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Selfsimilar Perturbation near a Corner: Matching and Multiscale Expansions

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After a paper with

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Solve an elliptic BVP (e.g. Laplace, elasticity) in a domain with two scales:

- A macro scale at which the domain looks like a polygon,
- A micro scale concerning regions of size ε at which the domain can be rounded, or cracked, or punctured,...

Keep memory of size ε for the notation of the domain: ω_ε .

The macro-micro structure can be present also in material laws, defining sub-regions subject to different laws.

Model problem: As usual Dirichlet-Laplace!

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in} \quad \omega_\varepsilon \\ u_\varepsilon = 0 & \text{on} \quad \partial\omega_\varepsilon. \end{cases}$$

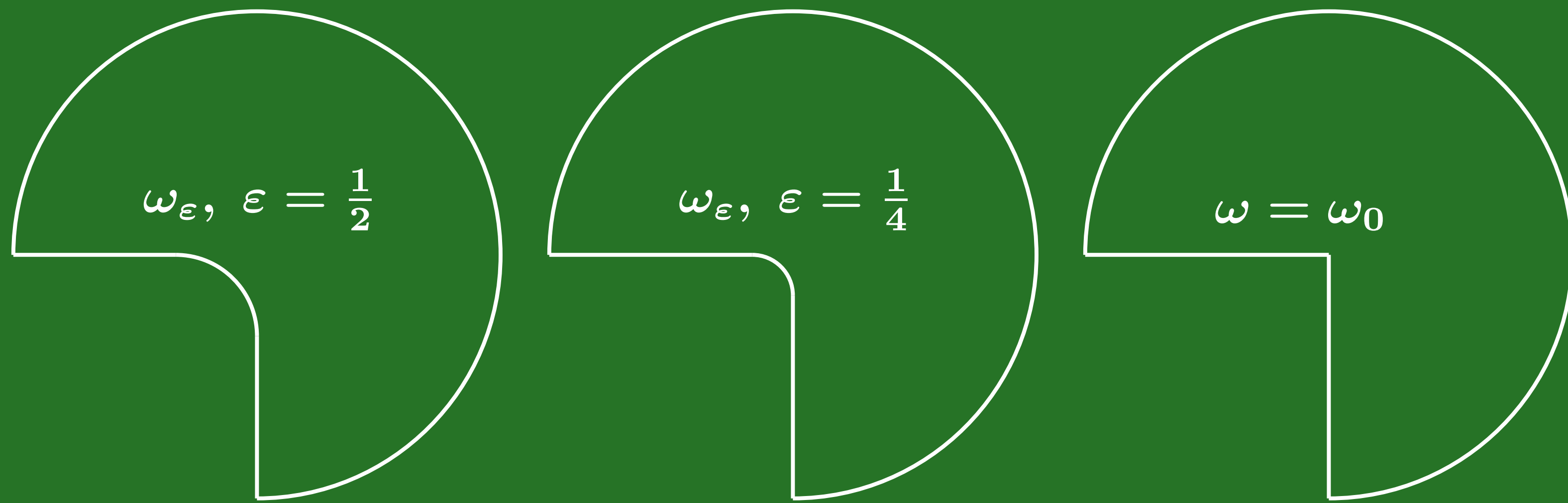
u_ε is the solution of the variational problem on ω_ε

$$u_\varepsilon \in H^1_0(\omega_\varepsilon), \quad \forall v \in H^1_0(\omega_\varepsilon), \quad \int_{\omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v = \int_{\omega_\varepsilon} f v.$$

Problem: structure of u_ε as ε gets small...

A typical example: the rounded corner

Example of a circular sector of opening 270° with a rounded corner:



As $\varepsilon \longrightarrow 0$, the domain $\omega_\varepsilon \longrightarrow \omega$, with $\omega = \omega_0$.

u_0 has the Dirichlet-Laplace singularities $r^{2k/3} \sin \frac{2k}{3}\,\theta$, $k = 1, 2, \ldots$

In principle u_ε has no singularities near the rounded corner if $\varepsilon > 0$.

We expect that $u_\varepsilon \longrightarrow u_0$ in energy as $\varepsilon \longrightarrow 0$.

How are the singularities of u_0 hiding inside u_ε?

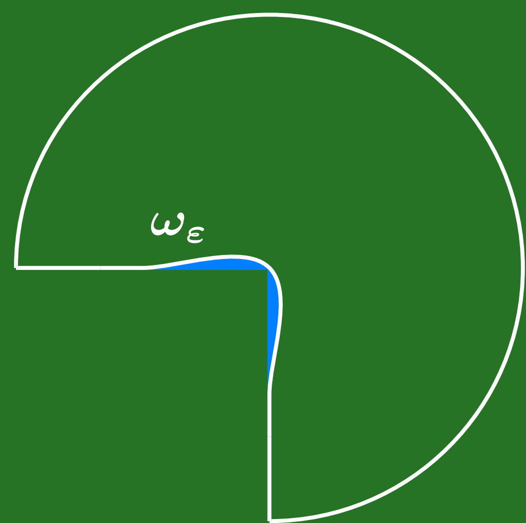
Rounded corners... from inside or outside?

Example of “the" corner of a circular sector of opening 270 $^\circ$, rounded by two different procedures.

From inside ($\mathcal{C}^1$ Bezier curve).

$\omega_\varepsilon \subset \omega_0$.

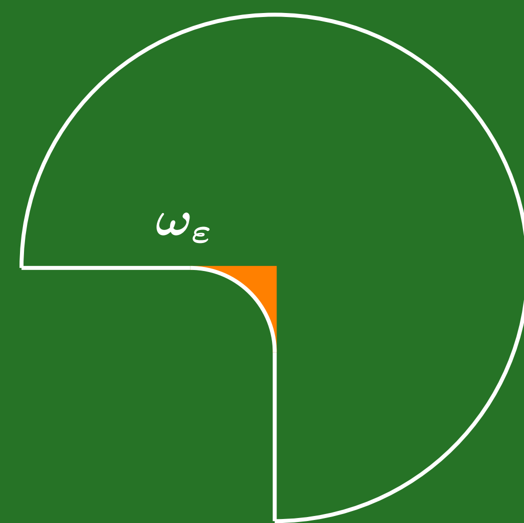
In blue $\omega_0 \setminus \omega_\varepsilon$.



From outside (exterior arc of circle).

$\omega_0 \subset \omega_\varepsilon$.

In orange $\omega_\varepsilon \setminus \omega_0$.



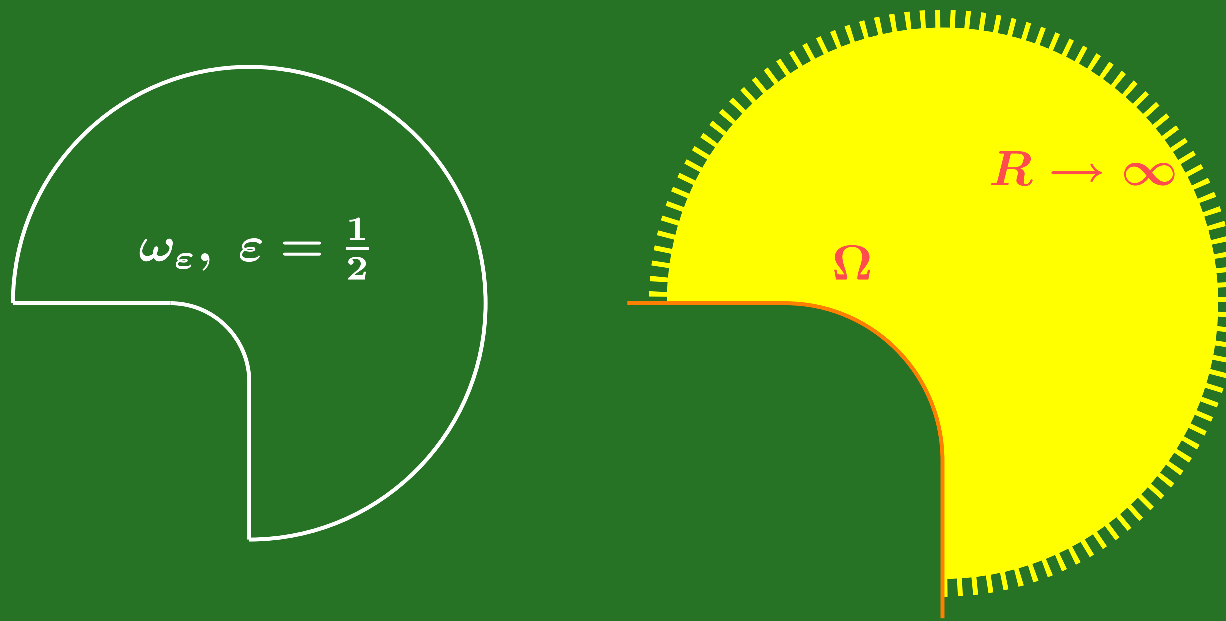
For convex angles (the exterior of the sector) the situation is reversed.

What is the meaning of a convergence of u_ε towards u_0 ?

Answers with the help of multi-scale expansions.

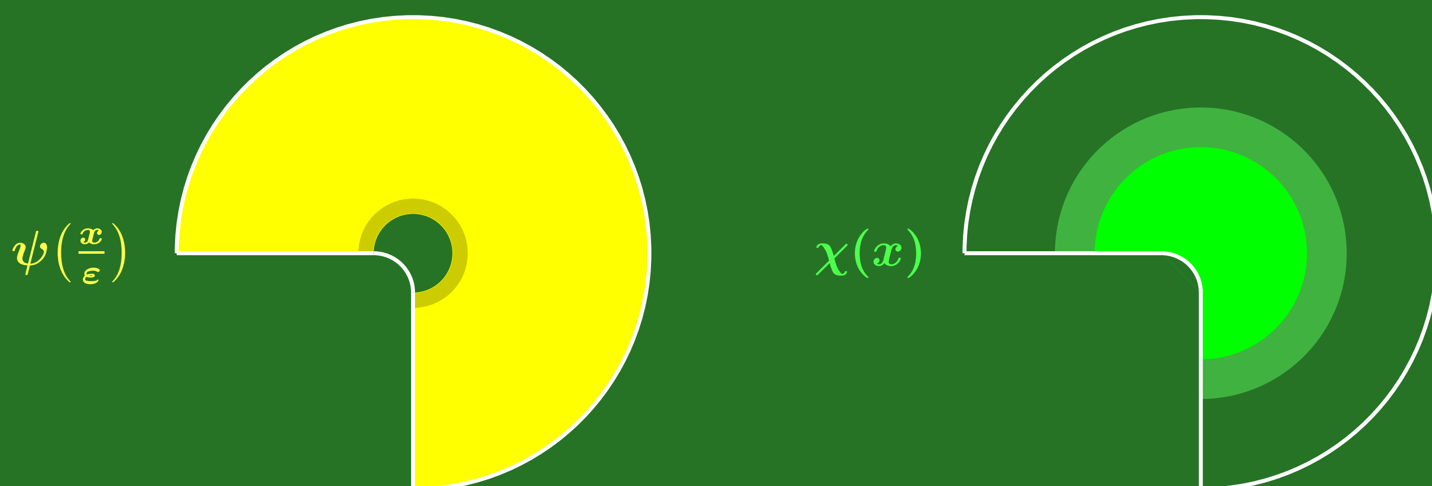
Multi-scale expansion for a rounded corner: Ingredients

A blow-up of the ε-pattern, generating the infinite domain Ω.



A “rapid" cut-off $x \mapsto \psi(\frac{x}{\varepsilon})$, so that $\psi(\frac{x}{\varepsilon})u_0$ is well defined on ω_ε.

A slow cut-off $x \mapsto \chi(x)$.



In slow variables x, over the corner domain ω:

$$u \in H_0^1(\omega), \quad \forall v \in H_0^1(\omega), \quad \int_\omega \nabla_x u \cdot \nabla_x v \, dx = \int_\omega f v \, dx.$$

Uniquely solvable for $f \in H^{-1}(\omega)$.

In scaled variables X, over the pattern domain Ω

$$U \in W_0^1(\Omega), \quad \forall V \in W_0^1(\Omega), \quad \int_\Omega \nabla_X U \cdot \nabla_X V \, dX = \int_\Omega F V \, dX.$$

Here

$$W_0^1(\Omega) = \{U; \quad \nabla U \in L^2(\Omega), \quad \frac{U(X)}{1+|X|} \in L^2(\Omega), \quad U\big|_{\partial\Omega} = 0\}.$$

Uniquely solvable for $F \in W^{-1}(\Omega)$, the dual space of $W_0^1(\Omega)$.

Refs : Oleinik - Shamaev - Yosifian (92) Maz'ya - Nazarov - Plamenevskij (94).

Opening of the limit angle $\frac{\pi}{\lambda}$. Assume $\lambda \notin \mathbb{Q}$.

Singularities $r^{k\lambda}\varphi_k(\theta)$ with $\varphi_k(\theta) = \sin k\lambda\theta$, for $k = 1, 2, \ldots$.

Assume $f \equiv 0$ close to the limit corner O.

Fix $K > 0$. The solution u_ε can be split according to

$$u_\varepsilon = \psi\Big(\frac{x}{\varepsilon}\Big)\, u_0(x) + \sum_{k=2}^{K} \varepsilon^{k\lambda}\, \psi\Big(\frac{x}{\varepsilon}\Big)\, v^{k\lambda}(x) + \sum_{k=1}^{K} \varepsilon^{k\lambda}\, \chi(r)\, V^{k\lambda}\Big(\frac{x}{\varepsilon}\Big) + \mathcal{O}(\varepsilon^{(K+1)\lambda}),$$

as $\varepsilon \to 0$.

Algorithm

- Variational solution u_0 in ω.
- Expansion of u_0 as $r \to 0$: $u_0 = a_1^0 r^\lambda \varphi_1(\theta) + a_2^0 r^{2\lambda}\varphi_2(\theta) + \cdots$
- Scaling to Ω of the residual $f - \Delta\big(\psi(\frac{x}{\varepsilon})u_0\big)$.
- Rhs $\varepsilon^\lambda F^\lambda$ in Ω: Variational solution $\varepsilon^\lambda V^\lambda$ in Ω.
- Expansion of V^λ as $R \to \infty$: $V^\lambda = A_1^1 R^{-\lambda}\varphi_1 + A_2^1 R^{-2\lambda}\varphi_2 + \ldots$
- Expression in ω of the residual of $\varepsilon^\lambda \chi(r) V^\lambda(\frac{x}{\varepsilon})$.
- Rhs $\varepsilon^{2\lambda} f^{2\lambda}$ in ω: Variational solution $\varepsilon^{2\lambda} v^{2\lambda}$ in ω
- ...

Matched asymptotics: The principle

Refs: VanDyke (75) Il'lin (92)

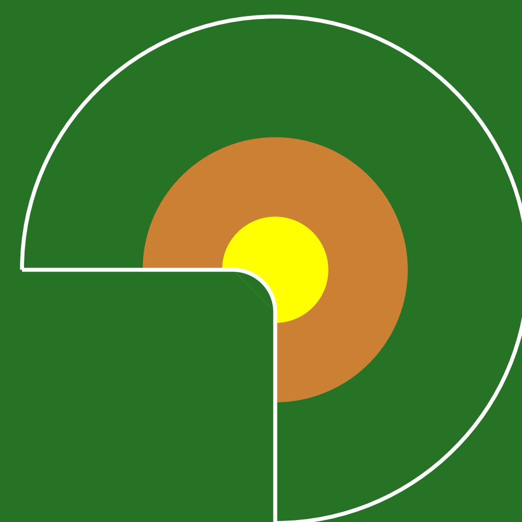
Look for two expansions of u_ε :

$$u_\varepsilon(x) \simeq u_0(x) + \sum_{k\geq 1} \varepsilon^{k\lambda}\, u^{k\lambda}(x) \qquad \text{for } x \in \omega_\varepsilon,\quad |x| \geq r_0 > 0$$

$$u_\varepsilon(x) \simeq \sum_{k\geq 0} \varepsilon^{k\lambda}\, U^{k\lambda}\Big(\frac{x}{\varepsilon}\Big) \qquad \text{for } x \in \omega_\varepsilon,\quad |x| \leq \varepsilon r_1$$

Support of outer exp.

Support of inner exp.



Somewhere in the annulus $\varepsilon r_1 < |x| < r_0$ is the transition region, where the outer expansion and the inner expansion can be matched.

- Variational solution u_0 in ω.
- Expansion of u_0 as $r \to 0$: $u_0 = a_1^0\, r^\lambda\, \varphi_1(\theta) + a_2^0\, r^{2\lambda}\, \varphi_2(\theta) + \cdots$
- Scaling to Ω of $a_1^0\, r^\lambda\, \varphi_1(\theta) = a_1^0\, \varepsilon^\lambda\, R^\lambda\, \varphi_1(\theta)$.
- Solve
  $\Delta U^\lambda = 0$ in Ω with $U^\lambda - a_1^0\, R^\lambda\, \varphi_1(\theta)$ in $W_0^1(\Omega)$.
- Expansion of U^λ as $R \to \infty$:
  $U^\lambda = a_1^0\, R^\lambda\, \varphi_1(\theta) + A_1^1\, R^{-\lambda}\, \varphi_1(\theta) + A_2^1\, R^{-2\lambda}\, \varphi_2(\theta) + \dots$
- Scaling to ω of $\varepsilon^\lambda\, A_1^1\, R^{-\lambda}\, \varphi_1(\theta) = \varepsilon^{2\lambda}\, A_1^1\, r^{-\lambda}\, \varphi_1(\theta)$.
- Solve
  $\Delta u^{2\lambda} = 0$ in ω with $u^{2\lambda} - A_1^1\, r^{-\lambda}\, \varphi_1(\theta)$ in $H_0^1(\Omega)$.
- ...

The terms $u^{k\lambda}$ and $U^{k\lambda}$ are more and more singular

The algorithm can be solved, but is formal.

How to transform it into an actual expansion?

Tordeux's trick :

Use an intermediate cut-off at $\sqrt{\varepsilon}$ scale

Let ϕ a smooth cut-off, $\phi \equiv 0$ for $|x| \leq 1$, $\phi \equiv 1$ for $|x| \geq 2$.
Fix $K > 0$. The solution u_ε can be split according to

$$u_\varepsilon = \phi\Big(\frac{x}{\sqrt{\varepsilon}}\Big)\, u_0(x) + \sum_{k=2}^{K} \varepsilon^{k\lambda}\, \phi\Big(\frac{x}{\sqrt{\varepsilon}}\Big)\, u^{k\lambda}(x)$$

$$+ \sum_{k=1}^{K} \varepsilon^{k\lambda}\, \Big(1-\phi\Big)\Big(\frac{x}{\sqrt{\varepsilon}}\Big)\, U^{k\lambda}\Big(\frac{x}{\varepsilon}\Big)$$

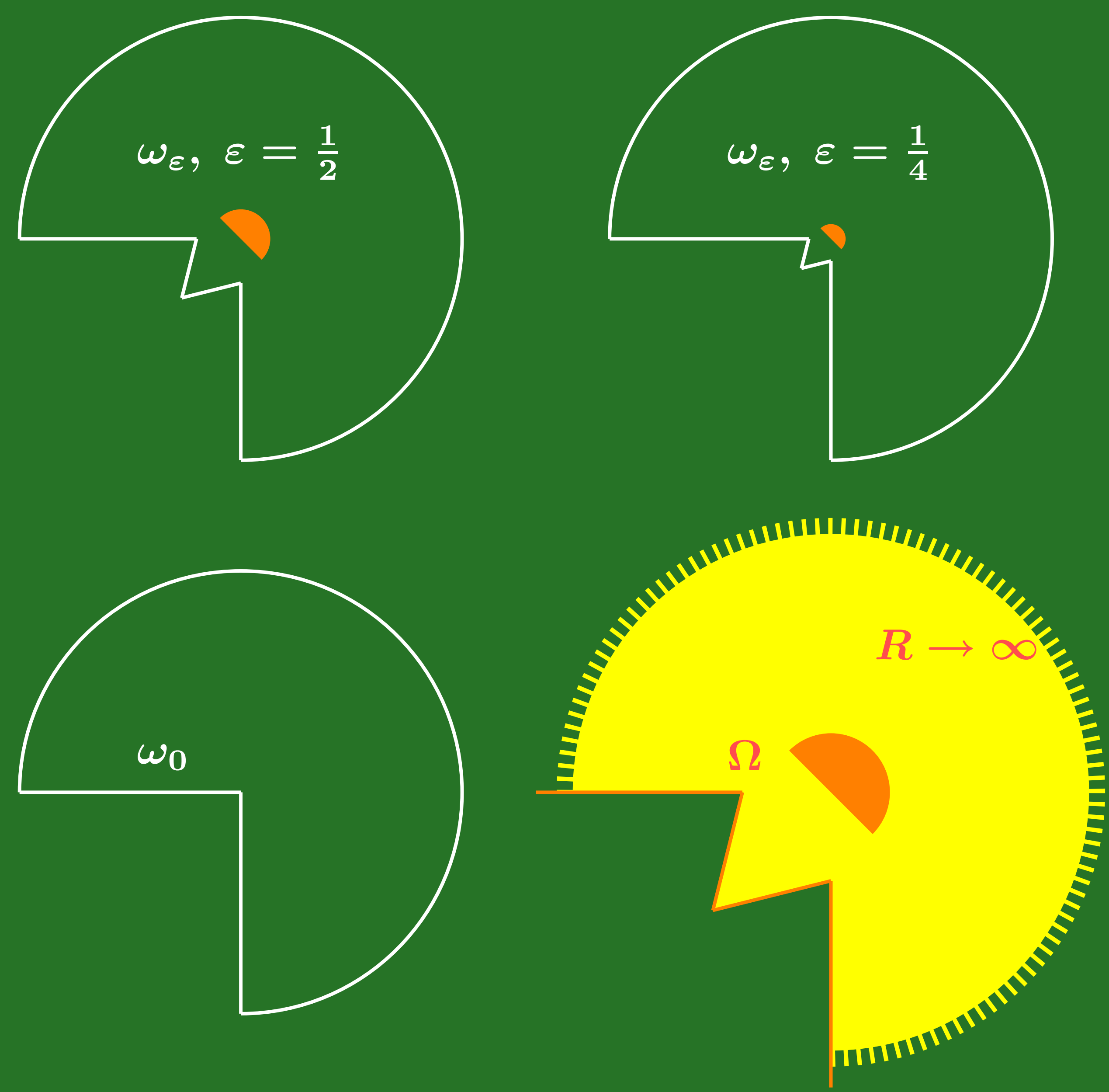
$$+\ \mathcal{O}(\varepsilon^{(K+1)\lambda/2}) \qquad \text{as} \quad \varepsilon \longrightarrow 0.$$

Translation table:

$$u^{k\lambda}(x) = v^{k\lambda}(x) + \chi(x) \sum_{p=1}^{k-1} A_p^{k-p}\, r^{-p\lambda} \varphi_p(\theta)$$

$$U^{k\lambda}(X) = V^{k\lambda}(X) + \psi(X) \sum_{p=1}^{k} a_p^{k-p}\, R^{p\lambda} \varphi_p(\theta).$$

The perturbed domain for two values of ε, the limit domain, the profile domain.



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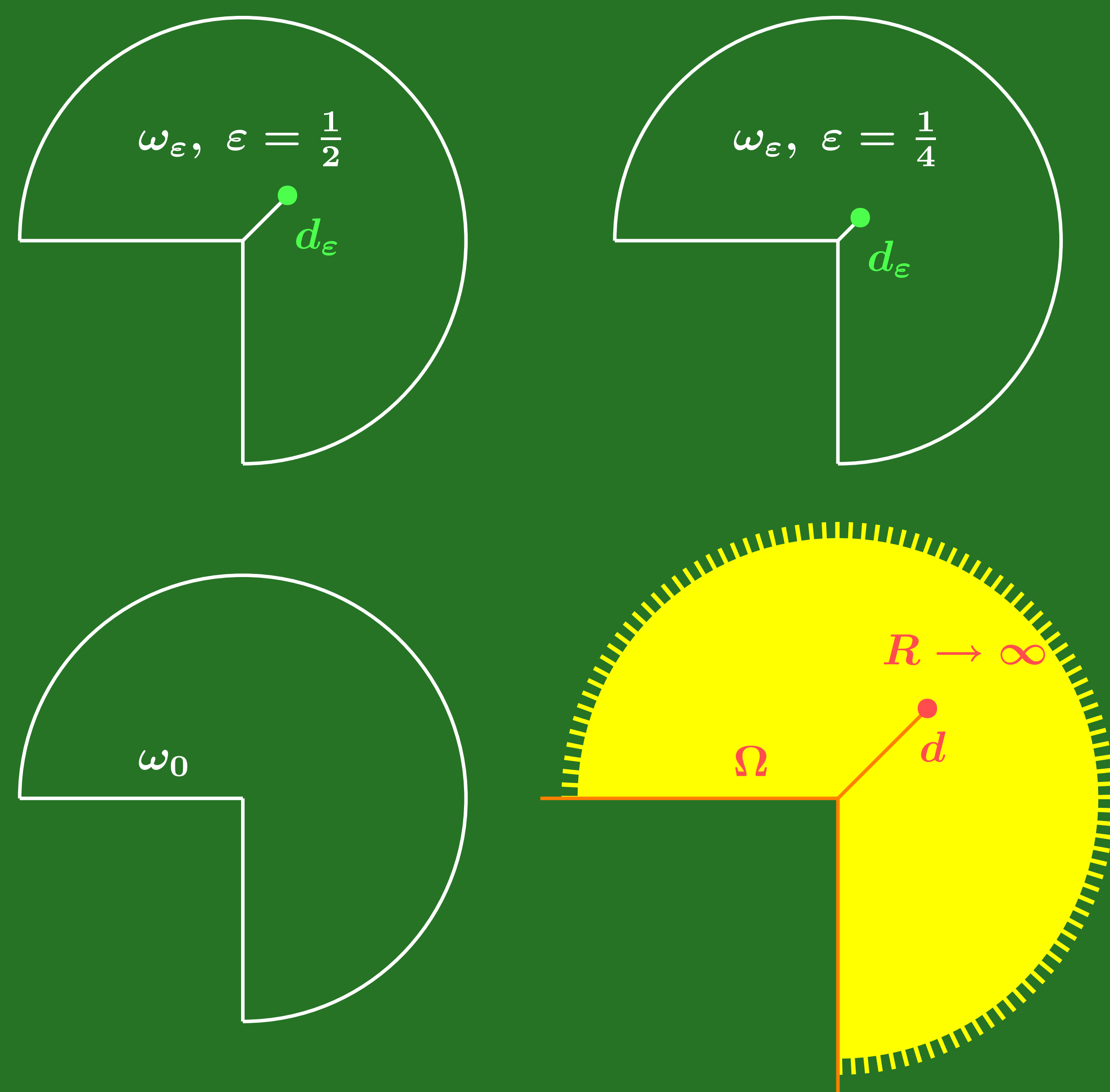
The perturbed domain for two values of ε, the limit domain, the profile domain.



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To each crack tip d of Ω corresponds a corner d_ε of ω_ε.

Let (R_d, Θ_d) the polar coordinates at d in Ω.

The polar coordinates at d_ε in ω_ε are

$$\rho_\varepsilon := \varepsilon\, R_d \quad \text{and} \quad \vartheta := \Theta_d.$$

The crack singularity of the solution u_ε

$$b_\varepsilon\, \rho_\varepsilon^{\frac{1}{2}} \sin \frac{\vartheta}{2}.$$

What is the behavior of b_ε as $\varepsilon \longrightarrow 0$?

The first profiles U^λ and V^λ have the same crack singularity at d:

$$\gamma R_d^{\frac{1}{2}} \sin \frac{\Theta_d}{2} \quad \text{with} \quad \gamma = a_1^0 \gamma^1.$$

But, near d_ε :

$$u_\varepsilon = \varepsilon^\lambda V^\lambda\Big(\frac{x}{\varepsilon}\Big) + \text{higher order terms}$$

Therefore

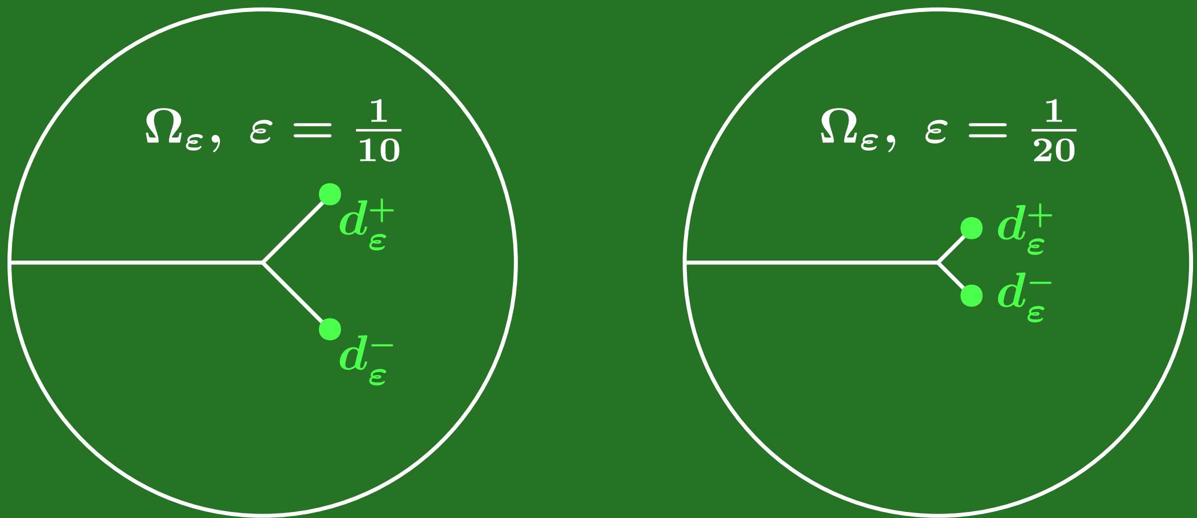
$$\boxed{b_\varepsilon = a_1^0 \gamma^1\, \varepsilon^\lambda \varepsilon^{-\frac{1}{2}}}$$

Example: $\lambda = \frac{2}{3}$. Hence: $b_\varepsilon = \mathcal{O}(\varepsilon^{\frac{1}{6}})$.

Exercises: Find coefficient asymptotics at crack tips

- Problem
- Rounded corn.
- Convergence?
- Starting
- Variational
- Multiscale
- Matching
- Matched
- Zig-zag
- Cracked
- Coefficients

EX 1. *Trouver l'asymptotique au fond des fissures.*



EX 2. *Considérer successivement les conditions de Dirichlet et de Neumann.*