega]),$$

$$\frac{\| (\varphi(\omega)(0) - \varphi(\omega)) v \|_{H^{-m}(\mathbb{R} \times [0, \omega])}}{\| v \|_{H^m(\mathbb{R} \times [0, \omega])}} \leq c \omega \| v \|_{H^m(\mathbb{R} \times [0, \omega])}. \tag{2.6}$$

From (2.4)-(2.6), we deduce that for $\omega$ small enough, $\varphi(\omega)$ is an isomorphism satisfying

$$\exists c > 0, \ \forall \omega, \ \forall v \in \tilde{H}^m(\mathbb{R} \times [0, \omega]),$$

$$\frac{\| v \|_{H^m(\mathbb{R} \times [0, \omega])}}{\| \varphi(\omega) v \|_{H^{-m}(\mathbb{R} \times [0, \omega])}} \leq c \omega \frac{\| \varphi(\omega) v \|_{H^{-m}(\mathbb{R} \times [0, \omega])}}{\| v \|_{H^m(\mathbb{R} \times [0, \omega])}}. \tag{2.7}$$

With (2.2) and (2.5), (2.7) yields that $\varphi_{\eta}(\omega)$ is an isomorphism if $\omega (1 + |\eta|)^{2m}$ is small enough. With (2.1), this gives the existence of a constant $c_0 > 0$ such that

$$\forall \eta \in \mathbb{R}, \ \forall \omega \leq c_0 (1 + |\eta|)^{-2m}, \ \varphi_{\eta}(\omega) \text{ isomorphism.}$$

Therefore $\mu_1(\omega)$ satisfies

$$\omega > c_0 (1 + \Re \mu_1(\omega))^{-2m} \ \text{i.e.} \ \Re \mu_1(\omega) > \left( \frac{\omega}{c_0} \right)^{-\frac{1}{2m}} - 1.$$
3 OUTGOING CUSP POINTS : CASE OF FLAT FUNCTIONS

We assume that in a neighborhood \([-a, a] \times [-a, a]\) of \(O\), \(\Omega\) is determined by the inequalities

\[(x, y) \in \Omega \cap [-a, a] \times [-a, a] \iff 0 < x < a \text{ and } 0 < y < \varphi(x), \quad (3.1)\]

where \(\varphi\) is a function \(C^\infty([-a, a])\), such that

\[\varphi(0) = 0, \quad \varphi'(0) = 0 \text{ and } \varphi > 0 \text{ on } [0, a]. \quad (3.2)\]

We assume moreover that \(\varphi\) is not infinitely flat in \(0\) and let \(p \in \mathbb{N}\) be the smallest integer such that

\[\varphi^{(p)}(0) \neq 0. \quad (3.3)\]

An example is given by the equation of a circle tangent to the \(x\) axis at \(O\): if the radius is equal to \(R\)

\[\varphi(x) = R \left(1 + \sqrt{1 - \frac{x^2}{R^2}}\right) = \frac{x^2}{2R^2} + \mathcal{O}(x^4).\]

We will see later (cf Remark 4.4) that our results can be applied to any domain \(\Omega = \mathbb{R}^2 \setminus \mathcal{U}\) exterior to the domain \(\mathcal{U}\) formed by two tangent domains with analytic boundaries (for instance, \(\mathcal{U}\) is the union of two tangent disks, or a disk tangent to a half plane.

As it has been proved in various frameworks by K. Ibuki [6], A. Khelif [7], V.G. Mazya & B.A. Plamenevskii [10], P. Grisvard [4] and J.-L. Steux [14], the operator of the Dirichlet problem (1.1) acts smoothly between spaces of flat functions: for any \(j \in \mathbb{N}\), let

\[V^j(\Omega) = \{u \in L^2(\Omega) \mid \forall \alpha \in \mathbb{N}^2, |\alpha| \leq j, \varphi^{(|\alpha|-j)} \partial^\alpha u \in L^2(\Omega)\}.

Moreover, the space \(V^{-j}(\Omega)\) is defined as the dual space of \(\hat{V}^j(\Omega)\), where \(\hat{V}^j(\Omega)\) is the closure of \(\mathcal{D}(\Omega)\) in \(V^j(\Omega)\).
Theorem 3.1 Let $L$ be a properly elliptic operator satisfying (2.9). In particular, $L$ can be any strongly elliptic operator. Let $j \in \mathbb{Z}$, $j > -m$. Then any solution $u$ of the Dirichlet problem (1.1) with right hand side $f \in V^j(\Omega)$ satisfies the optimal regularity property

$$u \in V^{2m+j}(\Omega).$$

The proof of this theorem relies on the change of variables

$$(x, y) \rightarrow (t, \theta) \quad \text{where} \quad \theta = \frac{y}{\varphi(x)} \quad \text{and} \quad t = -\int_x^a \frac{d\sigma}{\varphi(\sigma)},$$

which transforms

$$\Omega_a := \Omega \cap \{ (x, y) \mid 0 < x < a \} \quad \text{onto} \quad \Sigma := \{ (t, \theta) \mid t < 0, \theta \in ]0, 1[ \}.$$ 

The spaces $V^j(\Omega)$ are transformed in a simple way: we set

$$\tilde{\varphi}(t) = (p-1)p|t|^{-1}\left(|t|^{-\frac{1}{p-1}} + \mathcal{O}(|t|^{-\frac{2}{p-1}})\right) \quad \text{when} \quad t \to -\infty.$$ 

The transformation law of the operator $L$ is

$$\varphi^{2m}(x)L(x, y; \partial_x, \partial_y) := \tilde{\mathcal{L}}(t, \theta; \partial_t, \partial_\theta) = L_{\text{princ}}(\mathcal{O}; \partial_t, \partial_\theta) + M(t, \theta; \partial_t, \partial_\theta)$$

where the coefficients of $M$ are smooth functions behaving like $\mathcal{O}(|t|^{-\frac{1}{p-1}})$ when $t \to -\infty$. For any $\eta \in \mathbb{R}$, let $B_\eta$ be the operator

$$\mathcal{L}(t, \theta; \partial_t, \partial_\theta) \quad \text{acting from} \quad \tilde{H}^m(\Sigma) \quad \text{into} \quad H^{-m}(\Sigma) \quad \text{we have:}$$

$$\mathcal{L} = L_{\text{princ}}(\mathcal{O}; \partial_t, \partial_\theta) + M(t, \theta; \partial_t, \partial_\theta)$$

with

$$\|M_\eta\|_{\tilde{H}^m(-\infty, T[\times[0,1]) \rightarrow H^{-m}(-\infty, T[\times[0,1])} = \mathcal{O}(T^{-\frac{1}{p-1}}).$$

Since (2.9) allows for proving that $L_{\text{princ}}(\mathcal{O}; \partial_t, \partial_\theta)$ induces an isomorphism from $\tilde{H}^m(\Sigma)$ onto $H^{-m}(\Sigma)$, the proof of the Theorem is a consequence of (3.5) and (3.7).

The fundamental difference between the present case of a cusp point and an acute plane sector where $\varphi(x)$ would be equal to $\gamma x$ and $\tilde{\varphi}(t)$ behave like $e^t$ — compare with (3.6), is the decay property of the splitting (3.7) which does not hold for a plane sector.
4 OUTGOING CUSP POINTS : CASE OF SMOOTH FUNCTIONS

Following [14], we now intend to study the regularity of \( u \) solution of problem (1.1) when \( f \in C^\infty(\Omega) \). We can easily prove:

**Lemma 4.1** For \( f \in C^\infty(\Omega) \), we set for any \( \ell \geq 1 \):

\[
f_\ell(x,y) = \sum_{|\alpha| < \ell} x^{\alpha_1} y^{\alpha_2} \partial^\alpha f(\mathcal{O}).
\]

Then

\[
\forall \ell \geq p_j, \quad f - f_\ell \in V_j(\Omega).
\]

In view of Theorem 3.1, it remains to investigate the polynomial resolution. For \( \ell \geq 0 \), let \( C^\infty(\Omega_a) \) denote the space of functions:

\[
C^\infty(\Omega_a) = \{ f \in C^\infty(\Omega_a) \mid \forall \alpha, |\alpha| \leq \ell, \partial^\alpha f(\mathcal{O}) = 0 \}.
\]

We note that

\[
\forall \ell \geq p_j, \quad C^\infty(\Omega_a) \subset V_j(\Omega_a).
\]

**Lemma 4.2** We assume that \( L \) is elliptic. Let \( \alpha \in \mathbb{N}^2 \).

There exists a function \( U_\alpha \in C^\infty(\Omega_a) \) and constants \( d_{\alpha,1}, \ldots, d_{\alpha,\alpha_1} \) such that

\[
LU_\alpha - x^{\alpha_1} y^{\alpha_2} - \sum_{k=1}^{\alpha_1} d_{\alpha,k} x^{\alpha_1-k} y^{\alpha_2+k} \in C^\infty_{|\alpha|}(\Omega_a) \quad \text{and} \quad \chi U_\alpha \in \dot{H}^m(\Omega),
\]

where \( \chi \) is a smooth cut-off function \( \equiv 1 \) if \( x \leq \frac{a}{2} \) and \( \equiv 0 \) if \( x \geq a \).

In particular, if \( \alpha_1 = 0 \), the \( d_{\alpha,k} \) are not there and the function \( U_\alpha \) satisfies

\[
LU_\alpha - x^{\alpha_1} y^{\alpha_2} \in C^\infty_{|\alpha|}(\Omega_a) \quad \text{and} \quad \chi U_\alpha \in \dot{H}^m(\Omega),
\]

**Proof.** The method consists in solving the boundary value problem with respect to the variable \( y \) and considering \( x \) as a parameter.

The operator \( \partial_\theta^{2m} \) is continuous between the spaces of polynomials:

\[
\mathbb{P}_{2m+j} \cap \dot{H}^m([0,1]) \longrightarrow \mathbb{P}_j.
\]

It is one to one and since the dimensions of the two spaces \( \mathbb{P}_{2m+j} \cap \dot{H}^m([0,1]) \) and \( \mathbb{P}_j \) are equal (to \( j + 1 \)), \( \partial_\theta^{2m} \) is onto. Thus, there exists a unique polynomial \( P_{\alpha_2}(\theta) \) such that

\[
\partial_\theta^{2m} P_{\alpha_2} = \theta^{\alpha_2} \quad \text{and} \quad P_{\alpha_2} \in \dot{H}^m([0,1]).
\]
With $a_{0.2m}(x, y)$ the coefficient of $\partial_y^{2m}$ in $L$ and $b_0 := 1/a_{0.2m}(O)$, we set

$$U_\alpha(x, y) = b_0 \varphi(x)^{2m} x^{\alpha_1} \varphi(x)^{\alpha_2} P_{\alpha_2}(\frac{y}{\varphi(x)}).$$

Since $P_{\alpha_2}$ is a polynomial of degree $\leq \alpha_2 + 2m$, we check that $U_\alpha$ has the form

$$U_\alpha(x, y) = x^{\alpha_1} \sum_{\alpha'_2, \alpha''_2 \in \mathbb{N}, \alpha'_2 + \alpha''_2 = \alpha_2 + 2m} c_{\alpha'_2, \alpha''_2} \varphi(x)^{\alpha'_2} y^{\alpha''_2}.$$

Thus $U_\alpha$ is smooth in a neighborhood of $O$. By construction $U_\alpha$ satisfies the boundary conditions and

$$a_{0.2m}(O) \partial_y^{2m} U_\alpha(x, y) = x^{\alpha_1} y^{\alpha_2}.$$

A simple calculation proves (4.2) and (4.1).

The main result of this section is the regularity result [14]:

**Theorem 4.3** Let $L$ be a properly elliptic operator satisfying (2.9). In particular, $L$ can be any strongly elliptic operator. Then any solution $u$ of the Dirichlet problem (1.1) with right hand side $f \in C^\infty(\Omega)$ satisfies the optimal regularity property

$$u \in C^\infty(\Omega).$$

**Proof.** If suffices to prove that for any $j \in \mathbb{N}$, $u$ can be written as the sum of a function $u_j$ belonging to $C^\infty(\Omega)$ and of a flat function $v_j \in V^{2m+j}(\Omega)$.

Let $j \in \mathbb{N}$. We begin with the following algorithm of polynomial resolution for the polynomial part $f_{pj}$ of $f$ given by Lemma 4.1. We start with $\alpha = (0, 0)$, use Lemma 4.2 with $\alpha_1 = 0$ and put the remainder into the right hand side. Then we apply Lemma 4.2 for $(\alpha_1, \alpha_2) = (1, 0)$, put the remainder into the right hand side and apply Lemma 4.2 for $(\alpha_1, \alpha_2) = (0, 1)$, etc... The order in which the multi-indices $\alpha$ have to be treated is $|\alpha|$ increasing and $\alpha_2$ increasing. In this way, we construct $u_j$ in $\dot{H}^m(\Omega) \cap C^\infty(\Omega)$ such that

$$L u_j = f_{pj} + g_j \quad \text{with} \quad g_j \in C^\infty_{pj}.$$

We conclude with Theorem 3.1 since $L(u - u_j) = (f - f_{pj}) - g_j \in V^j(\Omega)$.

**Remark 4.4** If in a neighborhood $[-a, a] \times [-a, a]$ of $O$, $\Omega$ is determined by the inequalities

$$(x, y) \in \Omega \cap [-a, a] \times [-a, a] \iff -a < x < a \quad \text{and} \quad \varphi_1(x) < y < \varphi_2(x),$$

where $\varphi_1$ and $\varphi_2$ are $C^\infty([-a, a])$ such that

$$\varphi_1(0) = \varphi_2(0) = 0, \quad \varphi'_1(0) = \varphi'_2(0) = 0, \quad \varphi_2 - \varphi_1 > 0 \text{ on } [-a, 0[ \cup ]0, a],$$

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and such that $\varphi = \varphi_2 - \varphi_1$ is not infinitely flat in 0, then Theorem 4.3 still holds: the two Taylor expansions of $u$ in $\mathcal{O}$ in the half-planes $x > 0$ and $x < 0$ are linear functions of the Taylor expansions of $f$, $\varphi_1$ and $\varphi_2$; therefore, one can prove that they fit together.

**Remark 4.5** If $\varphi$ has an asymptotics in non integer powers of $x$:

$$\varphi(x) = x^p + \gamma_1 x^{p_1} + \cdots + \gamma_N x^{p_N} + \mathcal{O}(x^{p_N+1}),$$

with $p_n$ an increasing sequence tending to $+\infty$ (for example the profile of Joukowski has such a form with $p = \frac{3}{2}$ and $p_n = \frac{3}{2} + \frac{n}{2}$), Theorem 3.1 still holds [6, 7, 10, 4, 14], but the proofs of Lemma 4.2 and Theorem 4.3 yield the construction of a non smooth asymptotics for $u$, see [14].

## 5 OTHER BOUNDARY CONDITIONS

While conditions corresponding to (2.9) are satisfied, we have the analogue of Theorems 3.1 and 4.3 for any other elliptic boundary problem. Namely, if $B_1(x, y; \partial_x, \partial_y)$ and $B_2(x, y; \partial_x, \partial_y)$ are two systems of boundary conditions on the sides $\theta = 0$ and $\theta = 1$ of $\Omega_n$ respectively, each of them covering $L$, then the behavior near $\mathcal{O}$ of the solutions of the boundary problem

$$\begin{cases}
Lu = f \text{ in } \Omega, \\
B_1 u = g_1 \text{ if } y = 0, \\
B_2 u = g_2 \text{ if } y = \varphi(x), \\
\partial_x^k u = 0, \text{ } k = 0, \ldots, m-1 \text{ if } x = a,
\end{cases} \quad (5.1)$$

depend on the problem $\left( L_{\text{princ}}(\mathcal{O}; \partial_x, \partial_y), B_{1, \text{princ}}(\mathcal{O}; \partial_x, \partial_y), B_{2, \text{princ}}(\mathcal{O}; \partial_x, \partial_y) \right)$ on the infinite strip $R \times ]0, 1[$. The condition replacing (2.9) is now

$$\forall \xi \in \mathbb{R}, \quad \left( L_{\text{princ}}(\mathcal{O}; i\xi, \partial_y), B_{1, \text{princ}}(\mathcal{O}; i\xi, \partial_y), B_{2, \text{princ}}(\mathcal{O}; i\xi, \partial_y) \right) : \quad H^m([0, 1]) \longrightarrow H^{-m}([0, 1]) \times \mathbb{C}^{2m} \text{ isomorphism.} \quad (5.2)$$
Under this assumption, the real part of the exponent $\mu_1^{(\omega)}$ tends to $+\infty$ when $\omega \to 0$, the regularity results in spaces of flat functions and in spaces of smooth functions still hold — note that, however, if $B_1$ and $B_2$ contain operators of the same order, the boundary data $g_1$ and $g_2$ have to satisfy a countable number of compatibility conditions at $\mathcal{O}$.

Examples are given by $B_1 = \text{Id}$, $B_2 = \partial_n$ for $L = \Delta$, or for $B_1 = B_2 = (\text{Id}, \Delta)$ for $L = \Delta^2$.

At the opposite, for Neumann problem, condition (5.2) is always violated in $\xi = 0$: all polynomials $v$ in $P_{m-1}$ satisfy
\[
\left( L_{\text{princ}}(\mathcal{O}; 0, \partial_\theta), B_{1,\text{princ}}(\mathcal{O}; 0, \partial_\theta), B_{2,\text{princ}}(\mathcal{O}; 0, \partial_\theta) \right) v = (0, 0, 0).
\]
That fact induces severe difficulties to handle flat right hand sides. Anyway for the Laplace operator for instance, it is still possible to construct an ansatz for the asymptotics of $u$ from the Taylor expansion of the right hand side by alternating double integrations with respect to $x$ of mean values of the type
\[
x \mapsto \frac{1}{\varphi(x)} \left( g_1(x) - g_2(x) + \int_0^{\varphi(x)} f(x, y) \, dy \right)
\]
and double integrations in $y$ like for Dirichlet, see S.A. Nazarov & O.R. Polyakova [13]. Such methods can be compared with what is done in elasticity for asymptotics in thin plates.

6 CUSPIDAL EDGES

Let now $\mathcal{W}$ be a three-dimensional domain with a cuspidal edge: this means that the boundary of $\mathcal{W}$ is smooth, except in the neighborhood of a smooth curve $\mathcal{E}$, the edge of $\mathcal{W}$, where $\mathcal{W}$ is locally diffeomorphic to $\mathbb{R} \times \Omega$, where $\Omega$ is a plane domain with an outgoing cusp in $\mathcal{O}$ as in §3-5.

Let $M$ be a strongly elliptic operator with $C^\infty$ coefficients in $\mathbb{R}^3$. We are interested in the regularity of the solutions of the Dirichlet problem:
\[
\begin{aligned}
Mu &= f \quad \text{in } \mathcal{W}, \\
u &\in \hat{H}^m(\mathcal{W}).
\end{aligned}
\]
\]
We study first the localized problem and prove that it is regular in spaces of flat and $C^\infty$ functions respectively. Our method of proof is classical and relies on differential quotients.
Let \((x, y)\) be the variables in \(\Omega\) and \(z\) the variable in \(\mathbb{R}\). Let \(\varphi\) be the function defining the boundary of \(\Omega\) according to (3.1)-(3.3). The spaces \(V^j(\mathbb{R} \times \Omega)\) are defined for \(j \in \mathbb{N}\) by:

\[
V^j(\mathbb{R} \times \Omega) = \{ u \in L^2(\mathbb{R} \times \Omega) \mid \forall \alpha \in \mathbb{N}^3, |\alpha| \leq j, \varphi^{[\alpha]} \partial^\alpha u \in L^2(\mathbb{R} \times \Omega) \},
\]

and by duality if \(j < 0\).

We have the tensorization properties for all \(j \in \mathbb{N}\):

\[
V^j(\mathbb{R} \times \Omega) = H^j(\mathbb{R}, V^0(\Omega)) \cap L^2(\mathbb{R}, V^j(\Omega)) \quad (6.2)
\]

and

\[
V^{-j}(\mathbb{R} \times \Omega) = H^{-j}(\mathbb{R}, V^0(\Omega)) + L^2(\mathbb{R}, V^{-j}(\Omega)). \quad (6.3)
\]

**Proposition 6.1** Let \(M\) be a strongly elliptic operator of order 2. Let \(j \in \mathbb{N}\). Then any solution \(u\) of the Dirichlet problem (6.1) with compact support and right hand side \(f \in V^j(\mathbb{R} \times \Omega)\) satisfies the optimal regularity property

\[
u \in V^{2+j}(\mathbb{R} \times \Omega).
\]

P. Grisvard [4] proved this result for \(L = \Delta\) and \(j = 0\) in \(L^p\) Sobolev spaces by a completely different technique.

**Proof.** Thanks to the strong ellipticity of \(M\), we have the a priori estimate

\[
\|u\|_{H^1(\mathbb{R} \times \Omega)} \leq c \left( \|Mu\|_{H^{-1}(\mathbb{R} \times \Omega)} + \|u\|_{L^2(\mathbb{R} \times \Omega)} \right)
\]

where \(c\) depends only on the support of \(u\). This estimate can also be written as

\[
\|u\|_{H^1(\mathbb{R}, L^2(\Omega))} + \|u\|_{L^2(\mathbb{R}, H^1(\Omega))} \leq c \left( \|Mu\|_{H^{-1}(\mathbb{R}, L^2(\Omega)) + L^2(\mathbb{R}, H^{-1}(\Omega))} + \|u\|_{L^2(\mathbb{R} \times \Omega)} \right). \quad (6.4)
\]

Considering for \(h > 0\) small enough the function \((u(x, y, z + h) - u(x, y, z))h^{-1}\) and letting \(h \to 0\), we deduce from (6.4) by recurrence over \(\ell \in \mathbb{N}\) that there holds:

\[
\|u\|_{H^{\ell+1}(\mathbb{R}, L^2(\Omega))} + \|u\|_{H^\ell(\mathbb{R}, H^1(\Omega))} \leq c \left( \|Mu\|_{H^{-1}(\mathbb{R}, L^2(\Omega)) + L^2(\mathbb{R}, H^{-1}(\Omega))} + \|u\|_{L^2(\mathbb{R} \times \Omega)} \right). \quad (6.5)
\]

Integrating in \(y\) from the side \(y = 0\), we easily prove that

\[
\hat{H}^1(\Omega) = \hat{V}^1(\Omega). \quad (6.6)
\]

Thus, (6.5) writes

\[
\|u\|_{H^{\ell+1}(\mathbb{R}, V^0(\Omega))} + \|u\|_{H^\ell(\mathbb{R}, V^1(\Omega))} \leq c \left( \|Mu\|_{H^{-1}(\mathbb{R}, V^0(\Omega)) + H^\ell(\mathbb{R}, V^{-1}(\Omega))} + \|u\|_{L^2(\mathbb{R} \times \Omega)} \right). \quad (6.7)
\]
Thus, for \( f \in V^j(\mathbb{R} \times \Omega) \), the above estimate for \( \ell = j + 1 \) yields that
\[
  u \in H^{2+j}(\mathbb{R}, V^0(\Omega)) \cap H^{1+j}(\mathbb{R}, V^1(\Omega)).
\]  
(6.8)

Let \( L \) be the operator
\[
  L(x, y, z; \partial_x, \partial_y) = M(x, y, z; \partial_x, \partial_y, 0),
\]  
(6.9)

so that there exists an operator \( P \) of order \( \leq 1 \) such that
\[
  M(x, y, z; \partial_x, \partial_y, \partial_z) = L(x, y, z; \partial_x, \partial_y) + P(x, y, z; \partial_x, \partial_y, \partial_z) \partial_z.
\]  
(6.10)

Since \( Mu = f \) belongs to \( H^j(\mathbb{R}, V^0(\Omega)) \), we deduce from (6.8) and (6.10) that
\[
  Lu \in H^j(\mathbb{R}, V^0(\Omega)).
\]  
(6.11)

Theorem 3.1 applied for each \( z \) combined with an argument of differential quotients yields that (6.11) implies
\[
  u \in H^j(\mathbb{R}, V^2(\Omega)).
\]

In that way, we prove by induction over \( \ell = 0, \ldots, j \) that
\[
  Lu \in H^{j-\ell}(\mathbb{R}, V^\ell(\Omega))
\]
which implies
\[
  u \in H^{j-\ell}(\mathbb{R}, V^{\ell+2}(\Omega)).
\]  
(6.12)

(6.12) for \( \ell = j \) combined with (6.8) gives the Proposition. \( \blacksquare \)

For general operators of order \( 2m \), one encounters a technical difficulty in handling the norms with negative exponents. We have

**Lemma 6.2** Let \( M \) be a strongly elliptic operator of order \( 2m \) with \( m \geq 2 \). Let \( j \in \mathbb{Z} \), \( j > -m \). There exists an integer \( k = k(m) > 0 \) such that any solution \( u \) of the Dirichlet problem (6.1) with compact support and right hand side
\[
  f \in H^{j+k}(\mathbb{R}, V^0(\Omega)) \cap H^k(\mathbb{R}, V^j(\Omega))
\]  
(6.13)
satisfies the regularity property
\[
  u \in V^{2m+j}(\mathbb{R} \times \Omega).
\]

**Proof.** Thanks to the strong ellipticity of \( M \), just like above we obtain the a priori estimate for any \( \ell \in \mathbb{N} \) — compare with (6.7)
\[
  \|u\|_{H^{\ell+m}(\mathbb{R}, V^0(\Omega))} + \|u\|_{H^\ell(\mathbb{R}, V^m(\Omega))}
  \leq c \left( \|Mu\|_{H^{\ell-m}(\mathbb{R}, V^0(\Omega))} + H^\ell(\mathbb{R}, V^{-m}(\Omega)) + \|u\|_{L^2(\mathbb{R} \times \Omega)} \right).
\]  

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Thus, for \( f \) satisfying (6.13) the above estimate for \( \ell = j + m + k \) yields that

\[
 u \in H^{2m+j+k}(\mathbb{R}, V^0(\Omega)) \cap H^{m+j+k}(\mathbb{R}, V^m(\Omega)).
\]

The operator \( L \) is defined by (6.9) so that there exists an operator \( P \) of order \( \leq 2m - 1 \) satisfying (6.10). Since \( Mu = f \) belongs to \( H^{j+k}(\mathbb{R}, V^0(\Omega)) \), we deduce that

\[
 Lu \in H^{j+k}(\mathbb{R}, V^{1-m}(\Omega)).
\]

Theorem 3.1 yields that

\[
 u \in H^{j+k}(\mathbb{R}, V^{1+m}(\Omega)).
\]

In that way, we prove by induction over \( \ell = 0, \ldots, j + m \) that

\[
 Lu \in H^{j+m+\ell+k_0}(\mathbb{R}, V^{\ell-m}(\Omega))
\]

with \( k_0 = k, \ k_1 = k_0 - (m - 1), \ k_2 = k_1 - (m - 2), \ldots, \ k_{m-1} = k_m = \ldots = k_m + j \), and

\[
 u \in H^{j+m+\ell+k_0}(\mathbb{R}, V^{\ell+m}(\Omega)).
\]

It suffices to choose \( k_0 \) such that \( k_m \geq 0 \) to obtain finally for \( \ell = m + j \) that \( u \) belongs to \( L^2(\mathbb{R}, V^{2m+j}(\Omega)) \). As \( u \) also belongs to \( u \in H^{2m+j}(\mathbb{R}, V^0(\Omega)) \), the Lemma is proved.

**Theorem 6.3** Let \( M \) be a strongly elliptic operator. Then any solution \( u \) of the Dirichlet problem (6.1) with compact support and right hand side \( f \in C^\infty(\mathbb{R} \times \overline{\Omega}) \) satisfies the optimal regularity property

\[
 u \in C^\infty(\mathbb{R} \times \overline{\Omega}).
\]

**Proof.** We denote by \( H^\infty \) the intersection of all spaces \( H^k \) for \( k \in \mathbb{N} \). Since \( f \) belongs to \( H^\infty(\mathbb{R}, V^0(\Omega)) \), Proposition 6.1 and Lemma 6.2 yield that

\[
 u \in H^\infty(\mathbb{R}, V^{2m}(\Omega)). \quad (6.14)
\]

The proof runs as above, using as spaces on \( \Omega \) the spaces \( V^j(\Omega) \) augmented by the spaces of polynomials:

\[
 \tilde{V}^j(\Omega) = V^j(\Omega) + P_{pj-1}(\Omega).
\]

Indeed, Theorem 3.1 and Lemma 4.2 give that

\[
 \left\{ \begin{array}{l}
 Lu \in \tilde{V}^j(\Omega), \\
 u \in \tilde{H}^m(\Omega) 
\end{array} \right. \implies u \in \tilde{V}^{2m+j}(\Omega).
\]

So, starting from (6.14), we have \( Lu \in H^\infty(\mathbb{R}, \tilde{V}^1(\Omega)) \). Thus \( u \in H^\infty(\mathbb{R}, \tilde{V}^{2m+1}(\Omega)) \). Going on, we prove by induction that \( \forall \ell \in \mathbb{N} \),

\[
 u \in H^\infty(\mathbb{R}, \tilde{V}^{2m+\ell}(\Omega)).
\]
Similarly to Remark 4.4, we deduce from all these statements a local regularity result in the space $C^\infty(W)$ for any domain $W = \mathbb{R}^3 \setminus \overline{U}$ where the domain $U$ is the disjoint union of two (or more) domains with analytic boundaries $U_1$ and $U_2$ such that $\overline{U_1} \cap \overline{U_2}$ is a curve $E$. As examples, we can take for $U_1$ and $U_2$: a cylinder and a half-space, two cylinders, a torus and a cylinder, a torus and a ball, a torus and a half-space, two tori, etc...

**Remark 6.4** Let us give a short description of the case where there is only one contact point: we take as $U_1$ the half-space $y < 0$ and as $U_2$ the ball of radius 1 and of center $(x_1, x_2, y) = (0, 0, 1)$ and we consider the Dirichlet problem for the Laplace operator. Then it is possible to prove that for a right hand side flat enough in $O = (0, 0, 0)$, the solution is as flat as desired. The idea is to work in cylindrical coordinates $(r, \theta, y)$ with $x_1 = r \cos \theta$ and $x_2 = r \sin \theta$ and to combine arguments of differential quotients with respect to the variable $\theta$ and estimates on $u_r$ by considering $\Delta(u_r)$. It is still possible to construct the asymptotics of $u$ by the polynomial resolution, but the outcome of the construction gives not only polynomial terms but also non smooth terms. The most singular that we have found are

\[
\frac{x_1 y^4}{r^2} \quad \text{and} \quad \frac{x_2 y^4}{r^2}.
\]

**NOTE**

During the Conference, V.G. Mazya pointed out to us a work by V.I. Feigin. We only found in the literature the short note [3], where are stated results very similar to ours.

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