

Eigen-Frequencies in Thin Elastic 3-D Domains and Reissner-Mindlin Plate Models

Monique DAUGE[&] and Zohar YOSIBASH[#]

[&] Institut Mathématique, UMR 6625 du CNRS, Université de Rennes 1,
Campus de Beaulieu, 35042 Rennes, France

[#] Pearlstone Center for Aeronautical Engineering Studies, Mechanical
Engineering Dept., Ben-Gurion University of the Negev, Beer-Sheva, Israel

Abstract. *The eigen-frequencies of elastic three-dimensional thin plates are addressed and compared to the eigen-frequencies of two-dimensional Reissner-Mindlin plate models obtained by dimension reduction. The qualitative mathematical analysis is supported by quantitative numerical data obtained by the p-version finite element method.*

The mathematical analysis establishes asymptotic expansion for the eigen-frequencies in power series of the thickness parameter. Such results are new for orthotropic materials and for the Reissner-Mindlin model. The 3-D and R-M asymptotics have a common first term but differ in their second terms.

Numerical experiments for clamped plates show that for isotropic materials and relatively thin plates the Reissner-Mindlin eigen-frequencies provide a good approximation to the three-dimensional eigen-frequencies. However, for some anisotropic materials this is no longer the case, and relative errors of order of 30% are obtained even for relatively thin plates. Moreover, we showed that no shear correction factor is known to be optimal in the sense that it provides the best approximation of the R-M eigen-frequencies to their 3-D counterparts uniformly (for all relevant thicknesses range).

1 INTRODUCTION

Accurate computation of eigen-frequencies of elastic domains, is of high engineering importance because the smallest frequencies have to be higher compared with any frequency of a dynamically applied load. Otherwise resonance may occur, resulting in the amplification of the eigen-mode and finally the destruction of the elastic structure.

This is of particular importance in three-dimensional thin domains, such as three-dimensional plates, where the smallest eigen-frequencies are proportional to the thickness. It has been shown in [4, 5] that for isotropic thin plates the smallest frequencies are associated with bending modes. Therefore, we address herein these bending eigen-frequencies

in a fully three-dimensional domain.

Due to the complexity of a three-dimensional eigen-analysis, much attention has been given historically to the derivation of plate-models, which can be understood as an application of dimensional reduction principles. These plate models are aimed to approximately solve the three-dimensional *bending* eigen-problem by a two-dimensional formulation. In engineering practice the Reissner-Mindlin (R-M) plate model is frequently used as an approximation of the 3-D thin plate domain, and is assumed to be valid for thicknesses of plates under 5% compared with other dimensions.

In this paper, our aim is a twofold investigation of the quality of approximation of 3-D eigen-frequencies by R-M eigen-frequencies:

1. By a theoretical asymptotic analysis, we obtain after [9, 5] a power series expansion in the thickness parameter ε of both 3-D eigen-frequencies and R-M eigen-frequencies.
2. By a numerical implementation, we quantify the deviation from R-M with respect to 3-D, in a variety of situations for small but non-zero ε 's.

The outcome of point 1. is that the 3-D and R-M eigen-frequency asymptotics have *the same first terms* as $\varepsilon \rightarrow 0$, namely ε times the corresponding eigen-frequency λ_1 of the limit Kirchhoff-Love model (K-L). But *the second terms* in the asymptotics, as far as lateral clamped boundary conditions are considered, are *always different*: indeed we prove in this paper that this second term for the 3-D eigen-frequency has the form ε^2 times a positive quantity λ_2 , whereas for the R-M model, the term in ε^2 is zero.

Thus, from a strict mathematical point of view, the R-M approximation is not better than the K-L approximation, and in both cases, this is an order 1 approximation. From a practical point of view, the order of magnitude of the quantity λ_2 with respect to λ_1 is of importance, since the relative error of approximating a 3-D plate by the R-M model is equivalent to $\varepsilon\lambda_2/\lambda_1$ as ε approaches zero, but is still non-zero.

The quantity λ_2 is a coupling term between inner (boundary layer) and outer (regular) parts of a two-scale expansion of eigen-vectors. Moreover, in λ_2 there is a multiplicative factor c^b which depends only on the material law of the 3-D plates. For example, for isotropic materials, we show that this factor is an increasing function of the Poisson coefficient ν . For special non-isotropic materials, we show that this factor is much larger.

That is why we extend the results of [9, 5] to orthotropic materials. The multiplicative factor appears now as a density along the lateral boundary. We also present the corresponding construction of R-M model, and the asymptotics of eigen-frequencies. What is known about asymptotics for the R-M model is [1, 2] where it is proved that in the loading case, there is no boundary layer at the order 1, but that it starts at the order 2. This is exactly the reason for the absence for the term of order ε^2 in the eigen-frequency asymptotics.

Numerical experiments with clamped plates are provided. These results quantify the influence of the material law and the shape of the mid-surface on the R-M eigen-frequencies as compared to their 3-D counterparts. It is shown that for isotropic materials and relatively thin plates, the difference is very small, whereas for orthotropic materials the difference in the first eigen-frequency of the R-M plate model and the corresponding eigen-frequency of

the fully 3-D plate can be as large as 30%. Moreover, we showed that no shear correction factor is known to be optimal in the sense that it provides the best approximation of the R-M eigen-frequencies to their 3-D counterparts uniformly (for all thicknesses range).

In section 2 we provide the mathematical description of the plate models (3-D and R-M model), and in particular summarize the asymptotic behavior of the eigen-frequencies (in terms of ε). Details about the mathematical analysis on the derivation of the asymptotics of the eigen-frequencies for orthotropic 3-D plates is given in Appendix A and for orthotropic R-M model in Appendix B. The model problems used in the numerical experimentation are described in section 3, followed by the numerical results in section 4. Summary and conclusions are given in section 5.

2 3-D PLATE VIS REISSNER-MINDLIN PLATE MODEL

2.1 Eigen-Frequencies of the 3-D Plate

Consider a thin elastic domain Ω of thickness 2ε as shown in Figure 1, defined as follows:

$$\Omega = \omega \times (-\varepsilon, +\varepsilon), \quad \text{with } \omega \subset \mathbb{R}^2 \text{ a regular domain.}$$

The coordinates are $\mathbf{x} = (x_1, x_2, x_3)$ and the mid surface ω lies in the plane $x_1 - x_2$, while x_3 is in the normal direction. As a representative case we restrict our attention to clamped

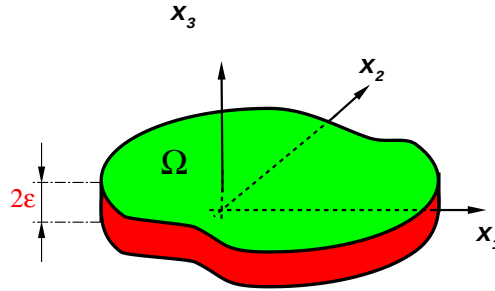


Figure 1: Typical plate of interest and notations.

boundary conditions on the lateral edges. More precisely, we assume that the upper and lower surfaces of the plate, i.e. $\omega \times \{\pm\varepsilon\}$, are traction free, and that the lateral edges of the plate $\partial\omega \times (-\varepsilon, \varepsilon)$ are clamped, i.e. $\mathbf{u} = \mathbf{0}$. Here $\mathbf{u} = (u_1(\mathbf{x}), u_2(\mathbf{x}), u_3(\mathbf{x}))^T$, denotes the displacement field.

We denote by $\mathbf{e}$ the *engineering notation* of the linear strain tensor. Accordingly, we use the index notation: $(e_1, e_2, e_3, e_4, e_5, e_6) = (e_{11}, e_{22}, e_{33}, 2e_{23}, 2e_{13}, 2e_{12})$ where $e_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, and $\partial_i \equiv \partial/\partial x_i$.

Orthotropic (or isotropic) homogeneous materials are considered, hence the stress tensor $\boldsymbol{\sigma}$ is given by Hooke's law

$$\boldsymbol{\sigma} = [C] \mathbf{e}, \tag{2.1}$$

where $[C]$ is a 6×6 (positive) symmetric matrix associated with the 3-D material constituting the domain, in the form:

$$[C] = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{pmatrix} \quad (2.2)$$

In case the material is isotropic then

$$C_{11} = C_{22} = C_{33} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \quad C_{12} = C_{13} = C_{23} = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

and

$$C_{44} = C_{55} = C_{66} = \frac{E}{2(1+\nu)}.$$

Here E is the Young modulus and ν is the Poisson ratio.

Denoting the material density by ρ , the sought vibration eigen-frequencies λ of the three-dimensional plate are the solutions of the following weak eigen-problem:

Seek $\lambda > 0$ and $\mathbf{0} \neq \mathbf{u} \in \mathbf{V}(\Omega)$, such that $\forall \mathbf{v} \in \mathbf{V}(\Omega)$,

$$\iint_{\omega} \int_{-\varepsilon}^{\varepsilon} \mathbf{e}(\mathbf{u})^T [C] \mathbf{e}(\mathbf{v}) dx_1 dx_2 dx_3 = \lambda^2 \iint_{\omega} \int_{-\varepsilon}^{\varepsilon} \rho \mathbf{u}^T \cdot \mathbf{v} dx_1 dx_2 dx_3 \quad (2.3)$$

where

$$\mathbf{V}(\Omega) = \{ \mathbf{v} \in [H^1(\Omega)]^3 ; \quad \mathbf{v} = 0 \text{ on } \partial\omega \times (-\varepsilon, \varepsilon) \},$$

$H^1(\Omega)$ being the usual Sobolev space.

The eigen-frequencies form an increasing sequence $n \mapsto \lambda_n^\varepsilon$ of positive numbers where we agree to repeat each eigen-frequency according to its multiplicity. There is no finite accumulation point. The superscript ε indicates that we fix the mid-surface ω and that we consider the eigen-frequencies as functions of the thickness ε .

Let S be the transverse symmetry operator defined for $\mathbf{u} = (u_1, u_2, u_3)$ by:

$$\mathbf{u} \xrightarrow{S} \left((x_1, x_2, x_3) \mapsto (u_1(x_1, x_2, -x_3), u_2(x_1, x_2, -x_3), -u_3(x_1, x_2, -x_3)) \right). \quad (2.4)$$

We note that

$$\iint_{\omega} \int_{-\varepsilon}^{\varepsilon} \rho \mathbf{u}^T \cdot \mathbf{v} dx_1 dx_2 dx_3 = \iint_{\omega} \int_{-\varepsilon}^{\varepsilon} \rho (S\mathbf{u})^T \cdot S\mathbf{v} dx_1 dx_2 dx_3$$

and that for orthotropic materials

$$\iint_{\omega} \int_{-\varepsilon}^{\varepsilon} \mathbf{e}(\mathbf{u})^T [C] \mathbf{e}(\mathbf{v}) dx_1 dx_2 dx_3 = \iint_{\omega} \int_{-\varepsilon}^{\varepsilon} \mathbf{e}(S\mathbf{u})^T [C] \mathbf{e}(S\mathbf{v}) dx_1 dx_2 dx_3.$$

We deduce that each eigen-space determined by (2.3) can be split as the direct sum of a *bending* subspace (i.e. where the elements $\mathbf{u}$ satisfy $S\mathbf{u} = -\mathbf{u}$, in other words $\mathbf{u}$ are anti-symmetric in respect to the mid plane ω) and a *membrane* or *stretching* subspace (where $\mathbf{u}$ satisfy $S\mathbf{u} = \mathbf{u}$). Therefore for each fixed ε , each eigen-frequency in the sequence $(\lambda_n^\varepsilon)_n$ is either bending or membrane.

2.2 Isotropic 3-D Plate

For ease of presentation we fix the material density $\rho \equiv 1$. For isotropic materials it has been shown in [4, 5] that for each fixed rank n , for ε small enough λ_n^ε is a bending eigen-frequency and that $\lambda_n^\varepsilon/\varepsilon$ tends to the n^{th} eigen-frequency, denoted λ_n^{KL} , of the *Kirchhoff-Love dimensionally reduced model*

$$L^{\text{KL}} := \frac{E \Delta^2}{3(1 - \nu^2)} \quad : \quad H_0^2(\omega) \longrightarrow H^{-2}(\omega). \quad (2.5)$$

In [5] it is proved that λ_n^ε has a power series expansion in terms of ε , i.e. there exist real coefficients $\lambda_{n,i}$ for $i \geq 1$ such that for any k :

$$\lambda_n^\varepsilon = \varepsilon \lambda_{n,1} + \varepsilon^2 \lambda_{n,2} + \dots + \varepsilon^k \lambda_{n,k} + \mathcal{O}(\varepsilon^{k+1}). \quad (2.6)$$

Here, of course, $\lambda_{n,1}$ coincides with λ_n^{KL} . Following the analyses of [5] (and see also [9, §8]) we find the first two terms of the expansion of the *scaled eigen-values* Λ_n^ε

$$\Lambda_n^\varepsilon := (\lambda_n^\varepsilon/\varepsilon)^2 = \Lambda_{n,0} + \varepsilon \Lambda_{n,1} + \dots + \varepsilon^k \Lambda_{n,k} + \mathcal{O}(\varepsilon^{k+1}). \quad (2.7)$$

The first term $\Lambda_{n,0}$ is the n^{th} eigen-value of the Kirchhoff-Love model, i.e. $(\lambda_n^{\text{KL}})^2$. As for the second term $\Lambda_{n,1}$, it is equal to

$$\Lambda_{n,1} = \frac{E}{3} \int_{\partial\omega} c^b(\nu) |(\partial_1^2 + \partial_2^2) \zeta_{n,0}|^2 ds, \quad (2.8)$$

where $\zeta_{n,0} \in H_0^2(\omega)$ is the n^{th} eigen-function* of the Kirchhoff-Love model L^{KL} normalized in $L^2(\omega)$, and c^b is a coupling constant related to the structure of the first boundary layer term in the expansion of 3-D eigen-vectors. This coupling constant is positive and depends only on ν . See details in Appendix A.

We immediately deduce that $\lambda_{n,2} = \Lambda_{n,1}/(2\sqrt{\Lambda_{n,0}})$, and obtain:

$$\lambda_{n,2} = \frac{E}{6\lambda_{n,1}} \int_{\partial\omega} c^b(\nu) |(\partial_1^2 + \partial_2^2) \zeta_{n,0}|^2 ds. \quad (2.9)$$

2.3 Orthotropic 3-D Plate

Stricto sensu, the eigenvalue expansion (2.7) is not known for non-isotropic plates. Nevertheless, for *clamped lateral boundary conditions*, it is possible to use all constructions in [5] and to extend the results of isotropic plates to orthotropic plates. Details are provided in Appendix A.

We still fix the density to $\rho \equiv 1$. Then for fixed n , the n^{th} eigen-frequency has a power series expansion with respect to ε in the form (2.6) where $\lambda_{n,1} = \lambda_n^{\text{KL}}$ with the n^{th}

*When λ_n^{KL} is a multiple eigen-frequency of L^{KL} , according to the constructions of [5], extra criteria have to be applied for the determination of $\zeta_{n,0}$: the pair $(\zeta_{n,0}, \Lambda_{n,1})$ has to be an eigen-pair of a finite positive symmetric eigen-problem, see Appendix A.

eigen-frequency λ_n^{KL} of the *Kirchhoff-Love dimensionally reduced model* L^{KL} associated with material matrix $[C]$ on ω , cf [6]:

$$\begin{aligned} L^{\text{KL}} : H_0^2(\omega) &\longrightarrow H^{-2}(\omega) \\ \zeta &\longmapsto \frac{1}{3}(\tilde{C}_{11}\partial_1^4 + \tilde{C}_{22}\partial_2^4 + 2(\tilde{C}_{12} + 2\tilde{C}_{66})\partial_1^2\partial_2^2)\zeta, \end{aligned} \quad (2.10)$$

where for $i, j \in \{1, 2, 6\}$, $\tilde{C}_{ij}$ is defined as

$$\tilde{C}_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}}. \quad (2.11)$$

Thus, according to (2.6), $\lambda_n^\varepsilon/\varepsilon \rightarrow \lambda_{n,1}$, as $\varepsilon \rightarrow 0$, i.e. the n^{th} eigen-frequency over ε must approach a constant value as the plate thickness tends to zero. This will be visualized by numerical examples.

As for the second term $\lambda_{n,2}$ in (2.6), it has an expression like (2.9) but with a *positive coupling function* defined on $\partial\omega$ and depending on the material matrix $[C]$:

$$\lambda_{n,2} = \frac{1}{2\lambda_{n,1}} \int_{\partial\omega} c^b([C]; s) |(\partial_1^2 + \partial_2^2)\zeta_{n,0}|^2 ds. \quad (2.12)$$

Therefore we expect, as noticed in [9], that $\varepsilon \mapsto \lambda_n^\varepsilon/\varepsilon$ is increasing for ε small enough. This will be visualized by numerical examples.

2.4 Eigen-frequencies of the Reissner-Mindlin Plate Model

For completeness of presentation, we provide herein the derivation of the weak eigen-problem for the R-M plate model made of orthotropic materials.

The displacement field under the R-M assumptions can be represented as:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -x_3\beta_1(x_\alpha) \\ -x_3\beta_2(x_\alpha) \\ w(x_\alpha) \end{pmatrix} \quad (2.13)$$

where the index α denotes the values $\{1, 2\}$ corresponding to the in-plane variables. The in-plane functions $\beta_1(x_\alpha)$ and $\beta_2(x_\alpha)$ have the physical interpretation of rotations, while the function $w(x_\alpha)$ denotes the deflection of the mean surface of the plate. R-M displacement field assumption implies that *normals to the mean surface ω remain straight lines after deformation, but not necessary normals*. Thus, the strain vector e is:

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} = \begin{pmatrix} -x_3\partial_1\beta_1 \\ -x_3\partial_2\beta_2 \\ 0 \\ (\partial_2w - \beta_2) \\ (\partial_1w - \beta_1) \\ -x_3(\partial_2\beta_1 + \partial_1\beta_2) \end{pmatrix} \quad (2.14)$$

and as noticed $e_3 = 0$.

The second important assumption for constructing the R-M plate model, is that σ_3 is *negligible*. Therefore one introduces the constraint $\sigma_3 = 0$ in the model. Writing (2.1) in index notation, we obtain:

$$\sigma_i = C_{ij}e_j \quad i, j = 1, \dots, 6. \quad (2.15)$$

Due to the constraint $\sigma_3 = 0$, i.e. $C_{3j}e_j = 0$, we substitute e_3 by $-\sum_{k \neq 3} C_{3k}e_k/C_{33}$ in (2.15) except for $i = 3$, and we obtain:

$$\sigma_i = \sum_{j \neq 3} \left(C_{ij} - \frac{C_{i3}C_{3j}}{C_{33}} \right) e_j, \quad i, j = 1, 2, 4, 5, 6, \quad (2.16)$$

We note that for $i, j = 1, 2, 6$, the new material coefficients $C_{ij} - \frac{C_{i3}C_{3j}}{C_{33}}$ are nothing else than $\tilde{C}_{ij}$ already introduced in notation (2.11). Based on (2.16) we obtain a modified Hooke's law, connecting the stress vector $(\sigma_1, \sigma_2, \sigma_4, \sigma_5, \sigma_6)$ with the strain vector $(e_1, e_2, e_4, e_5, e_6)$. The 5×5 material matrix $[\tilde{C}] = (\tilde{C}_{ij})_{i,j=1,2,4,5,6}$ obtained is:

$$[\tilde{C}] = \begin{pmatrix} C_{11} - \frac{C_{13}^2}{C_{33}} & C_{12} - \frac{C_{13}C_{23}}{C_{33}} & 0 & 0 & 0 \\ & C_{22} - \frac{C_{23}^2}{C_{33}} & 0 & 0 & 0 \\ & & C_{44} & 0 & 0 \\ & & & C_{55} & 0 \\ & & & & C_{66} \end{pmatrix} \quad (2.17)$$

To account for a correction in the shear stresses σ_{13} and σ_{23} to better represent the fully 3-D stresses, the material matrix entries $\tilde{C}_{44}$ and $\tilde{C}_{55}$ are changed by introducing the so-called shear correction factor κ :

$$\tilde{C}_{44} = \kappa C_{44} \quad \tilde{C}_{55} = \kappa C_{55}$$

By properly chosen κ , either the energy of the R-M solution, or the deflection w can be optimized with respect with the fully 3-D plate. The smallest the ε , the smallest the influence of κ on the results. For the isotropic case two possible κ 's are (see details in [3]):

$$\kappa_{\text{Energy}} = \frac{5}{6(1-\nu)}$$

$$\kappa_{\text{Deflection}} = \frac{20}{3(8-3\nu)}$$

For modal analysis, it is not clear that an optimal value of κ is available. A value of $\kappa = 5/6$ is frequently used in engineering practice. It is important to note that for orthotropic materials the value of κ is set to 1.

Defining the vector $\boldsymbol{\beta} = (\beta_1(x_\alpha), \beta_2(x_\alpha))^T$ and the vector $\boldsymbol{\eta} = (\eta_1(x_\alpha), \eta_2(x_\alpha))^T$, and assuming $\rho \equiv 1$ as in the previous subsections, we can introduce the following two

bi-linear forms for the R-M plate model:

$$\begin{aligned} \mathcal{A}(\boldsymbol{\beta}, w, \boldsymbol{\eta}, v) = & \frac{\varepsilon^3}{3} \iint_{\omega} \tilde{C}_{11} \partial_1 \beta_1 \partial_1 \eta_1 + \tilde{C}_{12} (\partial_1 \beta_1 \partial_2 \eta_2 + \partial_2 \beta_2 \partial_1 \eta_1) \\ & + \tilde{C}_{22} \partial_2 \beta_2 \partial_2 \eta_2 + \tilde{C}_{66} (\partial_2 \beta_1 + \partial_1 \beta_2) (\partial_2 \eta_1 + \partial_1 \eta_2) \, dx_1 dx_2 \\ & + \varepsilon \iint_{\omega} \tilde{C}_{44} (\partial_2 w - \beta_2) (\partial_2 v - \eta_2) + \tilde{C}_{55} (\partial_1 w - \beta_1) (\partial_1 v - \eta_1) \, dx_1 dx_2 \end{aligned} \quad (2.18)$$

$$\mathcal{M}(\boldsymbol{\beta}, w, \boldsymbol{\eta}, v) = \frac{\varepsilon^3}{3} \iint_{\omega} (\beta_1 \eta_1 + \beta_2 \eta_2) \, dx_1 dx_2 + \varepsilon \iint_{\omega} w v \, dx_1 dx_2 \quad (2.19)$$

Using the above definitions, the weak eigen-problem for the *clamped* R-M plate model is:

$$\begin{aligned} \text{Seek } \tilde{\lambda} > 0 \text{ and } \mathbf{0} \neq (\boldsymbol{\beta}, w) \in \mathbf{V}^{\text{RM}}(\omega), \text{ such that} \\ \mathcal{A}(\boldsymbol{\beta}, w, \boldsymbol{\eta}, v) = \tilde{\lambda}^2 \mathcal{M}(\boldsymbol{\beta}, w, \boldsymbol{\eta}, v), \quad \forall (\boldsymbol{\eta}, v) \in \mathbf{V}^{\text{RM}}(\omega), \end{aligned} \quad (2.20)$$

where

$$\mathbf{V}^{\text{RM}}(\omega) = \{ (\boldsymbol{\beta}, w) \in [H^1(\omega)]^3 ; \quad \beta_1 = \beta_2 = w = 0 \text{ on } \partial\omega \}.$$

The solution of (2.20) provides the Reissner-Mindlin eigen-frequencies, which form for each ε an increasing sequence denoted by $(\tilde{\lambda}_n^\varepsilon)_n$, where we do the same convention of repeating according to the multiplicity as before.

By similar (and much easier) constructions than in [5], we can prove that the n^{th} R-M eigen-frequency has a power series expansion in ε

$$\tilde{\lambda}_n^\varepsilon = \varepsilon \tilde{\lambda}_{n,1} + \varepsilon^2 \tilde{\lambda}_{n,2} + \dots + \varepsilon^k \tilde{\lambda}_{n,k} + \mathcal{O}(\varepsilon^{k+1}). \quad (2.21)$$

Here, the first coefficient $\tilde{\lambda}_{n,1}$ is equal to its counterpart $\lambda_{n,1}$ in (2.6), i.e. the n^{th} K-L eigen-frequency.

In the 3-D eigen-frequency asymptotics (2.6), the presence of a non-zero second coefficient $\lambda_{n,2}$ is a consequence of the presence of the in-plane boundary layer components Φ_α^1 at the second rank in (A.2). But, for the *clamped R-M model*, it is known, see [1, 2], that the boundary layer series starts one rank further. That is why we can prove, see Appendix B, that

$$\tilde{\lambda}_{n,2} = 0. \quad (2.22)$$

Therefore, from the asymptotic expansion point of view, the approximation of the 3-D eigen-frequencies by R-M eigen-frequencies is only correct at the order 1.

3 FRAMEWORK OF NUMERICAL INVESTIGATION

Of major interest is the question whether the R-M eigen-frequencies are a good approximation of the 3-D eigen-frequencies. To address this question we define the *Relative Error* for the n^{th} eigen-frequency as follows:

$$\delta_n^\varepsilon \stackrel{\text{def}}{=} \frac{\lambda_n^\varepsilon - \tilde{\lambda}_n^\varepsilon}{\lambda_n^\varepsilon}. \quad (3.1)$$

- (a) How close are the R-M eigen-frequencies to their 3-D counterparts? Which shear correction factor should be used so to obtain the “best” eigen-frequencies for the R-M plate models? How small should be the ratio between ε and the other plate dimensions so that the R-M eigen-frequencies approximate well the 3-D ones?
- (b) From asymptotics (2.12) and (2.22), we see that

$$\delta_n^\varepsilon \simeq \frac{\varepsilon \lambda_{n,2}}{\lambda_{n,1}}.$$

Thus the question is how big is $\varepsilon\lambda_{n,2}$ compared with $\lambda_{n,1}$ for small, but non-zero ε ? Especially, are there material properties so that $\lambda_{n,1}$ is of the same order of magnitude as $\varepsilon\lambda_{n,2}$ for a given small ε ?

- (c) In this connection, what is the influence of the geometrical shape of the plate boundary and the material properties on $\lambda_{n,2}$?

To address the above questions, which are of quantitative nature, three-dimensional plates are considered and analyzed from the numerical approximation point of view by the p -version of the finite element method.

3.1 Three Dimensional Plates

We consider a rectangular plate, with or without a circular hole, having dimensions of $2 \times 1 \times 2\varepsilon$, as shown in Figure 2. The lateral boundaries of the plate are clamped, $\mathbf{u} = \mathbf{0}$

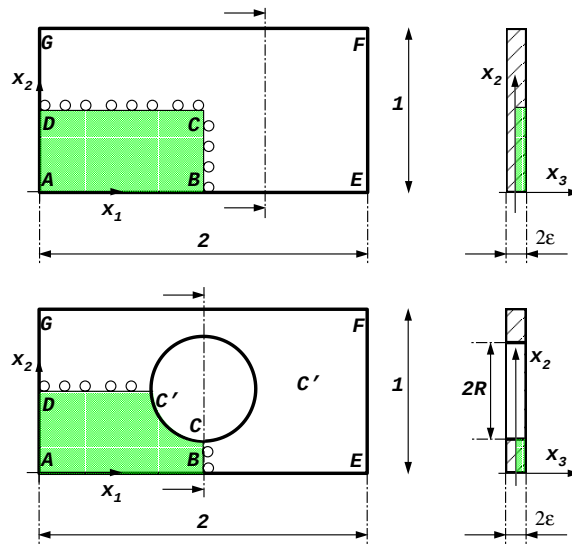


Figure 2: *Rectangular plate under consideration.*

along $AEFG$, and in the case of the plate with the hole also the boundaries along the circular hole are clamped. Since only bending eigen-frequencies are sought, we have to

consider anti-symmetric modes in respect to x_3 . This is accomplished by taking as the computational domain only the upper half of the plate $\varepsilon > x_3 > 0$, with anti-symmetric boundary condition $u_1 = u_2 = 0$ on the entire mid surface $x_3 = 0$. The first bending eigen-frequency of the whole plate is symmetric in respect to $x_1 = 1$ and $x_2 = 1/2$ therefore it is exactly the one obtained as the first eigen-frequency of the one-eighth computational domain. Thus, we further reduce the computational cost and compute the first three eigen-frequencies which are symmetric along $x_1 = 1$ and $x_2 = 1/2$. The computational domain is therefore one eighth of the original plate, the shaded domain $ABCD$ (respectively $ABCC'D$ for the plate with a circular hole), with clamped boundary conditions along the lateral boundaries AB and AD (and CC' respectively), symmetric boundary conditions along the other two lateral boundaries, i.e. $u_1 = 0$ along BC and $u_2 = 0$ along CD (resp. $C'D$), and anti-symmetric boundary conditions on the plane $x_3 = 0$.

Four different material properties are investigated, two of which are isotropic and the other two orthotropic. For all cases the density is taken as $\rho = 1$.

Name	Young Modulus E		Poisson ratio ν		
Isotropic	0.854		0.382		
Almost Incompressible	1		0.49		

Name	$C_{11} = C_{22}$	C_{12}	$C_{13} = C_{23}$	C_{33}	$C_{44} = C_{55} = C_{66}$
Strongly orthotropic	11	9.9	1	0.1	1
Orthotropic	3	1.5	1	0.5	1

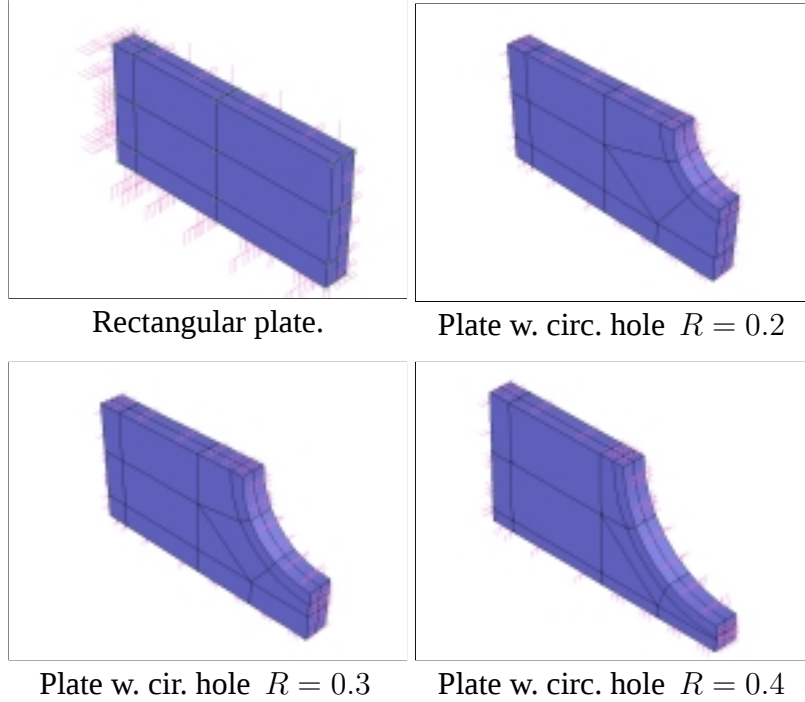


Figure 3: Finite element meshes for 3-D analysis with $\varepsilon = 0.1$.

A three dimensional p -version finite element model is constructed having two elements in the thickness direction, x_3 , and three elements in the x_1 and x_2 directions. In the

neighborhood of the edges, the mesh is graded so that there is an element of dimension ε each. The finite element model is constructed parametrically so that the value of ε may vary, and we change it from 0.1 to 0.001. For the plate with a circular hole, three radii are analyzed: $R = 0.2$, $R = 0.3$ and $R = 0.4$. In Figure 3 the mesh used for the various problems of interest is presented for plates of half thickness $\varepsilon = 0.1$.

The p -level over each element has been increased from 1 to 8 and the trunk space has been used (see [10]). For the rectangular plate (with 18 hexahedral elements) there are 4068 degrees of freedom at $p = 8$, and the eigen-frequencies are converged to within a relative error of less than 0.5% at $p = 8$ for most ε . An advantage of using p -version finite element methods is the possibility of having “needle elements” in the boundary layer zone with aspect ratios as large as 1000 without significant degradation in the numerical performance. An *exponential convergence rate* is obtained (due to the use of high-order elements) and the convergence of the eigen-frequencies (the first three of them) has been examined in order to evaluate the reliability of the numerical results.

In Figure 4 we present the finite element mesh for the case where $\varepsilon = 0.01$. As may be noticed the boundary layer elements are changed according to the plate thickness. Although almost not visible, there are two elements across the thickness and one element each of dimension ε in the neighborhood of the boundary.

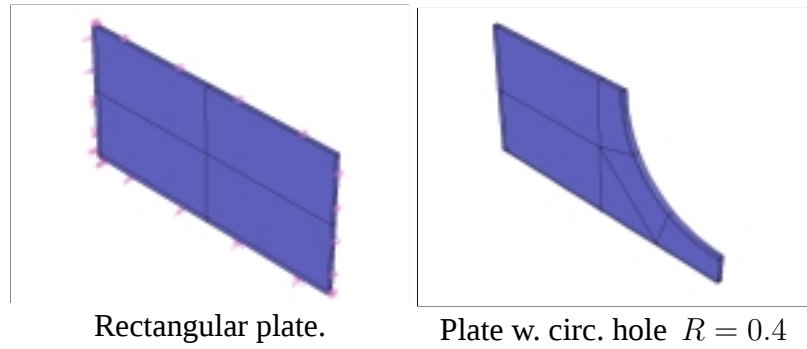


Figure 4: Finite element meshes for 3-D analysis with $\varepsilon = 0.01$.

All finite element computations are done within the finite element code Stress Check[†].

3.2 R-M Plate Models

Finite element models of mid-surface of the 3-D plate have been constructed similar to the 3-D meshes, where one fourth of the plate has been considered for computing the symmetric eigen-frequencies. Again, a layer of one element along the plate boundary of width equal to ε is present. An example of the meshes for the R-M plate with $\varepsilon = 0.1$ and $\varepsilon = 0.01$ is shown in Figure 5. One can specify three different shear correction factors used in conjunction with the R-M plate model made of an isotropic material: κ_{Energy} , $\kappa_{\text{Deflection}}$ or $\kappa = 1$. For orthotropic material only $\kappa = 1$ is available in the code that we

[†] Stress Check is a trade mark of Engineering Software Research and Development, Inc., 10845 Olive Blvd., Suite 170, St. Louis, MO 63141, USA

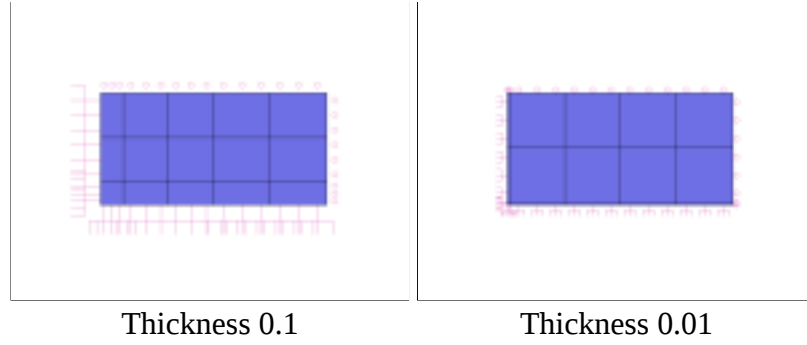


Figure 5: *Finite element meshes for the rectangular R-M plate model.*

use. Again, we increase the polynomial degree over each element from 1 to 8 and observe an exponential convergence in the eigen-frequencies (usually less than 0.1% relative error for most ε values at $p = 8$ with 2816 degrees of freedom).

4 QUANTITATIVE COMPARISON BETWEEN 3-D AND R-M EIGEN-FREQUENCIES

We give the results of numerical experimentations in the rectangular plate for the four different material laws introduced in the previous section. Then we provide a few results on the influence of the shape of the mid-surface.

4.1 The standard isotropic material

This subsection is devoted to the rectangular plate with the “isotropic” material properties. We investigate which of the three κ s, used for the R-M plate model, provide the closest eigen-frequencies compared with their 3-D counterparts. We plot the first three symmetric eigen-frequencies $/\varepsilon$ vis the plate half thickness ε in Figure 6.

As expected by the mathematical analysis, indeed $\lambda_n/\varepsilon \rightarrow \lambda_{n,1}$ as $\varepsilon \rightarrow 0$, and also $\tilde{\lambda}_n \rightarrow \lambda_n$ regardless of the κ used.

To better visualize the difference between $\tilde{\lambda}_n$ and λ_n we provide in Figure 7 the relative error δ_1^ε for the first eigen-frequency for the three choices of κ . The representation of relative errors δ_2^ε and δ_3^ε for the second and third eigen-frequencies is very similar.

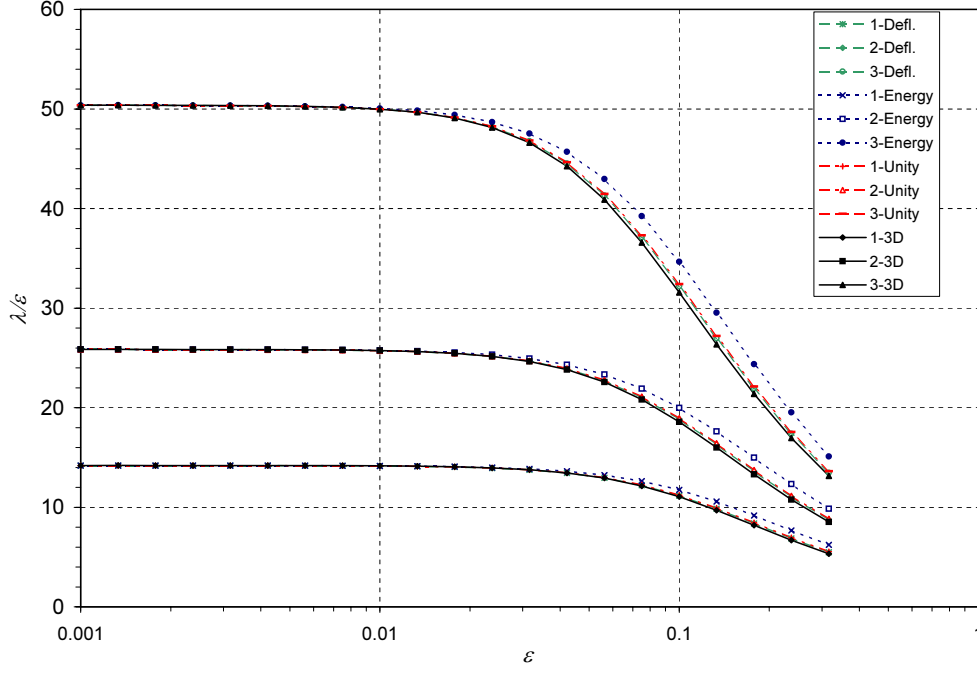


Figure 6: First 3 symmetric eigen-frequencies for “isotropic” rectangular plate: 3-D vis R-M model with different κ s.

At the observation scale of Figure 7, it seems that $\kappa_{\text{Deflection}}$ is the “best” κ from the three investigated, in the sense that $\tilde{\lambda}_n$ is the closest to λ_n for all investigated ε . Again at this observation scale, no linear dependence is visible (we cannot “see” $\lambda_{n,2}$).

Using $\kappa_{\text{Deflection}}$ for the “isotropic” material, the relative error for the first three eigen-frequencies is negligible (less than 0.12%) for thin plates with slander ratio of less than 1%, and for moderately thick plates (thickness about 5% compared to other plate dimensions) is smaller than 0.2%.

Now, if we magnify the ordinates of Figure 7 by a factor 100, we obtain Figure 8, where the expected increase of the eigen-frequency with ε (due to the non-zero factor $\lambda_{1,2}$) is clearly visible. In this “asymptotic” region, the shear correction factor κ_{Energy} is the best one.

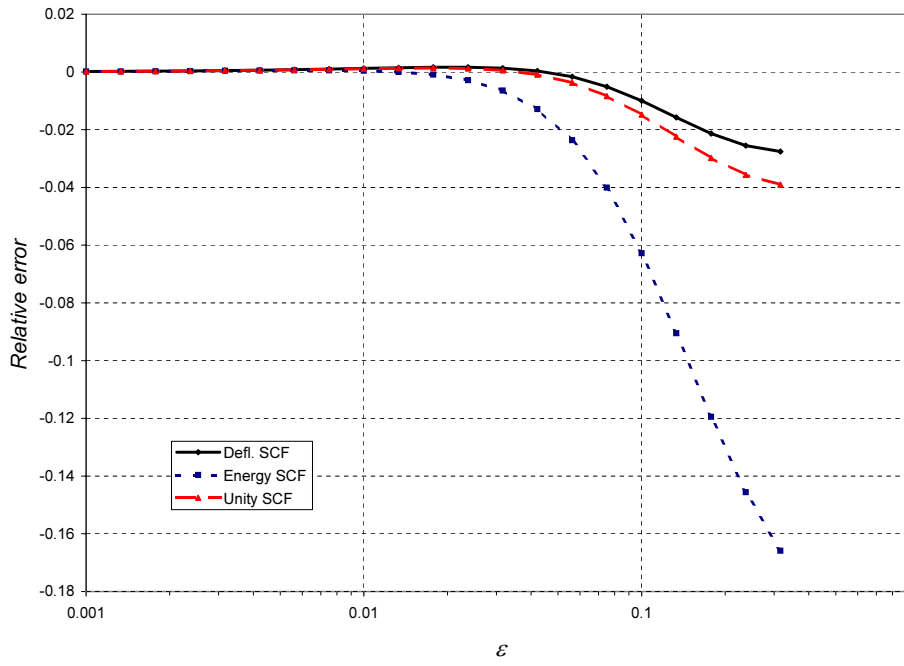


Figure 7: Relative error for first eigen-frequency: 3-D vis R-M model with different κ 's.

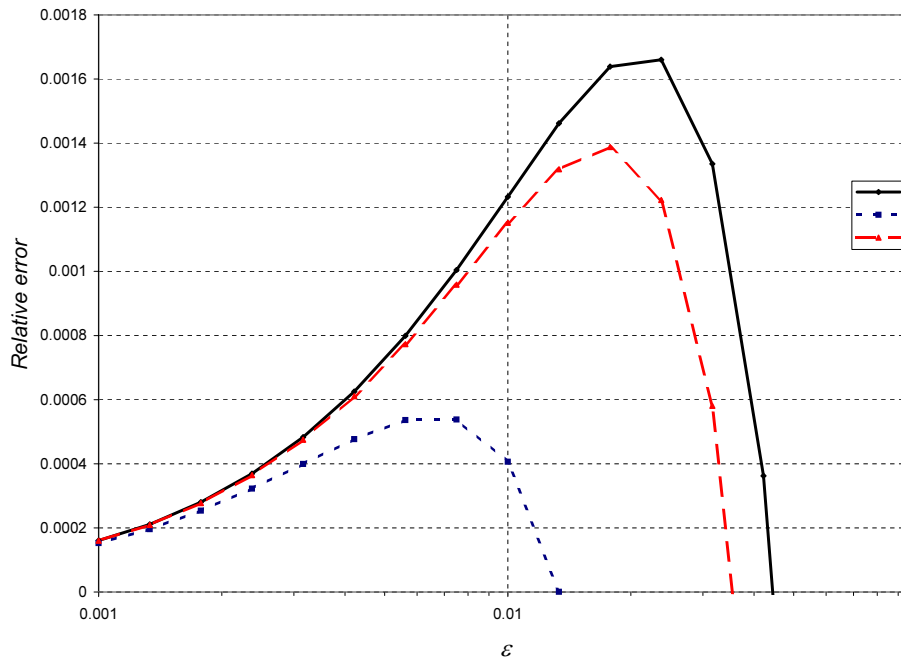


Figure 8: Zoom view of the relative error for first eigen-frequency: 3-D vis R-M model with different κ 's.

4.2 Almost Incompressible Materials.

The numerical values of the function $\nu \mapsto c^b(\nu)$, see Appendix A, suggest that the difference between λ_n and $\tilde{\lambda}_n$ should be much more pronounced as ν approaches 0.5. Herein we investigate the rectangular plate discussed in subsection 4.1 only with “almost incompressible” material properties. We provide in Figure 9 the relative error for the first, second and third eigen-frequencies with the $\kappa_{\text{Deflection}}$ which, for the standard isotropic material, was the best one at medium scale. Comparing with Figure 7 one notices the more

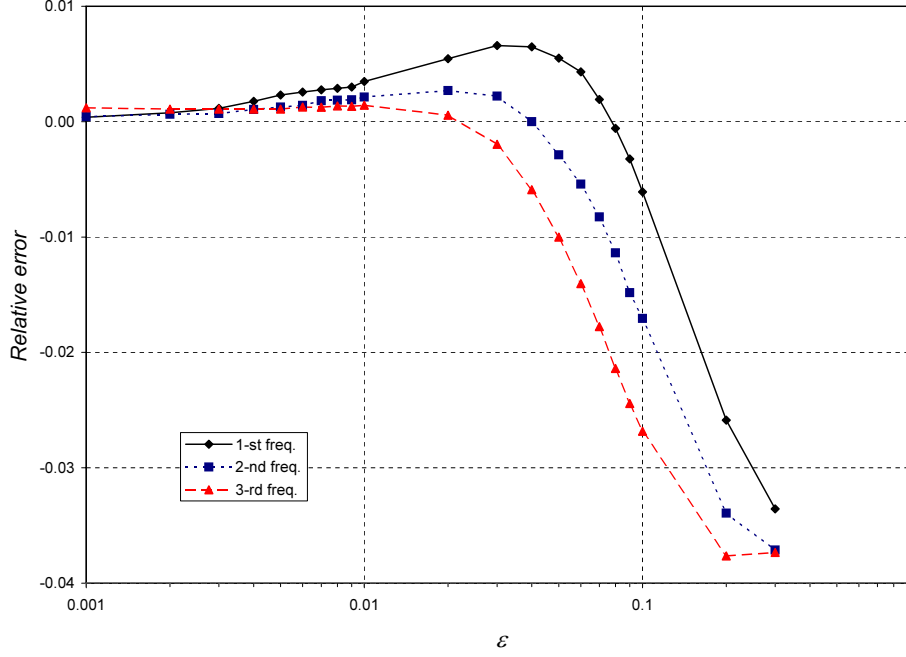


Figure 9: Relative error for first eigen-frequency: 3-D vis R-M model with different κ ’s of “almost incompressible material”.

pronounced difference between the R-M and 3-D eigen-frequencies as expected. In this case, a relative error of almost 0.7% is obtained for the first eigen-frequency at $\varepsilon \approx 0.03$.

As $\varepsilon \rightarrow 0$, and as we compute higher eigen-frequencies by the finite element method of a 3-D domain, the numerical errors increase. This numerical error is visible in Figure 9 for the third eigen-frequency at $\varepsilon < 0.003$.

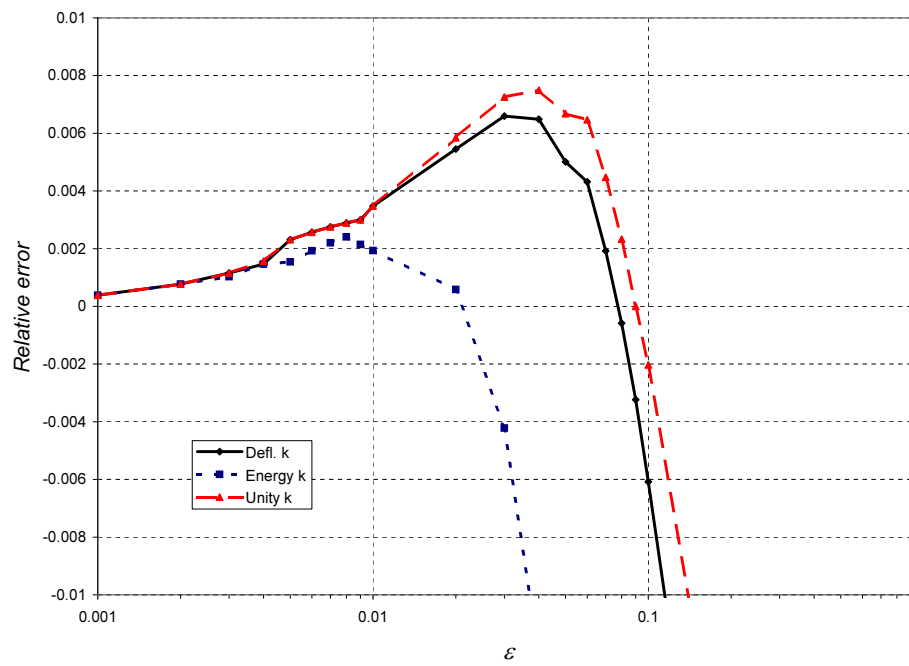


Figure 10: Zoom view of relative errors for first eigen-frequency of “almost incompressible material”.

4.3 Orthotropic Materials.

We herein investigate the influence of the material law (orthotropicity) on the eigen-frequencies, based on the rectangular plate model. The material properties are these defined as “orthotropic”, and in Figure 11 we present the first symmetric three eigen-frequencies over ε vis ε for the 3-D and R-M plate. The relative error is presented in Figure 12. The

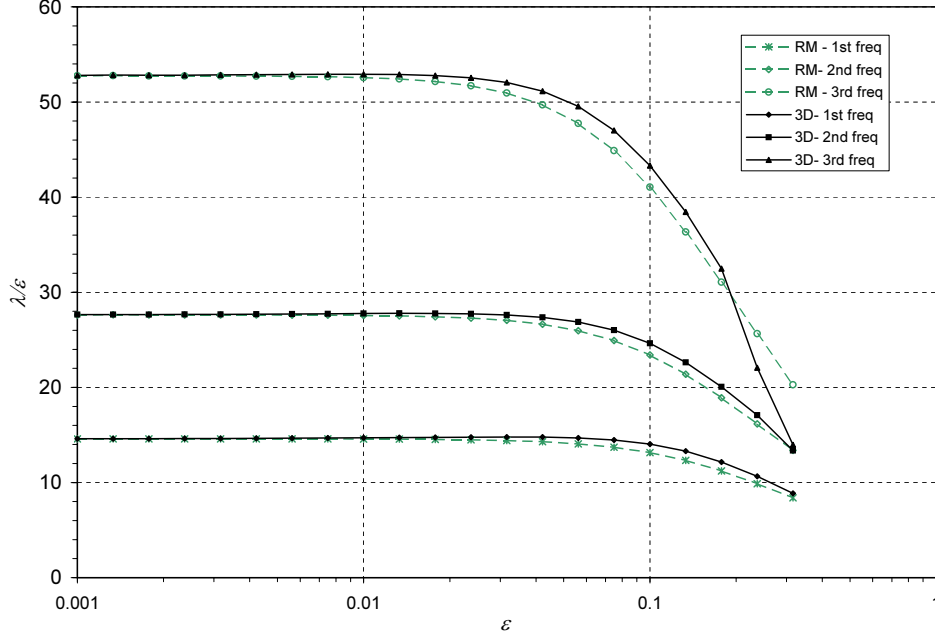


Figure 11: First 3 symmetric eigen-frequencies for “orthotropic” rectangular plate: 3-D vis R-M model.

relative error for the first eigen-frequency is around 5% at $\varepsilon \approx 0.07$, and increases up to 8% around $\varepsilon \approx 0.18$. Although the behavior of the eigen-frequencies compared to the “isotropic” plates is the same, we obtain herein a relative error which is larger by an order of magnitude compared to the isotropic case.

Plotting the relative error vis ε (not logarithmic scale), for $\varepsilon < 0.03$, a linear curve for $\varepsilon \rightarrow 0$ is visible as seen in Figure 13, supporting the asymptotic behavior (2.6).

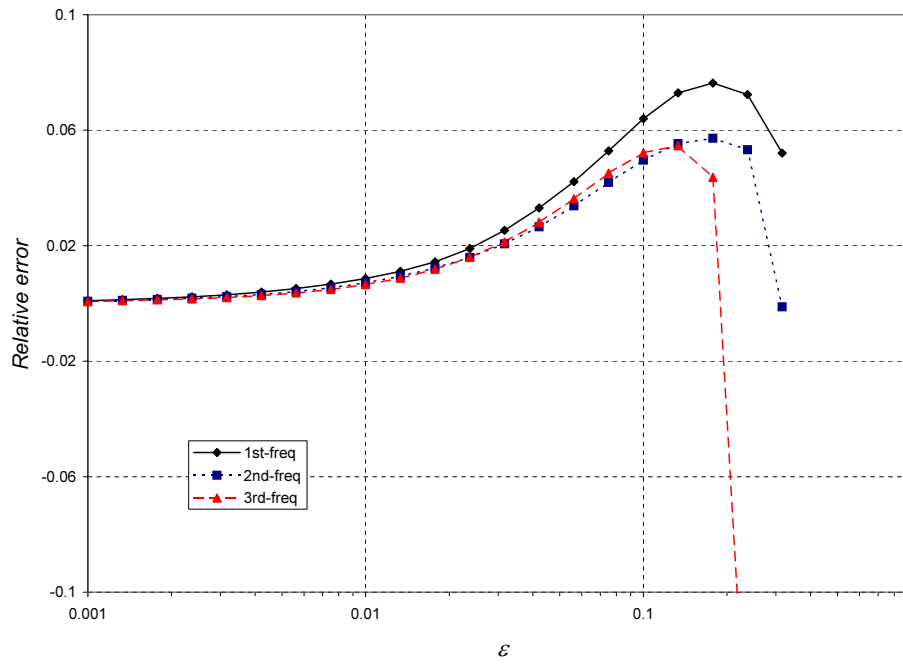


Figure 12: Relative error of first 3 symmetric eigen-frequencies: 3-D vis R-M model made of “orthotropic material”.

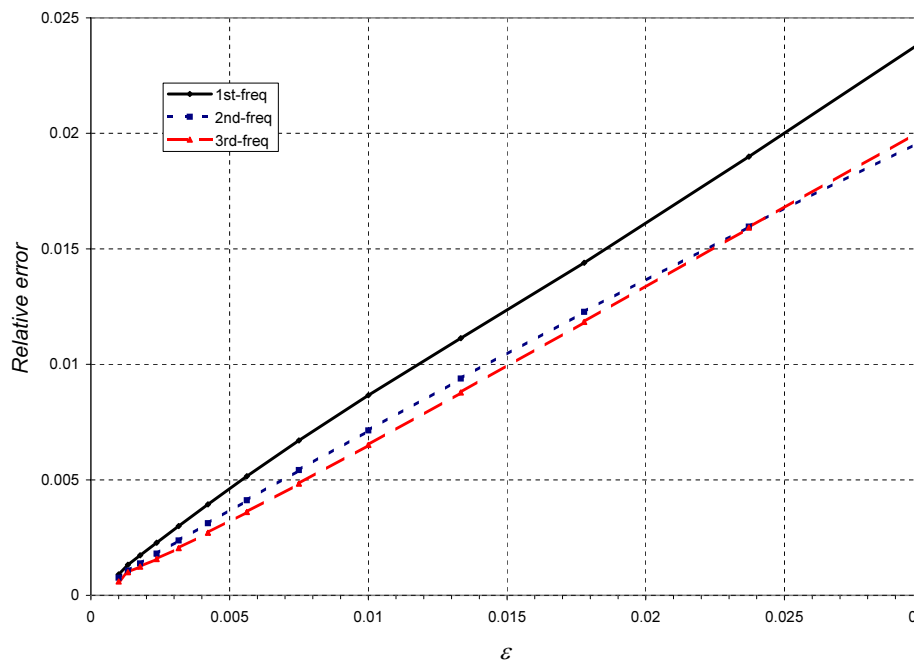


Figure 13: Relative error (linear graph) of first 3 symmetric eigen-frequencies: 3-D vis R-M model made of “orthotropic material”.

4.4 Strongly Orthotropic Materials.

For the “strongly orthotropic” plate, the boundary layer effect on the eigen-frequencies should be the most pronounced. We present in Figure 14 the first three symmetric eigen-frequencies over ε vis ε , as obtained from the 3-D analysis compared to these obtained by the R-M plate analysis with $\kappa=1$. The 3-D eigen-frequencies over ε clearly tend to

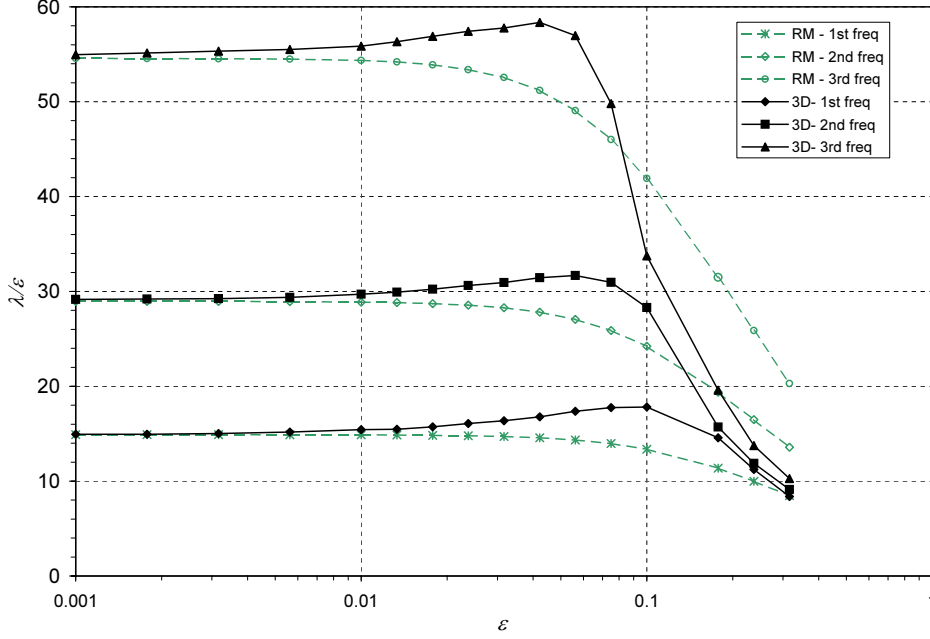


Figure 14: First 3 symmetric eigen-frequencies over ε for “strongly orthotropic” rectangular plate: 3-D vis R-M model.

a limit as $\varepsilon \rightarrow 0$, and the values λ_n/ε increase as ε increase up to $\varepsilon \approx 0.05$. The relative error is presented in Figure 15. A very large relative error of 25% is visible for the first eigen-frequency at $\varepsilon = 0.1$. This is a significant deviation whereas the R-M model underestimates the “true” 3-D by 25%, and is attributed to the boundary layer effect.

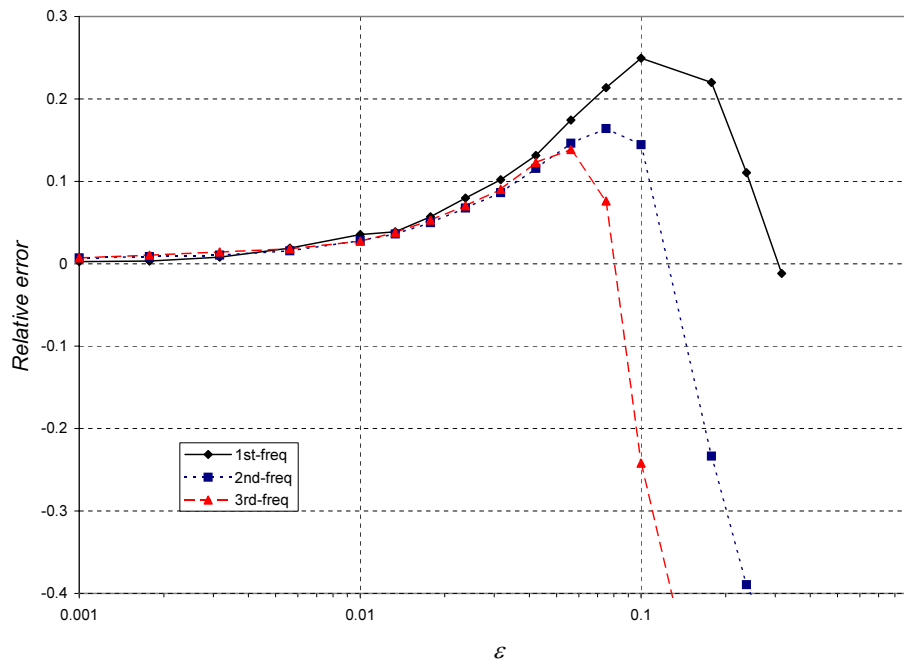


Figure 15: Relative error of first 3 symmetric eigen-frequencies: 3-D vis R-M model made of “strongly orthotropic material”.

4.5 The Effect of the Shape of $\partial\omega$.

The mathematical analysis demonstrates that the 3-D eigen-frequencies are clearly sensitive to the shape of the mid-surface boundary. To quantify this effect we analyze the rectangular plate having a hole in its center of increasing radii, as presented in Figure 3. Since the most pronounced effect between the 3-D and R-M plate eigen-frequencies is observed for the "strongly orthotropic" material, we herein use same material properties. The lowest (first) eigen-frequency over ε vis ε is plotted in Figure 16 for the rectangular plate without the whole, and for the plate with a hole of radius $R = 0.2, 0.3, 0.4$.

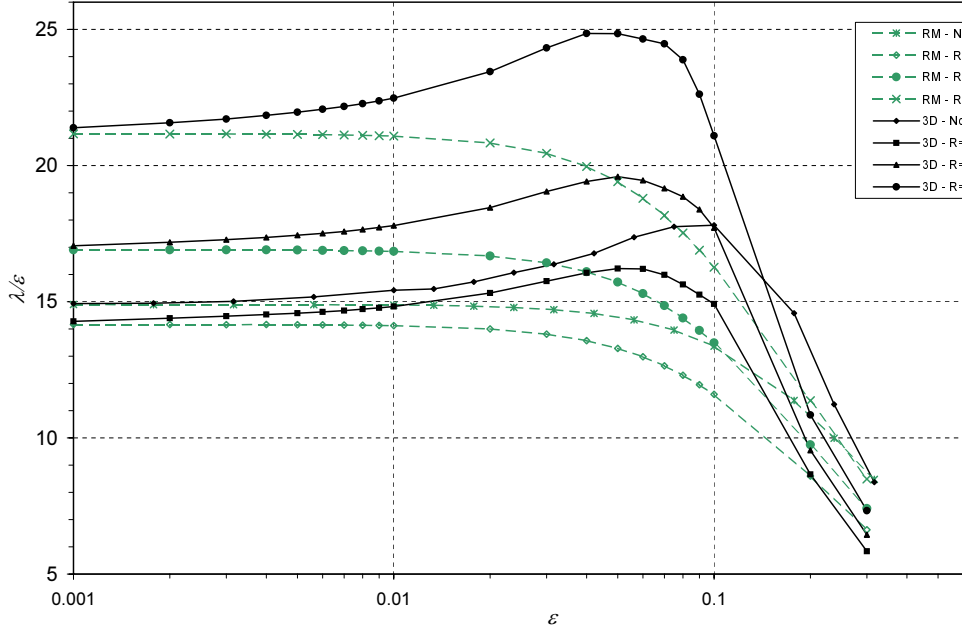


Figure 16: First eigen-frequency for "strongly orthotropic" plate with a hole of different radii: 3-D vis R-M model.

The influence of the hole (the boundary layer effect is increased) is better seen when observing the relative error in the first eigen-frequency as a function of ε in Figure 17.

One notices that for the plate with a hole with $R = 0.3$ the relative error may be beyond 30% for $\varepsilon \approx 0.07$. On the other hand, there does not seem to be a clear connection between the length of the clamped boundary layer and the increase in the relative error. In fact the relative error is larger for the plate without the hole in comparison with the plate having a hole of radius $R = 0.2$, however, the situation is reversed when comparing the rectangular plate with the one having a hole of radius $R = 0.3$.

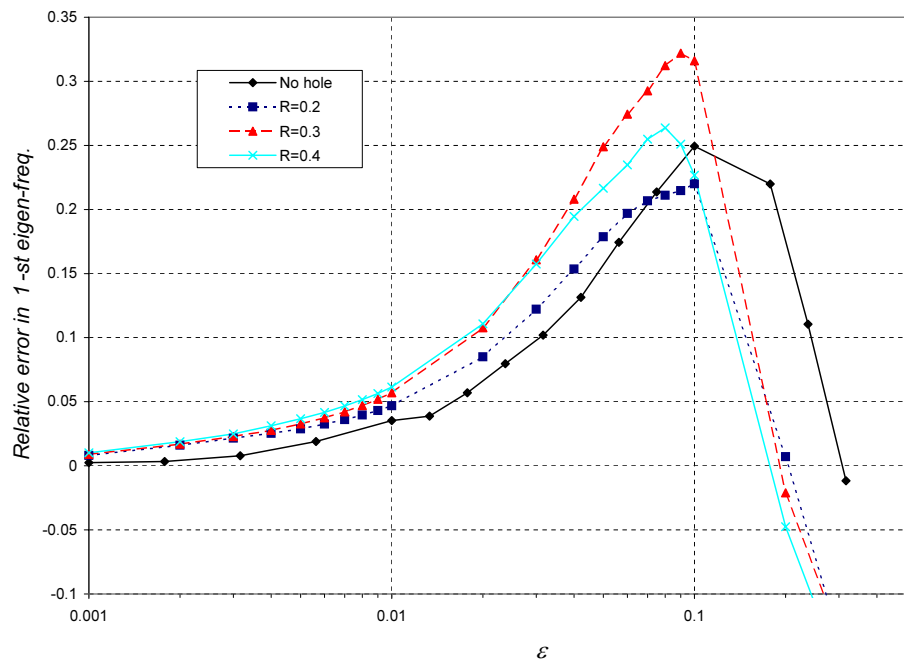


Figure 17: Relative error of first eigen-frequency for “strongly orthotropic” plate with a hole of different radii.

5 SUMMARY AND CONCLUSIONS

The eigen-frequencies computed using the Reissner-Mindlin plate model are frequently used in engineering practice as an approximation to the three dimensional plate eigen-frequencies for relatively thin elastic plates. In this paper, we have provided explicitly the asymptotic behavior of the eigen-frequencies as the thickness of the plate goes to zero, both for the R-M and 3-D plates made of either isotropic or orthotropic materials. The results for R-M and 3-D orthotropic plates are new. The 3-D and R-M asymptotics have a common first term but differ in their second terms.

The qualitative mathematical description is supported by quantitative numerical data obtained by the p -version finite element method for clamped plates. It was shown that for isotropic materials and relatively thin plates the Reissner-Mindlin eigen-frequencies provide a good approximation to the three-dimensional eigen-frequencies. It was also been demonstrated that for isotropic materials, no shear correction factor is known to be optimal in the sense that it provides the best approximation of the R-M eigen-frequencies to their 3-D counterparts uniformly (for all thicknesses range).

For some orthotropic materials we have shown relative errors between the R-M eigen-frequencies and their 3-D counterparts of an order of 30% even for relatively thin plates. This imposes a serious question mark on the relevance of the R-M eigen-analysis for plates made of orthotropic material. We also studied the influence of the shape of the mid-surface boundary on the errors in R-M eigen-frequencies.

A 3-D EIGEN-VALUES: ASYMPTOTICS FOR ORTHOTROPIC PLATES

Herein, the results and constructions of [5], which are valid for isotropic materials, are adapted to orthotropic materials. We explain in details the formula for the second term in the asymptotics, which is left implicit in [5].

A.1 Third order Ansatz

Let $X_3 := x_3/\varepsilon$ be the scaled transverse variable. Let (r, s) be boundary fitted coordinates along $\partial\omega$ in the mid-plane, with r the inward distance to $\partial\omega$ and s an arc-length coordinate along $\partial\omega$. Then $-\partial_r$ is the outward normal derivative along $\partial\omega$. Let furthermore $R := r/\varepsilon$ be the scaled distance to the boundary $\partial\omega$.

For ε small enough, there exists a 3-D eigenvector $\mathbf{u}^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ associated with λ_n^ε with a two-scale asymptotic expansion (outer and inner expansion terms) starting with

$$u_3^\varepsilon = \zeta[\varepsilon] + \varepsilon^2 \left(\frac{X_3^2}{2} - \frac{1}{6} \right) \frac{(C_{13}\partial_1^2 + C_{23}\partial_2^2)\zeta[\varepsilon]}{C_{33}} + \varepsilon^2 (\Phi_3^1 \zeta[\varepsilon])(s, R, X_3) + \mathcal{O}(\varepsilon^3) \quad (\text{A.1})$$

and for the in-plate components, $\alpha = 1, 2$

$$u_\alpha^\varepsilon = -\varepsilon X_3 \partial_\alpha \zeta[\varepsilon] + \varepsilon^2 (\Phi_\alpha^1 \zeta[\varepsilon])(s, R, X_3) + \mathcal{O}(\varepsilon^3) \quad (\text{A.2})$$

where the “generator” $\zeta[\varepsilon] = \zeta[\varepsilon](x_1, x_2) = \zeta_0 + \varepsilon\zeta_1 + \varepsilon^2\zeta_2 + \dots$ is a (scalar) *formal series solution* of the dimensionally reduced eigenvalue equation (A.3) below, and $\Phi^1 : \zeta \mapsto (\Phi^1\zeta)(s, R, X_3)$ is a 3-component boundary layer operator.

The PDE operator $P(\partial_1, \partial_2)$ in the second term of (A.1) can be found in the asymptotics for *monoclinic* materials in [6, §2.3]. Note that for isotropic materials, this operator reduces to:

$$\frac{C_{13}\partial_1^2 + C_{23}\partial_2^2}{C_{33}} = \frac{\nu}{1-\nu}(\partial_1^2 + \partial_2^2).$$

The equation to be satisfied by $\zeta[\varepsilon]$ in ω has the form

$$L[\varepsilon]\zeta[\varepsilon] = \Lambda[\varepsilon]\zeta[\varepsilon] \quad \text{in } \omega. \quad (\text{A.3})$$

Here $L[\varepsilon]$ is an operator formal series $\sum_{j \geq 0} \varepsilon^j L_j$. The term of order 0 is $L_0 = L^{\text{KL}}$ defined in (2.5), cf [6, §2.3] and $L_1 = 0$. Solution of the eigen-problem (A.3) provides the series $\Lambda[\varepsilon]$ (terms of the asymptotics of Λ_n^ε) as defined in (2.7). Equation (A.3) is completed by boundary conditions on $\partial\omega$ which can also be written as a formal series equation in the form $\gamma[\varepsilon]\zeta[\varepsilon] = 0$.

A.2 The first two eigen-problems for the generator

As we are interested in Λ_0 and Λ_1 , it is enough to consider the two first terms in $\zeta[\varepsilon]$, say $\zeta_0 + \varepsilon\zeta_1$. The boundary conditions on $\zeta_0 + \varepsilon\zeta_1$ are obtained by considering the Ansatz (A.1)-(A.2), and looking for zero Dirichlet traces up to the order ε^2 .

Considering the first terms in (A.1)-(A.2), we see that the boundary conditions on ζ_0 are $\zeta_0 = 0$ and $\partial_r \zeta_0 = 0$ on $\partial\omega$. With the first relation obtained from (A.3) we have

$$L^{\text{KL}}\zeta_0 = \Lambda_0\zeta_0 \quad \text{in } \omega \quad \text{and} \quad \zeta_0, \partial_r \zeta_0 = 0 \quad \text{on } \partial\omega. \quad (\text{A.4})$$

In order to obtain the boundary conditions for the next term in the asymptotics, ζ_1 , one has to recall the procedure for obtaining the first boundary layer term $\Phi^1\zeta_0$.

Let $B(\partial_1, \partial_2, \partial_3)$ and $G(\partial_1, \partial_2, \partial_3)$ be the interior operator and the traction operators on the upper and lower faces of the plate, associated with the left hand side of (2.3). Let moreover $(n_1(s), n_2(s))$ be the components of the inward unit normal to $\partial\omega$. Then there holds

$$\partial_1 = n_1\partial_r + n_2\partial_s \quad \text{and} \quad \partial_2 = n_2\partial_r - n_1\partial_s \quad \text{on } \partial\omega.$$

For each fixed $s \in \partial\omega$, the leading boundary layer operators are defined as

$$\mathcal{B}(s; \partial_R, \partial_{X_3}) := B(n_1(s)\partial_R, n_2(s)\partial_R, \partial_{X_3}) \quad \text{in the half-strip } (0, \infty) \times (-1, 1)$$

and

$$\mathcal{G}(s; \partial_R, \partial_{X_3}) := G(n_1(s)\partial_R, n_2(s)\partial_R, \partial_{X_3}) \quad \text{on upper and lower faces } (0, \infty) \times \{\pm 1\}.$$

For each fixed $s \in \partial\omega$, there exists a unique exponentially decreasing 3-component vector $\varphi^1(R, X_3) = \varphi^1(s; R, X_3)$ and two real numbers $\gamma^b = \gamma^b(s)$ and $\gamma'^b = \gamma'^b(s)$ such that the mixed boundary value problem in the half-strip

$$\begin{cases} \mathcal{B}(s)\psi = 0 & \text{in } (0, \infty) \times (-1, 1), \\ \mathcal{G}(s)\psi = 0 & \text{on } (0, \infty) \times \{\pm 1\}, \\ \psi = -(0, 0, \frac{X_3^2}{2} - \frac{1}{6})^T & \text{on } \{0\} \times (-1, 1). \end{cases}$$

has a solution of the form (when written in normal-, tangential-to- $\partial\omega$ and vertical local components)

$$\psi = \varphi^1 + \gamma^b \begin{pmatrix} -X_3 \\ 0 \\ R \end{pmatrix} + \gamma'^b \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then the first boundary layer operator $\zeta \mapsto \Phi^1\zeta$ is defined by

$$(\Phi^1\zeta)(s, R, X_3) = \frac{(C_{13}\partial_1^2 + C_{23}\partial_2^2)\zeta}{C_{33}} \Big|_{\partial\omega}(s) \varphi^1(s; R, X_3)$$

and the boundary conditions on ζ_1 are obtained by requiring that the Dirichlet trace of the components (A.1) and (A.2) of the Ansatz is a $\mathcal{O}(\varepsilon^2)$ and $\mathcal{O}(\varepsilon^3)$ respectively:

$$\zeta_1 = 0, \quad \partial_r \zeta_1 = \frac{(C_{13}\partial_1^2 + C_{23}\partial_2^2)\zeta_0}{C_{33}} \Big|_{\partial\omega}(s) \gamma^b(s) \quad \text{on } \partial\omega. \quad (\text{A.5})$$

These boundary conditions have to be combined with the 2nd equation arising from (A.3):

$$L^{\text{KL}}\zeta_1 = \Lambda_0\zeta_1 + \Lambda_1\zeta_0. \quad (\text{A.6})$$

A.3 Solution of the first two eigen-problems for the generator

The first eigen-problem is (A.4). Its solutions are obviously the eigen-pairs (ζ_0, Λ_0) of the fourth-order operator L^{KL} with Dirichlet conditions. Let us fix such a Λ_0 and let $\mathcal{E}$ be the corresponding eigen-space.

The solution of the next eigen-problem (A.5)-(A.6) is subject to *compatibility conditions* since $L^{\text{KL}} - \Lambda_0$ is not invertible. The Fredholm alternative can be formulated by integrating by parts equation (A.6) against any element $\zeta'_0 \in \mathcal{E}$, that is

$$0 = \iint_{\omega} L^{\text{KL}}\zeta_1\zeta'_0 - \Lambda_0\zeta_1\zeta'_0 - \Lambda_1\zeta_0\zeta'_0 \, dx_1 dx_2 \quad (\text{A.7})$$

where we integrate by parts four times the term $\iint L^{\text{KL}}\zeta_1\zeta'_0$. Since ζ'_0 belongs to $H_0^2(\omega)$ and ζ_1 to $H_0^1(\omega)$, we are left with the only term $-\int_{\partial\omega} \partial_r \zeta_1 P'\zeta'_0$, where P' is the second order operator[‡]

$$P'(s; \partial_1, \partial_2) = \frac{1}{3} \left(n_1^2 \tilde{C}_{11} \partial_1^2 + n_2^2 \tilde{C}_{22} \partial_2^2 + 2(\tilde{C}_{12} + 2\tilde{C}_{66}) n_1 n_2 \partial_1 \partial_2 \right).$$

[‡]We have used the relations $\partial_\alpha \zeta_1 = n_\alpha \partial_r \zeta_1$ valid on $\partial\omega$ for $\alpha = 1, 2$, since $\partial_s \zeta_1 = 0$ on $\partial\omega$.

And we know from (A.5) that

$$\partial_r \zeta_1 = \gamma^b(s) P(\partial_1, \partial_2) = \frac{C_{13} \partial_1^2 + C_{23} \partial_2^2}{C_{33}} \zeta_0 .$$

Then we deduce from (A.7), together with the equations $L^{\text{KL}} \zeta'_0 = \Lambda_0 \zeta'_0$:

$$0 = \int_{\partial\omega} \gamma^b(s) P \zeta_0 P' \zeta'_0 ds + \iint_{\omega} \Lambda_1 \zeta_0 \zeta'_0 dx_1 dx_2. \quad (\text{A.8})$$

Using that for any element $\zeta \in H_0^2(\omega)$, there holds $\partial_\alpha \partial_\beta \zeta = n_\alpha n_\beta \partial_r^2 \zeta$ on $\partial\omega$, and setting

$$c^b([C]; s) := -\gamma^b(s) \frac{n_1^4 \tilde{C}_{11} + n_2^4 \tilde{C}_{22} + 2(\tilde{C}_{12} + 2\tilde{C}_{66}) n_1^2 n_2^2}{3} \frac{n_1^2 C_{13} + n_2^2 C_{23}}{C_{33}} \quad (\text{A.9})$$

we obtain that, to be associated with a pair (ζ_1, Λ_1) , an element $\zeta_0 \in \mathcal{E}$ has to satisfy for all $\zeta'_0 \in \mathcal{E}$

$$\int_{\partial\omega} c^b([C]; s) \partial_r^2 \zeta_0 \partial_r^2 \zeta'_0 ds = \Lambda_1 \iint_{\omega} \zeta_0 \zeta'_0 dx_1 dx_2, \quad (\text{A.10})$$

which means that (ζ_0, Λ_1) is an eigen-pair of the symmetric eigen-problem (A.10). The positivity of the function $-\gamma^b(s)$ can be proved by the same arguments as in [7, §6.2]. The positivity of the other functions in (A.9) constituting c^b results from the positivity of the material matrix $[C]$.

Noting that for $\zeta \in H_0^2(\omega)$, there holds $\partial_r^2 \zeta = \Delta \zeta$ on $\partial\omega$, we obtain formula (2.12) in the case when Λ_0 is a simple eigenvalue. For isotropic materials, the “constant” $c^b(\nu)$ introduced in (2.8) is plotted for $\nu \in [0, 0.5]$ in Figure 18. It looks very much like the one plotted in [9].

For non-isotropic materials, $c^b([C]; s)$ also depends on s via the orientation of the material fibers with respect to the boundary at point s . Considering $(n_1, n_2) = (\cos \theta, \sin \theta)$, we may consider $c^b([C])$ as a function $c^b([C]; \theta)$ of θ . For both orthotropic and strongly orthotropic materials used in our computations, the function $\theta/\pi \mapsto c^b([C]; \theta)$ is plotted for $\theta/\pi \in [0, 1]$ in Figure 19. We note that c^b takes larger values for the orthotropic material than for isotropic materials, and still larger values for the strongly orthotropic material. The period $1/2$ is due to the symmetries of our materials.

The computations for Figures 18 and 19 are done with the Finite Element library MÉLINA [8].

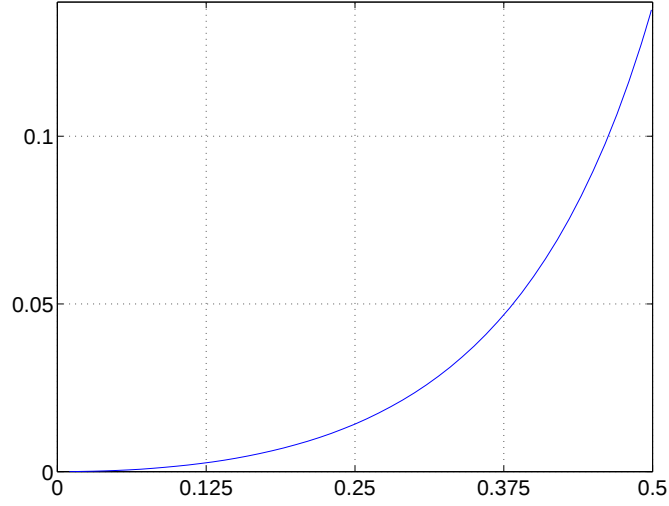


Figure 18: The “constant” $c^b(\nu)$ as a function of ν for isotropic materials.

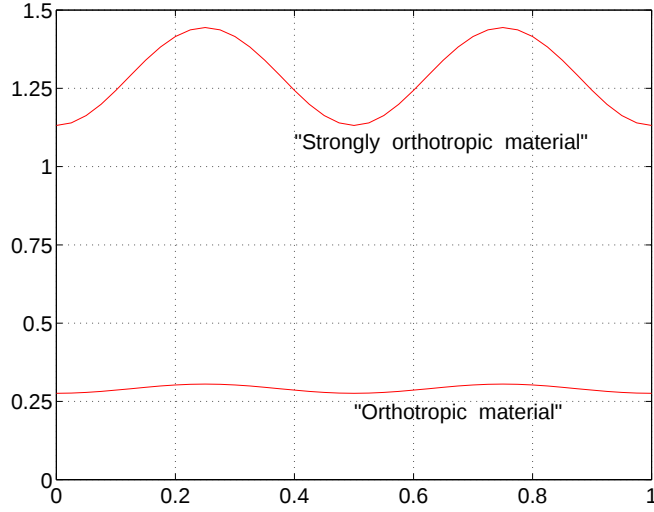


Figure 19: The function $\theta/\pi \mapsto c^b([C]; \theta)$ for both orthotropic materials.

B R-M EIGEN-VALUES: ASYMPTOTICS FOR ORTHOTROPIC PLATES

In this Appendix, we describe the first steps of the asymptotic analysis for the R-M eigenpairs, solution of (2.20). In order to have simpler formulations of the underlying operators, we assume, which introduces no essential restriction,

$$\tilde{C}_{44} = \tilde{C}_{55} = 1.$$

On the other hand we define the following 2×2 bi-dimensional elasticity system M^{KL}

$$\boldsymbol{\beta} \longmapsto M^{\text{KL}}\boldsymbol{\beta} = \frac{1}{3} \begin{cases} \tilde{C}_{11}\partial_1^2\beta_1 + \tilde{C}_{12}\partial_1\partial_2\beta_2 + \tilde{C}_{66}(\partial_2^2\beta_1 + \partial_1\partial_2\beta_2) \\ \tilde{C}_{22}\partial_1^2\beta_2 + \tilde{C}_{12}\partial_1\partial_2\beta_1 + \tilde{C}_{66}(\partial_1^2\beta_2 + \partial_1\partial_2\beta_1). \end{cases} \quad (\text{B.1})$$

We note that the system M^{KL} has the following relation with the K-L operator L^{KL} defined in (2.5)

$$\operatorname{div} M^{\text{KL}}\nabla = L^{\text{KL}}. \quad (\text{B.2})$$

Then the eigen-problem (2.20) can be written in the form

$$\begin{cases} -\varepsilon^3 M^{\text{KL}}\boldsymbol{\beta} - \varepsilon(\nabla w - \boldsymbol{\beta}) &= \frac{1}{3}\varepsilon^3\tilde{\lambda}^2\boldsymbol{\beta}, \\ -\varepsilon \operatorname{div}(\nabla w - \boldsymbol{\beta}) &= \varepsilon\tilde{\lambda}^2 w, \end{cases} \quad (\text{B.3})$$

with the Dirichlet boundary conditions

$$\beta_1 = \beta_2 = w = 0 \quad \text{on} \quad \partial\omega.$$

Setting $\tilde{\lambda}^2 = \varepsilon^2\Lambda$, we obtain that (B.3) is equivalent to

$$\begin{cases} -\varepsilon^2 M^{\text{KL}}\boldsymbol{\beta} - (\nabla w - \boldsymbol{\beta}) &= \frac{1}{3}\varepsilon^4\Lambda\boldsymbol{\beta}, \\ -\operatorname{div}(\nabla w - \boldsymbol{\beta}) &= \varepsilon^2\Lambda w. \end{cases} \quad (\text{B.4})$$

As there are only *even* powers of ε in (B.4) we start with even Ansätze: for Λ ,

$$\Lambda = \Lambda_0 + \varepsilon^2\Lambda_2 + \dots$$

and for $\boldsymbol{\beta}$ and w

$$\boldsymbol{\beta} = \boldsymbol{\beta}_0 + \varepsilon^2\boldsymbol{\beta}_2 + \varepsilon^4\boldsymbol{\beta}_4 + \dots \quad \text{and} \quad w = w_0 + \varepsilon^2w_2 + \varepsilon^4w_4 + \dots$$

Using these Ansätze in (B.4) and identifying the powers of ε we find systems of equations on Λ_k , $\boldsymbol{\beta}_k$ and w_k , the three first ones of which are

$$\begin{cases} -(\nabla w_0 - \boldsymbol{\beta}_0) &= 0, \\ -\operatorname{div}(\nabla w_0 - \boldsymbol{\beta}_0) &= 0, \end{cases} \quad (\text{B.5})$$

$$\begin{cases} -M^{\text{KL}}\boldsymbol{\beta}_0 - (\nabla w_2 - \boldsymbol{\beta}_2) &= 0, \\ -\operatorname{div}(\nabla w_2 - \boldsymbol{\beta}_2) &= \Lambda_0 w_0, \end{cases} \quad (\text{B.6})$$

$$\begin{cases} -M^{\text{KL}}\boldsymbol{\beta}_2 - (\nabla w_4 - \boldsymbol{\beta}_4) &= \frac{1}{3}\Lambda_0\boldsymbol{\beta}_0, \\ -\operatorname{div}(\nabla w_4 - \boldsymbol{\beta}_4) &= \Lambda_0 w_2 + \Lambda_2 w_0. \end{cases} \quad (\text{B.7})$$

Here follow all the equations satisfied by Λ_0 , Λ_2 , β_0 , β_2 , w_0 and w_2 so that (B.5)-(B.7) hold:

$$\beta_0 = \nabla w_0, \quad (\text{B.8})$$

$$\beta_2 = \nabla w_2 + M^{\text{KL}} \nabla w_0, \quad (\text{B.9})$$

$$(L^{\text{KL}} - \Lambda_0)w_0 = 0, \quad (\text{B.10})$$

$$(L^{\text{KL}} - \Lambda_0)w_2 = \Lambda_2 w_0 + (\tfrac{1}{3}\Lambda_0 \Delta + \text{div } M^{\text{KL}} M^{\text{KL}} \nabla)w_0. \quad (\text{B.11})$$

Equations (B.8) and (B.10) combined with the boundary conditions $w_0 = 0$ and $\beta_0 = 0$ are solved by the eigen-pairs (w_0, Λ_0) of the operator L^{KL} with Dirichlet boundary conditions $w_0 = 0$ and $\partial_n w_0 = 0$ on $\partial\omega$.

As for equation (B.11), for any pair of fixed traces g and h , we can find[§] (w_2, Λ_2) satisfying (B.11) together with the boundary conditions $w_2 = g$ and $\partial_n w_2 = h$ on $\partial\omega$. In order to satisfy the Dirichlet conditions β_2 , w_2 are zero on $\partial\omega$, and taking (B.9) into account, we may choose

$$g = 0 \quad \text{and} \quad h = -\partial_n M^{\text{KL}} \nabla w_0. \quad (\text{B.12})$$

Then w_2 and the normal component of β_2 are zero on $\partial\omega$. As for the tangential component of β_2 , since $\partial_s w_2 = 0$ on $\partial\omega$, its trace is equal to the tangential component χ of $M^{\text{KL}} \nabla w_0$, which has no reason to be zero.

In the *isotropic case*, the trace $\varepsilon^2 \chi$ of the tangential component of $\beta_0 + \varepsilon^2 \beta_2$ is compensated by the first boundary layer term in the expansion of the eigenvector: this term has the form $\varepsilon^2(\Psi, \varepsilon W)(s, R)$, where we recall that R is the scaled variable r/ε . Here, we simply obtain that $W = 0$, the normal component Ψ_n of Ψ is zero, and for each $s \in \partial\omega$ the tangential component $\Psi_s(s, \cdot)$ is the exponentially decreasing solution of the boundary value problem on $\mathbb{R}_+$:

$$-\tilde{C}_{66} \partial_R^2 \Psi_s + \Psi_s = 0, \quad \forall R > 0, \quad \text{and} \quad \Psi_s(s, 0) = -\chi(s). \quad (\text{B.13})$$

This fits perfectly with the loading case in [1, 2].

In the general orthotropic case, we do not have any more this uncoupling between normal and tangential components of Ψ . The first boundary layer term has still the form $\varepsilon^2(\Psi, \varepsilon W)(s, R)$ and the triple $(\Psi, W)(s, \cdot)$ solves the system

$$\mathcal{R}(s; \partial_R)(\Psi, W) = 0, \quad \forall R > 0,$$

where, if $R(\partial_1, \partial_2)$ denotes the system $(\beta, w) \mapsto (-M^{\text{KL}} \beta + \beta - \nabla w, \text{div } \beta - \Delta w)$,

$$\mathcal{R}(s; \partial_R) := R(n_1(s) \partial_R, n_2(s) \partial_R).$$

Then there exists a unique exponentially decreasing solution of the boundary value problem on $\mathbb{R}_+$:

$$\mathcal{R}(s; \partial_R)(\Psi, W) = 0, \quad \forall R > 0, \quad \text{and} \quad \Psi_s(s, 0) = -\chi(s). \quad (\text{B.14})$$

[§] If Λ_0 is a *simple eigenvalue* of L^{KL} . The case of multiple eigenvalues has to be treated like above, see Appendix A.

But now, in general, Ψ_n and W are not $\equiv 0$. As the power of ε in front of W is 3, the trace of W does not disturb the trace of $w_0 + \varepsilon^2 w_2$. This means that we keep the choice of the boundary condition $g = 0$ for w_2 in (B.12). But we have to modify h by setting instead of $h = -\partial_n M^{\text{KL}} \nabla w_0$:

$$h = -\partial_n M^{\text{KL}} \nabla w_0 - \Psi_n(s, 0).$$

Then the expansion of the R-M eigenvector starts with

$$\beta_0 + \varepsilon^2 \beta_2 + \varepsilon^2 \Psi(s, R) + \mathcal{O}(\varepsilon^3) \quad \text{and} \quad w_0 + \varepsilon^2 w_2 + \mathcal{O}(\varepsilon^3)$$

and that of the eigenvalue with

$$\Lambda_0 + \varepsilon^2 \Lambda_2 + \mathcal{O}(\varepsilon^3),$$

where Λ_2 is influenced by the boundary layer only in the non-isotropic case (when h has to be modified).

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