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STATIONARY STOKES AND NAVIER–STOKES SYSTEMS ON TWO- OR THREE-DIMENSIONAL DOMAINS WITH CORNERS. PART I: LINEARIZED EQUATIONS*

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Abstract. The H^s-regularity (s being real and nonnegative) of solutions of the Stokes system in domains with corners is studied. In particular, a H^2-regularity result on a convex polyhedron that generalizes Kellogg and Osborn's result on a convex polygon to three-dimensional domains is stated. Sharper regularity on a cube and on other domains with corners is attained. Conditions for the problem to be Fredholm are also given, and its singular functions along with those of the nonlinear problem are studied in the second part of this paper.

Key words. Stokes, corner, edge, polyhedron, regularity of solutions

AMS(MOS) subject classifications. 35Q, 76N

1. Introduction. The linearized equations corresponding to the Navier–Stokes system describing gas-dynamics consist of the following Stokes system in $\mathbf{R}^n$ ($n=2$ or 3):

$$-\Delta\vec{u}+\nabla p=\vec{f},\qquad \operatorname{div}\vec{u}=g \tag{1.1}$$

where $\vec{u}=(u_1,\cdots,u_n)$ is the speed of the fluid, p its pressure, and $\vec{f}$ the strength field. On a domain Ω, the boundary conditions are

$$\vec{u}_{|\partial\Omega}=0. \tag{1.2}$$

The problem (1.1)–(1.2) can be approached as an elliptic boundary value problem as in the paper by Agmon, Douglis, and Nirenberg [1]. On the other hand, it may be proved by a variational method (see Temam [22]) that for a bounded domain Ω and data $(\vec{f},g)$ in the product of Sobolev spaces $[H^{-1}(\Omega)]^n\times L^2(\Omega)$ with the compatibility condition

$$\int_\Omega g\,dX=0, \tag{1.3}$$

there exists a unique solution $(\vec{u},p)$ of (1.1)–(1.2) in the space $[\mathring{H}^1(\Omega)]^n\times[L^2(\Omega)/C]$. Here, as usual, $\mathring{H}^1(\Omega)$ denotes the H^1-space with null traces on the boundary, and H^{-1} is its dual with respect to the L^2-duality.

Thus, if $(\vec{f},g)$ is more regular, let us say

$$\vec{f}\in[H^{s-1}(\Omega)]^n\quad\text{and}\quad g\in H^s(\Omega),\quad s>0, \tag{1.4}$$

then, when Ω has a smooth boundary, we draw from [1] and interpolation (cf. [23]), that

$$\vec{u}\in[H^{s+1}(\Omega)]^n\quad\text{and}\quad p\in H^s(\Omega). \tag{1.5}$$

But, in the case of physical domains, or for partition of domains in numerical analysis, it is natural to study the case when Ω has corners.

In two-dimensional domains (2D), when Ω is a polygon, we have Kondrat'ev's [12] and Grisvard's [10] results for the divergence-free system ($g=0$: incompressible fluid) in spaces with integer exponents; for the general system (1.1), we have Osborn's

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results [19], Dauge's results [5] in weighted Sobolev spaces, and the regularity result of Kellogg and Osborn [11].

In three-dimensional domains (3D), Maz'ja and Plamenevskii study the problem (1.1)–(1.2) for a large class of domains in weighted Sobolev spaces: the results are announced in [15] and proved in [16], [17a], [17b]. The spaces are general L^p-Sobolev spaces with weight (of Kondrat'ev type) and also Hölder classes with weight. Merigot [18] and Grisvard [10] have also used L^p-Sobolev spaces in the 2D divergence-free problem on a polygon.

In this paper we state precise results of regularity in the ordinary spaces (1.4), (1.5). Among other things, the Sobolev spaces with real exponents are useful for studying the nonlinear Navier–Stokes system (Part II of this work is forthcoming), and for successive approximation schemes (see [20]).

Theorems 5.4 and 5.5 in 2D are a generalization of [10] and [11]. In 3D we get new results. For several examples of domains, we hereafter indicate a condition on s under which the solution $(\vec{u},p)$ of (1.1)–(1.2) with $(\vec{f},g)$ in the space (1.4) has the regularity of (1.5), provided g is zero at the singular points of Ω if $s\geqq 1$ (cf. [11] and the definition (9.17)):

(1.6) If Ω is any domain in our class of domains with corners $\mathcal{O}_3$ (introduced in § 2 below), $s<0.5$.

(1.7) If $\Omega=Q_1\backslash Q_2$ where Q_1 and Q_2 are two rectangular parallelepipeds with the same axes, $s\leqq 0.544$ (approximate value).

(1.8) If Ω is any convex domain in our class $\mathcal{O}_3$, $s\leqq 1$.

(1.9) If Ω is any convex domain with wedge angles $\leqq 2\pi/3$, $s<3/2$.

(1.10) If Ω is any cylinder with convex polygonal base, and angles $<2\pi/3$, $s\leqq 3/2$.

(1.11) If Ω is any cylinder with smooth base, $s<2$.

(1.12) If Ω is a half-ball, $s<2$.

When we say a cylinder, we mean a bounded cylinder truncated perpendicularly to its generating lines.

The plan of this paper is as follows. In § 2 we introduce our classes of domains and the functional spaces. In § 3 we recall general results from Dauge's works [6] and [9], and we apply them to the problem (1.1)–(1.2). As these results are based on a special condition of injectivity about tangent problems, in § 4 we link that condition to the usual one used by Kondrat'ev in [12]. In § 5 we recall some properties of the characteristic equation $\sin^2\lambda\omega-\lambda^2\sin^2\omega=0$, we give a graph and tables of values for its roots, and we state results in 2D. In § 6 we study the domains in 3D that have edges, but no vertices. In § 7 we study the tangent problem in a three-dimensional cone, which gives rise to a quantity linked with the Laplace–Beltrami operator that we estimate in § 8. Finally, we state 3D results in § 9.

2. Classes of domains and functional spaces. Our classes of domains contain various curvilinear polygons (in 2D) and polyhedra or domains with piecewise-smooth boundary (in 3D).

2.1. Plane and spherical domains. Our class $\mathcal{O}_2(\mathbf{R}^2)$ of plane domains consists of all curvilinear polygons, possibly with cracks but without cusps (or turning points): Ω is in $\mathcal{O}_2(\mathbf{R}^2)$ if and only if it enjoys the following properties:

(i) Ω is bounded and connected.

(ii) The boundary of Ω consists of a finite number of smooth closed arcs $\Gamma_1,\cdots,\Gamma_N$, $\Gamma_{N+1}=\Gamma_1$.

(iii) Let A_j and A_{j+1} be the ends of Γ_j; the A_j for $j=1,\cdots,N$ are the vertices of Ω and at the neighborhood of A_j, Ω is locally diffeomorphic to a neighborhood of zero in a plane sector Γ_{A_j}.

In the case when one of the sectors Γ_{A_j} has its opening equal to 2π, we have a *crack* and we dissociate the two sides of the crack as in Fig. 1.

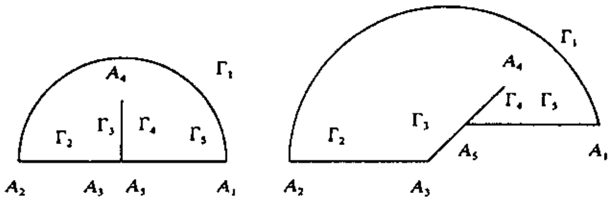

FIG. 1

In both cases, A_4 is at the bottom of the crack and Γ_3 and Γ_4 coincide in the neighborhood of A_4.

Let us note that condition (iii) may be rewritten in the following form. If the tangents of Γ_j and Γ_{j+1} coincide in A_j, then Γ_j and Γ_{j+1} coincide in a neighborhood of A_j.

We denote by $A_0(\Omega)$ the set $\{A_1,\cdots,A_N\}$ and denote simply by x any element of $A_0(\Omega)$. Thus, for $x=A_j$, Γ_{A_j} is denoted by Γ_x.

In the same way we define the class $\mathcal{O}_2(S^2)$ of curvilinear polygons on the unit sphere of $\mathbf{R}^3$.

2.2. Three-dimensional domains. Ω belongs to $\mathcal{O}_3(\mathbf{R}^3)$ if and only if it satisfies the following conditions:

(i) Ω is bounded and connected.

(ii) At each point x of its "stretched" boundary, Ω is locally diffeomorphic to a neighborhood of zero in one of the following three kinds of domains:

(1) A half-space: then x is a **regular** point;

(2) A dihedron isomorphic to $\mathbf{R}\times\Gamma_x$, with Γ_x a plane sector with an opening ω_x different from π: then x belongs to an **edge**;

(3) A cone Γ_x with vertex zero (which is not a dihedron), such that its intersection G_x with S^2 belongs to $\mathcal{O}_2(S^2)$: then x is a **vertex**.

Let $A_0(\Omega)$ be the set of vertices and $A_1(\Omega)$ be the union of the edges.

The **stretched boundary** is the notion corresponding to the doubling of the boundary when there is a crack in 2D. This is more completely explained in § 2 of [9].

Note that if Ω has a piecewise-smooth boundary, and its faces meet two by two or three by three with independent normals at meeting points, then Ω belongs to our class $\mathcal{O}_3(\mathbf{R}^3)$.

2.3. Sobolev spaces. For a positive integer s, $H^s(\Omega)$ is the usual Sobolev space of all distributions u in $\mathcal{D}'(\Omega)$ such that each derivative $D^\alpha u$ with length $|\alpha|\leq s$, in Ω, belongs to $L^2(\Omega)$. For a positive noninteger real number s, let $[s]$ be the integer part of s and $\sigma=s-[s]$. $H^s(\Omega)$ is the space of all u in $H^{[s]}(\Omega)$ that satisfy

$$\forall\alpha,|\alpha|=[s]\quad\iint_\Omega |D^\alpha u(x)-D^\alpha u(y)|^2\,d(x,y)^{-n-2s}\,dx\,dy<+\infty$$

where $d(x,y)$ is the infimum of length of the paths joining x to y and included in Ω.

$\mathring{H}^s(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$ and $H^{-s}(\Omega)$ its dual with respect to the L^2-duality.

2.4. Stokes operators. We denote by $D_n^s(\Omega)$ the product of Sobolev spaces $[\mathring{H}^1\cap H^{s+1}(\Omega)]^n\times H^s(\Omega)$ (cf. (1.5)) and by $E_n^s(\Omega)$ the product $[H^{s-1}(\Omega)]^n\times H^s(\Omega)$. We then denote by $\mathcal{S}_n$ the operator (1.1) applying $(\vec{u},p)$ on $(\vec{f},g)$, and we write especially $\mathcal{S}_n[s,\Omega]$ for $\mathcal{S}_n$ acting from $D_n^s(\Omega)$ to $E_n^s(\Omega)$. We suppose everywhere that $s\neq\frac{1}{2}$.

3. General Fredholm properties. General Fredholm properties rely on general statements of [9] that we apply here to the Stokes system (1.1)-(1.2).

In [9], we develop general conditions for a strongly elliptic operator to be Fredholm between Sobolev spaces H^s (in the above sense; § 2.3), e.g., with Dirichlet conditions. Moreover, we extend that theory to strongly elliptic systems, and other ones satisfying a weaker ellipticity property that holds in particular for the Stokes system (see (7.7) in [9]).

We will apply those results. To do so, we recall the characteristic conditions concerning the operator and the domain. When Ω is a polygon, it is well known that such conditions are related to the angle openings of Ω and to associated discriminant functions (cf. [10], [12]). In fact it is related with the spectrum of a holomorphic operator family; in three dimensions the condition may be written only in that form. We show in [9] that those "spectral" conditions are fully convenient for "homogeneous" weighted Sobolev spaces, and that, for ordinary Sobolev spaces, they must be replaced with a new type of condition we call "injectivity modulo polynomials."

Although that distinction is of lesser use for regularity properties than for Fredholm properties, we introduce it in anticipation of the forthcoming Part II of this paper where we will describe the singularities of solutions.

Our conditions are related to tangent (or frozen) operators at each singular point of Ω.

3.1. Frozen operators at a vertex. Let Ω be a domain in $\mathcal{O}_n(\mathbf{R}^n)$, $n=2,3$ and $x\in A_0(\Omega)$. We will suppose that the diffeomorphism χ, which implies a neighborhood of x in Ω on a neighborhood of zero in Γ_x, is such that

$$D\chi(x)=I \text{ is the identity matrix.}$$

Then, the operator L_x, obtained by taking the principal part of the operator $\chi\circ\mathcal{S}_n\circ\chi^{-1}$ frozen in zero, just coincides with $\mathcal{S}_n$ on the cone Γ_x.

3.2. Frozen operators along an edge. As in the case of a vertex, if $x\in A_1(\Omega)$, the frozen operator on the wedge $\mathbf{R}\times\Gamma_x$ is $\mathcal{S}_3$. But, we have to define a new frozen operator L_x on the plane sector Γ_x (cf. [9, (3.3)]). Let (y,z) be coordinates such that $y\in\mathbf{R}$ and $z\in\Gamma_x$. The operator L_x is defined as

$$L_x(D_z)=\mathcal{S}_3(0,D_z)$$

(we remove tangential derivatives along the edge). Thus, we have

$$L_x(u_1,u_2,u_3,p)=(f_1,f_2,f_3,g) \tag{3.1}$$

if and only if

$$\mathcal{S}_2(u_1,u_2,p)=(f_1,f_2,g)\quad\text{and}\quad\Delta u_3=f_3. \tag{3.2}$$

3.3. Injectivity modulo polynomials. For $\lambda\in\mathbf{C}$, $S_n^\lambda(\Gamma_x)$ denotes the set of vector functions $(u_1,\cdots,u_n,p)$ of the form:

$$u_j=r^\lambda\sum_{0\leq q\leq Q}u_{jq}(\Psi)\log^q r\quad\text{with } u_{jq}\in\mathring{H}^1(G_x),$$

$$p=r^{\lambda-1}\sum_{0\leq q\leq Q}p_q(\Psi)\log^q r\quad\text{with } p_q\in L^2(G_x)$$

where $(r,\Psi)=(|z|, z/|z|)$ are the polar coordinates and G_x is the intersection of Γ_x with the unit sphere S^{n-1}.

We say that L is injective modulo polynomial on $S_n^\lambda(\Gamma_x)$ if

$(\vec{u},p)\in S_n^\lambda(\Gamma_x)$ and $L(\vec{u},p)$ is polynomial implies that $(\vec{u},p)$ is polynomial.

Here "polynomial" means polynomial with respect to *cartesian variable* $z=(z_1,z_2)$ or (z_1,z_2,z_3). For instance, $r^\alpha \sin\alpha\theta$ is polynomial in $\mathbf{R}^2$ for $\alpha\in\mathbf{Z}$. Of course, the zero function is polynomial.

3.4. Index and regularity results.

THEOREM 3.3. *Let $\Omega\in\mathcal{O}_n(\mathbf{R}^n)$. The Stokes operator $\mathcal{S}_n[s,\Omega]$ is a Fredholm operator if and only if both the following conditions are satisfied*:

(3.4) $\forall x\in A_0(\Omega)$, $\forall\lambda$ *with* $\operatorname{Re}\lambda=s+1-n/2$, L_x *is injective modulo polynomials on* $S_n^\lambda(\Gamma_x)$;

(3.5) $\exists\varepsilon>0$, $\forall x\in A_1(\Omega)$, $\forall\lambda$ *with* $\operatorname{Re}\lambda\in[0,s+\varepsilon]$, L_x *is injective modulo polynomials on* $S_3^\lambda(\Gamma_x)$.

This statement is derived from (7.15) in [9], with the variant (6.8) in [9].

If Ω has only conical points (which is the case when $n=2$), the condition (3.5) is void. If Ω has no vertex (cf. examples (1.11), (1.12)), the condition (3.4) is void and (3.5) may be replaced with (3.5′):

(3.5′) $\forall x\in A_1(\Omega)$, $\forall\lambda$ with $\operatorname{Re}\lambda\in[0,s]$, L_x is injective modulo polynomials on $S_3^\lambda(\Gamma_x)$.

If Ω is a three-dimensional polyhedron with plane faces, (3.5) may still be replaced with (3.5′): the ε in (3.5) is useful in the case when Ω is a three-dimensional domain with smooth curved faces; that ε allows an easier formulation without introducing "subsections" or "singular chains," which describe the limit geometrical behavior at the neighborhood of a vertex.

THEOREM 3.6. *Assume that the conditions* (3.5) *and* (3.7) *are fulfilled*:

(3.7) $\forall x\in A_0(\Omega)$, $\forall\lambda$ *with* $\operatorname{Re}\lambda\in[1-n/2, s+1-n/2]$, L_x *is injective modulo polynomials on* $S_n^\lambda(\Gamma_x)$.

Then, each solution $(\vec{u},p)\in D_n^0(\Omega)$ of (1.1) *with $(\vec{f},g)$ in $E_n^s(\Omega)$ has the regularity $D_n^s(\Omega)$.*

When (3.4) is satisfied, and not (3.7), there are singular functions. We will study these in Part II of this paper, along with the nonlinear Navier-Stokes system.

Now, we will study (3.5) and (3.7) in order to give more precise regularity results in two and three dimensions.

4. The link between the injectivity condition and the usual spectral condition.

4.1. Generalities. Let us study condition (3.4). In view of § 3.1, $L_x=\mathcal{S}_n$. If we consider $\vec{u}$ of the form $r^\lambda\vec{\mathfrak{u}}(\Psi)$ and $p=r^{\lambda-1}\mathfrak{p}(\Psi)$, then we get

$$\mathcal{S}_n(\vec{u},p)=(\vec{f},g)$$

where $\vec{f}=r^{\lambda-2}\vec{\mathfrak{f}}(\Psi)$ and $g=r^{\lambda-1}\mathfrak{g}(\Psi)$, with

$$\mathcal{L}_n(\lambda)(\vec{\mathfrak{u}},\mathfrak{p})=(\vec{\mathfrak{f}},\mathfrak{g}), \tag{4.1}$$

$\mathcal{L}_n(\lambda)$ being a system on the sphere S^{n-1}, depending in a polynomial way on λ. As in [17], we can derive from the writing of $\mathcal{S}_n$ in polar coordinates that (4.1) may be written in the form

$$[\delta_n-\lambda(\lambda+1)]\vec{\mathfrak{u}}+[(\lambda-1)\vec{\Psi}+\vec{\nabla}_s]\mathfrak{p}=\vec{\mathfrak{f}},\qquad \langle\lambda\vec{\Psi}+\vec{\sigma},\vec{\mathfrak{u}}\rangle=\mathfrak{g}$$

where δ_n is the positive Laplace-Beltrami operator on S^{n-1}, $\vec{\Psi}$ is the vector $x/|x|$ in $\mathbf{R}^n$, and $\vec{\nabla}_s$ is the tangential component of the gradient on the sphere

$$\vec{\nabla}_s=\vec{\nabla}-\vec{\Psi}\partial/\partial_r.$$

$\mathcal{L}_n(\lambda)$ gives rise to an operator acting from $D_n^0(G_x)$ to $E_n^0(G_x)$. It is almost everywhere invertible. The set of λ for which $\mathcal{L}_n(\lambda)$ is not invertible is called the **spectrum** of $\mathcal{L}_n$, and the condition used by Kondrat'ev [12] or Maz'ja and Plamenevskii is that the straight line $\operatorname{Re}\lambda=s+1-n/2$ does not meet the spectrum of $\mathcal{L}_n$. As we have already said, this type of condition is correct for weighted Sobolev spaces (of the type $r^{\gamma+|\alpha|}D^\alpha v\in L^2$), but it is not always suitable for ordinary Sobolev spaces. Nevertheless, we have (cf. (4.2) for $s=0$ and (4.6) in [9]):

LEMMA 4.2. *If λ is not a positive integer, $\mathcal{S}_n$ is injective modulo polynomials on $S^\lambda(\Gamma_x)$ if and only if λ does not belong to the spectrum of $\mathcal{L}_n$ on G_x.*

If λ is an integer number, the comparison depends on the difference $d(\lambda)$ between the dimensions of two spaces of polynomial functions:

$$d(\lambda)=\dim P^\lambda(\Gamma_x)-\dim Q^{\lambda-2}$$

where $P^\lambda(\Gamma_x)$ is the set of the elements of $S^\lambda(\Gamma_x)$ that are polynomials *in cartesian variables*, and $Q^{\lambda-2}$ is the set of the $(\vec{f},g)$ with f_j (respectively, g) homogeneous polynomial of degree $\lambda-2$ (respectively, $\lambda-1$) in z. $d(\lambda)$ depends only on Γ_x. According to [9, Annex D], there exists a homogeneous polynomial A that is zero on the boundary of Γ_x and such that if B is a polynomial that is zero on $\partial\Gamma_x$, then A divides B (i.e., $P^\lambda(\Gamma_x)$ is a principal ideal).

If the degree of A is two, then $d(\lambda)=0$; and according to (4.9) and (7.14) in [9], we have the following lemma.

LEMMA 4.3. *If $d^0A=2$, for each integer λ we have the same equivalence as in* (4.2).

According to (4.8) and (7.14) in [9], we have the following lemma.

LEMMA 4.4. *If $d^0A\geqq 3$, for $\lambda=1$ we have the same equivalence as in* (4.2); *but for each integer $\lambda\geqq 2$, $\mathcal{S}_n$ is not injective modulo polynomials on $S^\lambda(\Gamma_x)$.*

According to (4.10) and (7.14) in [9], we have Lemma 4.5.

LEMMA 4.5. *If $d^0A=1$, and $\lambda\in\mathbf{N}^*$, then $\mathcal{S}_n$ is injective modulo polynomials on $S^\lambda(\Gamma_x)$ if and only if $\mathcal{L}_n(\mu)^{-1}$ has a pole of order one in $\mu=\lambda$ and if*

$$\dim\operatorname{Ker}\mathcal{L}_n(\lambda)=d(\lambda).$$

4.2. Application to two-dimensional cones. It is well known that the poles $\mathcal{L}_2(\lambda)^{-1}$ coincide with the roots of the following equations:

$$F_\omega(\lambda)=0\quad\text{where}\quad F_\omega(\lambda)=\frac{\sin^2\lambda\omega-\lambda^2\sin^2\omega}{\lambda^2} \tag{4.6}$$

because, more precisely, $\mathcal{L}_2(\lambda)^{-1}F_\omega(\lambda)$ is holomorphic on $\mathbf{C}$ (cf. [12], [10], [11], [5]).

When the opening ω of the plane sector Γ_x is not 2π, then the two sides of Γ_x are independent and $d^0A=2$. So, the condition (3.4) is that F_ω has no zero with real part s.

When $\omega=2\pi$, then $d^0A=1$ and $d(\lambda)>0$ for all positive integer numbers. As in § 15.B in [9] for fourth-order operators, we show in the Annex that, according to Lemma 4.5, $\mathcal{S}_2$ is injective modulo polynomials on S^λ for each $\lambda\in\mathbf{N}^*$ (including $\lambda=1$). As the roots of (4.6) are the half integers, we find that condition (3.4) is reduced to

$$s\neq k+\tfrac{1}{2},\quad \forall k\in\mathbf{N}.$$

4.3. Application to three-dimensional cones. If Γ_x is a revolution cone, then $d^0A=2$ and we apply Lemmas 4.2 and 4.3.

If Γ_x is a polyhedral cone, let D be the number of distinct planes containing at least one side of Γ_x. For a cube, $D=3$. For a pyramid with a square basis, $D=4$. If $D \geqq 3$, we apply Lemmas 4.2 and 4.4.

5. Precise results in two-dimensional domains.

5.1. More about the discriminant function F_ω. The roots of (4.6) have been studied by Seif [21], Lozi [13], Dauge [7], Bernardi and Raugel [2], [3], and Maslovskaya [14]. Bernardi and Raugel give a table for the roots of (4.6) with the lowest positive real part. Here, we complete that work by a table for the roots of (4.6) with their real parts $\xi \in [0, 4]$ and by the corresponding graph (Fig. 2) of ξ in function of ω.

Let us denote λ by $\xi + i\eta$, with ξ, η real. We are interested in roots of (4.6) with $\xi \geqq 0$. $\lambda = 1$ is always a root of (4.6) and plays a particular role (see § 5.2).

We denote by $\xi_k(\omega)$ the real part of the kth root of

$$(\lambda - 1)^{-1} F_\omega(\lambda) = 0$$

the roots being ordered with increasing real part and repeated according to their multiplicities. The following can be shown (cf. [7], [2]):

(a) If $\omega \in]0, \pi[$, $\xi_1(\omega) > \pi/\omega$.

(b) If $\omega \in]\pi, 2\pi[$, $\xi_1(\omega) \in]\sup(\frac{1}{2}, \omega_1/\omega), \pi/\omega[$, where $\omega_1 = 0.812825\pi$; ω_1 is the root of

$$\omega \in]0, \pi[\quad \text{and} \quad \frac{\sin \omega}{\omega} = -\cos \omega_0 \quad \text{with } \tan \omega_0 = \omega_0.$$

Tables 1 and 2 and Fig. 2 give values for $\xi_1, \cdots, \xi_{14}$ that occur in $[0, 4]$. A dash means a value greater than four.

For $j \geqq 1$, let I_j be the set of ω such that $\xi_{2j}(\omega)$ and $\xi_{2j+1}(\omega)$ coincide. In the interior of I_j, ξ_{2j} and ξ_{2j+1} are the real parts of two conjugate nonreal numbers. For $\omega \in \partial I_j$ and $\omega \neq 0$, there is a real double root and the bifurcation of two real roots.

TABLE 1

ω	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6
0.4	3.397	3.397	-	-	-	-
0.5	2.740	2.740	4.808	4.808	-	-
0.6	2.307	2.307	4.022	4.022	-	-
0.7	2.004	2.004	3.464	3.464	-	-
0.8	1.783	1.783	3.051	3.051	4.312	4.312
0.9	1.252	1.988	2.542	2.932	3.853	3.853
1	1	2	2	3	3	4
1.1	0.834	1.662	2.012	2.475	3.096	3.215
1.2	0.718	1.408	2.045	2.045	2.883	2.883
1.3	0.637	1.207	1.882	1.882	2.657	2.657
1.4	0.581	1.044	1.745	1.745	2.465	2.465
1.5	0.544	0.909	1.629	1.629	2.301	2.301
1.6	0.522	0.796	1.530	1.530	2.159	2.159
1.7	0.509	0.701	1.444	1.444	2.035	2.035
1.8	0.503	0.622	1.258	1.480	1.927	1.927
1.9	0.500	0.555	1.111	1.498	1.670	1.994
2	0.5	0.5	1	1.5	1.5	2

TABLE 2

ω	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}
1.1	4.067	4.067	-	-	-	-
1.2	3.720	3.720	-	-	-	-
1.3	3.430	3.430	4.203	4.203	-	-
1.4	3.184	3.184	3.901	3.901	-	-
1.5	2.972	2.972	3.641	3.641	-	-
1.6	2.788	2.788	3.415	3.415	4.042	4.042
1.7	2.626	2.626	3.216	3.216	3.806	3.806
1.8	2.484	2.484	3.041	3.041	3.597	3.597
1.9	2.233	2.485	2.808	2.964	3.413	3.413
2	2	2.5	2.5	3	3	3.5

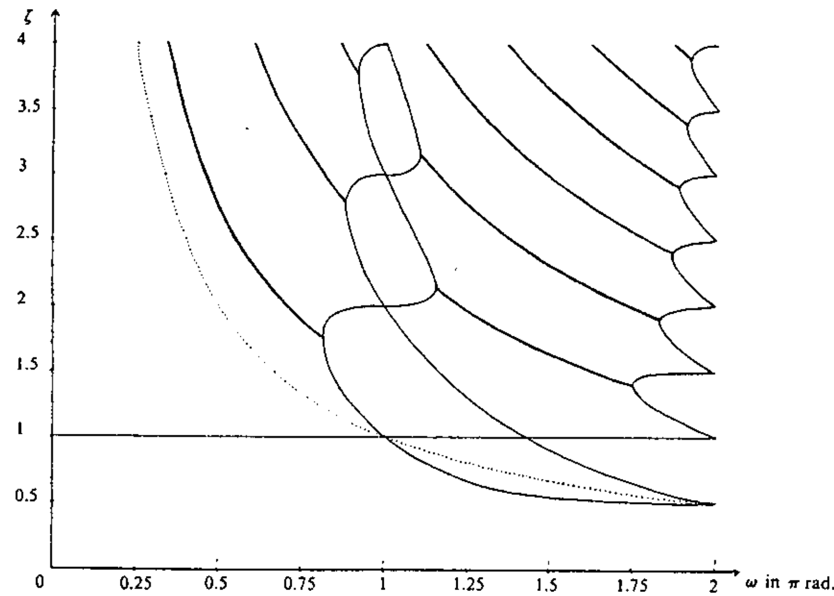

FIG. 2

I_1 has only one connected component: $I_1 =]0, \omega_1]$. For $j \geqq 2$, I_j has two connected components: $]0, \omega_j]$ and $[\omega_j', \omega_j'']$. When $j \to +\infty$, $\omega_j \to \pi$ and $\omega_j'' \to 2\pi$ increase, while $\omega_j' \to \pi$ decreases. All integers are double roots for $\omega = \pi$, and all half integers are double roots for $\omega = 2\pi$.

On the graph, the dotted line is the graph of $\omega \to \pi/\omega$. The heavy lines represent a double value for the ξ_k (conjugate roots), and the ordinary lines represent real roots.

Table 3 gives the values of the ω_j, ω_j', ω_j'' that occur in Fig. 2.

5.2. The special case of the pole $\lambda = 1$. As we have already shown $\lambda = 1$ is always a pole for $\mathscr{L}_2(\lambda)^{-1}$. But, if the opening of the cone Γ is $\omega = 2\pi$, $\mathscr{S}_2$ is injective modulo polynomials on $S^\lambda(\Gamma)$. If $\omega \neq 2\pi$, this is not so for $\mathscr{S}_2$.

It is easy to show that $\operatorname{Ker} \mathscr{L}_2(1)$ is one-dimensional and is generated by $(\vec{0}, 1)$ (see [11], [5]). As a consequence, we get the following lemma.

TABLE 3

	ω	ω'	ω''
1	0.813	–	–
2	0.884	1.154	1.751
3	0.915	1.102	1.825

LEMMA 5.1. *Let s be such that $1 \leqq s < \xi_1(\omega)$. Let $(\vec{u}, p)$ be in $D_2^0(\Gamma)$ such that $\mathscr{S}_2(\vec{u}, p) = (\vec{f}, g) \in E_2^s(\Gamma)$. Moreover, if $s > 1$, we suppose that $g(0) = 0$; if $s = 1$ we suppose that $r^{-1}g \in L^2(\Gamma)$. Then, if B denotes the unit ball, we have*

$$(\vec{u}, p) \in D_2^s(\Gamma \cap B).$$

Proof. We derive the proof from the methods of [9]. For $s = 1$, it is the result of [11].

By a cut-off, we assume that $(\vec{u}, p)$ has compact support. We use the Mellin transform $\mathcal{M}$ of $(\vec{u}, rp)$, which is defined for $\operatorname{Re} \lambda \leqq 0$; we have

$$\mathscr{L}_2(r\partial_r)(\vec{u}, rp) = (r^2\vec{f}, rg), \tag{5.2}$$

and thus using the Mellin transform we have

$$\mathscr{L}_2(\lambda)(\vec{U}(\lambda), P(\lambda)) = (\vec{F}(\lambda), G(\lambda)). \tag{5.3}$$

But $(\vec{F}(\lambda), G(\lambda))$ is defined for $\operatorname{Re} \lambda < s$. If $s \neq 1$, then we deduce from [9] (see the condensed proof in [8]) that there exists $(\vec{u}_0, p_0) \in D_2^s(\Gamma \cap B)$ such that for $\operatorname{Re} \lambda = 1 + \varepsilon$ with $\varepsilon \in]0, \min(1, s-1)[$,

$$\mathcal{M}(\vec{u}_0, rp_0)(\lambda) = (\vec{U}(\lambda), P(\lambda))$$

where $(\vec{U}, P)(\lambda)$ is the extension determined by (5.3). And we have

$$(\vec{u}, p) - (\vec{u}_0, p_0) = \sum_{0 < \operatorname{Re}\lambda < 1+\varepsilon} \operatorname{Res}\,[r^\lambda \vec{U}(\lambda), r^{\lambda-1}P(\lambda)].$$

Since $s < \xi_1(\omega)$, the sum is reduced to $\lambda = 1$. And we have

$$\begin{aligned}\mathscr{S}_2 \operatorname{Res}(r^\lambda \vec{U}(\lambda), r^{\lambda-1}P(\lambda)) &= \operatorname{Res} \mathscr{S}_2(r^\lambda \vec{U}(\lambda), r^{\lambda-1}P(\lambda)) \\ &= \operatorname{Res}_{\lambda=1}(r^{\lambda-2}\vec{F}(\lambda), r^{\lambda-1}G(\lambda)) \\ &= (\vec{0}, g(0)).\end{aligned}$$

The second equality is given by (5.3) and by the equivalence of $\mathscr{S}_2(\vec{u}, p) = (\vec{f}, g)$ with (5.2). Since $g(0) = 0$, we get

$$\mathscr{S}_2 \operatorname{Res}_{\lambda=1}(r^\lambda \vec{U}(\lambda), r^{\lambda-1}P(\lambda)) = 0.$$

Therefore

$$\mathscr{L}_2(1) \operatorname{Res}_{\lambda=1}(\vec{U}(\lambda), P(\lambda)) = 0.$$

The residue belongs to the kernel of $\mathscr{L}_2(1)$. Thus it is equal to $(\vec{0}, c)$, with c a constant. We finally get

$$(\vec{u}, p) - (\vec{u}_0, p_0) = (\vec{0}, c).$$

Thus $(\vec{u}, p) \in D_2^s(\Gamma \cap B)$.

If $s = 1$, since $r^{-1}g \in L^2$, $G(\lambda)$ is defined up to $\operatorname{Re} \lambda = 1$ (such is not the case if $g \in H^1$ only). Since $\operatorname{Res}_{\lambda=1} \mathscr{L}_2(\lambda)^{-1} = (\vec{0}, 1)\chi$, where χ is a linear form, we get that $\Pi\mathscr{L}_2(\lambda)^{-1}$ is holomorphic in the neighborhood of $\lambda = 1$, where Π is the projection on

the speed component. Thus, $\Pi\mathscr{L}_2(\lambda)^{-1}(F, G)(\lambda)$ is defined up to $\operatorname{Re} \lambda = 1$, with suitable estimates. Then, we deduce that $\vec{u} \in H^2(\Gamma \cap B)$. Since $p \in L^2$, and $\vec{\nabla} p = \Delta\vec{u} + \vec{f} \in L^2(\Gamma \cap B)$, then $p \in H^1(\Gamma \cap B)$. □

5.3. Index and regularity results.

THEOREM 5.4. *Let $\Omega \in \mathcal{O}_2(\mathbf{R}^2)$, and let $s > 0$. $\mathscr{S}_2(\Omega, s)$ is Fredholm if and only if the three following conditions are fulfilled:*

(a) $s \neq 1$;

(b) $\forall x \in A_0(\Omega)$ *such that* $\omega_x \neq 2\pi$, $\forall k$, $s \neq \xi_k(\omega_x)$;

(c) $\forall x \in A_0(\Omega)$ *such that* $\omega_x = 2\pi$, $\forall k$, $s \neq k + \frac{1}{2}$.

Let us recall that ξ_k is defined in § 5.1. It is a straightforward consequence of Theorem 3.3 and §§ 4.2, 4.3, and 5.1. From Theorem 3.6 and § 5.2 we derive Theorem 5.5.

THEOREM 5.5. *Let $\Omega \in \mathcal{O}_2(\mathbf{R}^2)$ and $s > 0$. Let $(\vec{u}, p) \in D_2^0(\Omega)$ be the solution of* (1.1) *with $(\vec{f}, g) \in E_2^s(\Omega)$:*

(a) If $s < 1$ *and* $s < \min_{x \in A_0(\Omega)} \xi_1(\omega_x)$, *then* $(\vec{u}, p) \in D_2^s(\Omega)$;

(b) If $s > 1$ *and moreover* $g(x) = 0$ *for each vertex* x, *and if* $s < \min_{x \in A_0(\Omega)} \xi_1(\omega_x)$, *then* $(\vec{u}, p) \in D_2^s(\Omega)$;

(c) If $s = 1$ *and moreover* $r_x^{-1}g \in L^2(\Omega)$ *for each vertex* x, *and if* $1 < \min_{x \in A_0(\Omega)} \xi_1(\omega_x)$, *then* $(\vec{u}, p) \in D_2^2(\Omega)$.

As $\xi_1(\pi) = 1$ and ξ_1 is a decreasing function, $1 < \min_{x \in A_0(\Omega)} \xi_1(\omega_x)$ holds if Ω is convex. It coincides with the result in [11].

6. Precise results in three-dimensional domains when there are edges, but no vertex.

6.1. The statements. In such a case, we study condition (3.5): since, for $x \in A_1(\Omega)$, L_x is given by (3.1)-(3.2), it is obvious that L_x is injective modulo polynomials on $S_3^\lambda(\Gamma_x)$ and only if we have (6.1) and (6.2):

(6.1) $\mathscr{S}_2$ is injective modulo polynomials on $S_2^\lambda(\Gamma_x)$,

(6.2) Δ is injective modulo polynomials on $S_0^\lambda(\Gamma_x)$,

where $S_0^\lambda(\Gamma) = \{v = r^\lambda \sum v_q(\Psi) \log^q r,\ v_q \in \mathring{H}^1(G)\}$.

If $\omega_x \neq 2\pi$, (6.2) is equivalent to $\lambda \notin \{(k\pi/\omega_x)/k \in \mathbf{N}^*\}$ and if $\omega_x = 2\pi$, (6.2) is equivalent to $\lambda \notin \{k + \frac{1}{2},\ k \in \mathbf{N}\}$ (for $\operatorname{Re} \lambda \geqq 0$). Our statement follows.

THEOREM 6.3. *Let $\Omega \in \mathcal{O}_3(\mathbf{R}^3)$ such that $A_0(\Omega) = \varnothing$. Let $(\vec{u}, p) \in D_3^0(\Omega)$ be the solution of* (1.1) *with $(\vec{f}, g) \in E_3^s(\Omega)$.*

(a) If $s < 1$ *and* $s < \min_{x \in A_1(\Omega)} \xi_1(\omega_x)$, *then* $(\vec{u}, p) \in D_3^s(\Omega)$;

(b) If $s > 1$ *and* $s < \min_{x \in A_1(\Omega)} \pi/\omega_x$, *and moreover* $g(x) = 0$ *for each* $x \in A_1(\Omega)$, *then* $(\vec{u}, p) \in D_3^s(\Omega)$;

(c) If $s = 1$ *and Ω is convex, and moreover* $\rho_1^{-1}g \in L^2(\Omega)$, *where ρ_1 is the distance from* $A_1(\Omega)$, *then* $(\vec{u}, p) \in D_3^2(\Omega)$.

Proof. First, here are the main arguments of the proof.

(a) If $s < 1$, then (6.1) (respectively, (6.2)) is true for all λ such that $\operatorname{Re} \lambda \in [0, s]$ if $s < \xi_1(\omega_x)$ (respectively, $s < \pi/\omega_x$). But, if $\xi_1(\omega_x) < 1$, then $\xi_1(\omega_x) < \pi/\omega_x$. Thus (a) is derived from (3.6).

(b) As $s > 1$, if $s < \pi/\omega_x$, then $\omega_x < \pi$ and $\xi_1(\omega_x) > \pi/\omega_x$. Thus, for each $\lambda \neq 1$ in the strip $\operatorname{Re} \lambda \in [0, s]$, and for each $x \in A_1(\Omega)$, L_x is injective modulo polynomials on $S_3^\lambda(\Gamma_x)$. We have that (b) is an adaptation of the proof of (5.12) in [9] by taking advantage of Lemma 5.1 above. See some details that follow this proof.

(c) When Ω is convex, $\xi_1(\omega_x)$ and π/ω_x are greater than one for each vertex x. Like (b), (c) is derived from Lemma 5.1.

6.2. A more detailed proof of Theorem 6.3. Because of the special role of the pole $\lambda = 1$ (cf. Lemma 5.1) we are led to revise the proofs in § 12.C of [9] to take into account the cancellation assumptions concerning g.

Let Γ be a plane sector. We first study $\mathscr{S}_3$ on $\mathbf{R}\times\Gamma$. We denote by y the coordinate in $\mathbf{R}$ and by z the coordinates in Γ; $r=|z|$. We introduce the space $H_0^1(\mathbf{R}\times\Gamma)$ as the set of functions v such that

$$\forall\alpha, |\alpha|\leqq 1 \quad r^{|\alpha|-1}D^\alpha v\in L^2(\mathbf{R}\times\Gamma).$$

$\mathring{H}_0^1(\mathbf{R}\times\Gamma)$ is the closure of $\mathscr{D}(\mathbf{R}\times\Gamma)$ in $H_0^1(\mathbf{R}\times\Gamma)$ and $H_0^{-1}(\mathbf{R}\times\Gamma)$ is its dual space.

LEMMA 6.4. *$\mathscr{S}_3$ induces an isomorphism from $\mathring{H}_0^1\times\mathring{H}_0^1\times\mathring{H}_0^1\times L^2(\mathbf{R}\times\Gamma)$ to $H_0^{-1}\times H_0^{-1}\times H_0^{-1}\times L^2(\mathbf{R}\times\Gamma)$.*

Sketch of the Proof (cf. [9, (8.1), (12.6)]). It is sufficient to prove that $\mathscr{S}_3$ is injective and has a closed range because its adjoint has the same form. Let B_ρ be the ball with center zero and radius ρ. For $(\vec{u}, p)\in(\mathring{H}^1)^3\times L^2$ with its support in B_1, we have the following estimate:

$$\sum_{j=1}^3 \|u_j\|_{H^1}+\|p\|_{L^2}\leqq C\left(\sum_{j=1}^3 \|f_j\|_{H^{-1}}+\|g\|_{L^2}+\sum_{j=1}^3 \|u_j\|_{L^2}+\|p\|_{H^{-1}}\right).$$

But $\mathring{H}^1(B_1)\subset H_0^1(B_1)$ and $H_0^{-1}(B_1)\subset H^{-1}(B_1)$. Moreover, if supp $(\vec{u}, p)\subset B_\rho$, we have

$$\|u_j\|_{L^2}\leqq\rho\|u_j\|_{H_0^1} \quad\text{and by duality}\quad \|p\|_{H^{-1}}\leqq\rho\|p\|_{L^2}.$$

Thus, if supp $(\vec{u}, p)\subset B_\rho$, we have the following estimate, for $\rho>0$ small enough:

$$\sum\|u_j\|_{H_0^1}+\|p\|_{L^2}\leqq 2C\left(\sum\|f_j\|_{H_0^{-1}}+\|g\|_{L^2}\right). \tag{6.5}$$

We deduce the same estimates for all $(\vec{u}, p)$ in $(\mathring{H}_0^1)^3\times L^2$ by homogeneity and density of functions with compact support. □

$\mathscr{S}_3$ may be written $\mathscr{S}_3(D_y, D_z)$.

LEMMA 6.6. *$\mathscr{S}_3(\pm 1, D_z)$ induces an isomorphism from $\mathring{H}^1\times\mathring{H}^1\times\mathring{H}^1\times L^2(\Gamma)$ to $H^{-1}\times H^{-1}\times H^{-1}\times L^2(\Gamma)$.*

Sketch of the Proof. Let $(\vec{u}, p)\in(\mathring{H}^1)^3\times L^2(\Gamma)$. With φ a cut-off function in $\mathbf{R}$, $\varphi\equiv 1$ in the neighborhood of zero, we consider

$$(\vec{v}, q)(y, z)=\varphi(y)\, e^{i\rho y}(\vec{u}, p)(z),$$

for $\rho\geqq 1$ and apply the estimate (6.5) to $(\vec{v}, q)$. Denoting $(\vec{f}, g)=\mathscr{S}_3(\rho, D_z)(\vec{u}, p)$, we have

$$\mathscr{S}_3(\vec{v}, q)=\varphi(y)\, e^{i\rho y}(\vec{f}, g)(z)+e^{i\rho y}(\vec{f}_R, g_R).$$

Let us introduce the $H_0^1(\Gamma, \rho)$-norm on $\mathring{H}^1(\Gamma)$:

$$\|W\|^2_{H_0^1(\Gamma,\rho)}\equiv\rho^2\|W\|^2_{L^2}+\|r^{-1}W\|^2_{L^2}+\sum_{|\alpha|=1}\|D^\alpha W\|^2_{L^2} \tag{6.7}$$

and $H_0^{-1}(\Gamma, \rho)$ the dual norm. We have the equivalences:

$$\|v_j\|_{H_0^1(\mathbf{R}\times\Gamma)}\approx\|u_j\|_{H_0^1(\Gamma,\rho)} \quad\text{and}\quad \|\varphi\, e^{i\rho y}f_j\|_{H_0^{-1}(\mathbf{R}\times\Gamma)}\approx\|f_j\|_{H_0^{-1}(\Gamma,\rho)}.$$

So the estimate (6.5) implies that

$$\begin{aligned}\sum\|u_j\|_{H_0^1(\Gamma,\rho)}+\|p\|_{L^2(\Gamma)}\leqq C(&\sum\|f_j\|_{H_0^{-1}(\Gamma,\rho)}+\|e^{i\rho y}f_{R,j}\|_{H_0^{-1}(\mathbf{R}\times\Gamma)}\\ &+\|g\|_{L^2(\Gamma)}+\|e^{i\rho y}g_R\|_{L^2(\mathbf{R}\times\Gamma)}).\end{aligned} \tag{6.8}$$

The support of g_R is included in supp φ, and

$$g_R(y, z)=\sum a_j(y)u_j(z), \text{ with } a_j \text{ smooth}.$$

So,

$$\|e^{i\rho y}g_R\|_{L^2(\mathbf{R}\times\Gamma)}\leqq C\rho^{-1}(\sum\|u_j\|_{H_0^1(\Gamma,\rho)}+\|p\|_{L^2(\Gamma)}),$$

and we can prove the same estimate for $\|e^{i\rho y}f_{R,j}\|$ in (6.8). So, for ρ large enough,

$$\sum\|u_j\|_{H_0^1(\Gamma,\rho)}+\|p\|_{L^2(\Gamma)}\leqq 2C(\sum\|f_j\|_{H_0^{-1}(\Gamma,\rho)}+\|g\|_{L^2(\Gamma)}),$$

which is an a priori estimate for $\mathscr{S}_3(\rho; D_z)$. Using a suitable scaling, we get an estimate for $\mathscr{S}_3(1, D_z)$. The proof for $\mathscr{S}_3(-1, D_z)$ is the same, with $e^{-i\rho y}$ instead of $e^{i\rho y}$. □

For $s>1$, we replace $E_3^s(\Gamma)$ by $F_3^s(\Gamma)$, which is the space of the $(\vec{f}, g)\in E_3^s(\Gamma)$ such that $g(0)=0$. For $s=1$, $F_3^1(\Gamma)$ is characterized by $r^{-1}g\in L^2(\Gamma)$. When $\mathscr{S}_3(1, D_z)$-$(\vec{u}, p)=(\vec{f}, g)$, we have

$$\partial_1 u_1+\partial_2 u_2+iu_3=g.$$

Let us suppose that the opening ω of Γ is not 2π. When the u_j belong to $H^{s+1}(\Gamma)$, with $s>1$, since they are zero on $\partial\Gamma$, $\nabla u_j(0)=0$; thus $g(0)=0$. On the other hand, if $s=1$ and $u_j\in H^2\cap\mathring{H}^1(\Gamma)$, then $r^{|\alpha|-2}D^\alpha u_j\in L^2(\Gamma)$ for $|\alpha|\leqq 2$ (cf. [9 (AC.6)]; thus $r^{-1}g\in L^2(\Gamma)$.

Therefore $\mathscr{S}_3(\pm 1, D_z)$ operate from $D_3^s(\Gamma)$ to $F_3^s(\Gamma)$. As a consequence of Lemmas 6.6 and 5.1, we get Proposition 6.9.

PROPOSITION 6.9. *Let $s\geqq 1$ be such that $s<\pi/\omega$. Then $\mathscr{S}_3(\pm 1, D_z)$ is an isomorphism from $D_3^s(\Gamma)$ to $F_3^s(\Gamma)$.*

By partial Fourier transform with respect to y the equation $\mathscr{S}_3(D_y, D_z)(u, p)=(f, g)$ becomes

$$\mathscr{S}_3(\pm\rho, D_z)(\hat{u}, \hat{p})(\pm\rho, z)=(\hat{f}, \hat{g})(\pm\rho, z), \qquad \rho\geqq 0.$$

We define $F_3^s(\mathbf{R}\times\Gamma)$ by the condition $g(y, 0)=0$ if $s>1$ and $|z|^{-1}g(y, z)\in L^2(\mathbf{R}\times\Gamma)$ if $s=1$. If $(f, g)\in F_3^s(\mathbf{R}\times\Gamma)$, then for all ρ, $(\hat{f}, \hat{g})(\pm\rho)\in F_3^s(\Gamma)$. By a scaling argument, we deduce from Proposition 6.9 the uniform estimate for $\rho\geqq 1$:

$$\|(\hat{u}, \hat{p})(\pm\rho)\|_{D_3^s(\Gamma,\rho)}\leqq C\|(\hat{f}, \hat{g})(\pm\rho)\|_{F_3^s(\Gamma,\rho)}$$

where $H^s(\Gamma, \rho)$ means the norm $\|\ \|_{H^s}+\rho^s\|\ \|_{L^2}$ which obviously defines $D_3^s(\Gamma, \rho)$ and $F_3^s(\Gamma, \rho)$ when $s\neq 1$; $F_3^1(\Gamma, \rho)=L^2(\Gamma)^3\times H_0^1(\Gamma, \rho)$ (see (6.7)). We also have an a priori estimate for $\rho\leqq 1$.

So, in the same way as in 9.C of [9], we get the following lemma.

LEMMA 6.10. $1\leqq s<\pi/\omega$. $\Gamma_\rho\equiv\Gamma\cap B_\rho$. *Then the inverse operator $(\mathscr{S}_3)^{-1}$ of (6.4) induces a continuous operator from $F_3^s(\mathbf{R}\times\Gamma_2)$ to $D_3^s(\mathbf{R}\times\Gamma_1)$.*

Now, if we go back to the operator $\mathscr{S}_3$ on Ω, for each $x\in A_1(\Omega)$, we get an operator equal to $\mathscr{S}_3+\mathscr{T}$, $\mathscr{T}$ being a perturbation. It is important to note that the fourth equation of $(\mathscr{S}_3+\mathscr{T})(\vec{u}, p)=(\vec{f}, g)$ has the following form:

$$\sum_{1\leqq j,k\leqq 3} a_{j,k}(y, z)\partial_j u_k=g \quad \text{with } a_{j,k} \text{ smooth}.$$

Thus, if $(\vec{u}, p)\in D_3^s(\mathbf{R}\times\Gamma)$, then $(\vec{f}, g)\in F_3^s(\mathbf{R}\times\Gamma)$. So, we are able to use Lemma 6.10 along with the perturbation argument and Neumann series of 10.D in [9] in order to get the local regularity result in the neighborhood of each $x\in A_1(\Omega)$.

7. Study of the parametrical operator associated to the Stokes system in three-dimensional domains: Description of areas free of poles.

7.1. First identities. Let Γ be a cone in $\mathbf{R}^3$ and let G be its intersection with the unit sphere S^2.

In § 4.1 we introduced an operator $\mathcal{L}_3(\lambda)$ acting from $D_3^0(G)$ to $E_3^0(G)$. In view of Lemmas 4.2 and 4.3, we wish to find areas in $\mathbf{C}$ where $\mathcal{L}_3(\lambda)$ is everywhere invertible. As the index of $\mathcal{L}_3(\lambda)$ is zero, it is equivalent to find where $\mathcal{L}_3(\lambda)$ is injective. But, from the definition of $\mathcal{L}_3(\lambda)$ we deduce that

$$\mathcal{L}_3(\lambda)(\mathbf{u}, p) = 0 \Leftrightarrow \mathcal{S}_3(r^\lambda \mathbf{u}, r^{\lambda-1}p) = 0.$$

For $\operatorname{Re}\lambda = -\frac{1}{2}$, $\mathcal{L}_3(\lambda)$ is always injective; it is a consequence of Theorem 3.3 for $s=0$ (see also [16]). On the other hand, according to condition (3.7), we are interested in the strip $\operatorname{Re}\lambda \in [-\frac{1}{2}, s-\frac{1}{2}]$. Thus, we suppose the following:

$$\operatorname{Re}\lambda > -\tfrac{1}{2}, \tag{7.1}$$

$$(\mathbf{u}, p) \in (\mathring{H}^1(G))^3 \times L^2(G), \qquad (\mathbf{u}, p) \neq 0, \tag{7.2}$$

$$-\Delta(r^\lambda \mathbf{u}) + \nabla(r^{\lambda-1}p) = 0, \tag{7.3}$$

$$\operatorname{div}(r^\lambda \mathbf{u}) = 0. \tag{7.4}$$

We denote

$\xi = \operatorname{Re}\lambda$, $\eta = \operatorname{Im}\lambda$;

z the cartesian coordinates in $\mathbf{R}^3$; $\Psi = z/|z|$;

$\mathbf{u}_r = \langle \bar{\mathbf{u}}, \bar{\Psi}\rangle$, the radial component of $\mathbf{u}$;

δ the positive Laplace–Beltrami operator on $\mathring{H}^1(G)$;

∇_s the spherical part of the gradient.

If C is $\Gamma \cap \{1<r<2\}$, we get by integrating (7.3) with $r^{\bar\lambda}\bar{\mathbf{u}}$ and (7.4) with $r^{\bar\lambda-1}\bar p$:

$$\int_C \langle -\Delta(r^\lambda \mathbf{u}) + \nabla(r^{\lambda-1}p), r^{\bar\lambda}\bar{\mathbf{u}}\rangle + \operatorname{div} r^\lambda \mathbf{u} \cdot r^{\bar\lambda-1}\bar p = 0.$$

As in [16], it implies by integration by parts:

$$\int_G |\nabla_s \mathbf{u}|^2 - \lambda(\lambda+1)|\mathbf{u}|^2 + (2\xi+1)\int_G p\bar{\mathbf{u}}_r = 0. \tag{7.5}$$

And, again as in [16], we deduce from

$$\langle -\Delta(r^\lambda \mathbf{u}) + \nabla(r^{\lambda-1}p), z\rangle = 0$$

and from (7.4) that

$$\delta \mathbf{u}_r - (\lambda+1)(\lambda+2)\mathbf{u}_r + (\lambda-1)p = 0. \tag{7.6}$$

By integrating (7.6) on G with $\bar{\mathbf{u}}_r$, we get

$$\int_G |\nabla_s \mathbf{u}_r|^2 - (\lambda+1)(\lambda+2)|\mathbf{u}_r|^2 + (\lambda-1)p\bar{\mathbf{u}}_r = 0. \tag{7.7}$$

And from (7.4), which implies

$$\int_C \operatorname{div}(r^\lambda \mathbf{u}) = 0,$$

we get

$$(\lambda+2)\int_G \mathbf{u}_r = 0. \tag{7.8}$$

7.2. The case when λ is not real.

LEMMA 7.9. *Let us assume that we have (7.1)–(7.4) and moreover that $\eta \neq 0$. Then*

$$\xi \geqq \frac{1}{\sqrt{3}}(\Lambda_1 + 1 + \eta^2)^{1/2}$$

where Λ_1 is the first eigenvalue of δ.

Proof. We take the imaginary part of (7.5) and divide it by $(2\xi+1)$ and obtain

$$\operatorname{Im}\int_G p\bar{\mathbf{u}}_r = \eta \int_G |\mathbf{u}|^2. \tag{7.10}$$

We take the imaginary part of (7.7) and use (7.10); after we divide that by η we obtain

$$-(2\xi+3)\int_G |\mathbf{u}_r|^2 + (\xi-1)|\mathbf{u}|^2 + \operatorname{Re}\int_G p\bar{\mathbf{u}}_r = 0. \tag{7.11}$$

We take the real part of (7.5) and eliminate $\operatorname{Re}\int_G p\bar{\mathbf{u}}_r$ by using (7.11). Then we obtain

$$\int_G |\nabla_s \mathbf{u}|^2 + [\eta^2 - \xi(\xi+1) - (2\xi+1)(\xi-1)]|\mathbf{u}|^2 + (2\xi+1)(2\xi+3)\int_G |\mathbf{u}_r|^2 = 0.$$

Since $(2\xi+1)(2\xi+3) > 0$, it implies

$$\int_G |\nabla_s \mathbf{u}|^2 \leqq (3\xi^2 - 1 - \eta^2)\int_G |\mathbf{u}|^2. \tag{7.12}$$

If $\mathbf{u}$ were zero, then (7.3) would yield

$$\nabla(r^{\lambda-1}p) = 0.$$

Thus $r^{\lambda-1}p$ would be constant, which implies, as $\lambda \neq 1$, that p would be zero. Therefore, if we have (7.2), then $\mathbf{u} \neq 0$. So we have

$$\int_G |\nabla_s \mathbf{u}|^2 \geqq \Lambda_1 \int_G |\mathbf{u}|^2 \tag{7.13}$$

and, with (7.12) it implies that

$$3\xi^2 - 1 - \eta^2 \geqq \Lambda_1,$$

i.e.,

$$\xi \geqq \frac{1}{\sqrt{3}}(\Lambda_1 + 1 + \eta^2)^{1/2}.$$

As $\xi > -\frac{1}{2}$, it is equivalent to

$$\xi > \frac{1}{\sqrt{3}}(\Lambda_1 + 1 + \eta^2)^{1/2}. \qquad \square$$

7.3. The case when λ is real: $-\frac{1}{2} < \lambda < 1$. As $\lambda \neq 1$, we determine $\int_G p\bar{\mathbf{u}}_r$ by using (7.7), and putting that into (7.3), we get

$$\int_G |\nabla_s \mathbf{u}|^2 - \lambda(\lambda+1)|\mathbf{u}|^2 = \frac{2\lambda+1}{\lambda-1}\int_G |\nabla_s \mathbf{u}_r|^2 - (\lambda+1)(\lambda+2)|\mathbf{u}_r|^2. \tag{7.14}$$

But, with (7.13), we have

$$\begin{aligned}\int_G |\nabla_s \mathbf{u}|^2 - \lambda(\lambda+1)|\mathbf{u}|^2 &\geqq [\Lambda_1 - \lambda(\lambda+1)]\int_G |\mathbf{u}|^2 \\ &\geqq [\Lambda_1 - \lambda(\lambda+1)]\int_G |\mathbf{u}_r|^2\end{aligned} \tag{7.15}$$

whenever

(7.16) $$\Lambda_1 \geqq \lambda(\lambda+1).$$

So, with (7.15), (7.14) yields

$$\frac{2\lambda+1}{\lambda-1}\int_G |\nabla_s \mathbf{u}_r|^2-(\lambda+1)(\lambda+2)|\mathbf{u}_r|^2 \geqq [\Lambda_1-\lambda(\lambda+1)]\int_G |\mathbf{u}_r|^2.$$

As $\lambda-1$ is negative, this may be written in the form

(7.17) $$\int_G |\nabla_s \mathbf{u}_r|^2 \leqq \phi(\lambda)\int_G |\mathbf{u}_r|^2$$

with

(7.18) $$\phi(\lambda)=(\lambda+1)(\lambda+2)-[\Lambda_1-\lambda(\lambda+1)]\frac{1-\lambda}{2\lambda+1}.$$

Just as in [16], we introduce Definition 7.19.

DEFINITION 7.19. *Let Λ' be the minimum of $\int_G |\nabla_s v|^2$ when $v \in \mathring{H}^1(G)$, $\|v\|_{L^2(G)}=1$ and $\int_G v=0$.*

Formulae (7.17) and (7.8) imply that if $\phi(\lambda)<\Lambda'$, then $\mathbf{u}_r=0$. With (7.16), this implies that $\mathbf{u}=0$ and $p=0$ since $\lambda \neq 1$. Therefore, we get the following lemma.

LEMMA 7.20. *Let us suppose that we have (7.1)-(7.4) and moreover that λ is real and smaller than one. Then, with (7.18) and Definition 7.19, we have*

$$\lambda(\lambda+1)>\Lambda_1 \quad \textit{or} \quad \phi(\lambda)\geqq\Lambda'.$$

This statement is close to what is proved in [16].

7.4. The case when $\lambda=1$. Equations (7.6) and (7.8), respectively, give

(7.21) $$\delta\mathbf{u}_r-6\mathbf{u}_r=0,$$

(7.22) $$\int_G \mathbf{u}_r=0.$$

Then if $\mathbf{u}_r$ is nonzero, six is an eigenvalue of δ. But since the first eigenfunction has a constant sign (cf. [4] and (19.B) in [9]), six cannot be equal to Λ_1. Therefore, if $6<\Lambda_2$ (the second eigenvalue of δ), then $\mathbf{u}_r=0$. So (7.5) implies that

$$\int_G |\nabla_s \mathbf{u}|^2-2|\mathbf{u}|^2=0.$$

If $2<\Lambda_1$, then $\mathbf{u}=0$ and p is a constant. We have just proved Lemma 7.23.

LEMMA 7.23. *Let us suppose that $\lambda=1$ and that*

$$\Lambda_1>2 \quad \textit{and} \quad \Lambda_2>6.$$

Then the solutions of (7.2)-(7.4) are proportional to $(\vec{0},1)$.

Now, we are going to prove Lemma 7.24.

LEMMA 7.24. *If $\Lambda_1 \geqq 6$, then the pole of $\mathscr{L}_3(\lambda)^{-1}$ in $\lambda=1$ has the order one.*

Proof. Because no confusion is possible, we drop the index 3 in $\mathscr{L}_3(\lambda)$.

Since $\mathscr{L}(1)$ is not injective, $\mathscr{L}^{-1}(\lambda)$ has a pole in $\lambda=1$. Let us write the Laurent expansion of $\mathscr{L}^{-1}(\lambda)$ in the neighborhood of $\lambda=1$, and the power series of $\mathscr{L}(\lambda)$:

$$\mathscr{L}(\lambda)^{-1}=\sum_{j\geqq -J}(\lambda-1)^j A_j$$

where J is the order of the pole

$$\mathscr{L}(\lambda)=\sum_{j\geqq 0}(\lambda-1)^j\mathscr{L}^{(j)}(1)/j!$$

where $\mathscr{L}^{(j)}(\lambda)$ is the jth derivative of $\mathscr{L}$ with respect to λ. As $\mathscr{L}(\lambda)\mathscr{L}^{-1}(\lambda)=I$, we get the relation

(7.25) $$\mathscr{L}(1)A_{-J}=0,$$

and, only if $J\geqq 2$,

(7.26) $$\mathscr{L}^{(1)}(1)A_{-J}+\mathscr{L}(1)A_{-J+1}=0.$$

Thus, (7.25) implies that $A_{-J}=\Phi\cdot(0,1)$ where Φ is a linear form, and using (7.26) we get that if

(7.27) $$\mathscr{L}^{(1)}(1)(0,1)\notin\mathscr{L}(1)D^0(G),$$

then $J=1$. On the other hand

(7.28) $$\mathscr{L}^{(1)}(1)(0,1)=(\vec{\Psi},0),$$

and

(7.29) $$\mathscr{L}(1)D^0(G)=(\ker\mathscr{L}(1)^*)^\perp.$$

But, according to [16], we have

(7.30) $$\mathscr{L}(1)^*(\mathbf{u},p)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\mathscr{L}(-2)(\mathbf{u},-p).$$

So we search the kernel of $\mathscr{L}(-2)$, which is one-dimensional just like the kernel of $\mathscr{L}(1)$. We may suppose that a basis of ker $\mathscr{L}(-2)$ has the radial form $(\mathbf{u},p)=(v\vec{\Psi},p)$. According to (7.6), we have

(7.31) $$p=\tfrac{1}{3}\delta v.$$

We also have

$$\begin{aligned}\operatorname{div}(r^{-2}v\vec{\Psi})&=\langle\vec{\nabla},r^{-3}v\vec{z}\rangle\\&=r^{-3}v\langle\vec{\nabla},\vec{z}\rangle+\langle\vec{z},\vec{\nabla}\rangle(r^{-3}v)\\&=3r^{-3}v+(r\partial_r)(r^{-3}v)=0.\end{aligned}$$

As a consequence, relation (7.4) is satisfied for any v (see also (7.8)). To take into account relation (7.3), we notice that

$$\begin{aligned}\Delta(r^{-3}z_jv)&=z_j\Delta(r^{-3}v)+2\partial_j(r^{-3}v),\\-\Delta(r^3v)&=r^{-5}(\delta v-6v),\\\partial_j(r^{-3}v)&=-3z_jr^{-5}v+r^{-3}\partial_jv.\end{aligned}$$

So (7.3), which may be written as

$$-\Delta(r^{-3}z_jv)+\tfrac{1}{3}\partial_j(r^{-3}\delta v)=0,\qquad j=1,2,3,$$

is equivalent to

$$\partial_j(\delta v-6v)=0,\qquad j=1,2,3.$$

We have just proved Lemma 7.32. □

LEMMA 7.32. *A basis of* $\ker \mathcal{L}(-2)$ *is given by* $(v\vec{\Psi}, p)$, *where* $p=\delta v/3$ *and*

(i) *If six is an eigenvalue of* δ: v *is an eigenfunction of* δ *associated with six*;

(ii) *If not,* v *is the unique solution of* $\delta v - 6v = 1$.

End of the proof of Lemma 7.4. Using (7.28)-(7.30), we may rewrite condition (7.27):

$$\int_G \langle \vec{\Psi}, v\vec{\Psi}\rangle \neq 0,$$

i.e.,

$$\int_G v \neq 0. \tag{7.33}$$

If $\Lambda_1 = 6$, v has a constant sign and (7.33) is fulfilled. If $\Lambda_1 \neq 6$, according to the assumptions of Lemma 7.24, $\Lambda_1 > 6$. Let (Λ_k, v_k) be the eigenvalues and eigenfunctions sequence of δ. Using Lemma 7.32, we have

$$v = \sum_k c_k v_k \quad \text{with } c_k = (\Lambda_k - 6)^{-1} \int_G v_k.$$

So

$$\int_G v = \sum_k (\Lambda_k - 6)^{-1} \left(\int_G v_k\right)^2,$$

which is positive since $\Lambda_1 > 6$. Thus (7.33) is true. □

8. Study of the minimum value Λ'.

8.1. Minoration of $\Lambda'(G)$. We study the minimum $\Lambda'(G)$ of $\int_G |\nabla_s v|^2$ when $v \in \mathring{H}^1(G)$, $\|v\|_{L^2(G)} = 1$ and $\int_G v = 0$ (cf. Definition 7.19) occurring in Lemma 7.20. As the extension by zero preserves the above conditions of v, we get (as in [16]):

$$\text{if } G_1 \subset G_2, \text{ then } \Lambda'(G_1) \geqq \Lambda'(G_2). \tag{8.1}$$

The minimum $\Lambda'(G)$ is reached for some function v. We recall that (Λ_k, v_k) is the eigenvalues and eigenfunctions sequence of the Laplace-Beltrami operator δ on $\mathring{H}^1(G)$. We denote

$$\gamma_k = \gamma \int_G v_k \, d\Psi \quad \text{where } \gamma = \left(\int_G d\Psi\right)^{-1/2}$$

(we suppose that $\|v_k\|_{L^2(G)} = 1$). We have

$$v = \sum_k c_k v_k \quad \text{with} \sum c_k^2 = 1, \tag{8.2}$$

$$\sum_k c_k \gamma_k = 0, \tag{8.3}$$

$$\Lambda'(G) = \sum_k \Lambda_k c_k^2. \tag{8.4}$$

If $G = S^2$, as $\gamma_1 = 1$ and the other γ_k are zero, it is obvious that (cf. [16])

$$\Lambda'(S^2) = \Lambda_2 = 2. \tag{8.5}$$

So, by (8.1) and (8.5), we get

$$\Lambda'(G) \geqq 2. \tag{8.6}$$

We will obtain further information about $\Lambda'(G)$.

LEMMA 8.7. *Let* K *be* $(\int_G v_1 \, d\Psi)^2 (\int_G d\Psi)^{-1}$. *Then*

$$\Lambda'(G) \geqq (1-K)\Lambda_1(G) + K\Lambda_2(G).$$

Proof. Using (8.3), we get

$$c_1^2\gamma_1^2 \leqq \left(\sum_{k\geqq 2} c_k^2\right)\left(\sum_{k\geqq 2} \gamma_k^2\right) = (1-c_1^2)(1-\gamma_1^2).$$

So, we have

$$c_1^2 \leqq 1 - \gamma_1^2. \tag{8.8}$$

Equation (8.4) implies that

$$\Lambda'(G) \geqq \Lambda_1 c_1^2 + \Lambda_2(1-c_1^2).$$

Using (8.8), we get

$$\Lambda'(G) \geqq \Lambda_1(1-\gamma_1^2) + \Lambda_2\gamma_1^2.$$

And as K is exactly γ_1^2, we get the lemma. □

Now, if $\gamma_2 = 0$, instead of (8.8) we get

$$c_1^2 \leqq (1-\gamma_1^2)(1-c_2^2).$$

And, with (8.4), we have

$$\Lambda'(G) \geqq \Lambda_1 c_1^2 + \Lambda_2 c_2^2 + \Lambda_3(1-c_1^2-c_2^2).$$

Thus

$$\Lambda'(G) \geqq [(1-\gamma_1^2)\Lambda_1 + \gamma_1^2\Lambda_3](1-c_2^2) + \Lambda_2 c_2^2. \tag{8.9}$$

And, it is easy to show that, if moreover $\gamma_3, \cdots, \gamma_{N-1} = 0$, Λ_3 may be replaced by Λ_N in (8.9). So we get Lemma 8.10.

LEMMA 8.10. *If* $\gamma_2, \cdots, \gamma_{N-1} = 0$, *then*

$$\Lambda'(G) \geqq \min\{[(1-K)\Lambda_1 + K\Lambda_N], \Lambda_2\}.$$

8.2. The exact value of Λ' in some special cases. For $\omega \in]0, 2\pi]$ and (θ, φ) the spherical coordinates in $[0, \pi] \times [0, 2\pi[$, we denote by G_ω:

$$G_\omega = \{\Psi \in S^2 / \theta \in]0, \pi[, \varphi \in]0, \omega[\}.$$

The associated cone Γ_ω is a dihedron with interior angle ω. Since v_1 is proportional to $\sin(\pi/\omega)\varphi$, it is easy to compute the following:

$$K(G_\omega) = 8/\pi^2 \approx 0.81057. \tag{8.11}$$

The main result is Proposition 8.12.

PROPOSITION 8.12. $\Lambda'(G_\omega) = \Lambda_2(G_\omega)$.

Proof. We denote π/ω by ν. As a consequence of (18.6′) in [9] we obtain

$$\Lambda_k = \mu_k(\mu_k + 1),$$

(μ_k) being the increasing sequence of positive numbers

$$l\nu + d \quad \text{with } l \in \mathbf{N}^* \text{ and } d \in \mathbf{N}.$$

(The multiplicity of μ is given by the number of couples (l, d) providing μ.)

From (18.9) in [9], we derive that an eigenfunction associated with $\mu_k(\mu_k+1)$ with $\mu_k = l\nu + d$ has the following form:

$$v_k = \sum_{0 \leqq 2p \leqq d} \alpha_p \cos^{d-2p} \theta \cdot \sin l\nu\varphi$$

where the α_p are some constants.

As a consequence

(8.13) if $\mu_k = \nu + 1$, $v_k = \alpha \cos\theta \sin \nu\varphi$ and $\gamma_k = 0$,

(8.14) if $\mu_k = 2\nu$, $v_k = \alpha \sin 2\nu\varphi$ and $\gamma_k = 0$.

(a) If $\nu \in [\frac{1}{2}, 1]$: $\mu_1 = \nu$, $\mu_2 = 2\nu$, $\mu_3 = \nu+1$, $\mu_4 = 3\nu$.
So γ_2 and γ_3 are zero; and according to (8.10) it is sufficient to prove that

$$(1-K)\nu(\nu+1) + 3K\nu(3\nu+1) \geqq 2\nu(2\nu+1).$$

With (8.11), this is easy to check.

(b) If $\nu \in [1, 2]$: $\mu_1 = \nu$, $\mu_2 = \nu+1$, $\mu_3 = 2\nu$, $\mu_4 = \nu+2$.
So γ_2 and γ_3 are zero again. And we can prove that

$$(1-K)\nu(\nu+1) + K(\nu+2)(\nu+3) \geqq (\nu+1)(\nu+2). \tag{8.15}$$

Using Lemma 8.10 we get Proposition 8.12.

(c) If $\nu \geqq 2$: $\mu_1 = \nu$, $\mu_2 = \nu+1$, $\mu_3 = \nu+2$. $\gamma_2 = 0$ and as in case (b), (8.15) is true and implies Proposition 8.12 by using Lemma 8.10. □

COROLLARY 8.16. *If* $\omega \in [\pi, 2\pi]$, $\Lambda'(G_\omega) = (2\pi/\omega)(1 + 2\pi/\omega)$.

9. Precise regularity results in three-dimensional domains.

9.1. Strips free of poles. Let Ω be a domain in $\mathcal{O}_3(\mathbf{R}^3)$. If Ω has no vertex, it has been studied in § 6 (Theorem 6.3). If not, for each vertex x of Ω, we must check condition (3.7).

Let us assume that

$$s < \tfrac{3}{2}. \tag{9.1}$$

So, using Lemma 4.2, we have that (3.7) is equivalent to

$$\forall\lambda,\ \operatorname{Re}\lambda \in [-\tfrac{1}{2}, s - \tfrac{1}{2}],\ \mathcal{L}_3(\lambda) \text{ is invertible on } D_3^0(G_x). \tag{9.2}$$

We are going to determine $s(G_x)$ so that (9.2) is true for $s = s(G_x)$.

We denote

$$\xi_0 = s(G_x) - \tfrac{1}{2}.$$

As a consequence of Lemmas 7.9 and 7.20, if we have the three following conditions, for a $\xi \in]-\frac{1}{2}, 1[$:

$$\xi(\xi+1) + (2\xi+1)(\xi-1) < \Lambda_1(G), \tag{9.3}$$

$$\xi(\xi+1) < \Lambda_1(G), \tag{9.4}$$

$$\phi(\xi) < \Lambda'(G), \tag{9.5}$$

then

$$\forall\lambda,\ \operatorname{Re}\lambda = \xi,\ \mathcal{L}_3(\lambda) \text{ is invertible on } D_3^0(G). \tag{9.6}$$

Condition (9.4) implies (9.3), and (9.5) may be written as

$$(\xi+1)(\xi^2+6\xi+2) < (2\xi+1)\Lambda' + (1-\xi)\Lambda_1. \tag{9.7}$$

Using (8.6), $\Lambda' \geqq 2$ and $\Lambda_1 > 0$, we obtain (9.7) if

$$(\xi+1)(\xi^2+6\xi+2) \leqq 2(2\xi+1).$$

It is easy to check that for all $\xi \in [-\frac{1}{2}, 0]$.

So

$$\xi_0 \geqq 0, \quad \text{i.e.} \quad s(G) \geqq \tfrac{1}{2}. \tag{9.8}$$

For $\xi > 0$, we may use one of the following three conditions, each implying (9.7):

$$(\xi+1)(\xi^2+6\xi+2) < 2(2\xi+1) + (1-\xi)\Lambda_1, \tag{9.9}$$

$$(\xi+1)(\xi^2+6\xi+2) \leqq (2\xi+1)\Lambda', \tag{9.10}$$

$$(\xi+1)(\xi^2+6\xi+2) \leqq (\xi+1)\Lambda' + \Lambda_1 \tag{9.11}$$

(since $\Lambda' > \Lambda_1$). Each of these conditions has the form

$$\psi(\xi) \leqq 0,$$

ψ being a strictly convex function on $[-7/3, +\infty[$. To have $\psi(\xi) \leqq 0$ on $\xi \in [0, \xi_0]$, it is enough to check that

$$\psi(0) \leqq 0 \quad \text{and} \quad \psi(\xi_0) \leqq 0.$$

Using (9.9), we find Proposition 9.12, as in [16].

PROPOSITION 9.12. *$s(G)$ may be taken as $\frac{1}{2} + \mu/(\mu+4)$, where $\mu > 0$ is such that $\mu(\mu+1) = \Lambda_1(G)$.*

Now, if we consider condition (9.11) with $\Lambda' \geqq 2$, we get

$$(\xi+1)(\xi^2+6\xi) \leqq \Lambda_1,$$

and it is easy to prove the following proposition.

PROPOSITION 9.13. *If $\Lambda_1(G) \leqq 2$, $s(G)$ may be taken to be $\frac{1}{2}\Lambda_1(G)/8$.*

It is better than Proposition 9.12 if $\Lambda_1 \geqq 1.20$.

We notice that the right-hand side of (9.11) increases when the domain G decreases. So, we may determine ξ such that (9.11) is satisfied for G_ω (cf. 8.3), and then we are sure that (9.11) is also satisfied for all $G \subset G_\omega$. It is not difficult to check Proposition 9.14.

PROPOSITION 9.14. *If $G \subset G_\omega$ with $\omega \in]\pi, 2\pi]$, $s(G)$ may be taken as $6\pi/5\omega$.*

Here, we use Corollary 8.16:

$$\Lambda'(G_\omega) = 2\nu(\nu+1) \quad \text{and} \quad \Lambda_1(G_\omega) = \nu(\nu+1), \quad \text{with} \quad \nu = \pi/\omega.$$

In the important case when $\omega = \pi$, we have

$$\Lambda'(G_\pi) = 6$$

and we immediately see that (9.10) is true for all $\xi \leqq 1$. Therefore, we have (9.5) for $\xi < 1$ and as $\Lambda_1 = 2$, we have (9.4) also. Thus we have Proposition 9.15.

PROPOSITION 9.15. *If $G \subset G_\pi$, then we may take $s(G) = \frac{3}{2} - \varepsilon$, for all $\varepsilon > 0$.*

Finally, for $G_{\pi/2}$, $\Lambda_1 = 6$ and according to Lemma 7.24 we get Proposition 9.16.

PROPOSITION 9.16. *If $G \subset G_{\pi/2}$, $\mathcal{L}_3(\lambda)^{-1}$ has only one pole in the strip* $\operatorname{Re}\lambda \in [-\frac{1}{2}, 1]$. *That pole is $\lambda = 1$, it is simple, and* $\operatorname{Ker}\mathcal{L}(1)$ *is generated by $(\vec{0}, 1)$. Here let $s(G)$ be 3/2.*

9.2. Regularity results. Just as in Theorem 6.3, we make various assumptions concerning the behavior of g at the singular points of Ω, according to the value of s.

DEFINITION 9.17. Let g be in $H^{s-1}(\Omega)$ for $s \geqq 1$. g is said to be *zero at the singular points of* Ω

(i) When $s=1$: $\rho_1^{-1} g \in L^2(\Omega)$ with $\rho_1(x)$ the distance of x from $A_1(\Omega)$, the edges of Ω;

(ii) When $1<s<\frac{3}{2}$: $g=0$ on $A_1(\Omega)$;

(iii) When $s=\frac{3}{2}$: $g=0$ on $A_1(\Omega)$ and $\rho_0^{-3/2} g \in L^2(\Omega)$ where ρ_0 is the distance from $A_0(\Omega)$, the vertices of Ω;

(iv) When $s>\frac{3}{2}$: $g=0$ on $A_1(\Omega) \cup A_0(\Omega)$.

Remarks 9.18.

(1) If $g \in H^{s-1} \cap \mathring{H}^1(\Omega)$, then it is zero at the singular points of Ω.

(2) If each vertex x is in the closure of an edge, then the conditions concerning $A_0(\Omega)$ are implied by conditions concerning $A_1(\Omega)$; such is the case when Ω is a polyhedron.

This may be proved as is (AC.3) in [9].

DEFINITION 9.19. For each vertex $x \in A_0(\Omega)$, we denote by s_x the best value of $s(G_x)$ drawn from Propositions 9.12–9.16.

THEOREM 9.20. *Let* $\Omega \in \mathcal{O}_3(\mathbf{R}^3)$. *Let* $(\vec{u}, p) \in D_3^0(\Omega)$ *be the solution of* (1.1) *with* $(\vec{f}, g) \in E_3^s(\Omega)$.

(a) *If* $s<1$, $s \leqq \min_{x \in A_0(\Omega)} s_x$ *and* $s < \inf_{x \in A_1(\Omega)} \xi_1(\omega_x)$, *then* $(\vec{u}, p) \in D_3^s(\Omega)$.

(b) *If* $s \geqq 1$, $s \leqq \min_{x \in A_0(\Omega)} s_x$, $s < \inf_{x \in A_1(\Omega)} \pi/\omega_x$ *and moreover* g *is zero at the singular points of* Ω, *then* $(\vec{u}, p) \in D_3^s(\Omega)$.

For $s<1$, it is a straightforward consequence of Theorem 3.6, as in the case of Theorem 6.3.

For $s \geqq 1$, in view of Theorems 3.6 and 6.3, and the methods of proofs of [9], it is enough to note that the Mellin transform $(\vec{F}(\lambda), G(\lambda))$ of $(r^2\vec{f}, rg)$, after localization in the neighborhood of any vertex, is defined up to $\operatorname{Re} \lambda \leqq s-\frac{1}{2}$, with values in $F_3^s(G_x)$.

Now, we derive from Theorem 9.20 the statements (1.6)–(1.12).

Since $s_x > \frac{1}{2}$ and $\xi_1(\omega_x) \geqq \frac{1}{2}$, (1.6) is a straightforward consequence of Theorem 9.20a.

In situation (1.7), the openings of the edges are $\pi/2$ or $3\pi/2$. $\xi_1(3\pi/2) > 0.544$, so $0.544 < \inf_{x \in A_1(\Omega)} \xi_1(\omega_x)$. If x is a vertex of Ω, then the associated cone Γ_x is an octant or the complementary of an octant. In the first case, Proposition 9.16 yields $s(G_x) = \frac{3}{2}$. In the last case, we note that Γ_x is included in the revolution cone

$$\Gamma = \{z \in \mathbf{R}^3 / \theta \in [0, \theta_0[\}$$

with $\theta_0 = \pi - \theta_1$, $\theta_1 = \arcsin(1/\sqrt{3})$; so $\theta_0 \simeq 144.74°$. As a consequence of 18.D in [9], we find that $\Lambda_1(G_x) = \mu(\mu+1)$ with $\mu > 0.35$. By Proposition 9.12 we get

$$s(G_x) > \frac{1}{2} + \frac{0.35}{4.35} \approx 0.580 > 0.544.$$

When Ω is convex, we have $\pi/\omega_x > 1$ for any $x \in A_1(\Omega)$, and by Proposition 9.15 $s_x > 1$. Thus, we get (1.8).

If, for each $x \in A_1(\Omega)$, $\pi/\omega_x \geqq \frac{3}{2}$, as by Proposition 9.15 we have $s_x \geqq \frac{3}{2} - \varepsilon$, then for all $\varepsilon > 0$, we get (1.9).

In the situation (1.10), $\min_{x \in A_1(\Omega)} \pi/\omega_x > \frac{3}{2}$ and, for each vertex x, owing to Proposition 9.16, we have $s_x = \frac{3}{2}$.

In the situations (1.11) or (1.12), we have: $\min_{x \in A_1(\Omega)} \pi/\omega_x = 2$ and there is no vertex.

Appendix. Behavior of $\mathcal{L}_2(\lambda)^{-1}$ in the neighborhood of positive integer numbers, on the domain $]0, 2\pi[$ (the model for a crack). Given suitable changes of functions (see [11] and [5]), the problem

$$\mathcal{L}_2(\lambda)(u_1, u_2, p) = (f_1, f_2, g) \text{ with } u_1, u_2 \in \mathring{H}^1(]0, 2\pi[)$$

is equivalent to the other:

$$\text{(A1)} \qquad \begin{cases} u'' + (\lambda+1)^2 u + (1-\lambda) q = l_1, \\ (\lambda-1) u' + (1-\lambda^2) v + q' = l_2, \\ (1+\lambda) u + v' = l_3, \end{cases}$$

$$\text{(A2)} \qquad u, v \in \mathring{H}^1(]0, 2\pi[).$$

As in the above references, it can be proved that (A1) is holomorphically solvable. To solve (A1)-(A2), it remains to solve (A1) with $l_j = 0$, and for any $(c_1, c_2, c_3, c_4) \in \mathbf{C}^4$

$$\text{(A3)} \qquad \begin{matrix} v(0) = \gamma_1, & v(2\pi) = \gamma_2, \\ u(0) = \gamma_3, & u(2\pi) = \gamma_4. \end{matrix}$$

Problem (A1) with the zero right-hand side is equivalent to (A4)-(A6):

$$\text{(A4)} \qquad u = -v'(1+\lambda)^{-1},$$

$$\text{(A5)} \qquad q = (v^{(3)} + (1+\lambda)^2 v')(1-\lambda^2)^{-1},$$

$$\text{(A6)} \qquad v^{(4)} + 2(1+\lambda^2) v^{(2)} + (1-\lambda^2)^2 v = 0.$$

For $\operatorname{Re} \lambda > 0$, a basis of solutions of (A6) is given by

$$v_1 = \sin(\lambda+1)\theta, \quad v_2 = \frac{\sin(\lambda-1)\theta}{\lambda-1}, \quad v_3 = \cos(\lambda+1)\theta, \quad v_4 = \cos(\lambda-1)\theta.$$

Let $M(\lambda)$ be the four $\times$ four matrix, the columns of which are

$${}^t[v_j(0), v_j(2\pi), -v_j'(0)(1+\lambda)^{-1}, -v_j'(2\pi)(1+\lambda)^{-1}].$$

The solvability of (A3), (A4), and (A6) is equivalent to finding $\alpha = {}^t(\alpha_1, \cdots, \alpha_4)$ such that

$$\text{(A7)} \qquad M(\lambda)\alpha = \gamma$$

where $\gamma = {}^t(\gamma_1, \cdots, \gamma_4)$. Then the solution of (A3), (A4), and (A6) is

$$\text{(A8)} \qquad v = \sum \alpha_j v_j \quad \text{and} \quad u = -v'(\lambda+1)^{-1}.$$

The determinant of $M(\lambda)$ is

$$4 \sin^2 2\pi\lambda (\lambda-1)^{-1} (\lambda+1)^{-2}.$$

So, in $\lambda = 1$, $M(\lambda)^{-1}$ has a simple pole. On the other hand, when λ is an integer number and $\lambda \geqq 2$, it is easy to see that the first and the third rows (respectively, the second and the fourth) are equal; then the cofactors of $M(\lambda)$ are zero and the pole is simple again.

It is obvious that, for integer λ

$$\text{(A9)} \qquad \dim \operatorname{Ker} M(1) = 1 \quad \text{and} \quad \dim \operatorname{Ker} M(\lambda) = 2 \quad \text{when } \lambda \geqq 2.$$

To deduce the properties of $\mathscr{L}_2(\lambda)^{-1}$, we must take (A5) into account. For $\lambda \geqq 2$, it is holomorphic. It remains to study the case when $\lambda = 1$.

For $v = v_1$, or $v = v_3$, (A5) yields $q = 0$.

For $v = v_4$, (A5) yields $q = ((\lambda+1)^2 - (\lambda-1)^2) \sin(\lambda-1)\theta/(\lambda+1)$.

For $\lambda = 1$, it is zero again.

For $v = v_2$, (A5) yields $q_2 = ((\lambda+1)^2 - (\lambda-1)^2) \cos(\lambda-1)\theta/(1-\lambda^2)$.

With (A8), we have $q = \alpha_2 q_2$.

As q_2 has a simple pole in $\lambda = 1$, it remains to state that α_2 is holomorphic. That arises from the structure of $M(\lambda)$: the matrix $M_2(\lambda)$ obtained by removing the second column of $M(\lambda)$ has its rank equal to two, and the corresponding cofactors balance the determinant of $M(\lambda)$.

So, for any $\lambda \in \mathbf{N}$, $\lambda \geqq 1$, $\mathscr{L}(\lambda)^{-1}$ has a simple pole, and with (A9), it is clear that

$$\dim \operatorname{Ker} \mathscr{L}_2(\lambda) = d(\lambda).$$

With Lemma 4.5, we get that $\mathscr{S}_2$ is injective modulo polynomial on $S^\lambda(\mathbf{R}^2\backslash\mathbf{R}^+)$.

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