On the asymptotic behavior of the discrete spectrum in buckling problems for thin plates

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Abstract

We consider the buckling problem for a family of thin plates with thickness parameter ε. This involves finding the least positive multiple $\lambda^\varepsilon_{\min}$ of the load that makes the plate *buckle*, a value that can be expressed in terms of an eigenvalue problem involving a non-compact operator. We show that under certain assumptions on the load, we have $\lambda^\varepsilon_{\min} = \mathcal{O}(\varepsilon^2)$. This guarantees that provided the plate is thin enough, this minimum value can be numerically approximated without the spectral pollution that is possible due to the presence of the non-compact operator. We provide numerical computations illustrating some of our theoretical results.

Key Words: buckling, eigenvalues, essential spectrum, thin domain, plate, shell

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1 Introduction

An important problem in engineering is the determination of the *limit of elastic stability* of a body, or more informally, the point at which the body *buckles*. Linearization of the problem leads to this limit being expressed as that critical multiple $\lambda_{\min}$ of the applied load (or, more generally, a pre-existing stress) at which the equations fail to have a unique solution. For structures such as plates and shells that are *thin* in one direction, the classical approach is now to impose Kirchhoff-type hypotheses on the displacements, to give a dimensionally reduced model. This leads to the critical multipliers being formulated as the eigenvalues $\mu = \lambda^{-1}$ of a *compact* operator $\mathbb{X}$. As a result, $\lambda_{\min} = \mu_{\max}$ is well-separated from other eigenvalues and can be easily approximated using the finite element method (see e.g. [1]).

In [12], a method that uses the full three-dimensional equations (rather than their dimensionally reduced version) has been proposed, based on an underlying model derived classically by Trefftz [13]. This allows various loads, boundary conditions and topological details (e.g. stiffeners) that might have otherwise complicated the dimensional reduction to be taken into account for the buckling analysis. The disadvantage of this formulation (which has been implemented in the hp commercial code STRESS CHECK) is that the underlying operator $\mathbb{X}$ is no longer compact. As a result, the essential spectrum, $\sigma_{\mathsf{e}}(\mathbb{X})$ of $\mathbb{X}$ no longer coincides with $\{0\}$ (as it must for compact $\mathbb{X}$) — it can potentially contain eigenvalues of infinite multiplicity, accumulation points, a continuous spectrum, etc. This can cause serious problems such as spectral pollution in the finite element approximations (see [5] and the references therein).

Let us define the *essential numerical range* W_{e} of $\mathbb{X}$ by

$$W_{\mathsf{e}} = [\min \sigma_{\mathsf{e}}(\mathbb{X}), \max \sigma_{\mathsf{e}}(\mathbb{X})]. \tag{1.1}$$

The corresponding region *free of the essential spectrum* for the buckling problem is

$$\Lambda = \{\lambda = \mu^{-1} \mid \ \mu \in \mathbb{R} \setminus W_{\mathsf{e}}\}. \tag{1.2}$$

Our goal in this paper is to show that for a model problem of a family of thin plates the eigenvalues of interest $\lambda_{\min}$ lie *inside* Λ when the plate thickness $d = 2\varepsilon$ is small enough. Then, by a result of Descloux [6], the finite element method gives pollution-free approximations of these eigenvalues (see [5]). Our proof also bounds the asymptotic behavior of the smallest three-dimensional eigenvalues λ as $\varepsilon \to 0$ in terms of the smallest eigenvalues of the two-dimensional model based on the Kirchhoff hypothesis.

The outline of our paper is as follows.

- In Section 2, we define the model family of plates and describe the buckling formulation under consideration. We present a result from [5] that shows $\operatorname{diam}(\Lambda) \geq C > 0$ for all $\varepsilon \to 0$ and we prove that $\lambda_{\min}$ is larger than $\varepsilon^2 c_0$ for a constant $c_0 > 0$.

- In sections 3 to 5, by the construction of quasi-modes we prove that under some generic assumptions on the *pre-existing stresses* in the family of plates there holds

$$\lambda^{\varepsilon}_{\min} \le \varepsilon^2 \lambda^{\mathrm{KL}}_{\min} + \mathcal{O}(\varepsilon^3), \tag{1.3}$$

  with $\lambda^{\mathrm{KL}}_{\min}$ the smallest eigenvalue of the corresponding Kirchhoff limiting problem.
- Section 6 contains the results of numerical experiments.
- Section 7 is an appendix in which we discuss the choice of the family of loads that are applied to the family of plates to make them buckle: We find that any non-zero membrane load constant through the thickness and independent of ε yields a pre-existing stress which satisfies the hypotheses leading to (1.3).

Although our results here are proved rigorously only for the special case of a plate, we expect that more complicated thin domains, such as flexural shells, would demonstrate the same types of behavior, in contrast with clamped elliptic shells where we do not expect any $\mathcal{O}(\varepsilon^2)$ eigenvalue.

2 The buckling problem for thin plates

2.a The elasticity operator

We consider a family of plates $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$ where the mid-surface ω is a fixed domain in $\mathbb{R}^2$. The boundary $\partial\omega$ will be considered smooth. We assume that the plate is made of isotropic elastic material, with Lamé constants given by $\overline{\lambda}$ and $\overline{\mu}$. Then for the displacement field $\boldsymbol{u} = \{u_i\}$ on Ω^ε (Latin indices are in $\{1,2,3\}$, while Greek ones α, β are in $\{1,2\}$, with repeated indices indicating summation), we define the linearized strain tensor $e_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. By Hooke's law, the stress tensor is then given by

$$\sigma(\boldsymbol{u}) = A e(\boldsymbol{u}),$$

where $A = A_{ijkl}$, the tensor of elastic constants of the material is given by

$$A_{ijkl} = \overline{\lambda}\delta_{ij}\delta_{kl} + \overline{\mu}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

The plate is left free on the top and bottom faces $\Gamma^\varepsilon_\pm := \omega \times \{\pm\varepsilon\}$. On the lateral edge $\Gamma^\varepsilon_0 := \partial\omega \times (-\varepsilon, \varepsilon)$, we enforce *clamped* boundary conditions, $\boldsymbol{u} = 0$. Then the space of admissible displacements is given by

$$V_\varepsilon := \left\{ \boldsymbol{v} \in H^1(\Omega^\varepsilon)^3 \mid \ \boldsymbol{v} = 0 \text{ on } \Gamma^\varepsilon_0 = \partial\omega \times (-\varepsilon, \varepsilon) \right\}.$$

The space V_ε is endowed with the norm

$$\|\boldsymbol{u}\|_{V_\varepsilon} := \Big(\sum_{i,j=1}^{3} \|\partial_j u_i\|^2_{L_2(\Omega^\varepsilon)} \Big)^{1/2}.$$

For functions $\boldsymbol{u}, \boldsymbol{v} \in V_\varepsilon$, we now define the usual bilinear form for elasticity by

$$a^\varepsilon(\boldsymbol{u},\boldsymbol{v}) = \int_{\Omega^\varepsilon} \sigma(\boldsymbol{u}) : e(\boldsymbol{v}) \,\mathrm{d}x = \int_{\Omega^\varepsilon} \left\{ \overline{\lambda} e_{pp}(\boldsymbol{u}) e_{qq}(\boldsymbol{v}) + 2\overline{\mu} e_{ij}(\boldsymbol{u}) e_{ij}(\boldsymbol{v}) \right\} \mathrm{d}x. \tag{2.1}$$

Obviously, a^ε is coercive on V_ε, though with coercivity constant dependent on ε. The following theorem holds.

Theorem 2.1 Korn Inequalities. *(i) There exists a constant $K > 0$ such that the following inequality holds uniformly $\forall \varepsilon \in (0,1]$, $\forall \boldsymbol{u} \in V_\varepsilon$:*

$$\|\boldsymbol{u}\|^2_{V_\varepsilon} \le K\big(a^\varepsilon(\boldsymbol{u},\boldsymbol{u}) + \varepsilon^{-2}\|\boldsymbol{u}\|^2_{L_2(\Omega^\varepsilon)}\big). \tag{2.2}$$

(ii) There exists a constant $K' > 0$ such that the following inequality holds uniformly $\forall \varepsilon \in (0,1]$, $\forall \boldsymbol{u} \in V_\varepsilon$:

$$\|\boldsymbol{u}\|^2_{V_\varepsilon} \le K' \varepsilon^{-2} a^\varepsilon(\boldsymbol{u},\boldsymbol{u}). \tag{2.3}$$

Proof: The result of (i) is stated in [5, Lemma 5.1] and proved there.
The proof of (ii) follows from a scaling argument: On $\Omega = \omega \times (-1,1)$ with coordinates (x_1, x_2, X_3) and $\mathrm{X}_3 = x_3/\varepsilon$, let $\widetilde{\boldsymbol{u}}$ be defined as

$$\widetilde{u}_\alpha(x_1,x_2,\mathrm{X}_3) = u_\alpha(x_1,x_2,x_3) \quad \text{and} \quad \widetilde{u}_3(x_1,x_2,\mathrm{X}_3) = \varepsilon u_3(x_1,x_2,x_3). \tag{2.4}$$

Denoting the derivatives with respect to x_1, x_2, X_3 in Ω by $\widetilde{\partial}_1$, $\widetilde{\partial}_2$, $\widetilde{\partial}_3$, we have

$$\varepsilon^{-1}\|\boldsymbol{u}\|^2_{V_\varepsilon} = \sum_{\alpha,\beta} \|\widetilde{\partial}_\alpha \widetilde{u}_\beta\|^2 + \varepsilon^{-2} \sum_\alpha \big(\|\widetilde{\partial}_\alpha \widetilde{u}_3\|^2 + \|\widetilde{\partial}_3 \widetilde{u}_\alpha\|^2\big) + \varepsilon^{-4}\|\widetilde{\partial}_3 \widetilde{u}_3\|^2, \tag{2.5}$$

where the norm $\|\cdot\|$ denotes the $L^2(\Omega)$ norm. Using the the positivity of the Lamé material matrix and the scaling argument, we obtain

$$\varepsilon^{-1} a^\varepsilon(\boldsymbol{u},\boldsymbol{u}) \ge C \sum_{\alpha,\beta} \|e_{\alpha\beta}(\widetilde{\boldsymbol{u}})\|^2 + \varepsilon^{-2} \sum_\alpha \|e_{\alpha 3}(\widetilde{\boldsymbol{u}})\|^2 + \varepsilon^{-4}\|\widetilde{\partial}_3 \widetilde{u}_3\|^2 \tag{2.6}$$

The Korn inequality on Ω gives the estimate

$$\sum_{i,j} \|e_{ij}(\widetilde{\boldsymbol{u}})\|^2 \ge C' \sum_{i,j} \|\widetilde{\partial}_i \widetilde{u}_j\|^2 \tag{2.7}$$

for some positive constant C'. Noting that the term $\varepsilon^{-4}\|\widetilde{\partial}_3 \widetilde{u}_3\|^2$ is present in both (2.5) and (2.6), we can combine the three previous inequalities to obtain (2.3). $\square$

2.b Pre-existing stresses

Suppose now that we are given a family $\{\sigma_*^\varepsilon\}$ of pre-existing stress states in the body, such that σ_*^ε satisfies the equations of equilibrium on Ω^ε. In applications, $\{\sigma_*^\varepsilon\}$ might be a sequence of *residual* stresses created e.g. in the manufacture of the plate, but in our context, it is more convenient to assume it arises from a sequence of loadings, as is discussed in the appendix (see also the examples in Section 6). Then the buckling problem is to find the smallest multiple $\lambda_{\min}^\varepsilon$ of σ_*^ε (called the 'pre-buckling stress') for which the plate buckles. As shown in [12, 5] this can be formulated as the minimum positive spectral value $\lambda_{\min}^\varepsilon$ of the problem:

$$\textit{Find}\quad (\boldsymbol{u},\lambda)\in V_\varepsilon\times\mathbb{R}\quad \textit{satisfying:}\quad \forall \boldsymbol{v}\in V_\varepsilon,\quad a^\varepsilon(\boldsymbol{u},\boldsymbol{v})=\lambda\, b^\varepsilon(\boldsymbol{u},\boldsymbol{v}), \tag{2.8}$$

where the term $b^\varepsilon(\cdot,\cdot)$ in (2.8) represents the work done by σ_*^ε due to the product terms of the Green-Lagrange strain tensor:

$$b^\varepsilon(\boldsymbol{u},\boldsymbol{v})=\int_{\Omega^\varepsilon}(\sigma_*^\varepsilon)_{ij}\,\partial_i u_m\,\partial_j v_m\,\mathrm{d}x. \tag{2.9}$$

Unless otherwise stated, σ_*^ε will be bounded on Ω^ε, uniformly for $\varepsilon\in[0,1]$. Then it is clear that $b^\varepsilon(\cdot,\cdot)$ is a uniformly bounded bilinear form on V_ε for all $\varepsilon\in(0,1]$:

$$|b^\varepsilon(\boldsymbol{u},\boldsymbol{v})|\le 9M_0\|\boldsymbol{u}\|_{V_\varepsilon}\|\boldsymbol{v}\|_{V_\varepsilon}, \tag{2.10}$$

where

$$M_0=\max_{\varepsilon,x,i,j}|(\sigma_*^\varepsilon)_{ij}(x)|. \tag{2.11}$$

Remark 2.2 Suppose σ_*^ε is determined from a given loading on Ω^ε (as in the cases discussed in Section 6 and the appendix). Then we can expect σ_*^ε to be singular at the edges $(x,\pm\varepsilon), x\in\partial\omega$. However, these infinite values are normally discarded in the engineering analysis (because of the presence of plastic zones). That is the reason why we will impose an assumption on the family $\{\sigma_*^\varepsilon\}$ (Hypothesis 3.1 ahead) which ensures that these pre-existing stresses have no boundary layer present. (For actual stresses determined by a given loading, this amounts to taking only the asymptotic contribution to them into account, regardless of boundary layer effects.) Then (2.11) is satisfied with $M_0<\infty$.

Let us define the operator $\mathbb{X}^\varepsilon: V_\varepsilon\to V_\varepsilon$ by
For any $\boldsymbol{w}\in V_\varepsilon$, $\mathbb{X}^\varepsilon\boldsymbol{w}\in V_\varepsilon$ is the unique solution of

$$a^\varepsilon(\mathbb{X}^\varepsilon\boldsymbol{w},\boldsymbol{v})=b^\varepsilon(\boldsymbol{w},\boldsymbol{v})\quad\forall\boldsymbol{v}\in V_\varepsilon. \tag{2.12}$$

Then (2.8) is simply the variational formulation for finding the eigenvalues $\mu=\lambda^{-1}$ (and corresponding eigenvectors) of $\mathbb{X}^\varepsilon$. We note from the definition of a^ε and b^ε that for any $\varepsilon>0$,

$\mathbb{X}^\varepsilon$ is not compact as an operator from V_ε into itself, so that its spectrum $\sigma(\mathbb{X}^\varepsilon)$ may have other components besides isolated eigenvalues of finite multiplicity.

Let us define for any $\mu \in \mathbb{C}$, the operator

$$\mathbb{X}^\varepsilon_\mu = \mu I - \mathbb{X}^\varepsilon.$$

Then we define the following components of the spectrum as in [8]:

(1) *Discrete spectrum*

$$\sigma_{\mathsf{d}}(\mathbb{X}^\varepsilon) = \{\mu \in \mathbb{C},\ \ker \mathbb{X}^\varepsilon_\mu \neq \{0\} \text{ and } \mathbb{X}^\varepsilon_\mu \text{ is a Fredholm operator from } V_\varepsilon \text{ into } V_\varepsilon\}.$$

(2) *Essential spectrum*

$$\sigma_{\mathsf{e}}(\mathbb{X}^\varepsilon) = \{\mu \in \mathbb{C},\ \mathbb{X}^\varepsilon_\mu \text{ is not a Fredholm operator from } V_\varepsilon \text{ into } V_\varepsilon\}.$$

Then we have the following result [15] (see also [5, Theorem 3.3])

Theorem 2.3 $\sigma(\mathbb{X}^\varepsilon) \subset \mathbb{R}$ *and* $\sigma(\mathbb{X}^\varepsilon) = \sigma_{\mathsf{e}}(\mathbb{X}^\varepsilon) \cup \sigma_{\mathsf{d}}(\mathbb{X}^\varepsilon)$.

We now quote a result proved in [5, Theorem 5.2] that provides an estimate of the essential spectrum. This result relies on the Korn inequality (2.2) given in Theorem 2.1.

Theorem 2.4 *Let* K, M_0 *be the constants in the uniform Korn's inequality* (2.2) *and bound for the pre-buckling stresses* σ^ε_* (2.11) *respectively. Then* $\forall \varepsilon \in (0, 1]$,

$$\sigma_{\mathsf{e}}(\mathbb{X}^\varepsilon) \subset [-9KM_0, 9KM_0].$$

By the results of Descloux [6], spectral pollution will only occur for μ in the above interval. In other words, any $\lambda = \mu^{-1}$ belonging to the interval

$$\Lambda = \Big(-\frac{1}{9KM_0}\,,\ \frac{1}{9KM_0} \Big) \tag{2.13}$$

can be approximated without pollution by the finite element method (see [5] for details).

The other Korn inequality (2.3) yields a lower bound on $\lambda^\varepsilon_{\min}$:

Theorem 2.5 *Let* K', M_0 *be the constants in the second uniform Korn's inequality* (2.3) *and bound for the pre-buckling stresses* σ^ε_* (2.11) *respectively. Then* $\forall \varepsilon \in (0, 1]$,

$$\lambda^\varepsilon_{\min} \geq \frac{\varepsilon^2}{9K'M_0}\,.$$

Proof: We have $\lambda^\varepsilon_{\min} = (\mu^\varepsilon_{\max})^{-1}$ and by the mini-max principle based on the Rayleigh-Riesz quotient we have

$$\mu^\varepsilon_{\max} = \max_{\boldsymbol{u}\in V_\varepsilon} \frac{b^\varepsilon(\boldsymbol{u},\boldsymbol{u})}{a^\varepsilon(\boldsymbol{u},\boldsymbol{u})}\ .$$

Inequalities (2.3) and (2.11) then give $\mu^\varepsilon_{\max} \le 9K' M_0\, \varepsilon^{-2}$, hence the result. □

The results in the next sections show that under some quite general assumptions on the pre-existing stresses there holds $\lambda^\varepsilon_{\min} \le \varepsilon^2 \lambda^{\mathrm{KL}}_{\min} + \mathcal{O}(\varepsilon^3)$ with $\lambda^{\mathrm{KL}}_{\min}$ the smallest positive eigenvalue of a similar 2D problem. Since $\operatorname{diam}(\Lambda) = \mathcal{O}(1)$, independently of ε, we can be assured $\lambda^\varepsilon_{\min} \in \Lambda$ provided ε is small enough and will hence be accurately approximated.

3 An introduction to asymptotic analysis

A natural way to start the analysis is as follows: Scaling the domains Ω^ε in the x_3 direction, we get the ε-independent domain $\Omega = \omega \times (-1,1)$. The coordinates in Ω naturally split into $(x_\top, \mathrm{X}_3)$ where $x_\top$ denotes the in-plane variables (x_1,x_2) and X_3 the stretched transverse variable x_3/ε. Our assumption on the pre-existing stresses is the following:

Hypothesis 3.1 (i) *There exist smooth real functions $\overline{\sigma}_{ij}$ on $\overline{\Omega}$ such that for all $\varepsilon > 0$, the pre-existing stress σ^ε_* is given by*

$$\left\{\begin{array}{rcll} (\sigma^\varepsilon_*)_{\alpha\beta}(x) &=& \overline{\sigma}_{\alpha\beta}(x_\top,\mathrm{X}_3), & \alpha,\beta=1,2\\ (\sigma^\varepsilon_*)_{\alpha 3}(x) &=& \varepsilon\,\overline{\sigma}_{\alpha 3}(x_\top,\mathrm{X}_3), & \alpha=1,2\\ (\sigma^\varepsilon_*)_{33}(x) &=& \varepsilon^2\overline{\sigma}_{33}(x_\top,\mathrm{X}_3). & \end{array}\right. \tag{3.1}$$

(ii) *The coefficients $p^0_{\alpha\beta}$ defined as*

$$p^0_{\alpha\beta}(x_\top) = \frac{1}{2}\int_{-1}^{1} \overline{\sigma}_{\alpha\beta}(x_\top,\mathrm{X}_3)\ \mathrm{d}\mathrm{X}_3 \tag{3.2}$$

satisfy the non-negativity property: There exists $\zeta \in \mathcal{C}^\infty_0(\overline{\omega})$ such that

$$\int_\omega p^0_{\alpha\beta}(x_\top)\partial_\alpha\zeta(x_\top)\,\partial_\beta\zeta(x_\top)\ \mathrm{d}x_\top > 0. \tag{3.3}$$

We note that $\overline{\sigma}_{ij} = \overline{\sigma}_{ji}$.

Remark 3.2 The way ε scales in (3.1) ensures that there will be no boundary layers present in σ^ε_* (see Remark 2.2). The second hypothesis guarantees that there will be positive eigenvalues present (see Remark 5.2). Note that the weaker assumption

$$p^0_{\alpha\beta}(x_\top) \not\equiv 0 \text{ for some } \alpha,\beta \tag{3.4}$$

would already guarantee that (3.3) is non-zero, which in turn would assure the existence of eigenvalues that might be positive or negative. As specified in [12], the engineering problem requires one to find *positive* eigenvalues, which is why we need the stronger assumption (3.3). Except for some specialized cases, (3.3) can be generally expected to be true whenever the weaker condition (3.4) holds.

We postpone to section 7 the discussion of the kinds of loads under which the above assumption will hold.

The aim of the next sections is to prove that under Hypothesis 3.1, the least positive buckling eigenvalues belong to the interval Λ and have a power series expansion in ε. A powerful tool for this is the construction of quasi-modes (approximate eigen-pairs). The validation of this method requires, however, that the eigen-pairs we want to approximate are the eigen-pairs of a *self-adjoint operator*. Since this is not the case for the operator $\mathbb{X}^\varepsilon$, we begin by constructing a self-adjoint operator $\mathbb{Y}^\varepsilon$ with the same spectrum as $\mathbb{X}^\varepsilon$.

3.a Self-adjoint equivalent operator

The elasticity operator $\mathbb{A}^\varepsilon$ defined by $\mathbb{A}^\varepsilon(\boldsymbol{u}) : \boldsymbol{v} \mapsto a^\varepsilon(\boldsymbol{u}, \boldsymbol{v})$ has a fully discrete spectrum. Let its eigen-pair basis be denoted as $(\Lambda_\ell^\varepsilon, \boldsymbol{w}_\ell^\varepsilon)_{\ell\geq 1}$. There holds:

$$\mathbb{A}^\varepsilon(\boldsymbol{u}) = \sum_\ell \Lambda_\ell^\varepsilon \, \langle \boldsymbol{u}, \boldsymbol{w}_\ell^\varepsilon \rangle \, \boldsymbol{w}_\ell^\varepsilon$$

(here $\langle \boldsymbol{u}, \boldsymbol{w} \rangle$ denotes the scalar product in $L^2(\Omega^\varepsilon)$). We define the operator $\mathbb{Q}^\varepsilon$ as

$$\mathbb{Q}^\varepsilon(\boldsymbol{u}) = \sum_\ell (\Lambda_\ell^\varepsilon)^{-1/2} \langle \boldsymbol{u}, \boldsymbol{w}_\ell^\varepsilon \rangle \, \boldsymbol{w}_\ell^\varepsilon \, .$$

In fact $\mathbb{Q}^\varepsilon = (\mathbb{A}^\varepsilon)^{-1/2}$ and there holds

$$a^\varepsilon\big(\mathbb{Q}^\varepsilon(\boldsymbol{u}), \mathbb{Q}^\varepsilon(\boldsymbol{u})\big) = (\boldsymbol{u}, \boldsymbol{u}). \tag{3.5}$$

The operator $\mathbb{Q}^\varepsilon$ is bounded from $H_\varepsilon := L^2(\Omega^\varepsilon)$ into V_ε and from V_ε' into H_ε. The Korn inequality (2.3) together with identity (3.5) gives that

$$|||\mathbb{Q}^\varepsilon|||_{H_\varepsilon \to V_\varepsilon} \leq C\varepsilon^{-1} \quad \text{and} \quad |||\mathbb{Q}^\varepsilon|||_{V_\varepsilon' \to H_\varepsilon} \leq C\varepsilon^{-1}. \tag{3.6}$$

With $\mathbb{B}^\varepsilon(\boldsymbol{u}) : \boldsymbol{v} \mapsto b^\varepsilon(\boldsymbol{u}, \boldsymbol{v})$, continuous from V_ε into V_ε', we define

$$\mathbb{Y}^\varepsilon = \mathbb{Q}^\varepsilon \mathbb{B}^\varepsilon \mathbb{Q}^\varepsilon : H_\varepsilon \to H_\varepsilon. \tag{3.7}$$

Then it is clear that there holds:

Theorem 3.3 *The operator $\mathbb{Y}^\varepsilon$ is self-adjoint and bounded from H_ε into itself and its spectrum coincides with the spectrum of $\mathbb{X}^\varepsilon$. Thus, the inverse of its discrete spectrum*

$$\sigma_{\mathsf{d}}^{-1}(\mathbb{Y}^\varepsilon) := \{\lambda \in \mathbb{R} \mid \lambda = \mu^{-1}, \quad \mu \in \sigma_{\mathsf{d}}(\mathbb{Y}^\varepsilon)\}$$

gives back the buckling eigenvalues.

With $h > 0$, an h quasi-mode $(\mu, \boldsymbol{y})$ for $\mathbb{Y}^\varepsilon$ is a pair with real μ and non-zero $\boldsymbol{w}$ such that

$$\|\mathbb{Y}^\varepsilon \boldsymbol{y} - \mu \boldsymbol{y}\|_{H_\varepsilon} \leq h \|\boldsymbol{y}\|_{H_\varepsilon}. \tag{3.8}$$

Using spectral projectors according to [11], we can extend the result of [14, Lemmas 12 & 13] and obtain

Lemma 3.4 *Let $(\mu, \boldsymbol{y})$ be an h quasi-mode for $\mathbb{Y}^\varepsilon$. Then*

$$\operatorname{dist}\big(\mu, \sigma(\mathbb{Y}^\varepsilon)\big) \leq h. \tag{3.9}$$

Let us assume that $\sigma(\mathbb{Y}^\varepsilon) \cap [\mu - h, \mu + h]$ is contained in the discrete spectrum of $\mathbb{Y}^\varepsilon$ and let $E_{\mu,h}$ be the sum of corresponding eigenspaces. Then there exists $\boldsymbol{u} \in E_{\mu,h}$ such that

$$\|\boldsymbol{y} - \boldsymbol{u}\|_{H_\varepsilon} \leq \frac{h}{M} \|\boldsymbol{y}\|_{H_\varepsilon}, \tag{3.10}$$

where M is the distance of $\sigma(\mathbb{Y}^\varepsilon) \cap [\mu - h, \mu + h]$ to the remaining part of the spectrum, i.e. to $\sigma(\mathbb{Y}^\varepsilon) \cap (\mathbb{R} \setminus [\mu - h, \mu + h])$.

We are going to construct quasi-modes for $\mathbb{Y}^\varepsilon$ by an asymptotic method adapted from [3]. It is based on a scaled boundary value formulation of the buckling problem.

3.b Scaled boundary value formulation

Under Hypothesis 3.1, we consider problem (2.8): Find $(u^\varepsilon, \lambda^\varepsilon) \in V_\varepsilon \times \mathbb{R}$ satisfying

$$\forall \boldsymbol{v} \in V_\varepsilon, \quad a^\varepsilon(\boldsymbol{u}^\varepsilon, \boldsymbol{v}) = \lambda^\varepsilon\, b^\varepsilon(\boldsymbol{u}, \boldsymbol{v}).$$

We scale the unknowns, *cf* (2.4)

$$u_\alpha(\varepsilon)(x_\top, \mathrm{x}_3) = u^\varepsilon_\alpha(x) \quad \text{and} \quad u_3(\varepsilon)(x_\top, \mathrm{x}_3) = \varepsilon u^\varepsilon_3(x). \tag{3.11}$$

Then the variational space V_ε is transformed into

$$V := \big\{\boldsymbol{v} \in H^1(\Omega)^3 \mid \ \boldsymbol{v} = 0 \text{ on } \Gamma_0 = \partial\omega \times (-1, 1)\big\}$$

and the above eigenvalue problem becomes

$$\forall \boldsymbol{v} \in V, \quad a(\varepsilon)(\boldsymbol{u}(\varepsilon), \boldsymbol{v}) = \lambda^\varepsilon\, b(\varepsilon)(\boldsymbol{u}(\varepsilon), \boldsymbol{v}), \tag{3.12}$$

where

$$a(\varepsilon)(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \big\{\overline{\lambda}\kappa_{pp}(\varepsilon)(\boldsymbol{u})\kappa_{qq}(\varepsilon)(\boldsymbol{v}) + 2\overline{\mu}\kappa_{ij}(\varepsilon)(\boldsymbol{u})\kappa_{ij}(\varepsilon)(\boldsymbol{v})\big\}\,\mathrm{d}x, \tag{3.13}$$

$$b(\varepsilon)(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \big\{\overline{\sigma}_{ij}\,\partial_i u_\alpha\,\partial_j v_\alpha + \varepsilon^{-2}\overline{\sigma}_{ij}\,\partial_i u_3\,\partial_j v_3\big\}\,\mathrm{d}x, \tag{3.14}$$

and where the scaled strain $\kappa_{ij}(\varepsilon)$ is defined as

$$\kappa_{\alpha\beta}(\varepsilon) = e_{\alpha\beta}, \qquad \kappa_{\alpha 3}(\varepsilon) = \varepsilon^{-1} e_{\alpha 3}, \qquad \kappa_{33}(\varepsilon) = \varepsilon^{-2} e_{33}. \tag{3.15}$$

We integrate by parts and obtain the following boundary value problem on Ω

$$\begin{aligned} \mathsf{A}(\varepsilon)\boldsymbol{u}(\varepsilon) &= \lambda^{\varepsilon}\,\mathsf{B}(\varepsilon)\boldsymbol{u}(\varepsilon) \quad \text{in} \quad \Omega, &(3.16)\\ \mathsf{T}(\varepsilon)\boldsymbol{u}(\varepsilon) &= \lambda^{\varepsilon}\,\mathsf{S}(\varepsilon)\boldsymbol{u}(\varepsilon) \quad \text{on} \quad \Gamma_{\pm}, &(3.17)\\ \boldsymbol{u}(\varepsilon) &= 0 \quad \text{on} \quad \Gamma_0, &(3.18) \end{aligned}$$

where the interior operators $\mathsf{A}(\varepsilon)$, $\mathsf{B}(\varepsilon)$ and the traction operators $\mathsf{T}(\varepsilon)$, $\mathsf{S}(\varepsilon)$ are defined as follows

$$\begin{aligned} \mathsf{A}(\varepsilon) &= \mathsf{A}^0 + \varepsilon^2 \mathsf{A}^2, \quad \text{with} \\ & \mathsf{A}^0 = \begin{pmatrix} 2\overline{\mu}\partial_3 e_{13}(\boldsymbol{u}) + \overline{\lambda}\partial_{13}u_3 \\ 2\overline{\mu}\partial_3 e_{23}(\boldsymbol{u}) + \overline{\lambda}\partial_{23}u_3 \\ (\overline{\lambda}+2\overline{\mu})\partial_{33}u_3 \end{pmatrix}, \quad \mathsf{A}^2 = \begin{pmatrix} (\overline{\lambda}+\overline{\mu})\partial_1 \operatorname{div}_{\top} \boldsymbol{u}_{\top} + \overline{\mu}\Delta_{\top} u_1 \\ (\overline{\lambda}+\overline{\mu})\partial_2 \operatorname{div}_{\top} \boldsymbol{u}_{\top} + \overline{\mu}\Delta_{\top} u_2 \\ \overline{\lambda}\partial_3 \operatorname{div}_{\top} \boldsymbol{u}_{\top} + 2\overline{\mu}\partial_\beta e_{\beta 3}(\boldsymbol{u}) \end{pmatrix} \end{aligned} \tag{3.19}$$

$$\mathsf{B}(\varepsilon) = \varepsilon^2 \mathsf{B} \quad \text{with} \quad (\mathsf{B}\boldsymbol{u})_k = \partial_i(\overline{\sigma}_{ij}\partial_j u_k), \tag{3.20}$$

$$\begin{aligned} \mathsf{T}(\varepsilon) &= \mathsf{T}^0 + \varepsilon^2 \mathsf{T}^2, \quad \text{with} \\ & \mathsf{T}^0 = \begin{pmatrix} 2\overline{\mu} e_{13}(\boldsymbol{u}) \\ 2\overline{\mu} e_{23}(\boldsymbol{u}) \\ (\overline{\lambda}+2\overline{\mu})\partial_3 u_3 \end{pmatrix}, \quad \mathsf{T}^2 = \begin{pmatrix} 0 \\ 0 \\ \overline{\lambda}\operatorname{div}_{\top} \boldsymbol{u}_{\top} \end{pmatrix} \end{aligned} \tag{3.21}$$

$$\mathsf{S}(\varepsilon) = \varepsilon^2 \mathsf{S} \quad \text{with} \quad (\mathsf{S}\boldsymbol{u})_k = \overline{\sigma}_{3j}\partial_j u_k. \tag{3.22}$$

In (3.19) and (3.21), $\boldsymbol{u}_{\top} = (u_1, u_2)$ are the in-plane components, $\operatorname{div}_{\top} \boldsymbol{u}_{\top} = \partial_1 u_1 + \partial_2 u_2$ and $\Delta_{\top} = \partial_1^2 + \partial_2^2$.

Our analysis is organized in two main steps:

(i) The construction of quasi-modes as power series solutions of the boundary value problem (3.16)-(3.18).

(ii) The identification of all smallest eigenvalues of problem (3.16)-(3.18) with quasi-mode expansions.

4 Buckling quasi-modes: An outer expansion

In a similar way as [4, 3], the construction of quasi-modes is itself split into two steps:

(a) The solution of the boundary value problem (3.16)-(3.17) (without the lateral Dirichlet boundary condition) by the construction of *power series expansions*:

$$\begin{aligned} \lambda[\varepsilon] &= \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 \dots &(4.1)\\ \boldsymbol{u}[\varepsilon] &= \boldsymbol{u}^0 + \varepsilon\boldsymbol{u}^1 + \varepsilon^2\boldsymbol{u}^2 + \dots &(4.2) \end{aligned}$$

with $\lambda^\varepsilon \sim \varepsilon^2\lambda[\varepsilon]$ and $\boldsymbol{u}(\varepsilon) \sim \boldsymbol{u}[\varepsilon]$. Note that we start the expansion of λ^ε with the power ε^2 because of Theorem 2.5 according to which we cannot find eigenvalues smaller than $\mathcal{O}(\varepsilon^2)$. Step *(a)* is referred to as *outer expansion*.

(b) The solution of the whole problem (3.16)-(3.18) requires the introduction of an *inner expansion* including boundary layer terms.

4.a Formal series solution for the outer expansion

As in [7], step *(a)* consists of solving (3.16)-(3.17) in the sense of formal series:

$$\left\{\begin{array}{rcll} (\mathsf{A}^0+\varepsilon^2\mathsf{A}^2)\boldsymbol{u}[\varepsilon] &=& \varepsilon^4\lambda[\varepsilon]\,\mathsf{B}\boldsymbol{u}[\varepsilon] & \text{in}\quad \Omega,\\ (\mathsf{T}^0+\varepsilon^2\mathsf{T}^2)\boldsymbol{u}[\varepsilon] &=& \varepsilon^4\lambda[\varepsilon]\,\mathsf{S}\boldsymbol{u}[\varepsilon] & \text{on}\quad \Gamma_\pm. \end{array}\right. \tag{4.3}$$

Equating the terms with the same power of ε in front, we find successively for all $\ell = 0, 1, \dots$ (with the convention that $\boldsymbol{u}^{-1} = \boldsymbol{u}^{-2} = 0$)

$$\left\{\begin{array}{rcll} \mathsf{A}^0\boldsymbol{u}^\ell+\mathsf{A}^2\boldsymbol{u}^{\ell-2} &=& \sum_{k=0}^{\ell-4}\lambda_k\,\mathsf{B}\boldsymbol{u}^{\ell-4-k} & \text{in}\quad \Omega,\\ \mathsf{T}^0\boldsymbol{u}^\ell+\mathsf{T}^2\boldsymbol{u}^{\ell-2} &=& \sum_{k=0}^{\ell-4}\lambda_k\,\mathsf{S}\boldsymbol{u}^{\ell-4-k} & \text{on}\quad \Gamma_\pm. \end{array}\right. \tag{4.4}$$

The six first problems are

$$\mathsf{A}^0\boldsymbol{u}^0 = 0 \quad [\Omega], \qquad \mathsf{T}^0\boldsymbol{u}^0 = 0 \quad [\Gamma_\pm], \tag{4.5}$$

$$\mathsf{A}^0\boldsymbol{u}^1 = 0 \quad [\Omega], \qquad \mathsf{T}^0\boldsymbol{u}^1 = 0 \quad [\Gamma_\pm], \tag{4.6}$$

$$\mathsf{A}^0\boldsymbol{u}^2+\mathsf{A}^2\boldsymbol{u}^0 = 0 \quad [\Omega], \qquad \mathsf{T}^0\boldsymbol{u}^2+\mathsf{T}^2\boldsymbol{u}^0 = 0 \quad [\Gamma_\pm], \tag{4.7}$$

$$\mathsf{A}^0\boldsymbol{u}^3+\mathsf{A}^2\boldsymbol{u}^1 = 0 \quad [\Omega], \qquad \mathsf{T}^0\boldsymbol{u}^3+\mathsf{T}^2\boldsymbol{u}^1 = 0 \quad [\Gamma_\pm], \tag{4.8}$$

$$\mathsf{A}^0\boldsymbol{u}^4+\mathsf{A}^2\boldsymbol{u}^2 = \lambda_0\mathsf{B}\boldsymbol{u}^0 \quad [\Omega], \qquad \mathsf{T}^0\boldsymbol{u}^4+\mathsf{T}^2\boldsymbol{u}^2 = \lambda_0\mathsf{S}\boldsymbol{u}^0 \quad [\Gamma_\pm], \tag{4.9}$$

$$\mathsf{A}^0\boldsymbol{u}^5+\mathsf{A}^2\boldsymbol{u}^3 = \lambda_0\mathsf{B}\boldsymbol{u}^1+\lambda_1\mathsf{B}\boldsymbol{u}^0 \quad [\Omega], \qquad \mathsf{T}^0\boldsymbol{u}^5+\mathsf{T}^2\boldsymbol{u}^3 = \lambda_0\mathsf{S}\boldsymbol{u}^1+\lambda_1\mathsf{S}\boldsymbol{u}^0 \quad [\Gamma_\pm]. \tag{4.10}$$

4.b First steps

It is well known and easy to check that the solutions $\boldsymbol{u}^0$ and $\boldsymbol{u}^1$ to (4.5) and (4.6) respectively can be any *Kirchhoff-Love displacement*, i.e.:

Lemma 4.1 *(i) Let the operator $\mathsf{U}^0 : \boldsymbol{\zeta} \mapsto \mathsf{U}^0\boldsymbol{\zeta}$ be defined from $\mathcal{C}^\infty(\overline{\omega})^3$ into $\mathcal{C}^\infty(\overline{\Omega})^3$ by*

$$\mathsf{U}^0\boldsymbol{\zeta} := (\zeta_1 - \mathrm{x}_3\partial_1\zeta_3,\ \zeta_2 - \mathrm{x}_3\partial_2\zeta_3,\ \zeta_3), \quad \boldsymbol{\zeta} = (\zeta_1,\zeta_2,\zeta_3)(x_\top). \tag{4.11}$$

(ii) Any smooth solution $\boldsymbol{u}^0$ and $\boldsymbol{u}^1$ to (4.5) *and* (4.6) *are of the form*

$$\boldsymbol{u}^0 = \mathsf{U}^0\boldsymbol{\zeta}^0 \quad \textit{and} \quad \boldsymbol{u}^1 = \mathsf{U}^0\boldsymbol{\zeta}^1, \quad \textit{with} \quad \boldsymbol{\zeta}^0,\ \boldsymbol{\zeta}^1 \in \mathcal{C}^\infty(\overline{\omega})^3.$$

For the two next equations, for a fixed $\boldsymbol{\zeta}$, we look for $\boldsymbol{v}$ such that

$$\mathsf{A}^0\boldsymbol{v} = -\mathsf{A}^2(\mathsf{U}^0\boldsymbol{\zeta}) \quad [\Omega], \qquad \mathsf{T}^0\boldsymbol{v} = -\mathsf{T}^2(\mathsf{U}^0\boldsymbol{\zeta}) \quad [\Gamma_\pm].$$

Selecting the third components of the equations in Ω and on $\Gamma_\pm$, we find for v_3 a problem of the type

$$(\overline{\lambda}+2\overline{\mu})\partial_{33}v_3 = -F_3 \quad [\Omega], \qquad (\overline{\lambda}+2\overline{\mu})\partial_3 v_3 = -G_3^\pm \quad [\Gamma_\pm], \tag{4.12}$$

with $F_3 = (\mathsf{A}^2\mathsf{U}^0\boldsymbol{\zeta})_3$ and $G_3^\pm = (\mathsf{T}^2\mathsf{U}^0\boldsymbol{\zeta})_3$. The problem (4.12) is a Neumann problem on the interval $(-1,1)$ for each fixed $x_\top \in \omega$. It can be solved if and only if the following compatibility conditions are satisfied

$$\Phi\big(F_3(x_\top,\cdot), G_3^+(x_\top), G_3^-(x_\top)\big) = 0, \qquad \forall x_\top \in \omega, \tag{4.13}$$

where for $f \in L^1(-1,1)$ and $g^\pm \in \mathbb{R}$ the compatibility form Φ is given by

$$\Phi(f, g^+, g^-) = \int_{-1}^{1} f(\mathrm{x}_3)\mathrm{dx}_3 + g^+ - g^-. \tag{4.14}$$

With the actual value of F_3 and $G_3^\pm$, the compatibility condition (4.13) is satisfied. Then there is a unique solution v_3 with zero mean value on each fiber $x_\top \times (-1,1)$. That solution v_3 is the result of the action of an operator U^2 on $\boldsymbol{\zeta}$: we write that $v_3 =: (\mathsf{U}^2\boldsymbol{\zeta})_3$, see (4.19).

In a similar way the two first components of the equations for $\boldsymbol{v}$ can be written as

$$\overline{\mu}\partial_{33}v_\alpha = -F_\alpha \quad [\Omega], \qquad \overline{\mu}\partial_3 v_\alpha = -G_\alpha^\pm \quad [\Gamma_\pm], \quad \alpha = 1,2, \tag{4.15}$$

with $F_\alpha = (\overline{\lambda}+\overline{\mu})\partial_{\alpha 3}v_3 + (\mathsf{A}^2\mathsf{U}^0\boldsymbol{\zeta})_\alpha$ and $G_\alpha^\pm = \overline{\mu}\partial_\alpha v_3 + (\mathsf{T}^2\mathsf{U}^0\boldsymbol{\zeta})_\alpha$.

Computing the corresponding compatibility form $\Phi(F_\alpha, G_\alpha^\pm)$ for $\alpha = 1,2$, we find that for all $x_\top \in \omega$

$$\begin{cases} \Phi(F_1, G_1^\pm)(x_\top) &=& 2\big(\overline{\mu}\Delta_\top\zeta_1 + (\widehat{\lambda}+\overline{\mu})\partial_1 \operatorname{div}_\top \boldsymbol{\zeta}_\top\big)(x_\top) \\ \Phi(F_2, G_2^\pm)(x_\top) &=& 2\big(\overline{\mu}\Delta_\top\zeta_2 + (\widehat{\lambda}+\overline{\mu})\partial_2 \operatorname{div}_\top \boldsymbol{\zeta}_\top\big)(x_\top). \end{cases} \tag{4.16}$$

Here $\boldsymbol{\zeta}_\top = (\zeta_1, \zeta_2)$ and $\widehat{\lambda}$ denotes the Lamé coefficient of the plane stress model:

$$\widehat{\lambda} = 2\overline{\lambda}\overline{\mu}(\overline{\lambda}+2\overline{\mu})^{-1}.$$

We denote by L_m the 2×2 matrix of the right hand sides of (4.16) divided by 2:

$$\mathsf{L}_\mathsf{m}\boldsymbol{\zeta}_\top = \begin{pmatrix} \overline{\mu}\Delta_\top\zeta_1 + (\widehat{\lambda}+\overline{\mu})\partial_1 \operatorname{div}_\top \boldsymbol{\zeta}_\top \\ \overline{\mu}\Delta_\top\zeta_2 + (\widehat{\lambda}+\overline{\mu})\partial_2 \operatorname{div}_\top \boldsymbol{\zeta}_\top \end{pmatrix}. \tag{4.17}$$

L_m is the actual plane stress elasticity operator. We find that we can solve, instead of (4.15):

$$\overline{\mu}\partial_{33}v_\alpha = -F_\alpha + (\mathsf{L}_\mathsf{m}\boldsymbol{\zeta}_\top)_\alpha \quad [\Omega], \qquad \overline{\mu}\partial_3 v_\alpha = -G_\alpha^\pm \quad [\Gamma_\pm], \quad \alpha = 1,2,$$

because this new right hand sides satisfy the compatibility condition

$$\int_{-1}^{1} \Big(F_\alpha(x_\top, \mathrm{x}_3) - (\mathsf{L}_\mathsf{m}\boldsymbol{\zeta}_\top)_\alpha(x_\top)\Big)\, \mathrm{dx}_3 - G_\alpha^+(x_\top) + G_\alpha^-(x_\top) = 0, \qquad \forall x_\top \in \omega.$$

We can then compute $v_\alpha =: (\mathsf{U}^2\zeta)_\alpha$. Explicitly calculating the operator U^2, we obtain the result, *cf* [4, Lemma 3.2]:

Lemma 4.2 *Let* $\mathbf{U}^0$ *be as defined in* (4.11).
(i) Let the operator $\mathbf{L}^0 : \boldsymbol{\zeta} \mapsto \mathbf{L}^0\boldsymbol{\zeta}$ *be defined from* $\mathcal{C}^\infty(\overline{\omega})^3$ *into* $\mathcal{C}^\infty(\overline{\omega})^3$ *by*

$$(\mathbf{L}^0\boldsymbol{\zeta})_\top = \mathbf{L}_\mathsf{m}\boldsymbol{\zeta}_\top \ \text{ cf (4.17)}, \qquad \textit{and} \quad (\mathbf{L}^0\boldsymbol{\zeta})_3 = 0. \tag{4.18}$$

(ii) Let the operator $\mathbf{U}^2 : \boldsymbol{\zeta} \mapsto \mathbf{U}^2\boldsymbol{\zeta}$ *be defined from* $\mathcal{C}^\infty(\overline{\omega})^3$ *into* $\mathcal{C}^\infty(\overline{\Omega})^3$ *by*

$$\begin{aligned} (\mathbf{U}^2\boldsymbol{\zeta})_\alpha &= q_2\,\partial_\alpha\,\mathrm{div}_\top\,\boldsymbol{\zeta}_\top + q_3\,\partial_\alpha\Delta_\top\zeta_3 \\ (\mathbf{U}^2\boldsymbol{\zeta})_3 &= q_1\ \mathrm{div}_\top\,\boldsymbol{\zeta}_\top + q_2\,\Delta_\top\zeta_3 \end{aligned} \tag{4.19}$$

with q_1, q_2, q_3 *the polynomials in the variable* x_3 *defined as*

$$q_1(\mathrm{x}_3) = -\tfrac{\widehat{\lambda}}{2\overline{\mu}}\,\mathrm{x}_3, \qquad q_2(\mathrm{x}_3) = \tfrac{\widehat{\lambda}}{4\overline{\mu}}\,\big(\mathrm{x}_3^2 - \tfrac{1}{3}\big),$$

$$q_3(\mathrm{x}_3) = \tfrac{1}{12\overline{\mu}}\,\big((\widehat{\lambda} + 4\overline{\mu})\,\mathrm{x}_3^3 - (5\widehat{\lambda} + 12\overline{\mu})\,\mathrm{x}_3\big).$$

(iii) Let $\boldsymbol{\zeta}$ *belong to* $\mathcal{C}^\infty(\overline{\omega})^3$.
Then the field $\mathbf{U}^2\boldsymbol{\zeta}$ *is the unique solution with zero mean values on each fiber* $x_\top \times (-1,1)$ *of the problem*

$$\mathbf{A}^0(\mathbf{U}^2\boldsymbol{\zeta}) + \mathbf{A}^2(\mathbf{U}^0\boldsymbol{\zeta}) = \mathbf{L}^0\boldsymbol{\zeta} \quad [\Omega], \qquad \mathbf{T}^0(\mathbf{U}^2\boldsymbol{\zeta}) + \mathbf{T}^2(\mathbf{U}^0\boldsymbol{\zeta}) = 0 \quad [\Gamma_\pm]. \tag{4.20}$$

The outcome is that the general solution of (4.5) & (4.7) is

$$\boldsymbol{u}^0 = \mathbf{U}^0\boldsymbol{\zeta}^0, \quad \boldsymbol{u}^2 = \mathbf{U}^0\boldsymbol{\zeta}^2 + \mathbf{U}^2\boldsymbol{\zeta}^0 \quad \text{for any } \boldsymbol{\zeta}^0 \text{ with } \mathbf{L}^0\boldsymbol{\zeta}^0 = 0, \tag{4.21}$$

and the general solution of (4.6) & (4.8) is

$$\boldsymbol{u}^1 = \mathbf{U}^0\boldsymbol{\zeta}^1, \quad \boldsymbol{u}^3 = \mathbf{U}^0\boldsymbol{\zeta}^3 + \mathbf{U}^2\boldsymbol{\zeta}^1 \quad \text{for any } \boldsymbol{\zeta}^1 \text{ with } \mathbf{L}^0\boldsymbol{\zeta}^1 = 0. \tag{4.22}$$

4.c Next terms

To solve the next equation (4.9), we look for an operator $\mathbf{U}^4$ such that $\boldsymbol{v} = \mathbf{U}^4\boldsymbol{\zeta}$ solves

$$\mathbf{A}^0\boldsymbol{v} = -\mathbf{A}^2(\mathbf{U}^2\boldsymbol{\zeta}) + \lambda_0\mathbf{B}\mathbf{U}^0\boldsymbol{\zeta} \quad [\Omega], \qquad \mathbf{T}^0\boldsymbol{v} = -\mathbf{T}^2(\mathbf{U}^2\boldsymbol{\zeta}) + \lambda_0\mathbf{S}\mathbf{U}^0\boldsymbol{\zeta} \quad [\Gamma_\pm].$$

For v_3 we have still a problem of the form (4.12) with, now F_3 and $G_3^\pm$ the third components of the right hand sides in the above equation. The compatibility condition (4.13) is not satisfied in general. Instead, we compute the value of $\Phi(F_3, G_3^\pm)$ and find it equal to:

$$\frac{2}{3}(\widehat{\lambda} + 2\overline{\mu})\Delta_\top^2\zeta_3 + \lambda_0\int_{-1}^{1}\partial_\alpha\overline{\sigma}_{\alpha\beta}\partial_\beta\zeta_3\ \mathrm{dx}_3. \tag{4.23}$$

We go on to find (v_1, v_2) and obtain a problem of the form (4.15). Computing $\Phi(F_\alpha, G_\alpha^\pm)$, we find

$$2c_{\overline{\lambda},\overline{\mu}}\Delta_\top\partial_\alpha\,\mathrm{div}_\top\,\boldsymbol{\zeta}_\top + \lambda_0\int_{-1}^{1}\big(\partial_\delta\overline{\sigma}_{\delta\beta}\partial_\beta\zeta_\alpha - \mathrm{x}_3\partial_\delta\overline{\sigma}_{\delta\beta}\partial_{\beta\gamma}\zeta_3 - \partial_\delta\overline{\sigma}_{\delta3}\partial_\gamma\zeta_3\big)\ \mathrm{dx}_3, \tag{4.24}$$

where $c_{\overline{\lambda},\overline{\mu}}$ is a real constant. In (4.23) appears the standard operator $(\widehat{\lambda}+2\overline{\mu})\Delta^2_\top$ which is the bending operator of thin plates for the Lamé coefficients $\overline{\lambda}$ and $\overline{\mu}$. We have also obtained in (4.23)-(4.24) different *moments of the pre-existing stresses*

$$p^0_{\alpha j}(x_\top) = \frac{1}{2}\int_{-1}^{1} \overline{\sigma}_{\alpha j}(x_\top, \mathrm{x}_3)\,\mathrm{dx}_3 \quad \text{and} \quad p^1_{\alpha\beta}(x_\top) = \frac{1}{2}\int_{-1}^{1} \mathrm{x}_3\overline{\sigma}_{\alpha\beta}(x_\top, \mathrm{x}_3)\,\mathrm{dx}_3. \tag{4.25}$$

With all of these, we obtain a statement like Lemma 4.2 which provides the operators for the solution of (4.9):

Lemma 4.3 *(i) Let the operator $\mathsf{L}^2 : \boldsymbol{\zeta} \mapsto \mathsf{L}^2\boldsymbol{\zeta}$ be defined from $\mathcal{C}^\infty(\overline{\omega})^3$ into $\mathcal{C}^\infty(\overline{\omega})^3$ by*

$$(\mathsf{L}^2\boldsymbol{\zeta})_\top = -c_{\overline{\lambda},\overline{\mu}}\Delta_\top\partial_\alpha\,\mathrm{div}_\top\,\boldsymbol{\zeta}_\top \quad \textit{and} \quad (\mathsf{L}^2\boldsymbol{\zeta})_3 = -\tfrac{1}{3}\,\mathsf{L}_\mathsf{b}\zeta_3 := -\tfrac{1}{3}\,(\widehat{\lambda}+2\overline{\mu})\Delta^2_\top\zeta_3. \tag{4.26}$$

(ii) Let the operator $\mathsf{M}^0 : \boldsymbol{\zeta} \mapsto \mathsf{M}^0\boldsymbol{\zeta}$ be defined from $\mathcal{C}^\infty(\overline{\omega})^3$ into $\mathcal{C}^\infty(\overline{\omega})^3$ by

$$\mathsf{M}^0\boldsymbol{\zeta} = -\begin{pmatrix} \mathsf{P} & \mathsf{Q}\nabla_\top \\ 0 & \mathsf{P} \end{pmatrix}\begin{pmatrix} \boldsymbol{\zeta}_\top \\ \zeta_3 \end{pmatrix} \tag{4.27}$$

with

$$\mathsf{P}\eta = -\partial_\alpha p^0_{\alpha\beta}\partial_\beta\eta \quad \textit{and} \quad \mathsf{Q}\eta = \big(\partial_\alpha p^1_{\alpha\beta}\partial_\beta + \partial_\alpha p^0_{\alpha 3}\big)\eta$$

where $p^0_{\alpha\beta}$, $p^0_{\alpha 3}$ and $p^1_{\alpha\beta}$ are defined in (4.25).
(iii) Let $\boldsymbol{\zeta}$ belong to $\mathcal{C}^\infty(\overline{\omega})^3$. Then the problem of finding $\boldsymbol{v}$ such that

$$\left\{\begin{array}{rcll} \mathsf{A}^0\boldsymbol{v} + \mathsf{A}^2(\mathsf{U}^2\zeta) - \lambda_0\mathsf{B}\mathsf{U}^0\zeta &=& \mathsf{L}^2\boldsymbol{\zeta} - \lambda_0\mathsf{M}^0\boldsymbol{\zeta} & [\Omega], \\ \mathsf{T}^0\boldsymbol{v} + \mathsf{T}^2(\mathsf{U}^2\zeta) - \lambda_0\mathsf{S}\mathsf{U}^0\zeta &=& 0 & [\Gamma_\pm]. \end{array}\right. \tag{4.28}$$

has a unique solution $\boldsymbol{v} =: \mathsf{U}^4\boldsymbol{\zeta}$ with zero mean values on each fiber $x_\top \times (-1,1)$.

As a result of all previous calculations, we obtain that the general solution of (4.5), (4.7) and (4.9) is given by (4.21) and

$$\boldsymbol{u}^4 = \mathsf{U}^0\boldsymbol{\zeta}^4 + \mathsf{U}^2\boldsymbol{\zeta}^2 + \mathsf{U}^4\boldsymbol{\zeta}^0 \quad \text{for any } \boldsymbol{\zeta}^2 \text{ with } \mathsf{L}^0\boldsymbol{\zeta}^2 + \mathsf{L}^2\boldsymbol{\zeta}^0 = \lambda_0\mathsf{M}^0\boldsymbol{\zeta}^0. \tag{4.29}$$

For the solution of the next step, we prove in the same way that there exist operators L^3 and U^5 such that the general solution of (4.5)-(4.10) is given by the conjunction of (4.21), (4.22), (4.29) and

$$\begin{aligned} \boldsymbol{u}^5 = \mathsf{U}^0\boldsymbol{\zeta}^5 + \mathsf{U}^2\boldsymbol{\zeta}^3 + \mathsf{U}^4\boldsymbol{\zeta}^1 + \mathsf{U}^5\boldsymbol{\zeta}^0 \\ \text{for any } \boldsymbol{\zeta}^3 \text{ with } \mathsf{L}^0\boldsymbol{\zeta}^3 + \mathsf{L}^2\boldsymbol{\zeta}^1 + \mathsf{L}^3\boldsymbol{\zeta}^0 = \lambda_0\mathsf{M}^0\boldsymbol{\zeta}^1 + \lambda_1\mathsf{M}^0\boldsymbol{\zeta}^0. \end{aligned} \tag{4.30}$$

4.d Operator series solutions

Following the method of [3, 7], we solve problem (4.3) in the sense of *formal series*: Solving successively equations (4.4) for each ℓ as above we find that for *all formal series* $\lambda[\varepsilon]$ given by (4.1) and for all formal series

$$\boldsymbol{\zeta}[\varepsilon] = \boldsymbol{\zeta} + \varepsilon\boldsymbol{\zeta}^1 + \varepsilon^2\boldsymbol{\zeta}^2 + \dots, \quad \boldsymbol{\zeta}^j \in \mathcal{C}^\infty(\overline{\omega})^3$$

subject to the "residual" equations

$$\mathsf{L}[\varepsilon]\boldsymbol{\zeta}[\varepsilon] = \varepsilon^2\lambda[\varepsilon]\mathsf{M}[\varepsilon]\boldsymbol{\zeta}[\varepsilon], \tag{4.31}$$

the formal series $\boldsymbol{u}[\varepsilon]$ given by

$$\boldsymbol{u}[\varepsilon] = \mathsf{U}[\varepsilon]\boldsymbol{\zeta}[\varepsilon] \tag{4.32}$$

yields *all solutions of* (4.3) (compare with (4.21), (4.22), (4.29) and (4.30)). Here $\mathsf{L}[\varepsilon] = \mathsf{L}^0 + \varepsilon\mathsf{L}^1 + \varepsilon^2\mathsf{L}^2 + \dots$ is a formal series with operator coefficients, acting from $\mathcal{C}^\infty(\overline{\omega})^3$ into itself. According to the calculations above, L^0 is given by (4.18), $\mathsf{L}^1 = 0$ and L^2 is given by (4.26). Similarly $\mathsf{M}[\varepsilon] = \mathsf{M}^0 + \varepsilon\mathsf{M}^1 + \varepsilon^2\mathsf{M}^2 + \dots$ has operator coefficients acting from $\mathcal{C}^\infty(\overline{\omega})^3$ into itself, M^0 is given by (4.27) and $\mathsf{M}^1 = 0$.

Finally, the operator series $\mathsf{U}[\varepsilon] = \mathsf{U}^0 + \varepsilon\mathsf{U}^1 + \varepsilon^2\mathsf{U}^2 + \dots$ has coefficients acting from $\mathcal{C}^\infty(\overline{\omega})^3$ into $\mathcal{C}^\infty(\overline{\Omega})^3$ with U^0 given by (4.11), $\mathsf{U}^1 = 0$ and U^2 given by (4.19).

The existence of the next operators L^k, U^k and M^k is proved as above.

5 Final construction of bucking quasi-modes

Until now, we have discarded the lateral boundary conditions and have found the general solution of the remaining equations. If we are able to find conditions on the coefficients $\boldsymbol{\zeta}^k$ and λ_k of the formal series $\lambda[\varepsilon]$ and $\boldsymbol{\zeta}[\varepsilon]$ so that the coefficients $\boldsymbol{u}^k$ of $\boldsymbol{u}[\varepsilon]$ satisfy the lateral Dirichlet conditions, the whole problem will be solved. In fact we can do this only for the first terms $\boldsymbol{u}^0$ and $\boldsymbol{u}^1$. To proceed, we have to take the boundary layer terms into account. We will also describe them with the help of formal series.

5.a Lateral boundary conditions on the outer expansion

Now we try to have the $\boldsymbol{u}^k$ satisfy the lateral Dirichlet conditions and we study the solvability of the residual equations (4.31) on the surface generator $\boldsymbol{\zeta}^k$.

We know that $\boldsymbol{u}^0$ is the Kirchhoff-Love displacement $\mathsf{U}^0\boldsymbol{\zeta}^0$ with generator $\boldsymbol{\zeta}^0$. It is clear that $\boldsymbol{u}^0 = 0$ on Γ_0 if and only if

$$\zeta_j^0 = 0 \;\text{ on }\; \partial\omega, \;\; j = 1,2,3 \quad \text{and} \quad \partial_n\zeta_3^0 = 0 \;\text{ on }\; \partial\omega. \tag{5.1}$$

In order to proceed, it is useful to distinguish between *membrane* and *bending* displacements and their corresponding surface generators. Recall that a displacement $\boldsymbol{u} = (u_1, u_2, u_3)$

on Ω is a membrane displacement if the two in-plane components u_1 and u_2 are even in the transverse variable X_3, and if the third component u_3 is odd. The displacement $\boldsymbol{u}$ is a bending displacement if, conversely, u_1 and u_2 are odd and u_3 is even in X_3.

With $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$, we denote $(\zeta_1, \zeta_2, 0)$ by $\boldsymbol{\zeta}_{\mathsf{m}}$ and $(0, 0, \zeta_3)$ by $\boldsymbol{\zeta}_{\mathsf{b}}$. We see that $\mathbf{U}^0\boldsymbol{\zeta}_{\mathsf{m}}$ is membrane whereas $\mathbf{U}^0\boldsymbol{\zeta}_{\mathsf{b}}$ is bending. Thus $\mathbf{U}^0\boldsymbol{\zeta}_{\mathsf{m}} + \mathbf{U}^0\boldsymbol{\zeta}_{\mathsf{b}} = \mathbf{U}^0\boldsymbol{\zeta}$ is the splitting of $\mathbf{U}^0\boldsymbol{\zeta}$ into its membrane and bending parts.

The first residual equation, see (4.21), is $\mathbf{L}^0\boldsymbol{\zeta}^0 = 0$. With (4.18), this means that $\mathbf{L}_{\mathsf{m}}\boldsymbol{\zeta}^0_{\top} = 0$. Taking the boundary conditions into account, we have obtained

$$\mathbf{L}_{\mathsf{m}}\boldsymbol{\zeta}^0_{\top} = 0 \quad \text{and} \quad \boldsymbol{\zeta}^0_{\top} = 0 \ \text{ on } \ \partial\omega.$$

Therefore $\boldsymbol{\zeta}^0_{\top} = 0$, which means that $\boldsymbol{\zeta}^0 = \boldsymbol{\zeta}^0_{\mathsf{b}}$.

The Dirichlet boundary conditions on $\boldsymbol{u}^1$ and the residual equation (4.22) also yield that $\boldsymbol{\zeta}^1 = \boldsymbol{\zeta}^1_{\mathsf{b}}$.

The next residual equation is given in (4.29): $\mathbf{L}^0\boldsymbol{\zeta}^2 + \mathbf{L}^2\boldsymbol{\zeta}^0 = \lambda_0\mathbf{M}^0\boldsymbol{\zeta}^0$. As we know that $\boldsymbol{\zeta}^0 = \boldsymbol{\zeta}^0_{\mathsf{b}}$ the third component of the above equation yields that (see Lemmas 4.2 and 4.3) $\frac{1}{3}\mathbf{L}_{\mathsf{b}}\zeta^0_3 = \lambda_0\mathbf{P}\zeta^0_3$. Since the operator $\mathbf{L}_{\mathsf{b}}$ is invertible from $H^2_0(\omega) \to H^{-2}(\omega)$, the equation that we have obtained on ζ^0_3 is compatible with the Dirichlet boundary condition $\zeta^0_3 \in H^2_0(\omega)$ which we have found in (5.1). Summarizing what we have obtained so far, we have:

Lemma 5.1 *The first surface generators $\boldsymbol{\zeta}^0$ and $\boldsymbol{\zeta}^1$ are of bending type, i.e. $\boldsymbol{\zeta}^0 = (0, 0, \zeta^0_3)$ and $\boldsymbol{\zeta}^1 = (0, 0, \zeta^1_3)$. The generator $\boldsymbol{\zeta}^0$ is solution of the following problem*[1]*:*

$$\tfrac{1}{3}\mathbf{L}_{\mathsf{b}}\zeta^0_3 \;=\; \lambda_0\,\mathbf{P}\zeta^0_3, \qquad \textit{with} \quad \zeta^0_3 \in H^2_0(\omega) \tag{5.2}$$

Remark 5.2 We will find positive eigenvalues λ_0 for problem (5.2) if and only if $\mathbf{P}$ is not negative definite, in other words if the mean values $p_{\alpha\beta}$ of the pre-existing stresses $\overline{\sigma}_{\alpha\beta}$ satisfy Hypothesis 3.1 (ii).

We cannot go further, because from the expression for $\mathbf{U}^2\boldsymbol{\zeta}^0_{\mathsf{b}}$ it follows that the condition $\boldsymbol{u}^2 = 0$ on Γ_0 would impose $\partial^2_n\zeta^0_3 = \partial^3_n\zeta^0_3 = 0$ on $\partial\omega$, a condition that cannot be fulfilled in general. To go further we have to introduce boundary layer profiles in our analysis.

5.b Boundary layer terms

In order to fulfill the Dirichlet boundary condition on Γ_0, we have to combine the general outer expansion found in (4.30)-(4.32) with an *inner expansion* $\boldsymbol{w}[\varepsilon] = \varepsilon\boldsymbol{w}^1 + \varepsilon^2\boldsymbol{w}^2 + \cdots$ with exponentially decreasing profiles $\boldsymbol{w}^k$, which are the *boundary layer terms* naturally involved in the solution asymptotics, see [9, 10] and [4, 3].

[1]The third component of the residual equation in (4.30) gives $\frac{1}{3}\mathbf{L}_{\mathsf{b}}\zeta^1_3 - (\mathbf{L}^3\boldsymbol{\zeta}^0_{\mathsf{b}})_3 = \lambda_0\mathbf{P}\zeta^1_3 + \lambda_1\mathbf{P}\zeta^0_3$.

For this, we use local coordinates (r, s) in a plane neighborhood of the lateral boundary $\partial\omega$. Here r denotes the distance to $\partial\omega$ and s the arclength along $\partial\omega$. The local basis at each point in $\partial\omega$ is given by the unit inner normal n and the unit tangent vector τ. We let R be the scaled distance $\mathrm{R} = r\varepsilon^{-1}$.

The boundary layer Ansatz is $\sum_{k\geq 0} \varepsilon^k \boldsymbol{w}^k(\mathrm{R}, s, \mathrm{X}_3)$ where $\mathrm{R} \mapsto \boldsymbol{w}^k(\mathrm{R}, s, \mathrm{X}_3)$ is exponentially decreasing as $\mathrm{R} \to +\infty$. Here s belongs to $\partial\omega$ and $(\mathrm{R}, \mathrm{X}_3)$ to the half-strip $\Sigma^+ = \mathbb{R}^+ \times (-1, 1)$. We need suitable functional spaces of exponentially decreasing functions. We use the notations: $\Sigma^{a,b} := (a, b) \times (-1, 1)$ and $\rho = \min\{\rho^+, \rho^-\}$, with $\rho^\pm$ the distance to the corner $(0, \pm 1)$ of Σ^+. For $\delta > 0$ let $\mathfrak{H}_\delta(\Sigma^+)$ be the space of $L^2(\Sigma^+)$ functions φ, which are smooth up to any regular point of the boundary of Σ^+ and satisfy

$$
\begin{aligned}
\forall i, j \in \mathbb{N}, \qquad & e^{\delta\mathrm{R}}\, \partial_{\mathrm{R}}^i \partial_3^j \varphi \in L^2(\Sigma^{1,\infty}) \\
\forall i, j \in \mathbb{N},\ i + j \neq 0, \qquad & \rho^{i+j-1}\, \partial_{\mathrm{R}}^i \partial_3^j \varphi \in L^2(\Sigma^{0,2}).
\end{aligned}
$$

In order to preserve the homogeneity of the elasticity system, we scale the ansatz $\boldsymbol{w}[\varepsilon]$ back, cf (3.11), that is we set $\boldsymbol{\varphi}[\varepsilon] = \varepsilon\boldsymbol{\varphi}^1 + \varepsilon^2\boldsymbol{\varphi}^2 + \ldots$ with $\boldsymbol{\varphi}_\top^k = \boldsymbol{w}_\top^k$ and $\varphi_3^k = w_3^{k+1}$ for all $k \in \mathbb{N}$. In variables $(\mathrm{R}, s, \mathrm{X}_3)$ and unknowns $\boldsymbol{\varphi} = (\varphi_\mathrm{R}, \varphi_s, \varphi_3)$ the interior and horizontal boundary operators $\mathbf{A}(\varepsilon)$ and $\mathbf{T}(\varepsilon)$ are transformed into operators whose ε-expansion yields formal series, see [3, §4]:

$$
\mathfrak{A}[\varepsilon] = \sum\nolimits_k \varepsilon^k \mathfrak{A}^k \quad \text{and} \quad \mathfrak{T}[\varepsilon] = \sum\nolimits_k \varepsilon^k \mathfrak{T}^k
$$

where $\mathfrak{A}^k(\mathrm{R}, s\,;\, \partial_\mathrm{R}, \partial_s, \partial_3)$ is a partial differential system of order 2 in the stretched domain $\partial\omega \times \Sigma^+$ whereas $\mathfrak{T}^k(\mathrm{R}, s\,;\, \partial_\mathrm{R}, \partial_s, \partial_3)$ is a partial differential system of order 1 on the horizontal boundaries $\partial\omega \times \gamma_\pm$, with $\gamma_\pm = \mathbb{R}^+ \times \{x_3 = \pm 1\}$. Similarly, the operators $\mathbf{B}(\varepsilon)$ and $\mathbf{S}(\varepsilon)$ correspond to the formal series $\mathfrak{B}[\varepsilon]$ and $\mathfrak{S}[\varepsilon]$. The counterpart of problem (4.3) In variables $(\mathrm{R}, s, \mathrm{X}_3)$ and unknown $\boldsymbol{\varphi}$ is

$$
\left\{
\begin{array}{rcll}
\mathfrak{A}[\varepsilon]\boldsymbol{\varphi}[\varepsilon] & = & \varepsilon^2\lambda[\varepsilon]\, \mathfrak{B}[\varepsilon]\boldsymbol{\varphi}[\varepsilon] & \text{in} \quad \Omega, \\
\mathfrak{T}[\varepsilon]\boldsymbol{\varphi}[\varepsilon] & = & \varepsilon^2\lambda[\varepsilon]\, \mathfrak{S}[\varepsilon]\boldsymbol{\varphi}[\varepsilon] & \text{on} \quad \Gamma_\pm.
\end{array}
\right.
\tag{5.3}
$$

The first terms $\mathfrak{A}^0$ and $\mathfrak{T}^0$ of $\mathfrak{A}[\varepsilon]$ and $\mathfrak{T}[\varepsilon]$ split respectively into 2D-Lamé and 2D-Laplace operators in variables (R, x_3) with Neumann boundary conditions:

$$
\begin{aligned}
&(\mathfrak{A}^0\boldsymbol{\varphi})_\mathrm{R} = \mu\, \Delta_{\mathrm{R},3}\varphi_\mathrm{R} + (\lambda + \mu)\, \partial_\mathrm{R}\big(\mathrm{div}_{\mathrm{R},3}(\varphi_\mathrm{R}, \varphi_3)\big), && (\mathfrak{T}^0\boldsymbol{\varphi})_\mathrm{R} = \mu(\partial_3\varphi_\mathrm{R} + \partial_\mathrm{R}\varphi_3), \\
&(\mathfrak{A}^0\boldsymbol{\varphi})_3 = \mu\, \Delta_{\mathrm{R},3}\varphi_3 + (\lambda + \mu)\, \partial_3\big(\mathrm{div}_{\mathrm{R},3}(\varphi_\mathrm{R}, \varphi_3)\big), && (\mathfrak{T}^0\boldsymbol{\varphi})_3 = (\lambda + 2\mu)\partial_3\varphi_3 + \lambda\, \partial_\mathrm{R}\varphi_\mathrm{R} \\
&(\mathfrak{A}^0\boldsymbol{\varphi})_s = \mu\, \Delta_{\mathrm{R},3}\varphi_s, && (\mathfrak{T}^0\boldsymbol{\varphi})_s = \mu\partial_3\varphi_s\,.
\end{aligned}
$$

The following lemma states that, after the possible subtraction of a rigid motion, any trace on the lateral boundary Γ_0 has a lifting in exponential decreasing displacement with zero forces, see [4]:

Lemma 5.3
Let $\mathfrak{Z} = \operatorname{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}-\mathrm{X}_3\\0\\\mathrm{R}\end{pmatrix}\right\}$ *be the space of rigid motions on* Σ^+.
There exists $\delta > 0$ *such that there holds: For any* $\boldsymbol{v} \in \mathcal{C}^\infty(\overline{\Gamma}_0)^3$*, there exist a unique* $\boldsymbol{\varphi} \in \mathcal{C}^\infty\big(\partial\omega, \mathfrak{H}_\delta(\Sigma^+)^3\big)$ *and a unique* $\boldsymbol{Z} \in \mathcal{C}^\infty(\partial\omega, \mathfrak{Z})$ *such that*

$$\begin{cases} \mathfrak{A}^0(\boldsymbol{\varphi}) &=& 0 \quad \textit{in} \quad \partial\omega\times\Sigma^+, \\ \mathfrak{T}^0(\boldsymbol{\varphi}) &=& 0 \quad \textit{on} \quad \partial\omega\times\gamma_\pm, \\ (\boldsymbol{\varphi}-\boldsymbol{Z})\big|_{t=0} + \boldsymbol{v}\big|_{\Gamma_0} &=& 0, \end{cases}$$

Since the space of traces of $\mathcal{C}^\infty(\partial\omega,\mathfrak{Z})$ on Γ_0 coincides with the space of *Dirichlet traces* of the $\mathcal{C}^\infty(\omega)$ Kirchhoff-Love displacements, it is possible to match the outer and inner expansion via admissible boundary conditions on the surface generators $\boldsymbol{\zeta}^k$.

Here, by Dirichlet traces we mean those associated with the pair $(\mathsf{L}_\mathsf{m}, \mathsf{L}_\mathsf{b})$ of the first non-zero operators arising in the residual equations (4.31): the membrane operator L_m acts on $\boldsymbol{\zeta}_\top = (\zeta_1,\zeta_2)$ and the Dirichlet traces are $\boldsymbol{\gamma}^0_\mathsf{m}\boldsymbol{\zeta}_\top = \boldsymbol{\zeta}_\top\big|_{\partial\omega}$; the bending operator L_b is of order 4 and acts on ζ_3, its Dirichlet traces are $\boldsymbol{\gamma}^0_\mathsf{b}\zeta_3 = (\zeta_3, \partial_n\zeta_3)\big|_{\partial\omega}$. The whole trace operator $\boldsymbol{\gamma}^0$ is defined as $\boldsymbol{\gamma}^0\boldsymbol{\zeta} = (\boldsymbol{\gamma}^0_\mathsf{m}\boldsymbol{\zeta}_\top, \boldsymbol{\gamma}^0_\mathsf{b}\zeta_3)$, cf (5.1).

With the help of Lemma 5.3 we can prove as in [3] the existence of a boundary operator series $\boldsymbol{\gamma}[\varepsilon] = \boldsymbol{\gamma}^0 + \varepsilon\boldsymbol{\gamma}^1 + \ldots$ [2] such that if the generator series $\boldsymbol{\zeta}[\varepsilon]$ satisfies the boundary condition

$$\boldsymbol{\gamma}[\varepsilon]\boldsymbol{\zeta}[\varepsilon] = 0 \quad \text{on} \quad \partial\omega,$$

then there exists a boundary layer series $\boldsymbol{w}[\varepsilon]$ in $\mathcal{C}^\infty\big(\partial\omega, \mathfrak{H}_\delta(\Sigma^+)^3\big)$ such that

- the corresponding series $\boldsymbol{\varphi}[\varepsilon]$ solves (5.3),
- the traces of $\boldsymbol{w}[\varepsilon]$ on Γ^0 coincide with those of $\boldsymbol{u}[\varepsilon] = \mathsf{U}[\varepsilon]\boldsymbol{\zeta}[\varepsilon]$ in (4.32).

Summarizing, we obtain:

Lemma 5.4 *Any formal series solution* ($\boldsymbol{\zeta}[\varepsilon]$, $\lambda[\varepsilon]$) *of the residual boundary value problem*

$$\begin{cases} \mathsf{L}[\varepsilon]\boldsymbol{\zeta}[\varepsilon] &=& \varepsilon^2\lambda[\varepsilon]\mathsf{M}[\varepsilon]\boldsymbol{\zeta}[\varepsilon] \qquad [\omega], \\ \boldsymbol{\gamma}[\varepsilon]\boldsymbol{\zeta}[\varepsilon] &=& 0 \qquad\qquad [\partial\omega]. \end{cases} \tag{5.4}$$

yields a solution $\boldsymbol{u}[\varepsilon] = \mathsf{U}[\varepsilon]\boldsymbol{\zeta}[\varepsilon]$ *of* (4.3) *and a solution* $\boldsymbol{\varphi}[\varepsilon]$ *of* (5.3) *such that* $\boldsymbol{u}[\varepsilon] + \boldsymbol{w}[\varepsilon] = 0$ *on* Γ_0.

There exists a one to one correspondence between the solutions of (5.4) such that $\boldsymbol{\zeta}^0 \neq 0$ and the eigenpairs $(\eta, \lambda^{\mathrm{KL}})$ of problem (5.2):

$$\eta \in H^2_0(\omega) \quad \text{and} \quad \tfrac{1}{3}\mathsf{L}_\mathsf{b}\eta = \lambda^{\mathrm{KL}}\,\mathsf{P}\eta. \tag{5.5}$$

[2] The calculations in [4, §6] give that $\boldsymbol{\gamma}^1\boldsymbol{\zeta} = (\boldsymbol{\gamma}^1_\mathsf{m}\boldsymbol{\zeta}_\top, \boldsymbol{\gamma}^1_\mathsf{b}\zeta_3)$ with $\boldsymbol{\gamma}^1_\mathsf{m}\boldsymbol{\zeta}_\top = (c_\mathsf{m}\operatorname{div}\boldsymbol{\zeta}_\top, 0)\big|_{\partial\omega}$ and $\boldsymbol{\gamma}^1_\mathsf{b}\zeta_3 = (0, c_\mathsf{b}\Delta\zeta_3)\big|_{\partial\omega}$ with c_m and c_b non-zero constants only depending on the Lamé coefficients $\overline{\lambda}$ and $\overline{\mu}$.

Lemma 5.5 (i) *For each solution* $(\boldsymbol{\zeta}[\varepsilon], \lambda[\varepsilon])$ *of* (5.4) *with* $\boldsymbol{\zeta}^0 \neq 0$, *the pair* (ζ_3^0, λ_0) *is an eigenpair of problem* (5.2).
(ii) *Let* $(\eta, \lambda^{\mathrm{KL}})$ *be an eigenpair of problem* (5.5). *If* λ^{KL} *is a simple eigenvalue, there exists a unique solution* $(\boldsymbol{\zeta}[\varepsilon], \lambda[\varepsilon])$ *of* (5.4) *with* $\lambda_0 = \lambda^{\mathrm{KL}}$ *and* $\boldsymbol{\zeta}^0 = (0, 0, \eta)$. *If* λ^{KL} *is a multiple eigenvalue of multiplicity* d, *there exist* d *independent solutions* $(\boldsymbol{\zeta}[\varepsilon], \lambda[\varepsilon])$ *of* (5.4) *with* $\lambda_0 = \lambda^{\mathrm{KL}}$.

The proof of this result follows along the same lines as the proof of [3, Th.5.3].

5.c Quasi-mode estimates

The last step in the construction of quasi-modes is the cut-off of these series, leaving a finite number of terms. Let $\boldsymbol{\zeta}[\varepsilon]$ and $\lambda[\varepsilon]$ be as in Lemma 5.4, and let us consider the associated solutions $\boldsymbol{u}[\varepsilon]$ and $\boldsymbol{w}[\varepsilon]$. Let $\chi = \chi(r)$ be a smooth cut-off function which is equal to 1 for $0 < r < r_0$ and to 0 for $r > r_1 > r_0$, where r_1 is small enough so that the region $0 < r < r_1$ is a well-defined tubular neighborhood of $\partial\omega$.

Let $N \geq 0$ be an integer. We denote by $\boldsymbol{u}_{\{N\}}(\varepsilon)$ the displacement field on Ω,

$$\boldsymbol{u}_{\{N\}}(\varepsilon) := \sum_{k=0}^{N+5} \varepsilon^k \Big(\boldsymbol{u}^k(x_\top, \mathrm{x}_3) + \chi(r) \boldsymbol{w}^k(\frac{r}{\varepsilon}, s, \mathrm{x}_3) \Big).$$

We unscale $\boldsymbol{u}_{\{N\}}(\varepsilon)$ according to (3.11) and obtain the displacement $\boldsymbol{u}^\varepsilon_{\{N\}}$ on the thin plate Ω^ε. Let $\lambda^\varepsilon_{\{N\}}$ be the finite sum

$$\lambda^\varepsilon_{\{N\}} := \varepsilon^2 \sum_{k=0}^{N} \varepsilon^k \lambda_k$$

and let $\boldsymbol{\psi}^\varepsilon_{\{N\}}$ denote the residual

$$\boldsymbol{\psi}^\varepsilon_{\{N\}}(\boldsymbol{v}) := a^\varepsilon\big(\boldsymbol{u}^\varepsilon_{\{N\}}, \boldsymbol{v}\big) - \lambda^\varepsilon_{\{N\}}\, b^\varepsilon\big(\boldsymbol{u}^\varepsilon_{\{N\}}, \boldsymbol{v}\big), \quad \boldsymbol{v} \in V_\varepsilon.$$

With the notations of section 3.a we have

$$\boldsymbol{\psi}^\varepsilon_{\{N\}} = \big(\mathbb{A}^\varepsilon - \lambda^\varepsilon_{\{N\}}\, \mathbb{B}^\varepsilon\big) \boldsymbol{u}^\varepsilon_{\{N\}}.$$

Revisiting the construction of $\boldsymbol{u}(\varepsilon)$ and $\boldsymbol{w}(\varepsilon)$, see also [3, Th6.1] we can prove that, if the first term $\boldsymbol{\zeta}^0$ of $\boldsymbol{\zeta}[\varepsilon]$ is not zero, the residue satisfies

$$\|\boldsymbol{\psi}^\varepsilon_{\{N\}}\|_{V'_\varepsilon} \leq C\varepsilon^{N+3} \|\boldsymbol{u}^\varepsilon_{\{N\}}\|_{V_\varepsilon} \tag{5.6}$$

We come back to the operator $\mathbb{Y}^\varepsilon$ (3.7), cf Theorem 3.3. Let $\boldsymbol{y}^\varepsilon_{\{N\}}$ be defined as $(\mathbb{Q}^\varepsilon)^{-1} \boldsymbol{u}^\varepsilon_{\{N\}}$. Combining (5.6) with (3.6), we obtain

$$\|\lambda^\varepsilon_{\{N\}} \mathbb{Y}^\varepsilon \boldsymbol{y}^\varepsilon_{\{N\}} - \boldsymbol{y}^\varepsilon_{\{N\}}\|_{H_\varepsilon} \leq C\varepsilon^{N+1} \|\boldsymbol{y}^\varepsilon_{\{N\}}\|_{H_\varepsilon}.$$

Multipying by $\mu^\varepsilon_{\{N\}} := \big(\lambda^\varepsilon_{\{N\}}\big)^{-1}$, we deduce, since $\mu^\varepsilon_{\{N\}} = \mathcal{O}(\varepsilon^{-2})$,

$$\|\mathbb{Y}^\varepsilon \boldsymbol{y}^\varepsilon_{\{N\}} - \mu^\varepsilon_{\{N\}} \boldsymbol{y}^\varepsilon_{\{N\}}\|_{H_\varepsilon} \le C\varepsilon^{N-1} \|\boldsymbol{y}^\varepsilon_{\{N\}}\|_{H_\varepsilon}. \tag{5.7}$$

In other words, $(\mu^\varepsilon_{\{N\}}, \boldsymbol{y}^\varepsilon_{\{N\}})$ is a $\mathcal{O}(\varepsilon^{N-1})$ quasi-mode for $\mathbb{Y}^\varepsilon$. Lemma 3.4 gives that

$$\operatorname{dist}\big(\mu^\varepsilon_{\{N\}}, \sigma(\mathbb{Y}^\varepsilon)\big) \le C\varepsilon^{N-1}.$$

Therefore we have $\operatorname{dist}\big(\lambda^\varepsilon_{\{N\}}, \sigma^{-1}(\mathbb{Y}^\varepsilon)\big) \le C\varepsilon^{N+1}$ and we can drop the last two terms $\varepsilon^{N+1}\lambda_{N-1}$ and $\varepsilon^{N+2}\lambda_N$ in $\lambda^\varepsilon_{\{N\}}$ without modifying that conclusion:

$$\operatorname{dist}\Big(\sum_{k=0}^{N-2} \varepsilon^{2+k}\lambda_k, \sigma^{-1}(\mathbb{Y}^\varepsilon)\Big) \le C\varepsilon^{N+1}. \tag{5.8}$$

Putting together (5.8) and Lemma 5.5 we finally obtain:

Theorem 5.6 *Recall that* $\mathsf{L}_\mathsf{b} := (\widehat{\lambda} + 2\overline{\mu})\Delta^2_\top$ *and* $\mathsf{P} = -\partial_\alpha p^0_{\alpha\beta}\partial_\beta$ *with*

$$p^0_{\alpha\beta}(x_\top) = \frac{1}{2}\int_{-1}^{1} \overline{\sigma}_{\alpha\beta}(x_\top, \mathrm{x}_3)\, \mathrm{d}\mathrm{x}_3.$$

Under Hypothesis 3.1*, for each eigenvalue* λ^{KL} *of problem* (5.5)*:* $\frac{1}{3}\mathsf{L}_\mathsf{b}\eta = \lambda^{\mathrm{KL}}\,\mathsf{P}\eta$ *with* $\eta \in H^2_0(\omega)$*, there exists* $C > 0$ *such that there holds:*

$$\forall\varepsilon \in (0,1], \quad \operatorname{dist}\big(\varepsilon^2\lambda^{\mathrm{KL}}, \sigma^{-1}(\mathbb{Y}^\varepsilon)\big) \le C\varepsilon^3, \tag{5.9}$$

where $\sigma^{-1}(\mathbb{Y}^\varepsilon)$ *denotes the set of inverses of elements of* $\mathbb{Y}^\varepsilon$ *as in Theorem* 3.3*.*

Combining Theorems 2.4 and 5.6, we find

Corollary 5.6.1 *There exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *the minimum* $\beta^\varepsilon_{\min}$ *of the positive part of* $\sigma^{-1}(\mathbb{Y}^\varepsilon)$ *coincides with the smallest buckling eigenvalue* $\lambda^\varepsilon_{\min}$ *and belongs to the region* Λ *free of spectral pollution given by* (2.13)*.*

Proof: According to Theorem 2.4, the interval $\Lambda = (-\frac{1}{9KM_0}, \frac{1}{9KM_0})$ is free from the essential spectrum of $\mathbb{Y}^\varepsilon$ for all $\varepsilon \in (0,1]$. Let $\lambda^{\mathrm{KL}}_{\min}$ be the lowest positive eigenvalue of problem (5.5): According to Hyposesis 3.1 (ii) the positive part of that spectrum is not empty, see Remark 5.2. From (5.9) we deduce that $\beta^\varepsilon_{\min} \le \varepsilon^2\lambda^{\mathrm{KL}}_{\min} + C\varepsilon^3$. Therefore, for all ε such that

$$\varepsilon^2\lambda^{\mathrm{KL}}_{\min} + C\varepsilon^3 < \frac{1}{9KM_0}$$

we are sure that $\beta^\varepsilon_{\min}$ belongs to the discrete spectrum of $\mathbb{Y}^\varepsilon$. □

6 Numerical experiments

Let us now present numerical experiments that illustrate some of the results of the previous sections. We consider an isotropic unit disc of thickness 2ε, clamped on the circular lateral part, with λ, μ corresponding to Young's modulus $E = 3 \times 10^4$ and Poisson ratio $\nu = 0.3$. It is subjected to a body force of the form

$$F_* = (f_\alpha(x_\top), \varepsilon f_3(x_\top)), \tag{6.1}$$

together with tractions $T_*^\pm$ applied to the top and bottom surfaces, of the form

$$T_*^\pm = (\varepsilon t_\alpha^\pm(x_\top), \varepsilon^2 t_3^\pm(x_\top)). \tag{6.2}$$

Each pair (F_*, T_*) leads to a corresponding pre-existing stress state $\{\sigma_*^\varepsilon\}$. (We discuss, in the next section, when these stress states will satisfy Hypothesis 3.1.)

We perform four families of experiments, based on the following four choices of (F_*, T_*).

(L1) $F_* \equiv 0,\ T_*^+ = (\varepsilon, \varepsilon, 0),\ T_*^- = (-\varepsilon, -\varepsilon, 0)$.
(L2) $F_* \equiv 0,\ T_*^+ = (\varepsilon,\ \varepsilon, 0),\ T_*^- = (\varepsilon, \varepsilon, 0)$.
(L3) $F_* = (-1, -1, 0),\ T_*^+ = (2\varepsilon, 2\varepsilon, 0),\ T_*^- = (2\varepsilon, 2\varepsilon, 0)$.
(L4) $F_* = (-1, -1, 0),\ T_*^+ = (\varepsilon, \varepsilon, 0),\ T_*^- = (\varepsilon, \varepsilon, 0)$.

As in [5], for each of the above loads, we reduce the computation to a quarter of the plate by using symmetry boundary conditions on the plane lateral parts of the boundary $x_1 = 0$ and $x_2 = 0$. This enforces symmetry in the solution on the full domain across these planes, and we only compute approximations of those eigenvalues whose eigenvectors satisfy these symmetries (roughly a quarter of the total number).

Each of the loads above will lead to boundary layers in the pre-existing stress state σ_*^ε, and one of the factors we investigate is the sensitivity of the computed eigenvalues to the resolution of these layers. We therefore consider three different meshes, as shown in Figure 1, each with twelve elements in the quarter disc (six elements above the midsurface of the disc, and six below). In Mesh UNIF, the layers are of thickness 0.5, 0.3 and 0.2. In Mesh MID, the thicknesses are 0.65, 0.35 $- \varepsilon$ and ε, while in Mesh FIN, the refinement is even more concentrated at the boundary, with thicknesses $1 - 2\varepsilon$, ε, ε. For each loading, σ_*^ε is first numerically computed by the program STRESS CHECK, using one of the above meshes. Here, finer meshes will result in better resolution of the layers, i.e. in higher values of M_0 in (2.11). This computed σ_*^ε is then used as the pre-existing stress in each case, and the lowest 10 eigenvalues computed by STRESS CHECK, using the same mesh.

Let us begin with load L1, which results in a pre-existing stress that is of *pure bending* type. This load does not satisfy the requirement that the first term R_m^0 in the series expansion for the membrane resultant is non-zero, since

$$R_\mathsf{m}^0 := -\tfrac{1}{2}\big(2f_\alpha + t_\alpha^+ + t_\alpha^-\big)_{\alpha=1,2} = (0,0). \tag{6.3}$$

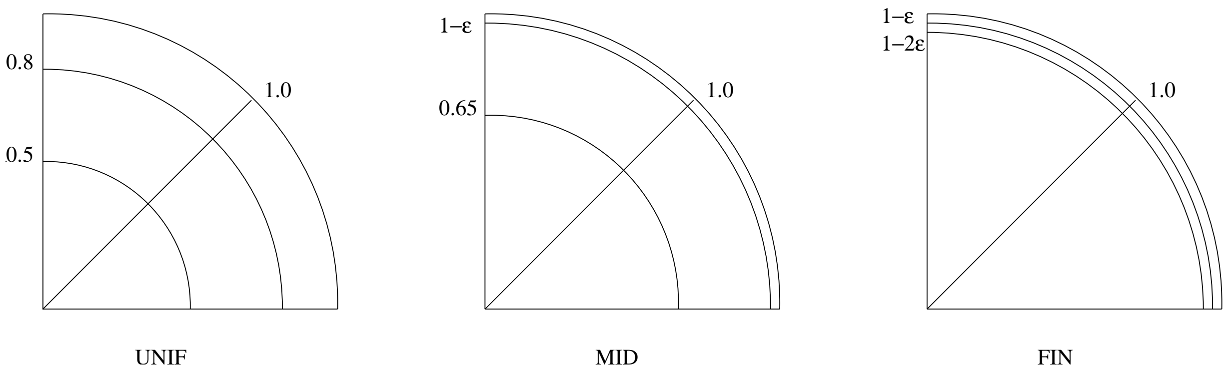

Figure 1: The three meshes on the quarter disc (top view)

Hence Hypothesis 3.1 (ii) will be violated (in fact, the terms in (3.2) will all be zero, as shown in the Appendix). Consequently, the lowest eigenvalue $\lambda^{\varepsilon}_{\min}$ will not necessarily satisfy (1.3). Hence, even for ε small enough, the condition $\lambda^{\varepsilon}_{\min} \in \Lambda$ may be violated, and we may get spectral pollution.

In Tables 1 and 2, we have tabulated the first ten eigenvalues computed using the meshes UNIF and FIN respectively. (In this and all the computations that follow, the polynomial degree used over each element is $p = 8$.)

0.004	0.008	0.016	0.032	0.064	0.128	0.256
2.566e3	9.481e3	2.582e4	2.194e4	1.149e4	1.124e4	9.697e3
5.275e3	1.730e4	3.225e4	2.273e4	1.507e4	1.168e4	1.007e4
9.091e3	2.437e4	3.271e4	2.275e4	1.510e4	1.179e4	1.017e4
9.460e3	2.600e4	3.327e4	2.435e4	1.603e4	1.224e4	1.018e4
1.400e4	3.035e4	3.344e4	2.599e4	1.703e4	1.242e4	1.072e4
1.441e4	3.083e4	3.492e4	2.657e4	1.720e4	1.302e4	1.075e4
1.824e4	3.223e4	3.520e4	2.800e4	1.755e4	1.308e4	1.084e4
1.898e4	3.408e4	3.573e4	2.934e4	1.857e4	1.341e4	1.108e4
2.003e4	3.428e4	3.642e4	2.948e4	1.928e4	1.348e4	1.143e4
2.068e4	3.469e4	3.726e4	2.974e4	1.931e4	1.418e4	1.149e4

Table 1: First ten computed eigenvalues for various ε, Load L1, mesh UNIF

We plot these values in Figure 2. We observe, first of all, that for large ε, the eigenvalues all coalesce together, which is a symptom typical of spectral pollution (indicating that what we are recovering are values from the essential spectrum — see [5]). In fact, when mesh FIN is used, this clumping together is observed for all values of ε, both large and small. The separation observed in the eigenvalues with mesh UNIF for smaller values of ε, shows, moreover, that these eigenvalues are highly dependent on the mesh. Finally, there is no $\mathcal{O}(\varepsilon^2)$ behavior observed, as will be seen for loads L2 and L3 ahead. The conclusion is that in the

0.004	0.008	0.016	0.032	0.064	0.128	0.256
8.141e3	8.212e3	8.325e3	8.495e3	8.686e3	8.940e3	9.380e3
8.172e3	8.242e3	8.353e3	8.520e3	8.760e3	9.022e3	9.451e3
8.191e3	8.255e3	8.355e3	8.539e3	8.858e3	9.402e3	1.016e4
8.263e3	8.327e3	8.425e3	8.575e3	8.884e3	9.443e3	1.020e4
8.471e3	8.536e3	8.634e3	8.789e3	9.070e3	9.613e3	1.052e4
8.530e3	8.596e3	8.696e3	8.858e3	9.152e3	9.665e3	1.075e4
8.760e3	8.810e3	8.876e3	8.990e3	9.241e3	9.923e3	1.079e4
8.780e3	8.854e3	8.951e3	9.064e3	9.299e3	9.964e3	1.081e4
8.808e3	8.874e3	8.985e3	9.213e3	9.586e3	1.028e4	1.091e4
8.861e3	8.917e3	9.037e3	9.261e3	9.648e3	1.035e4	1.116e4

Table 2: First ten computed eigenvalues for various ε, Load L1, mesh FIN

absence of Hypothesis 3.1, the recovered eigenvalues may not be physically relevant.

Next, we consider loads L2 and L3, each of which gives a $\{\sigma_*^\varepsilon\}$ of purely membrane type. As indicated in the next section, Hypothesis 3.1 is now satisfied with the *non-zero membrane resultant*

$$R_{\mathsf{m}}^0 := -\tfrac{1}{2}\big(2f_\alpha + t_\alpha^+ + t_\alpha^-\big)_{\alpha=1,2} = (-1,-1). \tag{6.4}$$

In this case, the presence or absence of body forces does not make much of a difference in the resulting eigenvalues, due to the fact that the resultant is the same for both cases. Tables 3 and 4, both based on mesh MID, illustrate this. Also, it turns out that using the meshes UNIF and FIN gives very similar results (the tables are not reproduced here). This lack of mesh-dependence suggests that even with the refinement used here, the boundary layer effects have still not been resolved sufficiently for the computed M_0 in (2.11) to cause a problem. (We note that the computational results in [5] dealt with a different case of infinite stresses — this time, due to corner singularities. It was shown that if the mesh is sufficiently refined around the corner, then the essential spectrum does eventually predominate, giving spectral pollution. A similar effect may be anticipated here, if we resolve the boundary layer sufficiently. Hence, paradoxically, too much refinement is detrimental to recovering the physical eigenvalues.)

We plot the first five eigenvalues for load L3 in figure 3. (The plot for L2 is similar.) Now the $\mathcal{O}(\varepsilon^2)$ behavior is clearly observed. Moreover, we see that when the thickness gets sufficiently large, the computed eigenvalues begin to coalesce, indicating that the required eigenvalues no longer lie in Λ.

As shown in [5], the difference between physical and non-physical eigenvalues can be quite clearly seen by examining their eigenvectors — non-physical eigenvectors have a markedly local character, where not much variation is observed over the domain. In Figure 4, we plot these eigenvectors corresponding to the first three computed eigenvalues for load L3, using the mesh MID. The plots are for the component u_3 on the midplane of the disc. We note that the

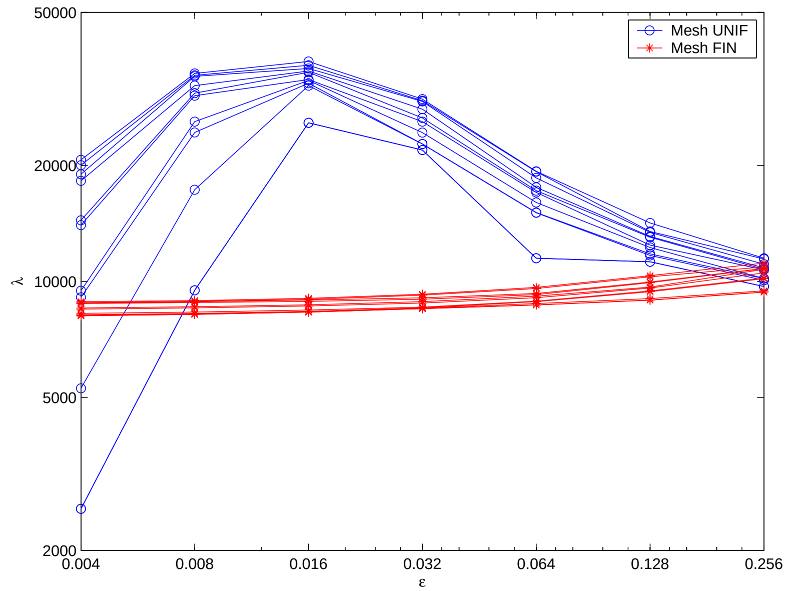

Figure 2: First 10 eigenvalues plotted against ε, load L1

0.004	0.008	0.016	0.032	0.064	0.128	0.256
1.861e0	7.444e0	2.975e1	1.187e2	4.694e2	1.798e3	6.130e3
7.383e0	2.951e1	1.177e2	4.662e2	1.795e3	6.246e3	1.460e4
8.956e0	3.574e1	1.423e2	5.605e2	2.114e3	6.884e3	1.621e4
1.174e1	4.672e1	1.857e2	7.309e2	2.758e3	9.011e3	1.717e4
1.604e1	6.346e1	2.500e2	9.670e2	3.500e3	1.028e4	1.742e4
1.786e1	7.104e1	2.816e2	1.098e3	4.023e3	1.144e4	1.759e4
2.033e1	8.059e1	3.179e2	1.221e3	4.267e3	1.215e4	1.777e4
2.530e1	9.938e1	3.793e2	1.402e3	4.808e3	1.303e4	1.873e4
2.572e1	1.009e2	3.981e2	1.536e3	5.447e3	1.430e4	1.917e4
3.108e1	1.225e2	4.779e2	1.792e3	5.961e3	1.517e4	1.972e4

Table 3: First ten computed eigenvalues for various ε, Load L2, mesh MID

plots on the top and bottom surfaces (not shown here) are very similar, as can be expected from the theory: From our construction of quasi-modes in Section 5, we know that each eigenvector $\eta = \eta(x_\top)$ of problem (5.5) gives rise to a buckling eigenvector with its transverse component $u_3(x) = \eta(x_\top) + O(\varepsilon)$.

The four rows (from top to bottom) give the results for $\varepsilon = 0.004$, 0.064, 0.128, 0.256 respectively. For $\varepsilon = 0.004$ and 0.064, the buckling modes are the physical ones. For $\varepsilon = 0.128$, however, the second eigenvector is clearly non-physical, as are the first and third eigenvectors for $\varepsilon = 0.256$. It is also interesting to note the presence of an axisymmetric mode at the first place (certainly corresponding to an axisymmetric 2-d eigenvector η as in (5.5)) for $\varepsilon = 0.004$,

0.004	0.008	0.016	0.032	0.064	0.128	0.256
1.861e0	7.442e0	2.975e1	1.187e2	4.697e2	1.802e3	6.114e3
7.381e0	2.950e1	1.177e2	4.662e2	1.796e3	6.244e3	1.480e4
8.954e0	3.574e1	1.423e2	5.604e2	2.113e3	6.864e3	1.505e4
1.174e1	4.671e1	1.857e2	7.309e2	2.760e3	9.002e3	1.515e4
1.604e1	6.344e1	2.497e2	9.669e2	3.499e3	1.024e4	1.543e4
1.785e1	7.103e1	2.816e2	1.098e3	4.025e3	1.138e4	1.560e4
2.033e1	8.058e1	3.178e2	1.221e3	4.265e3	1.212e4	1.586e4
2.529e1	9.935e1	3.792e2	1.402e3	4.808e3	1.297e4	1.754e4
2.571e1	1.009e2	3.981e2	1.536e3	5.449e3	1.422e4	1.763e4
3.107e1	1.224e2	4.779e2	1.792e3	5.959e3	1.512e4	1.787e4

Table 4: First ten computed eigenvalues for various ε, Load L3, mesh MID

0.064 and 0.128.

Finally, we consider load L4, which is designed to give a zero resultant,

$$R^0_{\mathsf{m}} := -\tfrac{1}{2}\big(2f_\alpha + t^+_\alpha + t^-_\alpha\big)_{\alpha=1,2} = (0,0). \tag{6.5}$$

leading to a cancellation of the right hand side in equation (7.7) ahead. As a result of this, the magnitude of σ^ε_* (and hence $\boldsymbol{u}$) will drop in order, leading to a corresponding increase in the magnitude of the eigenvalues, compared to the previous loadings. In Tables 5 and 6, we have tabulated the eigenvalues computed using meshes UNIF and FIN respectively, for this case. The mentioned increase in order due to there being a zero resultant is clearly seen.

0.004	0.008	0.016	0.032	0.064	0.128	0.256
5.809e3	2.364e4	7.842e4	1.231e5	1.319e5	1.307e5	1.200e5
1.974e4	7.740e4	1.927e5	2.214e5	2.176e5	2.016e5	1.536e5
4.371e4	1.583e5	2.789e5	2.933e5	2.871e5	2.501e5	1.602e5
4.565e4	1.603e5	2.980e5	3.083e5	3.083e5	2.604e5	1.605e5
6.859e4	2.228e5	3.231e5	3.405e5	3.405e5	2.833e5	1.758e5
8.057e4	2.683e5	3.837e5	3.770e5	3.770e5	2.850e5	1.782e5
9.257e4	2.763e5	3.990e5	4.282e5	4.282e5	3.102e5	1.817e5
1.185e5	3.391e5	4.267e5	4.382e5	4.382e5	3.272e5	1.826e5
1.249e5	3.780e5	4.511e5	4.537e5	4.537e5	3.285e5	1.838e5
1.549e5	3.967e5	4.911e5	4.765e5	4.765e5	3.355e5	1.845e5

Table 5: First ten computed eigenvalues for various ε, Load L4, mesh UNIF

We plot these numbers in figure 5. The leveling out of the eigenvalues once again occurs due to the limit of Λ being reached, leading to non-physical eigenvalues being recovered. It may be also noticed that for small values of ε, the eigenvalues are mesh-dependent.

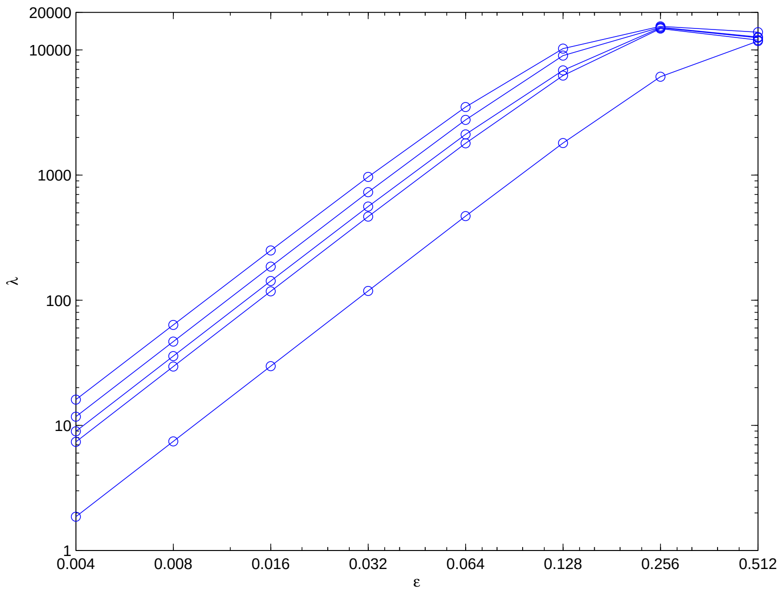

Figure 3: First 5 eigenvalues plotted against ε, load L3

0.004	0.008	0.016	0.032	0.064	0.128	0.256
7.108e3	2.858e4	9.220e4	1.331e5	1.355e5	1.318e5	9.957e4
2.829e4	1.061e5	2.127e5	2.238e5	2.190e5	1.983e5	1.016e5
4.251e4	1.501e5	2.805e5	2.932e5	2.883e5	2.021e5	1.134e5
9.629e4	2.739e5	3.238e5	3.180e5	2.959e5	2.024e5	1.150e5
1.650e5	2.937e5	3.480e5	3.500e5	3.333e5	2.258e5	1.202e5
2.462e5	4.128e5	4.001e5	3.934e5	3.592e5	2.290e5	1.323e5
4.628e5	4.580e5	4.294e5	4.455e5	3.954e5	2.507e5	1.334e5
5.538e5	5.189e5	4.795e5	4.673e5	4.038e5	2.613e5	1.353e5
6.961e5	5.662e5	5.183e5	4.931e5	4.169e5	2.626e5	1.354e5
8.280e5	5.930e5	5.604e5	5.346e5	4.270e5	2.650e5	1.368e5

Table 6: First ten computed eigenvalues for various ε, Load L4, mesh FIN

In Figure 6, we plot the corresponding eigenvectors, which more clearly illustrate the spectral pollution. The first two rows are for $\varepsilon = 0.004$, with the meshes MID and FIN respectively. We observe that the eigenvectors are now mesh-dependent, indicating that even for such low values of ε, the computed eigenvalues may be spurious. For $\varepsilon = 0.064$ (Row 3) and 0.128 (Row 4), the presence of spurious eigenvectors is clear. Our conclusion is that for this loading, spectral pollution starts at smaller values of ε compared to the cases of L2 and L3, in relation with the fact that the eigenvalues are much larger for L4. Note also that the shape invariance clearly visible in Figure 4 (related to the asymptotic limit of eigenvectors) disappears here anyway.

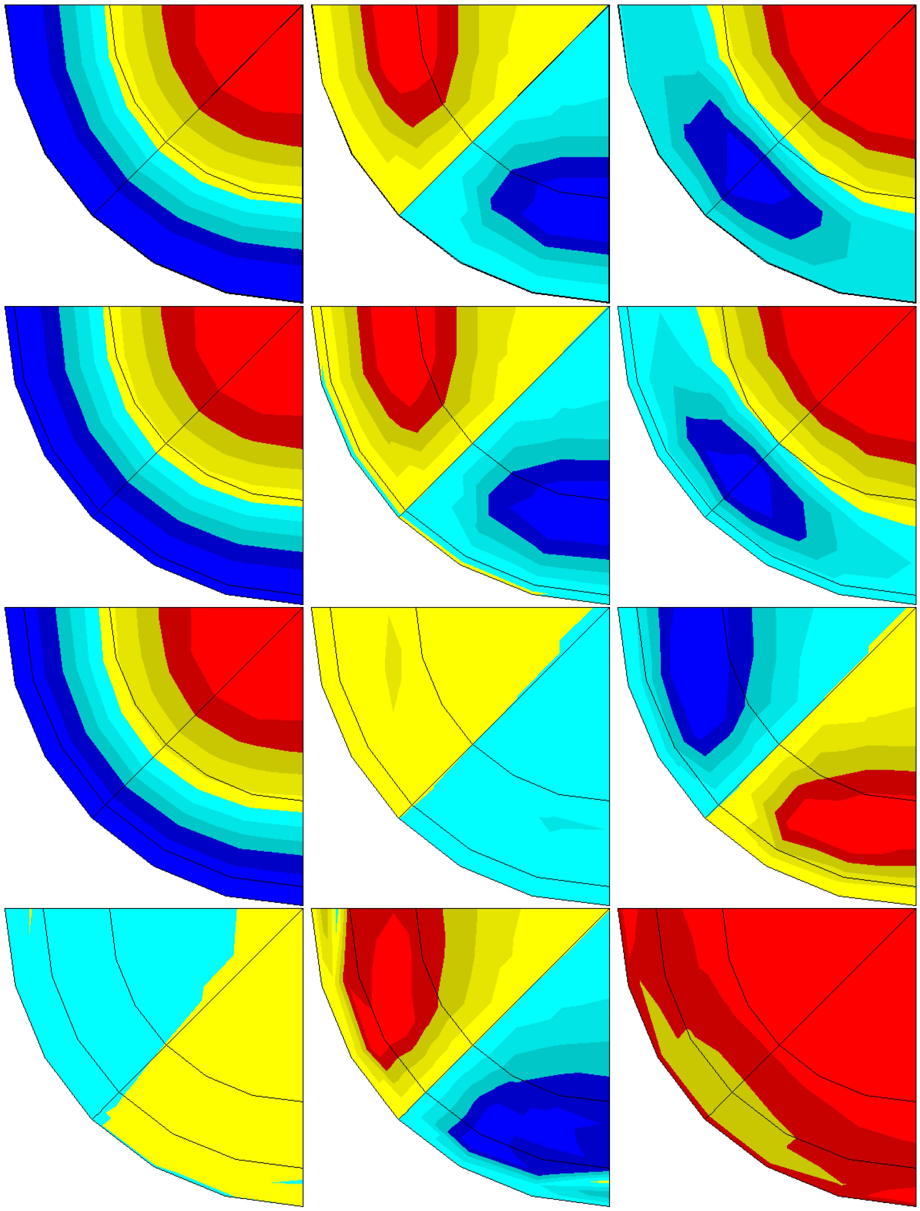

Figure 4: Midplane plot of u_3 for first three eigenvectors, load L3, mesh MID. $\varepsilon = 0.004$ (row 1), 0.064 (row 2), 0.128 (row 3), 0.256 (row 4)

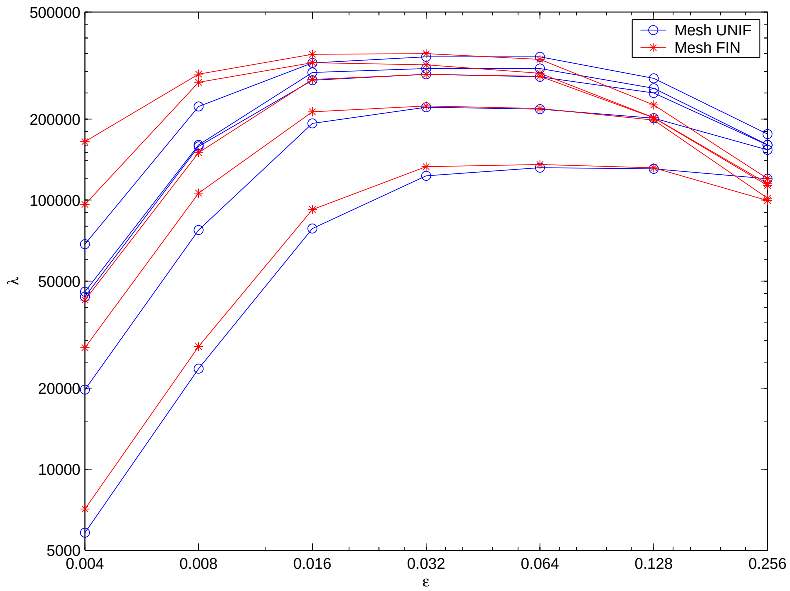

Figure 5: First 10 eigenvalues plotted against ε, load L4

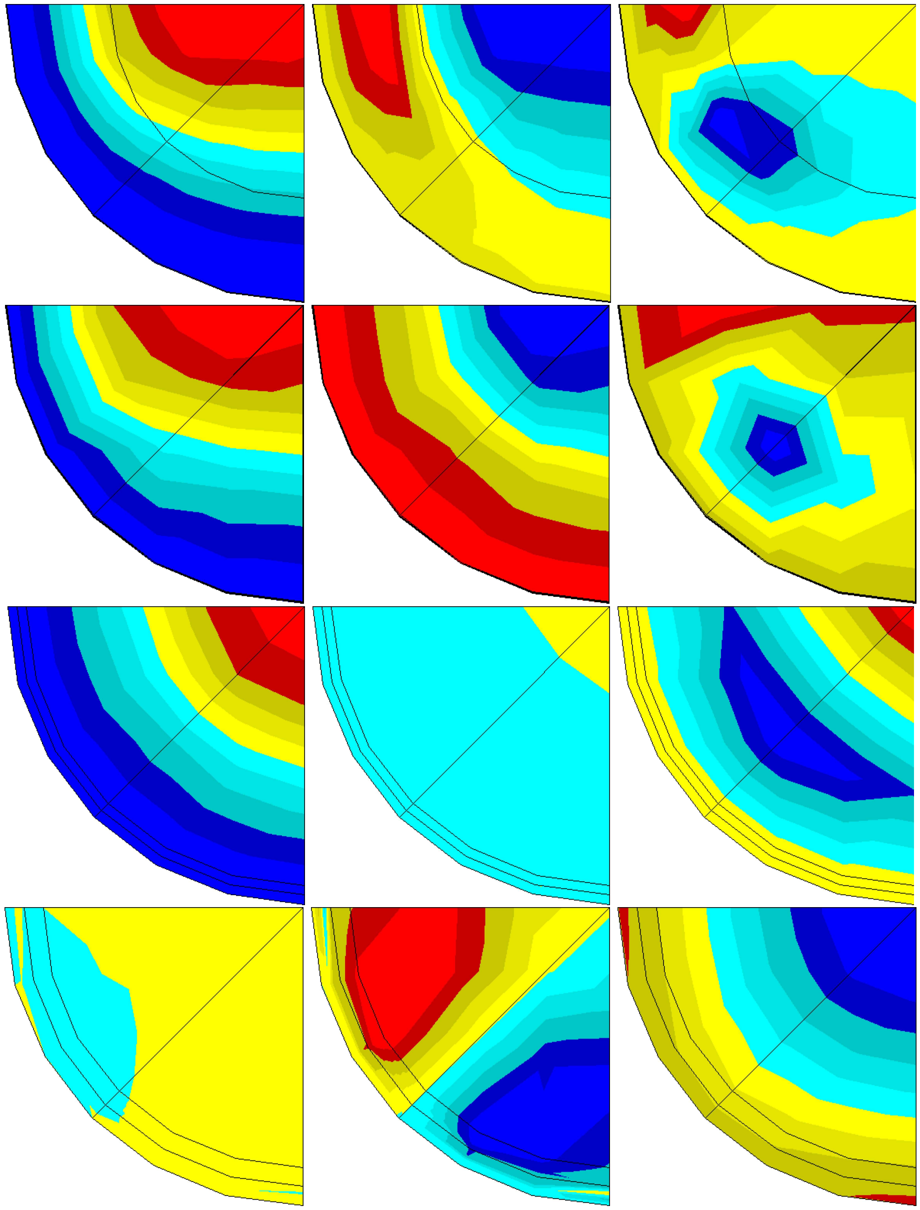

Figure 6: Midplane plot of u_3 for first three eigenvectors, load L4.
Row 1: $\varepsilon = 0.004$, mesh MID, Row 2: $\varepsilon = 0.004$, mesh FIN, Row 3: $\varepsilon = 0.064$, mesh FIN, Row 4: $\varepsilon = 0.128$, mesh FIN

7 Appendix

In this section, we discuss the form of the pre-buckling stress σ_*^ε that is induced by choosing a loading of the form (6.1),(6.2) used in the numerical experiments in Section 6. In particular, we investigate the limit σ_*^0, as $\varepsilon \to 0$, taken in the sense of regular terms (without taking the boundary layer into account).

We assume that the pre-existing loading has the form

$$F_* = \big(f_\alpha(x_\top, \mathrm{x}_3), \varepsilon f_3(x_\top, \mathrm{x}_3)\big), \quad T_*^\pm = (\varepsilon t_\alpha^\pm(x_\top), \varepsilon^2 t_3^\pm(x_\top)). \tag{7.1}$$

(Here x_3 is the scaled vertical variable $x_3\varepsilon^{-1}$.) The pre-buckling displacement $\boldsymbol{u}_*^\varepsilon$ solves

Find $\boldsymbol{u}_*^\varepsilon \in H^1(\Omega^\varepsilon)^3$ *such that* $\forall \boldsymbol{v} \in H^1(\Omega^\varepsilon)^3$

$$\int_{\Omega^\varepsilon} Ae(\boldsymbol{u}_*^\varepsilon) : e(\boldsymbol{v}) \,\mathrm{d}x = \int_{\Omega^\varepsilon} F_* \cdot \boldsymbol{v} \,\mathrm{d}x + \int_{\Gamma^+} T_*^+ \cdot \boldsymbol{v} \,\mathrm{d}x_\top + \int_{\Gamma^-} T_*^- \cdot \boldsymbol{v} \,\mathrm{d}x_\top. \tag{7.2}$$

According to [4], the outer part, i.e., outside the boundary layer, of the asymptotics of the solution $\boldsymbol{u}_*^\varepsilon$ as $\varepsilon \to 0$ takes the form

$$\begin{aligned} \boldsymbol{u}_*^\varepsilon \simeq \varepsilon^{-1}\boldsymbol{u}^0_{\mathrm{KL},\mathsf{b}} + \boldsymbol{u}^0_{\mathrm{KL},\mathsf{m}} + \boldsymbol{u}^1_{\mathrm{KL},\mathsf{b}} + \varepsilon(\boldsymbol{u}^1_{\mathrm{KL},\mathsf{m}} + \boldsymbol{u}^2_{\mathrm{KL},\mathsf{b}} + \boldsymbol{v}_*^1) + \ldots \\ \ldots + \varepsilon^k(\boldsymbol{u}^k_{\mathrm{KL},\mathsf{m}} + \boldsymbol{u}^{k+1}_{\mathrm{KL},\mathsf{b}} + \boldsymbol{v}_*^k) + \cdots \end{aligned} \tag{7.3}$$

where

- $\boldsymbol{u}^k_{\mathrm{KL},\mathsf{b}}$ and $\boldsymbol{u}^k_{\mathrm{KL},\mathsf{m}}$ are the bending and membrane parts on Ω^ε of the Kirchhoff-Love displacement with generator $\boldsymbol{\zeta}_*^k = \boldsymbol{\zeta}^k_{*,\mathsf{m}} + \boldsymbol{\zeta}^k_{*,\mathsf{b}}$, see sec. 5.a, namely

$$\boldsymbol{u}^k_{\mathrm{KL},\mathsf{b}} = (-\mathrm{x}_3\partial_1\zeta^k_{*,3},\ \mathrm{x}_3\partial_2\zeta^k_{*,3},\ \zeta^k_{*,3}) \quad \text{and} \quad \boldsymbol{u}^k_{\mathrm{KL},\mathsf{m}} = (\zeta^k_{*,1},\ \zeta^k_{*,2}, 0).$$

- $\boldsymbol{v}_*^k = \boldsymbol{v}_*^k(x_\top, \mathrm{x}_3)$, i.e. does not depend on ε in the scaled domain Ω.

The formula for $\boldsymbol{v}_*^1$ is

$$\boldsymbol{v}_*^1(x_\top, \mathrm{x}_3) = \frac{\overline{\lambda}}{6(\overline{\lambda} + 2\overline{\mu})}\Big(0,\ 0,\ -6\mathrm{x}_3 \,\mathrm{div}_\top \boldsymbol{\zeta}^0_{*,\top} + (3\mathrm{x}_3^2 - 1)\,\Delta_\top\zeta^0_{*,3}\Big). \tag{7.4}$$

Then we can check that the stresses σ_*^ε of the expansion (7.3) have the form,

$$\begin{aligned} (\sigma_*^\varepsilon)_{\alpha\beta}(x) &= \overline{\sigma}_{\alpha\beta}(x_\top, \mathrm{x}_3) + \mathcal{O}(\varepsilon), \\ (\sigma_*^\varepsilon)_{\alpha3}(x) &= \varepsilon\,\overline{\sigma}_{\alpha3}(x_\top, \mathrm{x}_3) + \mathcal{O}(\varepsilon^2), \\ (\sigma_*^\varepsilon)_{33}(x) &= \varepsilon^2\overline{\sigma}_{33}(x_\top, \mathrm{x}_3) + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{7.5}$$

which, when compared with (3.1), show that the first part of Hypothesis 3.1 will hold in the limit.

It remains to discuss the second part of Hypothesis 3.1, which is the core of our assumption which says that $\overline{\sigma}_{\alpha\beta}$ is a genuine principal part in the sense of (3.3). Accordingly, let us compute

$$\overline{\sigma}_{\alpha\beta}(x_\top, \mathrm{x}_3) = -\mathrm{x}_3(2\overline{\mu}\partial_{\alpha\beta} + \delta_{\alpha\beta}\widehat{\lambda}\Delta_\top)\zeta^0_{*,3} + 2\overline{\mu}e_{\alpha\beta}(\boldsymbol{\zeta}^0_{*,\mathsf{m}}) + \delta_{\alpha\beta}\widehat{\lambda}\,\mathrm{div}_\top \boldsymbol{\zeta}^0_{*,\mathsf{m}}. \tag{7.6}$$

We note that $-\mathrm{x}_3(2\overline{\mu}\partial_{\alpha\beta} + \delta_{\alpha\beta}\widehat{\lambda}\Delta_\top)\zeta^0_{*,3}$ is the bending contribution to $\overline{\sigma}_{\alpha\beta}$ and that it has a null average across the thickness, whereas $2\overline{\mu}e_{\alpha\beta}(\boldsymbol{\zeta}^0_{*,\mathsf{m}}) + \delta_{\alpha\beta}\widehat{\lambda}\,\mathrm{div}_\top\,\boldsymbol{\zeta}^0_{*,\mathsf{m}}$ is the membrane contribution. Hence, (3.2) gives

$$p^0_{\alpha\beta} = 2\overline{\mu}e_{\alpha\beta}(\boldsymbol{\zeta}^0_{*,\mathsf{m}}) + \delta_{\alpha\beta}\widehat{\lambda}\,\mathrm{div}_\top\,\boldsymbol{\zeta}^0_{*,\mathsf{m}}.$$

According to [4] again, the membrane Kirchhoff-Love generator $\boldsymbol{\zeta}^0_{*,\mathsf{m}}$ is the solution of the boundary value problem on the midsurface ω, – see (4.17) for the plain stress operator $\mathbf{L}_\mathsf{m}$:

Find $\boldsymbol{\zeta}^0_{*,\mathsf{m}} \in H^1_0(\omega)^2$ *such that*

$$\mathbf{L}_\mathsf{m}\boldsymbol{\zeta}^0_{*,\mathsf{m}}(x_\top) = -\frac{1}{2}\Big(\int_{-1}^{1} f_\alpha(x_\top,\mathrm{x}_3)\mathrm{dx}_3 + t^+_\alpha(x_\top) + t^-_\alpha(x_\top)\Big), \quad x_\top \in \omega. \tag{7.7}$$

Hence, $\boldsymbol{\zeta}^0_{*,\mathsf{m}}$, and consequently $p^0_{\alpha\beta}$, can be expected to be non-zero, provided the right hand side of (7.7) is non-zero. This is clearly the case when the resultant R^0_m is non-zero, as in the case of Loads L2 and L3 (see (6.4)). As explained in Remark 3.2, we can therefore conclude that Hypothesis 3.1 (ii) will (in general) hold for all such cases where the first term of the membrane resultant of the pre-existing load is not identically zero.

We note also that $p^0_{\alpha\beta}$ clearly vanishes when $R^0_\mathsf{m} = 0$ as for loads L1 and L4, showing that Hypothesis 3.1 (ii) is violated for these cases.

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