Plane waveguides with corners and small angle limit

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Abstract

We study the eigenpairs of the Dirichlet Laplacian for plane waveguides with corners. We prove that in presence of a non-trivial corner there exist eigenvalues under the essential spectrum. Moreover we provide accurate asymptotics for eigenpairs associated with the lowest eigenvalues in the small angle limit. For this, we also investigate the eigenpairs of a one-dimensional toy model related to Born-Oppenheimer approximation, and of the Dirichlet Laplacian on triangles with sharp angles.

Keywords: Discrete spectrum, Semi-classical limit, Born-Oppenheimer approximation, Quasi-mode, Agmon estimates.

1 Introduction and main results

1.1 Motivations

This is a well-known fact, from the papers [10, 7, 8], that curvature makes discrete spectrum to appear in waveguides. Moreover the analysis of this spectrum can be accurately performed in the thin tube limit (in dimension 2 and 3, see [10, Section 5]). In fact, this asymptotical regime corresponds to a semiclassical limit so that the standard techniques of [16] could have been used to investigate that problem.

Curvature inducing discrete spectrum, this is then a natural question to ask what happens in dimension 2 when there is corner (infinite curvature): do discrete spectrum always exist? This question is investigated with the L-shape waveguide in [11] where the existence of discrete spectrum is proved. For an arbitrary angle, this existence is proved in [3] and an asymptotic study of the ground energy is done when θ goes to $\frac{\pi}{2}$ (where θ is the semi-opening of the waveguide). This problem is also analyzed (through experiments and numerical simulations)

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in [6] in which this two-dimensional model is derived from the three-dimensional Maxwell equations. Another question which arises is the estimation of the lowest eigenvalues in the regime $\theta \to 0$.

For the case in dimension 3, we can cite the paper [12] which deals with the Dirichlet Laplacian in a conical layer. In this case, there is an infinite number of eigenvalues below the essential spectrum. The other initial motivation for the present investigation is our previous work [4] in which we study the Neumann realization on $\mathbb{R}^2_+ = \{(s,t) \in \mathbb{R}^2 : t > 0\}$ of the Schrödinger operator $-\partial_s^2 - \partial_t^2 + (t\cos\theta - s\sin\theta)^2$ in the regime $\theta \to 0$ (see also [18, 17]). It turns out that the lowest eigenfunctions of this operator are concentrated near the cancellation line of the potential as it is confirmed by numerical experiments which also enlighten the link between a confining electric potential and a strip with Dirichlet boundary conditions.

It will appear in the analysis of plane waveguides with corners (also called ”broken strips”), that we shall precisely study the Dirichlet problem on a triangle with a small angle. This subject is already dealt with in [13, Theorem 1] where three term asymptotics is proved for the two lowest eigenvalues by using the asymptotics of zeros of Bessel functions. Finally, we can mention the papers [14, 15] whose results provide the two terms asymptotics for the thin rhombi and also [5] which deals with a regular case (thin ellipse for instance).

Note sur l'état d'avancement de nos travaux Le document présent rassemble tout ce que nous avons démontré. Nous allons considérer la question de la fintude du nombre de valeurs propres (qui est évidente numériquement) et voir si nous pouvons le prouver. Ensuite nous ferons un plan de répartition entre l'article mathématique et l'article des proceedings du congrès SMAI.

Notation We denote by $\sigma_{\mathsf{ess}}(A)$ the essential spectrum of a self-adjoint operator A, and by $\sigma_{\mathsf{dis}}(A)$ its discrete spectrum. The L^2 norm will always be denoted by $\|\cdot\|$ without mention of the integration domain.

1.2 Definition of the operator and spectral questions

Let us denote by (x_1, x_2) the Cartesian coordinates of the plane and $\Delta = \partial_1^2 + \partial_2^2$ the Laplace operator. We investigate the spectrum of the Dirichlet Laplacian $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ on the “waveguide”

$$\Omega_\theta = \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \tan\theta < |x_2| < \left(x_1 + \frac{\pi}{\sin\theta}\right)\tan\theta\right\},$$

where $\theta \in \left(0, \frac{\pi}{2}\right)$. In particular, the width of Ω_θ is π. We will also need to introduce the triangular end of this waveguide:

$$\mathsf{Tri}_\theta = \left\{(x_1,x_2) \in \mathbb{R}_- \times \mathbb{R} : x_1 \tan\theta < |x_2| < \left(x_1 + \frac{\pi}{\sin\theta}\right)\tan\theta\right\}$$

and the corresponding Dirichlet Laplacian denoted by $-\Delta^{\mathsf{Dir}}_{\mathsf{Tri}_\theta}$.

Proposition 1.1 *For any $\theta \in (0, \frac{\pi}{2})$ the essential spectrum of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ coincides with $[1, +\infty)$.*

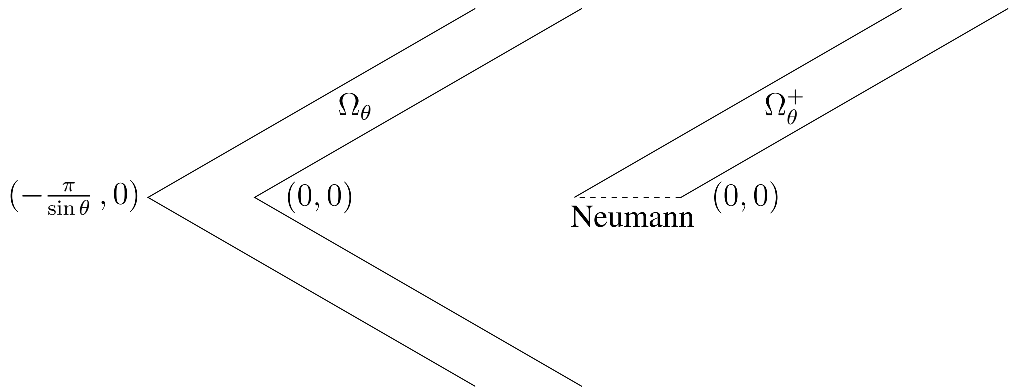

Figure 1: The waveguide Ω_θ and the half-guide Ω_θ^+ for $\theta = \frac{\pi}{6}$.

Proof: By Persson's theorem (see [19]), we obtain that the infimum of $\sigma_{\mathsf{ess}}(-\Delta^{\mathsf{Dir}}_{\Omega_\theta})$ is 1. The construction of appropriate Weyl sequences yields that any value $\lambda \in [1, \infty)$ belongs to $\sigma_{\mathsf{ess}}(-\Delta^{\mathsf{Dir}}_{\Omega_\theta})$. $\square$

Reduction to the half-guide It will be convenient to use Ω_θ^+ defined by, see Fig. 1:

$$\Omega_\theta^+ = \left\{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}_+ : x_1 \tan\theta < x_2 < \left(x_1 + \frac{\pi}{\sin\theta}\right)\tan\theta\right\}.$$

We define the Dirichlet part of the boundary by $\partial_{\mathsf{Dir}}\Omega_\theta^+ = \partial_{\mathsf{Dir}}\Omega_\theta \cap \partial\Omega_\theta^+$. Let us introduce $-\Delta^{\mathsf{Mix}}_{\Omega_\theta^+}$ as the Laplacian with mixed Dirichlet-Neumann conditions on Ω_θ^+ with domain:

$$\begin{aligned}\mathsf{Dom}(\Delta^{\mathsf{Mix}}_{\Omega_\theta^+}) = \big\{\psi \in H^1(\Omega_\theta^+) : \quad & \Delta\psi \in L^2(\Omega_\theta^+), \\ & \psi = 0 \text{ on } \partial_{\mathsf{Dir}}\Omega_\theta^+ \qquad \text{and} \quad \partial_2\psi = 0 \text{ on } x_2 = 0\big\}.\end{aligned}$$

Then $\sigma_{\mathsf{ess}}(-\Delta^{\mathsf{Mix}}_{\Omega_\theta^+})$ coincides with $\sigma_{\mathsf{ess}}(-\Delta^{\mathsf{Dir}}_{\Omega_\theta})$. Concerning the discrete spectrum we have:

Proposition 1.2 *For any $\theta \in (0, \frac{\pi}{2})$, $\sigma_{\mathsf{dis}}(-\Delta^{\mathsf{Dir}}_{\Omega_\theta})$ coincides with $\sigma_{\mathsf{dis}}(-\Delta^{\mathsf{Mix}}_{\Omega_\theta^+})$.*

Proof: The proof relies on the invariance of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ by the symmetry $x_2 \mapsto -x_2$.

(i) If (λ, u_λ) is an eigenpair of $-\Delta^{\mathsf{Mix}}_{\Omega_\theta^+}$, the even extension of u_λ to Ω_θ defines an eigenfunction of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ associated with the same eigenvalue λ. Therefore $\sigma_{\mathsf{dis}}(-\Delta^{\mathsf{Mix}}_{\Omega_\theta^+}) \subset \sigma_{\mathsf{dis}}(-\Delta^{\mathsf{Dir}}_{\Omega_\theta})$.

(ii) Conversely, let (λ, u_λ) be an eigenpair of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ with $\lambda < 1$. Splitting the odd part u_λ^{odd} and the even part $u_\lambda^{\mathsf{even}}$ of u_λ with respect to x_2, we obtain:

$$-\Delta^{\mathsf{Dir}}_{\Omega_\theta} u_\lambda^{\mathsf{odd}} = \lambda u_\lambda^{\mathsf{odd}}, \quad -\Delta^{\mathsf{Dir}}_{\Omega_\theta} u_\lambda^{\mathsf{even}} = \lambda u_\lambda^{\mathsf{even}}.$$

Let us check that $u_\lambda^{\mathsf{odd}} = 0$. If it is not the case, this would mean that λ is an eigenvalue for the Dirichlet Laplacian on the half-waveguide whose spectrum begins at 1. Thus, we have necessarily: $u_\lambda^{\mathsf{odd}} = 0$ and $u_\lambda^{\mathsf{even}}$ (which satisfies the Neumann condition on $x_2 = 0$ by symmetry) is an eigenfunction of $-\Delta^{\mathsf{Mix}}_{\Omega_\theta^+}$ associated with λ. $\square$

Rescaling of the half-guide In order to analyze the asymptotics $\theta \to 0$, it will be useful to rescale the integration domain and transfer the dependence on θ into the coefficients of the operator. For this reason, let us perform the following linear change of coordinates:

$$x = x_1\sqrt{2}\sin\theta, \quad y = x_2\sqrt{2}\cos\theta,$$

which maps Ω_θ^+ onto $\Omega_{\pi/4}^+$ which will serve as reference domain. That is why we set for simplicity

$$\Omega := \Omega_{\pi/4}^+ \quad \text{and} \quad \partial_{\mathsf{Dir}}\Omega = \partial_{\mathsf{Dir}}\Omega_{\pi/4}^+.$$

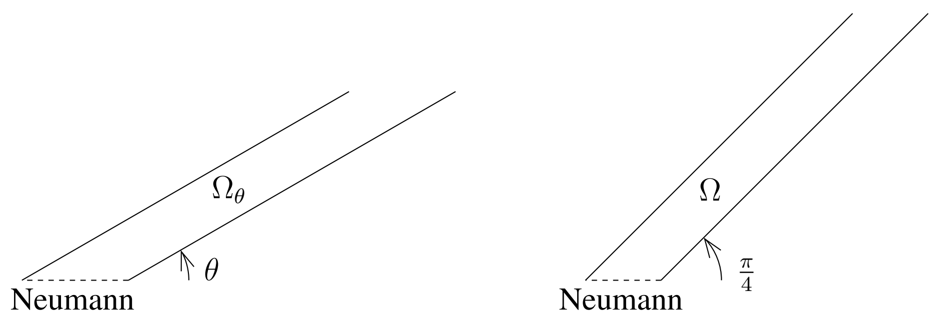

Figure 2: The half-guide Ω_θ^+ for $\theta = \frac{\pi}{6}$ and the reference domain Ω.

Then, $-\Delta^{\mathsf{Mix}}_{\Omega_\theta^+}$ is unitarily equivalent to the operator defined on $\Omega^+_{\frac{\pi}{4}}$ by:

$$\mathcal{D}_{\mathsf{Gui}}(\theta) := -2\sin^2\theta\,\partial_x^2 - 2\cos^2\theta\,\partial_y^2,$$

with Neumann condition on $y = 0$ and Dirichlet everywhere else on the boundary of Ω. We let $h = \tan\theta$; after a division by $2\cos^2\theta$, we get the new operator:

$$\mathcal{L}_{\mathsf{Gui}}(h) = -h^2\partial_x^2 - \partial_y^2,$$

with domain:

$$\begin{aligned}\mathsf{Dom}(\mathcal{L}_{\mathsf{Gui}}(h)) = \big\{\psi \in H^1(\Omega) : \quad & \mathcal{L}_{\mathsf{Gui}}(h)\psi \in L^2(\Omega), \\ & \psi = 0 \ \text{ on } \ \partial_{\mathsf{Dir}}\Omega \quad \text{and} \quad \partial_y\psi = 0 \ \text{ on } \ y = 0\big\}.\end{aligned}$$

As a preliminary investigation, we are to going to study $\mathcal{L}_{\mathsf{Tri}}(h)$ which denotes the same operator $-h^2\partial_x^2 - \partial_y^2$ with Dirichlet conditions on the triangular end Tri of the model waveguide $\Omega_{\pi/4}$

$$\mathsf{Tri} = \left\{(x, y) \in \mathbb{R}^2 : -\pi\sqrt{2} < x < 0 \text{ and } |y| < x + \pi\sqrt{2}\right\}.$$

1.3 Born-Oppenheimer approximation and models

In the analysis of $\mathcal{L}_{\mathsf{Tri}}(h)$ and $\mathcal{L}_{\mathsf{Gui}}(h)$, we will see that its so-called Born-Oppenheimer approximation will play an important role:

$$\mathcal{H}_{\mathsf{BO,Gui}}(h) = -h^2\partial_x^2 + V(x), \tag{1.1}$$

where

$$V(x) = \begin{cases} \dfrac{\pi^2}{4(x+\pi\sqrt{2})^2} & \text{when } x \in (-\pi\sqrt{2}, 0) \\ \dfrac{1}{2} & \text{when } x \geq 0 \end{cases}$$

This effective potential V is obtained by replacing $-\partial_y^2$ by its lowest eigenvalue on each slice at fixed x. When h goes to zero, the behavior of the ground eigenpairs of $\mathcal{H}_{\mathsf{BO,Gui}}(h)$ is driven by the structure of the potential near its minimum, attained at $x = 0$: In a neighborhood of $x = 0$, V can be approximated by its tangents, which provides the approximate potential V_{app} defined by

$$V_{\mathsf{app}}(x) = \begin{cases} \dfrac{1}{8} - \dfrac{1}{4\pi\sqrt{2}}\,x & \text{when } x \in (-\pi\sqrt{2}, 0) \\ \dfrac{1}{2} & \text{when } x \geq 0 \end{cases}$$

After the change of variables $z = \sqrt{2}x/(3\pi)$ and the change of parameter $\kappa = 4h/(3\pi\sqrt{3})$, we find the correspondence

$$-h^2\partial_x^2 + V_{\mathsf{app}}(x) \sim \frac{3}{8}\mathcal{H}_{\mathsf{toy}}(\kappa)[z;\partial_z] + \frac{1}{8} \tag{1.2}$$

where the toy model operator $\mathcal{H}_{\mathsf{toy}}(\kappa)[z;\partial_z]$ is defined as:

$$\mathcal{H}_{\mathsf{toy}}(\kappa) = -\kappa^2\partial_z^2 + W(z) \quad \text{with} \quad W(z) = \begin{cases} -z & \text{when } z \leq 0, \\ 1 & \text{when } z \geq 0. \end{cases} \tag{1.3}$$

This toy model invites us to recall the properties of the Airy operator.

The Airy function Let us recall the basic properties of the Airy operator, i.e. the Dirichlet realization on $L^2(\mathbb{R}_-)$ of $-\partial_x^2 - x$. This is standard that this (positive) operator has compact resolvent. Thus, its spectrum can be described as an increasing sequence of eigenvalues tending to $+\infty$. Let us use the traditional notation Ai for the Airy function. We recall that it satisfies:

$$-\mathsf{Ai}'' + x\mathsf{Ai} = 0.$$

All along this paper, we will use A the reverse Airy function, i.e. $\mathsf{A}(x) = \mathsf{Ai}(-x)$. We recall that A does not vanish on $\mathbb{R}_-$, is exponentially decreasing when $x \to -\infty$ and that its zeros (which are simple) form an increasing sequence of positive numbers tending to $+\infty$.

Notation 1.3 The n-th zero of A will be denoted by $z_{\mathsf{A}}(n)$.

If (λ, ψ_λ) is an eigenpair of the Airy operator, we have $-\psi_\lambda'' - x\psi_\lambda = \lambda\psi_\lambda$, hence the equation $-\psi_\lambda'' - (x+\lambda)\psi_\lambda = 0$. We deduce that there exists a number $c(\lambda)$ so that:

$$\psi_\lambda(x) = c(\lambda)\mathsf{A}(x+\lambda).$$

With those remarks, we can see that the spectrum of the Airy operator is $\{z_{\mathsf{A}}(n), n \geq 1\}$ and these eigenvalues are simple.

Finally, let us introduce the Dirichlet realization on $L^2((-\pi\sqrt{2}, 0))$ of:

$$\mathcal{H}_{\mathsf{BO,Tri}}(h) = -h^2\partial_x^2 + \frac{\pi^2}{4(x+\pi\sqrt{2})^2}. \tag{1.4}$$

This operator is the Born-Oppenheimer approximation of the operator $\mathcal{L}_{\mathsf{Tri}}(h)$ on the triangle Tri and will be the first order approximation of $\mathcal{H}_{\mathsf{BO,Gui}}(h)$ defined in (1.1).

1.4 Main results on eigenvalues

We can now state the main results of this paper. The first one proves that there is always discrete spectrum in a waveguide with corner:

Proposition 1.4 *For $\theta \in \left(0, \frac{\pi}{2}\right)$, $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ has at least one eigenvalue below 1.*

This result is already known (see [3]) but we will provide another proof related to a more general argument developed in [10, 7, 8] for waveguides with curvature. Moreover, the discrete spectrum is increasing with respect to θ:

Proposition 1.5 *The eigenvalues of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ are continuous increasing functions of θ.*

The lowest eigenvalues of the toy model admit analytic expansions with respect to $\kappa^{1/3}$ (when κ is small enough):

Theorem 1.6 *For all $N_0 \in \mathbb{N}$, there exists $\kappa_0 > 0$ such that, for $\kappa \in (0, \kappa_0)$, there exists at least N_0 eigenvalues of $\mathcal{H}_{\mathsf{toy}}(\kappa)$ below 1. Denoting by $\lambda_{\mathsf{toy},n}(\kappa)$ these eigenvalues, we have the converging expansions for $1 \leq n \leq N_0$ and κ small enough:*

$$\lambda_{\mathsf{toy},n}(\kappa) = \kappa^{2/3}\sum_{j=0}^{+\infty} \alpha_{j,n}\kappa^{j/3} \quad \textit{with first coefficient} \;\; \alpha_{0,n} = z_{\mathsf{A}}(n).$$

The corresponding eigenvectors have expansions in powers of $h^{1/3}$ with the scales $z/h^{2/3}$ when $z < 0$ and z/h when $z > 0$, see (3.6).

The lowest eigenvalues of the triangle admit expansions at any order[1] in powers of $h^{2/3}$:

Theorem 1.7 *The eigenvalues of $\mathcal{L}_{\mathsf{Tri}}(h)$, denoted by $\lambda_{\mathsf{Tri},n}(h)$, admit the expansions:*

$$\lambda_{\mathsf{Tri},n}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \beta_{j,n} h^{j/3} \quad \textit{with} \;\; \beta_{0,n} = \frac{1}{8}, \;\; \beta_{1,n} = 0, \;\; \textit{and} \;\; \beta_{2,n} = (4\pi\sqrt{2})^{-2/3} z_{\mathsf{A}}(n),$$

the terms of odd rank being zero for $j \leq 8$. The corresponding eigenvectors have expansions in powers of $h^{1/3}$ with both scales $x/h^{2/3}$ and x/h.

In terms of the physical domain Tri_θ, we deduce immediately from the previous theorem that the eigenvalues of $-\Delta^{\mathsf{Dir}}_{\mathsf{Tri}_\theta}$, denoted by $\mu_{\mathsf{Tri},n}(\theta)$, admit the expansions:

$$\mu_{\mathsf{Tri},n}(\theta) \underset{h\to 0}{\sim} \sum_{j\geq 0} \beta^{\Delta}_{j,n} \theta^{j/3} \quad \text{with} \;\; \beta^{\Delta}_{0,n} = \frac{1}{4}, \;\; \beta^{\Delta}_{1,n} = 0, \;\; \text{and} \;\; \beta^{\Delta}_{2,n} = 2(4\pi\sqrt{2})^{-2/3} z_{\mathsf{A}}(n),$$

with the same properties as above. Performing the scaling:

$$\tilde{x}_1 = 2\cos\theta\sin\theta\, x_1 \quad \tilde{x}_2 = 2\cos\theta\sin\theta\, x_2,$$

we get an isosceles triangle with angle $\alpha = 2\theta$ and side $c = 2\pi$ denoted by $\mu_{\widetilde{\mathsf{Tri}},n}(\alpha)$. With this scaling, the eigenvalues satisfy the relation:

$$\mu_{\mathsf{Tri},n}(\theta) = (\sin\alpha)^2 \mu_{\widetilde{\mathsf{Tri}},n}(\alpha),$$

so that we find back the result of [13, Theorem 1] and notice that the odd term after $O(\alpha^{2/3})$ in the asymptotics of $\mu_{\widetilde{\mathsf{Tri}},1}(\alpha)$ is not zero.

Remark 1.8 As it will be seen in the proof, the existence of a non-zero coefficient $\beta_{9,n}$ at the order 9 reduces to the evaluation of an integral, see (5.6). The numerical value of this integral for a few lowest values of n will be investigated in a further work.

Finally, the lowest eigenvalues of the waveguide admit expansions in powers of $h^{1/3}$:

Theorem 1.9 *For all N_0, there exists $h_0 > 0$, such that for $h \in (0, h_0)$ the N_0 first eigenvalues of $\mathcal{L}_{\mathsf{Gui}}(h)$ exist. These eigenvalues, denoted by $\lambda_{\mathsf{Gui},n}(h)$, admit the expansions:*

$$\lambda_{\mathsf{Gui},n}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \gamma_{j,n} h^{j/3} \quad \textit{with} \;\; \gamma_{0,n} = \frac{1}{8}, \;\; \gamma_{1,n} = 0, \;\; \textit{and} \;\; \gamma_{2,n} = (4\pi\sqrt{2})^{-2/3} z_{\mathsf{A}}(n)$$

and the term of order h is not zero. The corresponding eigenvectors have expansions in powers of $h^{1/3}$ with the scale x/h when $x > 0$, and both scales $x/h^{2/3}$ and x/h when $x < 0$, see (6.6).

[1] By the notation $\lambda(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} c_j h^{\rho j}$ we mean that for any positive integer J we have the estimate $|\lambda(h) - \sum_{0\leq j\leq J} c_j h^{\rho j}| \leq C_J\, h^{\rho(J+1)}$ for h small enough.

We get the obvious corollary concerning the eigenvalues in the waveguide Ω_θ:

Corollary 1.10 *For all N_0, there exists $\theta_0 > 0$, such that for $\theta \in (0, \theta_0)$ the N_0 first eigenvalues of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ exist. These eigenvalues, denoted by $\mu_{\mathsf{Gui},n}(\theta)$, admit the expansions:*

$$\mu_{\mathsf{Gui},n}(\theta) \underset{h\to 0}{\sim} \sum_{j\geq 0} \gamma^{\Delta}_{j,n}\theta^{j/3} \quad \textit{with } \gamma^{\Delta}_{0,n} = \frac{1}{4},\ \ \gamma^{\Delta}_{1,n} = 0,\ \ \textit{and } \gamma^{\Delta}_{2,n} = 2(4\pi\sqrt{2})^{-2/3} z_{\mathsf{A}}(n)$$

and the term of order θ is not zero. The corresponding eigenvectors have expansions in powers of $\theta^{1/3}$ with terms independent of θ, and with terms at the scale $x_1 h^{1/3}$ when $x_1 < 0$,.

1.5 Organization of the paper

In Section 2, we prove the existence of discrete spectrum for waveguides with corners and its monotonicity with respect to the opening θ. In Section 3 we investigate the toy model $\mathcal{H}_{\mathsf{toy}}(\kappa)$ through a construction of quasimodes and an ODE analysis. In Section 4 we study $\mathcal{H}_{\mathsf{BO,Tri}}(h)$ thanks to the construction of quasimodes and Agmon estimates in order to analyze the Dirichlet problem on a triangle with small angle. In Section 5 we apply the results of Section 4 through a projection method reducing the analysis to dimension 1. Finally, in Section 6, we perform again a construction of quasimodes for the waveguide and introduce in particular Dirichlet-to-Neumann operators to solve a transmission problem ; we conclude by comparing with the triangle case.

2 Elementary properties of the spectrum

In this section, we prove Propositions 1.4 and 1.5. For that purpose, we will work with another representation of $-\Delta^{\mathsf{Mix}}_{\Omega^+_\theta}$.

2.1 Preliminary

More precisely, let us define:

$$\widetilde{\Omega}^+_\theta = \left\{(x_1,x_2) \in \left(-\frac{\pi}{\tan\theta}, +\infty\right) \times (0,\pi) : x_2 < x_1 \tan\theta \text{ if } x_1 \in \left(-\frac{\pi}{\tan\theta}, 0\right)\right\}.$$

Figure 3: The reference half-guide $\widetilde{\Omega} := \widetilde{\Omega}^+_{\pi/4}$.

Let us denote $\partial_{\mathsf{Dir}}\widetilde{\Omega}_\theta^+$ the part of the boundary carrying the Dirichlet condition, i.e. : $x_2 = 0$ and on $x_2 = \pi$ if $x_1 \geq 0$. We put the Neumann condition everywhere else. This operator is unitarily equivalent to $-\Delta^{\mathsf{Mix}}_{\Omega_\theta^+}$. Let us now perform the change of variable:

$$x = x_1 \tan\theta, \quad y = x_2,$$

so that the new integration domain is $\widetilde{\Omega} := \widetilde{\Omega}^+_{\pi/4}$ and is independent of θ. The operator $-\Delta$ becomes :

$$\widetilde{\mathcal{L}}(\theta) = -\tan^2\theta\,\partial_x^2 - \partial_y^2,$$

with Dirichlet boundary condition on $\partial_{\mathsf{Dir}}\widetilde{\Omega}$ and Neumann condition on $\partial\widetilde{\Omega} \setminus \partial_{\mathsf{Dir}}\widetilde{\Omega}$. The form domain $\mathsf{Dom}(Q_\theta)$ associated with $\widetilde{\mathcal{L}}(\theta)$ is independent of θ:

$$\mathsf{Dom}(Q_\theta) = \{\psi \in H^1(\widetilde{\Omega}) : \quad \psi = 0 \ \text{ on } \ \partial_{\mathsf{Dir}}\widetilde{\Omega}\}.$$

The function $\theta \mapsto \tan^2\theta$ being increasing and continuous, the min-max principle (see [20, Chapter XIII]) implies that the Rayleigh quotients of $\widetilde{\mathcal{L}}(\theta)$ are increasing and continuous functions of θ, which proves Proposition 1.5. Let us prove that eigenvalues exist.

2.2 Existence of discrete spectrum

In this subsection, we prove Proposition 1.4 using an idea of [8, p. 104-105]. Let us introduce the following quadratic form, defined for $\psi \in \mathsf{Dom}(Q_\theta)$ by:

$$q_\theta(\psi) = Q_\theta(\psi) - \|\psi\|_2^2 = \int_{\widetilde{\Omega}} \left(\tan^2\theta|\partial_x\psi|^2 + |\partial_y\psi|^2\right) dxdy - \int_{\widetilde{\Omega}} |\psi|^2\, dxdy.$$

To prove our statement, this is enough to construct a function $\psi \in \mathsf{Dom}(Q_\theta)$ such that:

$$q_\theta(\psi) < 0.$$

In order to do that, we first consider the Weyl sequence defined as follows. Let χ be a smooth cutoff function equal to 1 for $x \leq 0$ and 0 for $x \geq 1$. We let, for $n \in \mathbb{N} \setminus \{0\}$:

$$\chi_n(x) = \chi\left(\frac{x}{n}\right) \quad \text{and} \quad \psi_n(x, y) = \chi_n(x)\sin y.$$

Estimate of $q_\theta(\psi_n)$ Using the support of χ_n, we find that $q_\theta(\psi_n)$ is equal to

$$\int_{-\pi}^0\int_0^{x+\pi}(\cos^2 y - \sin^2 y)\,dydx + \int_0^\infty\int_0^\pi\left(\tan^2\theta(\chi_n')^2\sin^2 y + \chi_n^2(\cos^2 y - \sin^2 y)\right)dydx.$$

Then, elementary computations provide:

$$\int_0^\pi(\cos^2 y - \sin^2 y)\,dy = 0 \quad \text{and} \quad \int_{-\pi}^0\int_0^{x+\pi}(\cos^2 y - \sin^2 y)\,dydx = 0.$$

Moreover, we have:

$$\int_0^\infty \int_0^\pi \tan^2\theta (\chi_n')^2 \sin^2 y\, dydx \leq \left(\int_0^1 |\chi'(u)|^2 du\right) \frac{\pi \tan^2\theta}{2n}.$$

Hence:

$$q_\theta(\psi_n) \leq \left(\int_0^1 |\chi'(u)|^2 du\right) \frac{\pi \tan^2\theta}{2n}\,. \tag{2.1}$$

Perturbation of ψ_n We introduce a smooth cutoff function η of x supported in $(-\pi, 0)$. We consider a function f of $y \in [0, \pi]$ to be determined later and satisfying $f(0) = 0$. We define $\phi(x, y) = \eta(x)f(y)$. For $\varepsilon > 0$ to be chosen small enough, we introduce:

$$\psi_{n,\varepsilon}(x, y) = \psi_n(x, y) + \varepsilon\phi(x, y).$$

We have:

$$q_\theta(\psi_{n,\varepsilon}) = q_\theta(\psi_n) + 2\varepsilon b_\theta(\psi_n, \phi) + \varepsilon^2 q_\theta(\phi),$$

where b_θ is the bilinear form associated to q_θ. Let us compute $b_\theta(\psi_n, \phi)$. We can write, thanks to support considerations:

$$b_\theta(\psi_n, \phi) = \int_{-\pi}^0 \int_0^{x+\pi} \eta(x)\big(\cos y f'(y) - \sin y f(y)\big)\, dydx = \int_{-\pi}^0 \int_0^{x+\pi} \eta(x)\big(\cos y f(y)\big)' dydx.$$

Using $f(0) = 0$, this leads to:

$$b_\theta(\psi_n, \phi) = \int_{-\pi}^0 \eta(x) \cos(x + \pi) f(x + \pi)\, dx.$$

We choose $f(y) = \eta(y - \pi) \cos(y - \pi)$ and we find:

$$b_\theta(\psi_n, \phi) = -\int_{-\pi}^0 \eta^2(x) \cos^2(x)\, dx = -\Gamma < 0.$$

This implies, using (2.1):

$$q_\theta(\psi_{n,\varepsilon}) \leq \left(\int_0^1 |\chi'(u)|^2 du\right) \frac{\pi \tan^2\theta}{2n} - \Gamma\varepsilon + D\varepsilon^2,$$

where $D = q_\theta(\phi)$ is a constant. There exists $\varepsilon > 0$ such that:

$$-\Gamma\varepsilon + D\varepsilon^2 \leq -\frac{\Gamma}{2}\varepsilon.$$

The angle θ being fixed, we can take N large enough so that

$$\left(\int_0^1 |\chi'(u)|^2 du\right) \frac{\pi \tan^2\theta}{2N} \leq \frac{\Gamma}{4}\,\varepsilon,$$

from which we deduce:

$$q_\theta(\psi_{N,\varepsilon}) \leq -\frac{\Gamma}{4}\,\varepsilon < 0,$$

which ends the proof of Proposition 1.4.

3 A toy model

This subsection is devoted to the proof of Theorem 1.6. This proof is divided into two steps. First, we construct quasimodes and quasi-eigenvalues for $\mathcal{H}_{\mathsf{toy}}(\kappa)$ and second, we show that the lowest quasi-eigenvalues are the approximations of the lowest eigenvalues of $\mathcal{H}_{\mathsf{toy}}(\kappa)$ of the same rank.

3.1 Construction of quasimodes

This subsection aims at proving the following proposition:

Proposition 3.1 *For all $N_0 \in \mathbb{N}^*$, there exists $\kappa_0 > 0$ and $C > 0$ such that for $\kappa \in (0, \kappa_0)$:*

$$\operatorname{dist}\big(\sigma_{\mathsf{dis}}(\mathcal{H}_{\mathsf{toy}}(\kappa),\ \kappa^{2/3} z_{\mathsf{A}}(n)\big) \le C\kappa, \quad n = 1, \cdots N_0. \tag{3.1}$$

Proof: The basic tool for the proof is the construction of quasimodes and the application of the spectral theorem. Convenient quasimodes are given by power series in $\kappa^{1/3}$ of *profiles* at the scales

$$s = \kappa^{-2/3} z \ \text{ when } \ z \le 0 \quad \text{and} \quad \sigma = \kappa^{-1} z \ \text{ when } \ z \ge 0.$$

More precisely we look for quasi-eigenfunctions ψ_κ in the form:

$$\psi_\kappa(z) \sim \begin{cases} \sum_{j\ge0} \Psi_{\mathsf{lef},j}(s)\, \kappa^{j/3} & \text{when } \ z \le 0 \\ \sum_{j\ge0} \Phi_{\mathsf{rig},j}(\sigma)\, \kappa^{j/3} & \text{when } \ z \ge 0\,, \end{cases} \tag{3.2}$$

and quasi-eigenvalues in the form:

$$\alpha_\kappa \sim \kappa^{2/3} \sum_{j\ge0} \alpha_j \kappa^{j/3} \quad \text{as} \quad \kappa \to 0. \tag{3.3}$$

The continuity conditions at $z = 0$ provide the formal identities:

$$\begin{cases} \sum_{j\ge0} \Psi_{\mathsf{lef},j}(0)\, \kappa^{j/3} & = & \sum_{j\ge0} \Phi_{\mathsf{rig},j}(0)\, \kappa^{j/3} \\ \kappa^{-2/3} \sum_{j\ge0} \partial_s \Psi_{\mathsf{lef},j}(0)\, \kappa^{j/3} & = & \kappa^{-1} \sum_{j\ge0} \partial_\sigma \Phi_{\mathsf{rig},j}(0)\, \kappa^{j/3}, \end{cases} \tag{3.4}$$

and the formal eigen-equation is

$$-\kappa^2 \psi_\kappa''(z) + W(z)\psi_\kappa(z) = \alpha_\kappa \psi_\kappa(z) \quad z \in \mathbb{R}. \tag{3.5}$$

Determination of α_0 Collecting the terms in $\kappa^{2/3}$ in (3.5) and using (3.2)-(3.4) we obtain:

$$\begin{cases} -\Phi''_{\mathsf{rig},0}(\sigma) + \Phi_{\mathsf{rig},0}(\sigma) = 0 & \text{for } \ \sigma > 0, & \text{and} & \Phi'_{\mathsf{rig},0}(0) = 0, \\ -\Psi''_{\mathsf{lef},0}(s) - s\Psi_{\mathsf{lef},0}(s) = \alpha_0 \Psi_{\mathsf{lef},0}(s) & \text{for } \ s < 0, & \text{and} & \Psi_{\mathsf{lef},0}(0) = \Phi_{\mathsf{rig},0}(0). \end{cases}$$

We deduce first that $\Phi_{\mathsf{rig},0} = 0$ and thus $\Psi_{\mathsf{lef},0}(0) = 0$. This implies that α_0 is a zero of the reverse Airy function A. At this stage we can choose a positive integer n, take $\alpha_0 = z_{\mathsf{A}}(n)$ and $\Psi_{\mathsf{lef},0}$ as the corresponding normalized eigenfunction $g_{(n)}$.

Determination of α_1 Collecting the terms in κ, we get the equations:

$$\begin{cases} -\Phi''_{\mathsf{rig},1} + \Phi_{\mathsf{rig},1} = 0 & \text{for } \sigma > 0, \quad \text{and } \ \Phi'_{\mathsf{rig},1}(0) = \Psi'_{\mathsf{lef},0}(0), \\ -\Psi''_{\mathsf{lef},1} - s\Psi_{\mathsf{lef},1} - \alpha_0\Psi_{\mathsf{lef},1} = \alpha_1\Psi_{\mathsf{lef},0} & \text{for } s < 0, \quad \text{and } \ \Psi_{\mathsf{lef},1}(0) = \Phi_{\mathsf{rig},1}(0). \end{cases}$$

We find first:

$$\Phi_{\mathsf{rig},1}(\sigma) = -\Psi'_{\mathsf{lef},0}(0)e^{-\sigma}.$$

Moreover we obtain the existence of a number α_1 and of an exponentially decreasing $\Psi_{\mathsf{lef},1}$ solution of the second equation with the help of the following lemma:

Lemma 3.2 *Let $n \geq 1$. We denote by $g_{(n)}$ an eigenvector of the operator $-\partial_s^2 - s$ associated with the eigenvalue $z_{\mathsf{A}}(n)$ and normalized in $L^2(\mathbb{R}_-)$. Let $f = f(s)$ be a real function with an exponential decay and let $c \in \mathbb{R}$. Then there exists a unique $\alpha \in \mathbb{R}$ such that the problem:*

$$\big(-\partial_s^2 - s - z_{\mathsf{A}}(n)\big)\, g = f + \alpha g_{(n)} \ \text{ in } \ \mathbb{R}_-, \text{ with } g(0) = c,$$

has a solution with an exponential decay. There holds

$$\alpha = c\, g'_{(n)}(0) - \int_{-\infty}^{0} f(s)\, g_{(n)}(s)\, ds.$$

Further terms A similar procedure can be reproduced at each step, providing the construction of $\Phi_{\mathsf{rig},j}$, then α_j and $\Psi_{\mathsf{lef},j}$, for any $j \geq 2$.

Expressions for quasimodes Relying on the previous iterative constructions we can set for all integer $J \geq 0$

$$\psi_\kappa^{[J]}(z) = \begin{cases} \displaystyle\sum_{j=0}^{J+2} \Psi_{\mathsf{lef},j}\Big(\frac{z}{\kappa^{2/3}}\Big)\,\kappa^{j/3} & \text{when } \ z \leq 0 \\ \displaystyle\sum_{j=0}^{J+2} \Phi_{\mathsf{rig},j}\Big(\frac{z}{\kappa}\Big)\,\kappa^{j/3} + \Psi'_{\mathsf{lef},J+2}(0)\,\kappa^{J/3} z\,\chi\Big(\frac{z}{\kappa}\Big) & \text{when } \ z \geq 0\,, \end{cases} \tag{3.6}$$

where χ is a smooth cutoff function equal to 1 near 0. By construction, $\psi_\kappa^{[J]}$ and its first derivative are continuous in $z = 0$. Moreover $\psi_\kappa^{[J]}$ is exponentially decreasing as $z \to \pm\infty$. Therefore it belongs to the domain of $\mathcal{H}_{\mathsf{toy}}(\kappa)$. With this remark and taking the error introduced by χ into account, we get for all $\kappa_0 > 0$:

$$\Big\|\Big(\mathcal{H}_{\mathsf{toy}}(\kappa) - \kappa^{2/3}\big(z_{\mathsf{A}}(n) + \sum_{j=1}^{J+2}\alpha_j\kappa^{J/3}\big)\Big)\psi_\kappa^{[J]}\Big\| \leq C(J,n,\kappa_0)\,\kappa^{1+J/3}, \quad \forall\kappa \leq \kappa_0.$$

Hence

$$\big\|\big(\mathcal{H}_{\mathsf{toy}}(\kappa) - \kappa^{2/3}z_{\mathsf{A}}(n)\big)\,\psi_\kappa\big\| \leq C(n,\kappa_0)\,\kappa, \quad \forall\kappa \leq \kappa_0,$$

and the spectral theorem applies. In particular, for κ small enough, the discrete spectrum of $\mathcal{H}_{\mathsf{toy}}(\kappa)$ is not empty since $\sigma_{\mathsf{ess}}(\mathcal{H}_{\mathsf{toy}}(\kappa)) = [1, +\infty)$. $\square$

3.2 Expansion of the lowest eigenvalues

We now want to refine Proposition 3.1 by proving that the $\lambda_{\mathsf{toy},n}(\kappa)$ are power series with respect to $\kappa^{1/3}$ and whose coefficients are given by (3.3). We begin to prove the following proposition:

Proposition 3.3 *For all $N_0 \in \mathbb{N}^*$, there exists $\kappa_0 > 0$ and $C > 0$ such that for $\kappa \in (0, \kappa_0)$:*

$$|\lambda_{\mathsf{toy},n}(\kappa) - \kappa^{2/3} z_{\mathsf{A}}(n)| \leq C\kappa, \quad n = 1, \cdots N_0. \tag{3.7}$$

Proof: Let $N_0 \in \mathbb{N}^*$. We have proved in particular that, for all $\kappa \in (0, \kappa_0)$, the N_0 first eigenvalues $\lambda_{\mathsf{toy},n}(\kappa)$ (denoted by λ_n for shortness) exist and that they satisfy:

$$|\lambda_n| \leq C(N_0)\,\kappa^{2/3}, \quad \kappa \in (0, \kappa_0), \quad n = 1, \cdots N_0. \tag{3.8}$$

Let us denote by ψ_n an eigenfunction associated with λ_n so that $\langle \psi_n, \psi_m \rangle = 0$ if $n \neq m$. For $z < 0$ we have:

$$-\kappa^2 \psi_n'' - z\psi_n = \lambda_n \psi_n.$$

Thus, there exists $c_n(\kappa) \neq 0$ such that, for $z < 0$:

$$\psi_n(z) = c_n(\kappa) \mathsf{A}(\kappa^{-2/3} z + \kappa^{-2/3} \lambda_n).$$

On the other side we obtain the existence of $d_n(\kappa) \neq 0$ such that, for $z > 0$:

$$\psi_n(z) = d_n(\kappa) e^{-\kappa^{-1} z \sqrt{1-\lambda_n}}.$$

The transmission conditions at $z = 0$ imply:

$$c_n(\kappa) \mathsf{A}(\kappa^{-2/3} \lambda_n) = d_n(\kappa), \quad c_n(\kappa) \kappa^{1/3} \mathsf{A}'(\kappa^{-2/3} \lambda_n) = -d_n(\kappa) \sqrt{1 - \lambda_n}.$$

This implies:

$$\mathsf{A}(\kappa^{-2/3} \lambda_n) = -\frac{\kappa^{1/3}}{\sqrt{1 - \lambda_n}} \mathsf{A}'(\kappa^{-2/3} \lambda_n). \tag{3.9}$$

We infer:

$$|\mathsf{A}(\kappa^{-2/3} \lambda_n)| \leq C(N_0)\,\kappa^{1/3}.$$

Since $\kappa^{-2/3} \lambda_n$ is bounded, see (3.8), and the zeros of the Airy function being isolated and simple, we deduce that for all $n \in \{1, \cdots, N_0\}$, there exists $p = p(n, \kappa)$ such that:

$$|\kappa^{-2/3} \lambda_n - z_{\mathsf{A}}(p)| \leq C(N_0) \kappa^{1/3}.$$

Note that p is bounded too. In view of Proposition 3.1, it suffices now to prove than if κ is small enough and $n \neq m$ (with n, $m \leq N_0$), the integers $p(n, \kappa)$ and $p(m, \kappa)$ are distinct. Let us prove this by contradiction. Since the considered sets of integers n, m and p are finite, the negation of what we want to prove can be written as

$$\exists\, m,n,p \in \mathbb{N}, \quad \forall \kappa_1 > 0, \quad \exists \kappa \in (0,\kappa_1) \quad \text{such that} \quad p(m,\kappa) = p(n,\kappa) = p.$$

The eigenfunctions can be taken in the form:

$$\psi_j(z) = \begin{cases} \mathsf{A}(\kappa^{-2/3} z + \kappa^{-2/3}\lambda_j) & \text{when } \ z \leq 0 \\ \mathsf{A}(\kappa^{-2/3}\lambda_j)\, e^{-\kappa^{-1} z \sqrt{1-\lambda_j}} & \text{when } \ z \geq 0\,, \end{cases} \quad \text{for} \quad j = m,n,$$

and we have

$$\langle \psi_n, \psi_m \rangle = \int_{z<0} \mathsf{A}(\kappa^{-2/3} z + \kappa^{-2/3}\lambda_n)\mathsf{A}(\kappa^{-2/3} z + \kappa^{-2/3}\lambda_m)\, dz + O(\kappa^{5/3}) = 0.$$

A rescaling leads to:

$$\left| \int_{z<0} \mathsf{A}(z + \kappa^{-2/3}\lambda_n)\, \mathsf{A}(z + \kappa^{-2/3}\lambda_m)\, dz \right| \leq C(N_0)\kappa.$$

By assumption, $\kappa^{-2/3}\lambda_n = z_{\mathsf{A}}(p) + O(\kappa^{1/3})$ and $\kappa^{-2/3}\lambda_m = z_{\mathsf{A}}(p) + O(\kappa^{1/3})$. For $j = n, m$, A being Lipschitz on $(-\infty, M]$ for all M, there exists $D(N_0) > 0$ such that for all $z < 0$:

$$|\mathsf{A}(z + \kappa^{-2/3}\lambda_j) - \mathsf{A}(z + z_{\mathsf{A}}(p))| \leq D(N_0)\kappa^{1/3}, \quad \text{for} \quad j = m,n,$$

so that:

$$\left| \int_{z<0} \mathsf{A}(z + \kappa^{-2/3}\lambda_n)\mathsf{A}(z + \kappa^{-2/3}\lambda_m)\, dz - \int_{z<0} \mathsf{A}^2(z + z_A(p))\, dz \right| \leq \tilde{D}(N_0)\kappa^{1/3}.$$

We deduce:

$$\forall \kappa_1 > 0, \quad \exists \kappa \in (0,\kappa_1) \quad \text{such that} \quad \left| \int_{z<0} \mathsf{A}^2(z + z_{\mathsf{A}}(p))\, dz \right| \leq \tilde{D}(N_0)\kappa^{1/3}$$

which leads to a contradiction and ends the proof of Proposition 3.3. $\square$

Proof of Theorem 1.6 Let us observe that Proposition 3.3 permits to separate the N_0 first eigenvalues when $\kappa < \kappa_0$. Let us write $\delta = \kappa^{1/3}$. We let:

$$\breve{\lambda}_n(\delta) = \delta^{-2}\lambda_{\mathsf{toy},n}(\delta^3),$$

so that $\breve{\lambda}_n(\delta)$ is uniformly bounded for $n = 1, \ldots, N_0$ and $\delta < \kappa_0^{1/3}$.

We deduce from (3.9):

$$\mathsf{A}(\breve{\lambda}_n(\delta)) = -\frac{\delta}{\sqrt{1 - \delta^2 \breve{\lambda}_n(\delta)}}\, \mathsf{A}'(\breve{\lambda}_n(\delta)). \tag{3.10}$$

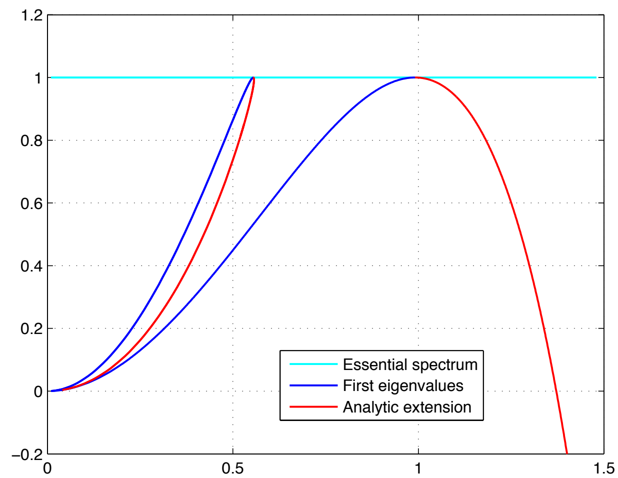

Figure 4: The first two eigenvalues $\lambda_{\mathsf{toy},1}$ and $\lambda_{\mathsf{toy},2}$ as functions of $\delta = \kappa^{1/3}$.

We know that A is analytic and, using again the simplicity of its zeros, we can apply the analytic implicit function theorem near $\delta = 0$ and for all $n \in \{1, \cdots, N_0\}$, which ends the proof of Theorem 1.6.

From (3.10), we can deduce that the $\breve{\lambda}_n(\delta)$ are solutions of the analytic equation:

$$(1 - \delta^2 \breve{\lambda})\mathsf{A}(\breve{\lambda})^2 - \delta^2\, \mathsf{A}'(\breve{\lambda})^2 = 0 \tag{3.11}$$

This equation provides an analytic extension of the functions $\delta \mapsto \breve{\lambda}_n(\delta)$, hence of $\lambda_{\mathsf{toy},n} = \delta^2 \breve{\lambda}_n(\delta)$, in the sense of analytic curves. We represent in Figures 4 and 5 the first two eigenvalues and their analytic extensions. Taking the continuity and monotonicity of the eigenvalues with respect to δ into account, we can see that any branch which starts by $\delta \mapsto \lambda(\delta) = \delta^2 z_{\mathsf{A}} + \mathcal{O}(\delta^3)$ represents an eigenvalue while $\lambda(\delta)$ is less that 1. Beyond 1, the Rayleigh quotient stays $\equiv 1$, but the curve $\lambda(\delta)$ has an analytic extension as a continuation of a branch of roots of the equation (3.11).

4 Born-Oppenheimer approximation for the triangle

This section is devoted to the analysis of $\mathcal{H}_{\mathsf{BO},\mathsf{Tri}}(h)$ defined in (1.4).

Proposition 4.1 *The eigenvalues of $\mathcal{H}_{\mathsf{BO},\mathsf{Tri}}(h)$, denoted by $\lambda_{\mathsf{BO},\mathsf{Tri},n}(h)$, admit the expansions:*

$$\lambda_{\mathsf{BO},\mathsf{Tri},n}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \hat{\beta}_{j,n} h^{2j/3}, \quad \textit{with} \quad \hat{\beta}_{0,n} = \frac{1}{8} \ \textit{and} \ \hat{\beta}_{1,n} = (4\pi\sqrt{2})^{-2/3} z_{\mathsf{A}}(n).$$

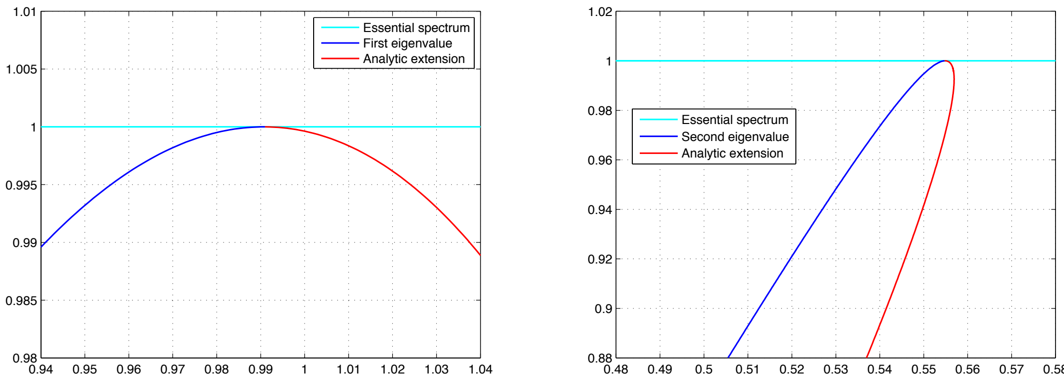

Figure 5: The eigenvalues $\lambda_{\mathsf{toy},1}$ and $\lambda_{\mathsf{toy},2}$ as functions of $\delta = \kappa^{1/3}$, zoom near the bottom of the essential spectrum.

4.1 Quasimodes

In this subsection, we construct quasimodes to prove the proposition:

Proposition 4.2 *For all $N_0 \in \mathbb{N}^*$, there exists $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$:*

$$\operatorname{dist}\Big(\sigma_{\mathsf{dis}}(\mathcal{H}_{\mathsf{BO,Tri}}(h)),\ \frac{1}{8} + h^{2/3}(4\pi\sqrt{2})^{-2/3} z_{\mathsf{A}}(n)\Big) \le Ch^{4/3}, \quad n = 1, \cdots N_0. \tag{4.1}$$

Proof: The proper scale in x is $h^{2/3}$ as can be seen by approximating the potential in $x = 0$ by its tangent and recognizing the Airy operator. Thus, we will construct quasimodes ψ_h as functions of $s = h^{-2/3}x$: We look for quasi-eigenpairs (λ_h, ψ_h) in the form of series

$$\lambda_h \sim \sum_{j\ge 0} \hat{\beta}_j h^{2j/3} \quad \text{and} \quad \psi_h(x) \sim \sum_{j \ge 0} \Psi_j(s) h^{2j/3}$$

in order to solve $\mathcal{H}_{\mathsf{BO,Tri}}(h)\psi_h = \lambda_h \psi_h$ in the sense of formal series. A Taylor expansion at $x = 0$ of the potential V yields:

$$\mathcal{H}_{\mathsf{BO,Tri}}(h) \sim -h^2\partial_x^2 + \sum_{j\ge 0} V_j x^j, \quad \text{with} \quad V_0 = \frac{1}{8} \text{ and } V_1 = -\frac{1}{4\pi\sqrt{2}},$$

which, in s variable, becomes

$$\mathcal{H}_{\mathsf{BO,Tri}}(h) \sim \frac{1}{8} + h^{2/3}\big(-\partial_s^2 + V_1 s\big) + \sum_{j\ge 2} h^{2j/3} V_j s^j. \tag{4.2}$$

The construction of the terms $\hat{\beta}_j$ and Ψ_j is similar (even simpler) than for Proposition 3.1.

- The expansion (4.2) yields that $\hat{\beta}_0 = \frac{1}{8}$, and collecting the terms in $h^{2/3}$ and we obtain:

$$-\Psi_0''(s) - \frac{s}{4\pi\sqrt{2}}\Psi_0(s) = \hat{\beta}_1\Psi_0(s) \quad \forall s < 0 \quad \text{and} \quad \Psi_0(0) = 0. \tag{4.3}$$

Thus for any chosen positive integer n we can take $\hat{\beta}_1 = (4\pi\sqrt{2})^{-2/3} z_{\mathsf{A}}(n)$ together with $\Psi_0(s) = \mathsf{A}((4\pi\sqrt{2})^{1/3}s + z_{\mathsf{A}}(n))$.

- Collecting the terms in $h^{4/3}$ we obtain

$$-\Psi_1''(s) + V_1 s\Psi_1(s) - \hat{\beta}_1\Psi_1(s) = \hat{\beta}_2\Psi_0 - V_2 s^2\Psi_0 \quad \forall s < 0 \quad \text{and} \quad \Psi_1(0) = 0.$$

The compatibility condition states that $\hat{\beta}_2\langle\Psi_0, \Psi_0\rangle = V_2\langle s^2\Psi_0, \Psi_0\rangle$. This determines $\hat{\beta}_2$ and implies the existence of a unique solution $\Psi_1 \in L^2(\mathbb{R}_-)$ such that $\langle\Psi_1, \Psi_0\rangle = 0$.

- This procedure can be continued at any order and determines $(\hat{\beta}_j, \Psi_j)$ at each step.

- To conclude, we take a cutoff function $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ equal to 1 near 0 and to 0 for $|x| \geq \varepsilon_0 > 0$ with $\varepsilon_0 < \pi\sqrt{2}$. We choose $n \geq 1$, $J \geq 0$ and introduce:

$$\psi_h^{[J]}(x) = \chi(x)\sum_{j=0}^{J}\Psi_j(h^{-2/3}x)h^{2j/3}.$$

Using the exponential decay of $x \mapsto \Psi_j(h^{-2/3}x)$ and the definition of Ψ_j and $\hat{\beta}_j$, we get for any $h_0 > 0$ the existence of $C(n, J, h_0) > 0$ such that:

$$\left\|\left(\mathcal{H}_{\mathsf{BO,Tri}}(h) - \sum_{j=0}^{J}\hat{\beta}_j h^{2j/3}\right)\psi_h^{[J]}\right\| \leq C(n, J, h_0)\, h^{2(J+1)/3}, \quad \forall h \in (0, h_0).$$

This proves the existence of quasimodes at any order and ends the proof of Proposition 4.2. □

4.2 Agmon estimates and consequences

In this subsection, we prove Agmon estimates (see [1, 2]) for the eigenfunctions of $\mathcal{H}_{\mathsf{BO,Tri}}(h)$ and deduce Proposition 4.1. There will be two kind of estimates: near $x = -\pi\sqrt{2}$ and near $x = 0$. In the analysis of the triangles (cf. Section 5), we will meet the same estimates. Let us consider an eigenpair (λ, ψ) of $\mathcal{H}_{\mathsf{BO,Tri}}(h)$. The Agmon identity writes, for some Lipschitz function Φ to be determined:

$$\int_{-\pi\sqrt{2}}^{0} h^2|\partial_x(e^\Phi\psi)|^2 + V(x)|e^\Phi\psi|^2 - h^2|\Phi' e^\Phi|^2 - \lambda|(e^\Phi\psi)|^2\, dx = 0. \tag{4.4}$$

It is a consequence of Proposition 4.2 that the lowest N_0 eigenvalues λ of $\mathcal{H}_{\mathsf{BO,Tri}}(h)$ satisfy:

$$|\lambda - \tfrac{1}{8}| \leq \Gamma_0\, h^{2/3}, \tag{4.5}$$

for some positive constant Γ_0 depending on N_0.

Agmon estimates near $x = 0$ We use (4.4) and the convexity of V to get the inequality:

$$\int_{-\pi\sqrt{2}}^{0} h^2|\partial_x(e^{\Phi}\psi)|^2 + \left(\frac{1}{8} - \frac{x}{4\pi\sqrt{2}}\right)|e^{\Phi}\psi|^2 - h^2|\Phi' e^{\Phi}|^2 - \lambda|(e^{\Phi}\psi)|^2\, dx \leq 0.$$

With (4.5), we deduce:

$$\int_{-\pi\sqrt{2}}^{0} -\frac{x}{4\pi\sqrt{2}}|e^{\Phi}\psi|^2 - h^2|\Phi' e^{\Phi}|^2 - Ch^{2/3}|(e^{\Phi}\psi)|^2\, dx \leq 0.$$

This leads to the choice

$$\Phi(x) = \eta\, h^{-1}|x|^{3/2},$$

for a number $\eta > 0$ to be chosen small enough. We get:

$$\int_{-\pi\sqrt{2}}^{0} \left(\frac{|x|}{4\pi\sqrt{2}} - \frac{9}{4}\eta^2|x| - Ch^{2/3}\right)|e^{\Phi}\psi|^2\, dx \leq 0.$$

For η small enough, we obtain the existence of $\tilde{\eta} > 0$ such that:

$$\int_{-\pi\sqrt{2}}^{0} \left(\tilde{\eta}|x| - Ch^{2/3}\right)|e^{\Phi}\psi|^2\, dx \leq 0.$$

Splitting the integral into the parts $-\pi\sqrt{2} < x < -Dh^{2/3}$ (where Φ is unbounded) and $-Dh^{2/3} < x < 0$ (where Φ is bounded) with $\tilde{\eta}D - C = d > 0$, we find:

$$\begin{aligned}\int_{-\pi\sqrt{2}}^{-Dh^{2/3}} d\, h^{2/3}|e^{\Phi}\psi|^2\, dx &\leq \int_{-\pi\sqrt{2}}^{-Dh^{2/3}} \left(\tilde{\eta}|x| - Ch^{2/3}\right)|e^{\Phi}\psi|^2\, dx \\ &\leq \int_{-Dh^{2/3}}^{0} \left(\tilde{\eta}|x| + Ch^{2/3}\right)|e^{\Phi}\psi|^2\, dx \leq \tilde{C}h^{2/3}\int_{-Dh^{2/3}}^{0} |\psi|^2\, dx.\end{aligned}$$

We deduce the proposition:

Proposition 4.3 *Let* $\Gamma_0 > 0$. *There exist* $h_0 > 0$, $C_0 > 0$ *and* $\eta_0 > 0$ *such that for* $h \in (0, h_0)$ *and all eigenpair* (λ, ψ) *of* $\mathcal{H}_{\mathsf{BO,Tri}}(h)$ *satisfying* $|\lambda - \frac{1}{8}| \leq \Gamma_0 h^{2/3}$, *we have:*

$$\int_{-\pi\sqrt{2}}^{0} e^{\eta_0 h^{-1}|x|^{3/2}}\left(|\psi|^2 + |h^{2/3}\partial_x\psi|^2\right) dx \leq C_0\|\psi\|^2.$$

Agmon estimates near $x = -\pi\sqrt{2}$ We use again (4.4) and (4.5):

$$\int_{-\pi\sqrt{2}}^{0} h^2|\partial_x(e^{\Phi}\psi)|^2 + \left(\frac{\pi^2}{4(x+\pi\sqrt{2})^2} - \frac{1}{8}\right)|e^{\Phi}\psi|^2 - h^2|\Phi' e^{\Phi}\psi|^2 - Ch^{2/3}|(e^{\Phi}\psi)|^2\, dx \leq 0.$$

We take:

$$\Phi(x) = -\rho h^{-1}\ln\left(D^{-1}(x + \pi\sqrt{2})\right),$$

where we choose $\rho \in (0, \frac{\pi}{2})$ so that there holds:

$$\int_{-\pi\sqrt{2}}^{0} \left(\left(\frac{\pi^2}{4} - \rho^2 \right) (x + \pi\sqrt{2})^{-2} - \frac{1}{8} \right) |e^{\Phi}\psi|^2 - Ch^{2/3} |(e^{\Phi}\psi)|^2 \, dx \leq 0,$$

and $D > 0$ large enough so that

$$\left(\frac{\pi^2}{4} - \rho^2 \right) D^2 - \frac{1}{8} > 0.$$

Then we split the integral into the parts $-\pi\sqrt{2} < x < -\pi\sqrt{2} + D$ (where Φ is unbounded) and $-\pi\sqrt{2} + D < x < 0$ (where Φ is bounded) and the same procedure as in the previous paragraph leads to the proposition:

Proposition 4.4 *Let $\Gamma_0 > 0$ and $\rho_0 \in (0, \frac{\pi}{2})$. There exist $h_0 > 0$, $C_0 > 0$ such that for any $h \in (0, h_0)$ and all eigenpair (λ, ψ) of $\mathcal{H}_{\mathsf{BO},\mathsf{Tri}}(h)$ satisfying $|\lambda - \frac{1}{8}| \leq \Gamma_0 h^{2/3}$, we have:*

$$\int_{-\pi\sqrt{2}}^{0} (x + \pi\sqrt{2})^{-\rho_0/h} \left(|\psi|^2 + |h\,\partial_x \psi|^2 \right) dx \leq C_0 \|\psi\|^2.$$

Proof of Proposition 4.1 Let us fix N_0 and consider the N_0 first eigenvalues of $\mathcal{H}_{\mathsf{BO},\mathsf{Tri}}(h)$ denoted by $\lambda_n = \lambda_{\mathsf{BO},\mathsf{Tri},n}(h)$. For each $n \in \{1, \cdots N_0\}$, we choose a normalized ψ_n in the eigenspace of λ_n so that $\langle \psi_n, \psi_m \rangle = 0$ for $n \neq m$. Let us introduce the space:

$$\mathfrak{E}_{N_0}(h) = \mathrm{span}(\psi_1, \ldots, \psi_{N_0}).$$

We recall that, for h small enough, (4.5) holds. We can write:

$$\mathcal{H}_{\mathsf{BO},\mathsf{Tri}}(h)\psi_n = \lambda_n \psi_n$$

so that (the ψ_n are orthogonal in L^2 and for the quadratic form), for all $\psi \in \mathfrak{E}_{N_0}(h)$:

$$Q_{\mathsf{BO},\mathsf{Tri},h}(\psi) \leq \lambda_{N_0} \|\psi\|^2.$$

For ε_0 small enough we introduce a smooth cutoff function χ being 0 for $|x + \pi\sqrt{2}| \leq \varepsilon_0$ and 1 for $|x + \pi\sqrt{2}| \geq 2\varepsilon_0$. Proposition 4.4 implies that:

$$Q_{\mathsf{BO},\mathsf{Tri},h}(\chi\psi) \leq (\lambda_{N_0} + O(h^\infty)) \|\chi\psi\|^2.$$

Then, Proposition 4.3 provides:

$$\left\langle \left(-h^2 \partial_x^2 - \frac{1}{4\pi\sqrt{2}} x + \frac{1}{8} \right) \chi\psi, \chi\psi \right\rangle \leq (\lambda_{N_0} + O(h^\infty)) \|\chi\psi\|^2,$$

where we have used the convexity. The dimension of $\chi\mathfrak{E}_{N_0}(h)$ is N_0 so that, with the properties of the Airy operator and the mini-max principle, we get:

$$\frac{1}{8} + (4\pi\sqrt{2})^{-2/3} z_{\mathsf{A}}(N_0) \leq \lambda_{N_0} + O(h^\infty).$$

This is true for all fixed N_0 and provides the separation of the lowest eigenvalues of $\mathcal{H}_{\mathsf{BO},\mathsf{Tri}}(h)$. Combined with Proposition 4.2, we obtain Proposition 4.1.

5 Triangle with Dirichlet boundary condition

The aim of this section is to prove Theorem 1.7. As usual, the proof will be divided into two main steps: a construction of quasimodes and the use of the true eigenfunctions of $\mathcal{L}_{\mathsf{Tri}}(h)$ as quasimodes for the Born-Oppenheimer approximation in order to obtain a lower bound for the true eigenvalues.

We first perform a change of variables to transform the triangle into a rectangle:

$$u = x \in (-\pi\sqrt{2}, 0), \quad t = \frac{y}{x + \pi\sqrt{2}} \in (-1, 1). \tag{5.1}$$

so that Tri is transformed into $\mathsf{Rec} = (-\pi\sqrt{2}, 0) \times (-1, 1)$. The operator $\mathcal{L}_{\mathsf{Tri}}(h)$ becomes:

$$\mathcal{L}_{\mathsf{Rec}}(h)(u, t; \partial_u, \partial_t) = -h^2 \Big(\partial_u - \frac{t}{u + \pi\sqrt{2}}\,\partial_t\Big)^2 - \frac{1}{(u + \pi\sqrt{2})^2}\,\partial_t^2,$$

with Dirichlet boundary conditions on $\partial\mathsf{Rec}$.

5.1 Quasimodes

This subsection is devoted to the proof of the following proposition.

Proposition 5.1 *There are sequences $(\beta_{j,n})_{j\geq 0}$ for any integer $n \geq 1$ so that there holds: For all $N_0 \in \mathbb{R}$ and $J \in \mathbb{N}$, there exists $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$*

$$\operatorname{dist}\Big(\sigma_{\mathsf{dis}}\big(\mathcal{L}_{\mathsf{Tri}}(h)\big), \sum_{j=0}^{J} \beta_{j,n} h^{j/3}\Big) \leq C h^{(J+1)/3}, \quad n = 1, \cdots N_0. \tag{5.2}$$

Moreover, we have: $\beta_{0,n} = \frac{1}{8}$, $\beta_{1,n} = 0$, and $\beta_{2,n} = (4\pi\sqrt{2})^{-2/3} z_{\mathsf{A}}(n)$.

Proof: We want to construct quasi-eigenpairs (β_h, ψ_h) for the operator $\mathcal{L}_{\mathsf{Tri}}(h)(\partial_x, \partial_y)$. It will be more convenient to work on the rectangle Rec with the operator $\mathcal{L}_{\mathsf{Rec}}(h)(u, t; \partial_u, \partial_t)$. We introduce the new scales $s = h^{-2/3}u$ and $\sigma = h^{-1}u$ and we look quasi-eigenpairs $(\beta_h, \hat{\psi}_h)$ in the form of series

$$\beta_h \sim \sum_{j\geq 0} \beta_j h^{j/3} \quad \text{and} \quad \hat{\psi}_h(u, t) \sim \sum_{j\geq 0} \big(\Psi_j(s, t) + \Phi_j(\sigma, t)\big) h^{j/3} \tag{5.3}$$

in order to solve $\mathcal{L}_{\mathsf{Rec}}(h)\hat{\psi}_h = \beta_h\hat{\psi}_h$ in the sense of formal series. As will be seen hereafter, an Ansatz containing the scale $h^{-2/3}u$ alone (like for the Born-Oppenheimer operator $\mathcal{H}_{\mathsf{BO},\mathsf{Tri}}(h)$) is not sufficient to construct quasi-modes for $\mathcal{L}_{\mathsf{Rec}}(h)$. Expanding the operator in powers of $h^{2/3}$, we obtain the formal series:

$$\mathcal{L}_{\mathsf{Rec}}(h)(h^{2/3}s, t; h^{-2/3}\partial_s, \partial_t) \sim \sum_{j\geq 0} \mathcal{L}_{2j} h^{2j/3} \quad \text{with leading term} \quad \mathcal{L}_0 = -\frac{1}{2\pi^2}\partial_t^2$$

and in powers of h:

$$\mathcal{L}_{\mathsf{Rec}}(h)(h\sigma, t; h^{-1}\partial_\sigma, \partial_t) \sim \sum_{j\geq 0} \mathcal{N}_{3j} h^j \quad \text{with leading term} \quad \mathcal{N}_0 = -\partial_\sigma^2 - \frac{1}{2\pi^2}\partial_t^2.$$

In what follows, in order to finally ensure the Dirichlet conditions on the triangle Tri, we will require for our Ansatz the boundary conditions, for any $j \in \mathbb{N}$:

$$\Psi_j(0,t) + \Phi_j(0,t) = 0, \quad -1 \leq t \leq 1 \tag{5.4}$$

$$\Psi_j(s,\pm 1) = 0, \;\; s < 0 \quad \text{and} \quad \Phi_j(\sigma, \pm 1) = 0, \;\; \sigma \leq 0. \tag{5.5}$$

More specifically, we are interested in the ground energy $\lambda = \frac{1}{8}$ of the Dirichlet problem for $\mathcal{L}_0$ on the interval $(-1,1)$. Thus we have to solve Dirichlet problems for the operators $\mathcal{N}_0 - \frac{1}{8}$ and $\mathcal{L}_0 - \frac{1}{8}$ on the half-strip $\mathsf{Hst} = \mathbb{R}_- \times (-1,1)$, and look for *exponentially decreasing solutions*. The situation is similar to that encountered in thin structure asymptotics with Neumann boundary conditions. The following lemma shares common features with the Saint-Venant principle, see for example [9, §2].

Lemma 5.2 *We denote the first normalized eigenvector of $\mathcal{L}_0$ on $H_0^1((-1,1))$ by c_0:*

$$c_0(t) = \cos\left(\frac{\pi}{2}t\right).$$

Let $F = F(\sigma,t)$ be a function in $L^2(\mathsf{Hst})$ with exponential decay with respect to σ and let $G \in H^{3/2}((-1,1))$ be a function of t with $G(\pm 1) = 0$. Then there exists a unique $\gamma \in \mathbb{R}$ such that the problem

$$\left(\mathcal{N}_0 - \frac{1}{8}\right)\Phi = F \;\; in \;\; \mathsf{Hst}, \quad \Phi(\sigma, \pm 1) = 0, \quad \Phi(0,t) = G(t) + \gamma c_0(t),$$

admits a (unique) solution in $H^2(\mathsf{Hst})$ with exponential decay. There holds

$$\gamma = -\int_{-\infty}^{0}\int_{-1}^{1} F(\sigma,t)\,\sigma c_0(t)\,d\sigma dt - \int_{-1}^{1} G(t)\,c_0(t)\,dt.$$

The following two lemmas are consequences of the Fredholm alternative.

Lemma 5.3 *Let $F = F(s,t)$ be a function in $L^2(\mathsf{Hst})$ with exponential decay with respect to s. Then, there exist solution(s) Ψ such that:*

$$\left(\mathcal{L}_0 - \frac{1}{8}\right)\Psi = F \;\; in \;\; \mathsf{Hst}, \quad \Psi(s,\pm 1) = 0$$

if and only if $\left\langle F(s,\cdot), c_0\right\rangle_t = 0$ for all $s < 0$. In this case, $\Psi(s,t) = \Psi^\perp(s,t) + g(s)c_0(t)$ where $\Psi^\perp$ satisfies $\left\langle \Psi(s,\cdot), c_0\right\rangle_t \equiv 0$ and has also an exponential decay.

Then, we will also need a rescaled version of Lemma 3.2.

Lemma 5.4 *Let $n \geq 1$. We recall that $z_{\mathsf{A}}(n)$ is the n-th zero of the reverse Airy function, and we denote by $g_{(n)}$ a normalized eigenvector of the operator $-\partial_s^2 - (4\pi\sqrt{2})^{-1}s$ with Dirichlet condition on $\mathbb{R}_-$ associated with the eigenvalue $(4\pi\sqrt{2})^{-2/3}z_{\mathsf{A}}(n)$. Let $f = f(s)$ be a function in $L^2(\mathbb{R}_-)$ with exponential decay and let $c \in \mathbb{R}$. Then there exists a unique $\beta \in \mathbb{R}$ such that the problem:*

$$\left(-\partial_s^2 - \frac{s}{4\pi\sqrt{2}} - (4\pi\sqrt{2})^{-2/3}z_{\mathsf{A}}(n)\right) g = f + \beta g_{(n)} \text{ in } \mathbb{R}_-, \text{ with } g(0) = c,$$

has a solution in $H^2(\mathbb{R}_-)$ with exponential decay.

Now we can start the construction of the terms of our Ansatz (5.3).

Terms in h^0 The equations provided by the constant terms are:

$$\mathcal{L}_0\Psi_0 = \beta_0\Psi_0(s,t), \quad \mathcal{N}_0\Phi_0 = \beta_0\Phi_0(s,t)$$

with boundary conditions (5.4)-(5.5) for $j = 0$, so that we choose $\beta_0 = \frac{1}{8}$ and $\Psi_0(s,t) = g_0(s)c_0(t)$. The boundary condition (5.4) provides: $\Phi_0(0,t) = -g_0(0)c_0(t)$ so that, with Lemma 5.2, we get $g_0(0) = 0$ and $\Phi_0 = 0$. The function $g_0(s)$ will be determined later.

Terms in $h^{1/3}$ Collecting the terms of order $h^{1/3}$, we are led to:

$$(\mathcal{L}_0 - \beta_0)\Psi_1 = \beta_1\Psi_0 - \mathcal{L}_1\Psi_1 = \beta_1\Psi_0, \quad (\mathcal{N}_0 - \beta_0)\Phi_1 = \beta_1\Phi_0 - \mathcal{N}_1\Phi_1 = 0$$

with boundary conditions (5.4)-(5.5) for $j = 1$. Using Lemma 5.3, we find $\beta_1 = 0$, $\Psi_1(s,t) = g_1(s)c_0(t)$, $g_1(0) = 0$ and $\Phi_1 = 0$.

Terms in $h^{2/3}$ We get:

$$(\mathcal{L}_0 - \beta_0)\Psi_2 = \beta_2\Psi_0 - \mathcal{L}_2\Psi_0, \quad (\mathcal{N}_0 - \beta_0)\Phi_2 = 0,$$

where $\mathcal{L}_2 = -\partial_s^2 + \frac{s}{\pi^3\sqrt{2}}\partial_t^2$ and with boundary conditions (5.4)-(5.5) for $j = 2$. Lemma 5.3 provides the equation in s variable

$$\left\langle(\beta_2\Psi_0 - \mathcal{L}_2\Psi_0(s,\cdot)), c_0\right\rangle_t = 0, \quad s < 0.$$

Taking the formula $\Psi_0 = g_0(s)c_0(t)$ into account this becomes

$$\beta_2 g_0(s) = \left(-\partial_s^2 - \frac{s}{4\pi\sqrt{2}}\right) g_0(s).$$

This equation leads to take $\beta_2 = (4\pi\sqrt{2})^{-2/3}z_{\mathsf{A}}(n)$ and for g_0 the corresponding eigenfunction $g_{(n)}$. We deduce $(\mathcal{L}_0 - \beta_0)\Psi_2 = 0$, then get $\Psi_2(s,t) = g_2(s)c_0(t)$ with $g_2(0) = 0$ and $\Phi_2 = 0$.

Terms in $h^{3/3}$ We get:

$$(\mathcal{L}_0 - \beta_0)\Psi_3 = \beta_3\Psi_0 + \beta_2\Psi_1 - \mathcal{L}_2\Psi_1, \quad (\mathcal{N}_0 - \beta_0)\Phi_3 = 0,$$

with boundary conditions (5.4)-(5.5) for $j = 3$. The scalar product with c_0 (Lemma 5.3) and then the scalar product with g_0 (Lemma 5.4) provide $\beta_3 = 0$ and $g_1 = 0$. We deduce: $\Psi_3(s,t) = g_3(s)c_0(t)$, and $g_3(0) = 0$, $\Phi_3 = 0$.

Terms in $h^{4/3}$ We get:

$$(\mathcal{L}_0 - \beta_0)\Psi_4 = \beta_4\Psi_0 + \beta_2\Psi_2 - \mathcal{L}_4\Psi_0 - \mathcal{L}_2\Psi_2, \quad (\mathcal{N}_0 - \beta_0)\Phi_4 = 0,$$

where

$$\mathcal{L}_4 = \frac{\sqrt{2}}{\pi}\, t\partial_t\partial_s - \frac{3}{4\pi^4}s^2\partial_t^2,$$

and with boundary conditions (5.4)-(5.5) for $j = 4$. The scalar product with c_0 provides an equation for g_2 and the scalar product with g_0 determines β_4. By Lemma 5.3 this step determines $\Psi_4 = \Psi_4^{\perp} + c_0(t)g_4(s)$ with a non-zero $\Psi_4^{\perp}$ and $g_4(0) = 0$. Since by construction $\left\langle \Psi_4^{\perp}(0,\cdot),\, c_0 \right\rangle_t = 0$, Lemma 5.2 yields a solution Φ_4 with exponential decay. Note that it also satisfies $\left\langle \Phi_4(\sigma,\cdot),\, c_0 \right\rangle_t = 0$ for all $\sigma < 0$.

Terms in $h^{5/3}$ We get:

$$(\mathcal{L}_0 - \beta_0)\Psi_5 = \beta_5\Psi_0 + \beta_2\Psi_3 - \mathcal{L}_2\Psi_3, \quad (\mathcal{N}_0 - \beta_0)\Phi_5 = 0,$$

and with boundary conditions (5.4)-(5.5) for $j = 5$. We find $\beta_5 = 0$, $g_3 = 0$, $\Psi_5 = g_5(s)c_0(t)$, $g_5(0) = 0$, $\Phi_5 = 0$.

Terms in $h^{6/3}$ We get:

$$(\mathcal{L}_0 - \beta_0)\Psi_6 = \beta_6\Psi_0 + \beta_4\Psi_2 + \beta_2\Psi_4 - \mathcal{L}_2\Psi_4 - \mathcal{L}_4\Psi_2, \quad (\mathcal{N}_0 - \beta_0)\Phi_6 = \beta_2\Phi_4,$$

and with boundary conditions (5.4)-(5.5) for $j = 6$. This determines β_6, g_4, $\Psi_6 = \Psi_6^{\perp} + c_0(t)g_6(s)$, $g_6(0) = 0$, and Φ_6 with exponential decay due to the orthogonality of Φ_4 to c_0.

Terms in $h^{7/3}$ We get:

$$(\mathcal{L}_0 - \beta_0)\Psi_7 = \beta_7\Psi_0 + \beta_2\Psi_5 - \mathcal{L}_2\Psi_5, \quad (\mathcal{N}_0 - \beta_0)\Phi_7 = -\mathcal{N}_3\Phi_4,$$

where

$$\mathcal{N}_3 = \frac{2}{\pi\sqrt{2}}t\partial_\sigma\partial_t + \frac{\sigma}{\pi^3\sqrt{2}}\partial_t^2,$$

and with boundary conditions (5.4)-(5.5) for $j = 7$. We take $\beta_7 = 0$, $g_5 = 0$, $\Psi_7 = g_7(s)c_0(t)$. Then, Lemma 5.2 induces a value for the trace $g_7(0)$ so that there exists Φ_7 with an exponential decay. Note that if there holds:

$$\int_{\mathsf{Hst}} (\mathcal{N}_3\Phi_4)(\sigma, t)\, \sigma c_0(t)\, d\sigma dt \neq 0, \tag{5.6}$$

we would deduce by Lemma 5.2 that $g_7(0) \neq 0$.

Terms in $h^{8/3}$ We get:

$$(\mathcal{L}_0 - \beta_0)\Psi_8 = \beta_8\Psi_0 + \beta_6\Psi_2 + \beta_4\Psi_4 + \beta_2\Psi_6 - \mathcal{L}_8\Psi_0 - \mathcal{L}_6\Psi_2 - \mathcal{L}_4\Psi_4 - \mathcal{L}_2\Psi_6,$$
$$(\mathcal{N}_0 - \beta_0)\Phi_8 = \beta_4\Phi_4 + \beta_2\Phi_6.$$

This determines β_8, g_6 and $\Psi_8 = \Psi_8^\perp + c_0 g_8$, the trace $g_8(0)$ and the exponentially decreasing solution Φ_8.

Terms in $h^{9/3}$ We get:

$$(\mathcal{L}_0 - \beta_0)\Psi_9 = \beta_9\Psi_0 + \beta_2\Psi_7 - \mathcal{L}_2\Psi_7, \quad (\mathcal{N}_0 - \beta_0)\Phi_9 = \beta_2\Phi_7 - \mathcal{N}_3\Phi_6.$$

We find β_9, g_7 and then $\Psi_9 = \Psi_9^\perp + c_0 g_9$ and $g_9(0)$, Φ_9. Note that if $g_7(0) \neq 0$, i.e. if (5.6) holds, we would deduce that $\beta_9 \neq 0$.

Continuation. The construction of the further terms goes on along the same lines. □

5.2 Agmon estimates

On our way to prove Theorem 1.7, we now state Agmon estimates like for $\mathcal{H}_{\mathsf{BO},\mathsf{Tri}}(h)$. Let us first notice that, due to Proposition 5.1, the lowest eigenvalues of $\mathcal{L}_{\mathsf{Tri}}(h)$ still satisfy an estimate like (4.5). It turns out that we have the following lower bound, for all $\psi \in \mathsf{Dom}(Q_{\mathsf{Tri},h})$:

$$Q_{\mathsf{Tri},h}(\psi) \geq \int_{\mathsf{Tri}} h^2|\partial_x\psi|^2 + \frac{\pi^2}{4(x+\pi\sqrt{2})^2}|\psi|^2\, dxdy.$$

Thus, the analysis giving Propositions 4.3 and 4.4 applies exactly in the same way and we obtain:

Proposition 5.5 *Let $\Gamma_0 > 0$. There exist $h_0 > 0$, $C_0 > 0$ and $\eta_0 > 0$ such that for $h \in (0, h_0)$ and all eigenpair (λ, ψ) of $\mathcal{L}_{\mathsf{Tri}}(h)$ satisfying $|\lambda - \frac{1}{8}| \leq \Gamma_0 h^{2/3}$, we have:*

$$\int_{\mathsf{Tri}} e^{\eta_0 h^{-1}|x|^{3/2}} \left(|\psi|^2 + |h^{2/3}\partial_x\psi|^2\right) dxdy \leq C_0\|\psi\|^2.$$

Proposition 5.6 *Let $\Gamma_0 > 0$. There exist $h_0 > 0$, $C_0 > 0$ and $\rho_0 > 0$ such that for $h \in (0, h_0)$ and all eigenpair (λ, ψ) of $\mathcal{L}_{\mathsf{Tri}}(h)$ satisfying $|\lambda - \frac{1}{8}| \leq \Gamma_0 h^{2/3}$, we have:*

$$\int_{\mathsf{Tri}} (x + \pi\sqrt{2})^{-\rho_0/h} \Big(|\psi|^2 + |h\partial_x \psi|^2\Big)\, dxdy \leq C_0 \|\psi\|^2.$$

5.3 Approximation of the first eigenfunctions

In this subsection, we will work with the operator $\mathcal{L}_{\mathsf{Rec}}(h)$ rather than $\mathcal{L}_{\mathsf{Tri}}(h)$. Let us consider the first N_0 eigenvalues of $\mathcal{L}_{\mathsf{Rec}}(h)$ (shortly denoted by λ_n). In each corresponding eigenspace, we choose a normalized eigenfunction $\hat{\psi}_n$ so that $\langle \hat{\psi}_n, \hat{\psi}_m \rangle = 0$ if $n \neq m$. As in Section 4.2, we introduce:

$$\mathfrak{E}_{N_0}(h) = \operatorname{span}(\hat{\psi}_1, \dots, \hat{\psi}_{N_0}).$$

Let us define Q^0_{Rec} the following quadratic form:

$$Q^0_{\mathsf{Rec}}(\hat{\psi}) = \int_{\mathsf{Rec}} \left(\frac{1}{2\pi^2} |\partial_t \hat{\psi}|^2 - \frac{1}{8} |\hat{\psi}|^2 \right) (u + \pi\sqrt{2})\, dudt,$$

associated with the operator $\mathcal{L}^0_{\mathsf{Rec}} = \mathrm{Id}_u \otimes \left(-\frac{1}{2\pi^2}\partial_t^2 - \frac{1}{8}\right)$ on $L^2(\mathsf{Rec}, (u + \pi\sqrt{2})dudt)$. We consider the projection on the eigenspace associated with the eigenvalue 0 of $-\frac{1}{2\pi^2}\partial_t^2 - \frac{1}{8}$:

$$\Pi_0 \hat{\psi} = \left\langle \hat{\psi}, c_0 \right\rangle_t c_0(t),$$

where we recall that $c_0(t) = \cos\left(\frac{\pi}{2} t\right)$. We can now state a first approximation result:

Proposition 5.7 *There exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$ and all $\hat{\psi} \in \mathfrak{E}_{N_0}(h)$:*

$$0 \leq Q^0_{\mathsf{Rec}}(\hat{\psi}) \leq C h^{2/3} \|\hat{\psi}\|^2$$

and

$$\|(\mathrm{Id} - \Pi_0)\hat{\psi}\| + \|\partial_t (\mathrm{Id} - \Pi_0)\hat{\psi}\| \leq C h^{1/3} \|\hat{\psi}\|.$$

Moreover, $\Pi_0 : \mathfrak{E}_{N_0}(h) \to \Pi_0(\mathfrak{E}_{N_0}(h))$ is an isomorphism.

Proof: If $\hat{\psi} = \hat{\psi}_n$, we have:

$$Q_{\mathsf{Rec},h}(\hat{\psi}_n) = \lambda_n \|\hat{\psi}_n\|^2.$$

From this we infer:

$$Q_{\mathsf{Rec},h}(\hat{\psi}_n) \leq \left(\frac{1}{8} + C h^{2/3}\right) \|\hat{\psi}_n\|^2.$$

The orthogonality of the $\hat{\psi}_n$ (in L^2 and for the quadratic form) allows to extend this inequality to $\hat{\psi} \in \mathfrak{E}_{N_0}(h)$:

$$Q_{\mathsf{Rec},h}(\hat{\psi}) \leq \left(\frac{1}{8} + C h^{2/3}\right) \|\hat{\psi}\|^2.$$

This clearly implies:

$$Q^0_{\mathsf{Rec}}(\hat{\psi}) \leq Ch^{2/3}\|\hat{\psi}\|^2.$$

$\Pi_0\hat{\psi}$ being in the kernel of $\mathcal{L}^0_{\mathsf{Rec}}$, we have:

$$Q^0_{\mathsf{Rec}}(\hat{\psi}) = Q^0_{\mathsf{Rec}}((\mathrm{Id}-\Pi_0)\hat{\psi}).$$

If we denote by μ_2 the second eigenvalue of the $1D$ operator $-\frac{1}{2\pi^2}\partial_t^2 - \frac{1}{8}$, we get by the min-max principle:

$$Q^0_{\mathsf{Rec}}((\mathrm{Id}-\Pi_0)\hat{\psi}) \geq \mu_2\|(\mathrm{Id}-\Pi_0)\hat{\psi}\|^2.$$

Now the conclusions are standard. □

5.4 Reduction to the Born-Oppenheimer approximation

In this section, we prove Theorem 1.7 by using the projections of the true eigenfunctions $(\Pi_0\psi_n)$ as test functions for the Born-Oppenheimer approximation. Let us consider an eigenpair (λ,ψ) of $\mathcal{L}_{\mathsf{Tri}}(h)$ such that (4.5) holds. We let $\hat{\psi}(u,t) = \psi(x,y)$. Then, $(\lambda,\hat{\psi})$ satisfies:

$$-h^2\left(\partial_u^2 - \frac{2t\partial_u\partial_t}{u+\pi\sqrt{2}} + \frac{2t\partial_t}{(u+\pi\sqrt{2})^2} + \frac{t^2\partial_t^2}{(u+\pi\sqrt{2})^2}\right)\hat{\psi} - \frac{1}{(u+\pi\sqrt{2})^2}\partial_t^2\hat{\psi} = \lambda\hat{\psi}.$$

The main idea is to determine the (differential) equation satisfied by $\Pi_0\hat{\psi}$. In other words we shall compute and control the commutator between the operator and the projection Π_0. For that purpose, a few lemmas will be necessary. The first one is an estimate established in the original coordinates (x,y) in the triangle Tri:

Lemma 5.8 *For all $k\in\mathbb{N}$, there exist $h_0>0$ and $C>0$ such that we have, for $h\in(0,h_0)$:*

$$\int_{\mathsf{Tri}} (x+\pi\sqrt{2})^{-k}|\partial_y\psi|^2\,dxdy \leq C\|\psi\|^2.$$

Proof: The equation satisfied by ψ is:

$$(-h^2\partial_x^2 - \partial_y^2)\psi = \lambda\psi.$$

Multiplying by $(x+\pi\sqrt{2})^{-k}$, taking the scalar product with ψ and integrating by parts we find:

$$\int_{\mathsf{Tri}} (x+\pi\sqrt{2})^{-k}|\partial_y\psi|^2\,dxdy \leq C\int_{\mathsf{Tri}} (x+\pi\sqrt{2})^{-k}\Big(|\psi|^2 + h^2(x+\pi\sqrt{2})^{-1}|\psi||\partial_x\psi|\Big)\,dxdy.$$

Using the Agmon estimates of Proposition 5.6 with $\rho_0/h \geq k+1$ we deduce the lemma. □

We can now prove:

Lemma 5.9 *There exist $h_0 > 0$ and $C > 0$ such that we have, for $h \in (0, h_0)$:*

$$\left\| \left\langle (u + \pi\sqrt{2})^{-1} t \partial_u \partial_t \hat{\psi},\, c_0(t) \right\rangle_t \right\|_{L^2(du)} \leq C h^{-1} \|\hat{\psi}\|.$$

Proof: Integrating by parts in t for any fixed $u \in (-\pi\sqrt{2}, 0)$, we find:

$$\begin{aligned}\left| \left\langle (u + \pi\sqrt{2})^{-1} t \partial_u \partial_t \hat{\psi},\, c_0(t) \right\rangle_t \right| &\leq C \int_{-1}^{1} (u + \pi\sqrt{2})^{-1} |\partial_u \hat{\psi}|\, dt \\ &\leq C \left(\int_{-1}^{1} (u + \pi\sqrt{2})^{-2} |\partial_u \hat{\psi}|^2\, dt \right)^{1/2}.\end{aligned}$$

To have the lemma, it remains to prove that

$$\int_{\mathsf{Rec}} (u + \pi\sqrt{2})^{-2} |\partial_u \hat{\psi}|^2\, dudt \leq C h^{-2} \int_{\mathsf{Rec}} |\hat{\psi}|^2\, dudt.$$

We have:

$$\int_{\mathsf{Rec}} (u + \pi\sqrt{2})^{-2} |\partial_u \hat{\psi}|^2\, dudt = \int_{\mathsf{Tri}} (x + \pi\sqrt{2})^{-3} \left| \left(\partial_x + \frac{y \partial_y}{x + \pi\sqrt{2}} \right) \psi \right|^2 dxdy$$

and we apply Lemma 5.8 to control the term in ∂_y. We end the proof using the Agmon estimates of Proposition 5.6. □

The same kind of computations yields:

Lemma 5.10 *There exist $h_0 > 0$ and $C > 0$ such that we have, for $h \in (0, h_0)$:*

$$\left\| \left\langle (u + \pi\sqrt{2})^{-2} t \partial_t \hat{\psi},\, c_0(t) \right\rangle_t \right\|_{L^2(du)} \leq C \|\hat{\psi}\|.$$

Finally, we have:

Lemma 5.11 *There exist $h_0 > 0$ and $C > 0$ such that we have, for $h \in (0, h_0)$:*

$$\left\| \left\langle (u + \pi\sqrt{2})^{-2} t^2 \partial_t^2 \hat{\psi},\, c_0(t) \right\rangle_t \right\|_{L^2(du)} \leq C \|\hat{\psi}\|.$$

From Lemmas 5.9, 5.10 and 5.11, and from Proposition 5.7, we infer:

Proposition 5.12 *Let $\Gamma_0 > 0$. There exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$ and all eigenpair (λ, ψ) of $\mathcal{L}_{\mathsf{Tri}}(h)$ satisfying $|\lambda - \frac{1}{8}| \leq \Gamma_0 h^{2/3}$, we have:*

$$\left\| \left(-h^2 \partial_u^2 + \frac{\pi^2}{4(u + \pi\sqrt{2})^2} - \lambda \right) \Pi_0 \hat{\psi} \right\| \leq C h \|\Pi_0 \hat{\psi}\|.$$

Proof of Theorem 1.7 We deduce, from Proposition 5.12, for all $n \in \{1, \cdots, N_0\}$:

$$\left\| \left(-h^2 \partial_u^2 + \frac{\pi^2}{4(u+\pi\sqrt{2})^2} \right) \Pi_0 \hat{\psi}_n \right\| \leq (\lambda_{\mathsf{Tri},N_0}(h) + Ch) \|\Pi_0 \hat{\psi}_n\|.$$

From this inequality, we infer, for all $\psi \in \mathfrak{E}_{N_0}(h)$:

$$\left\| \left(-h^2 \partial_u^2 + \frac{\pi^2}{4(u+\pi\sqrt{2})^2} \right) \Pi_0 \hat{\psi} \right\| \leq (\lambda_{\mathsf{Tri},N_0}(h) + Ch) \|\Pi_0 \hat{\psi}\|$$

and thus:

$$Q_{\mathsf{BO},\mathsf{Tri},h}(\Pi_0 \hat{\psi}) \leq (\lambda_{\mathsf{Tri},N_0}(h) + Ch) \|\Pi_0 \hat{\psi}\|.$$

It remains to apply the min-max principle to the N_0 dimensional space $\Pi_0 \mathfrak{E}_{N_0}(h)$ (see Proposition 5.7) and Proposition 4.1 to get the separation of eigenvalues. Then, the conclusion follows from Proposition 5.1.

6 Application to the waveguide

In this section, we prove Theorem 1.9. Firstly, we construct quasimodes and secondly we use Agmon estimates reduce to the triangle case. On the left, $\mathcal{L}_{\mathsf{Gui}}(h)$ writes, in the coordinates (u, t) defined in (5.1):

$$\mathcal{L}_{\mathsf{Gui}}^{\mathsf{lef}}(h) = \mathcal{L}_{\mathsf{Rec}}(h) = -h^2 \left(\partial_u - \frac{t}{u+\pi\sqrt{2}} \partial_t \right)^2 - \frac{1}{(u+\pi\sqrt{2})^2} \partial_t^2$$

and on the right, we let:

$$u = x, \quad \tau = \frac{y-x}{\pi\sqrt{2}}$$

and the operator writes:

$$\mathcal{L}_{\mathsf{Gui}}^{\mathsf{rig}}(h) = -h^2 \left(\partial_u - \frac{1}{\pi\sqrt{2}} \partial_\tau \right)^2 - \frac{1}{2\pi^2} \partial_\tau^2.$$

The integration domain is $(-\pi\sqrt{2}, +\infty) \times (0,1) = \Omega_{\mathsf{lef}} \cup \Omega_{\mathsf{rig}}$ where:

$$\Omega_{\mathsf{lef}} = (-\pi\sqrt{2}, 0) \times (0,1) \text{ and } \Omega_{\mathsf{rig}} = (0, +\infty) \times (0,1).$$

The boundary conditions are Dirichlet on $(0,\infty) \times \{0\} \cup (-\pi\sqrt{2}, \infty) \times \{1\}$ and Neumann on $(-\pi\sqrt{2}, 0) \times \{0\}$.

6.1 Quasimodes

The aim of this subsection is to prove the following proposition:

Proposition 6.1 *For any $n \geq 1$, there exists a sequence $(\gamma_{j,n})$ such that, for all $N_0 \in \mathbb{N}$ and $J \in \mathbb{N}$, there exists $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$:*

$$\operatorname{dist}\Big(\sigma_{\mathsf{dis}}\big(\mathcal{L}_{\mathsf{Gui}}(h)\big), \sum_{j=0}^{J} \gamma_{j,n} h^{j/3}\Big) \leq C h^{(J+1)/3}, \quad n = 1, \cdots N_0. \tag{6.1}$$

Moreover, we have: $\gamma_{0,n} = \frac{1}{8}$, $\gamma_{1,n} = 0$ and $\gamma_{2,n} = (4\pi\sqrt{2})^{-2/3} z_{\mathsf{A}}(n)$.

6.1.1 Preliminaries

Ansatz, boundary and transmission conditions In order to construct quasi-eigenpairs for $\mathcal{L}_{\mathsf{Gui}}(h)$ of the form (γ_h, ψ_h), we use the coordinates (u, t) on the left and (u, τ) on the right and look for quasimodes $\hat{\psi}_h(u, t, \tau) = \psi_h(x, y)$. Such quasimodes will have the form on the left:

$$\psi_{\mathsf{lef}}(u, t) \sim \sum_{j \geq 0} h^{j/3} \left(\Psi_{\mathsf{lef},j}(h^{-2/3}u, t) + \Phi_{\mathsf{lef},j}(h^{-1}u, t)\right), \tag{6.2}$$

and on the right:

$$\psi_{\mathsf{rig}}(u, \tau) \sim \sum_{j \geq 0} h^{j/3} \Phi_{\mathsf{rig},j}(h^{-1}u, \tau) \tag{6.3}$$

associated with quasi-eigenvalues:

$$\gamma_h \sim \sum_{j \geq 0} \gamma_j h^{j/3}.$$

We will denote $s = h^{-2/3}u$ and $\sigma = h^{-1}u$. Since ψ_h has no jump across the line $x = 0$, we find that ψ_{lef} and ψ_{rig} should satisfy two transmission conditions on the line $u = 0$:

$$\psi_{\mathsf{lef}}(0, t) = \psi_{\mathsf{rig}}(0, t) \quad \text{and} \quad \left(\partial_u - \frac{t}{\pi\sqrt{2}}\partial_t\right)\psi_{\mathsf{lef}}(0, t) = \left(\partial_u - \frac{\partial_\tau}{\pi\sqrt{2}}\right)\psi_{\mathsf{rig}}(0, t),$$

for all $t \in (0, 1)$. For the Ansätze (6.2)-(6.3) these conditions write for all $j \geq 0$

$$\Psi_{\mathsf{lef},j}(0, t) + \Phi_{\mathsf{lef},j}(0, t) = \Phi_{\mathsf{rig},j}(0, t) \tag{6.4}$$

and

$$\partial_\sigma \Phi_{\mathsf{lef},j}(0, t) + \partial_s \Psi_{\mathsf{lef},j-1}(0, t) - \frac{t\partial_t}{\pi\sqrt{2}} \Phi_{\mathsf{lef},j-3}(0, t) - \frac{t\partial_t}{\pi\sqrt{2}} \Psi_{\mathsf{lef},j-3}(0, t)$$
$$= \partial_\sigma \Phi_{\mathsf{rig},j}(0, t) - \frac{\partial_\tau}{\pi\sqrt{2}} \Phi_{\mathsf{rig},j-3}(0, t), \tag{6.5}$$

where we understand that the terms associated with a negative index are 0.

Notation 6.2 Like in the case of the triangle Tri, the operators written in variables (s,t) and (σ,t) expand in powers of $h^{2/3}$ and h, respectively. Now we have three operator series:

- $L_{\mathsf{Rec}}(h)(h^{2/3}s,t;h^{-2/3}\partial_s,\partial_t) \sim \sum_{j\geq 0}\mathcal{L}_{2j}h^{2j/3}$. The operators are the same as for Tri, but they are defined now on the half-strip $\mathsf{Hlef} = (-\infty,0)\times(0,1)$.
- $\mathcal{L}_{\mathsf{Rec}}(h)(h\sigma,t;h^{-1}\partial_\sigma,\partial_t) \sim \sum_{j\geq 0}\mathcal{N}^{\mathsf{lef}}_{3j}h^j$ also defined on Hlef.
- $\mathcal{L}_{\mathsf{Gui}}(h)(h\sigma,t;h^{-1}\partial_\sigma,\partial_t) \sim \sum_{j\geq 0}\mathcal{N}^{\mathsf{rig}}_{3j}h^j$ defined on $\mathsf{Hrig} = (0,\infty)\times(0,1)$.

We agree to incorporate the boundary conditions on the horizontal sides of Hlef in the definition of the operators $\mathcal{L}_j$, $\mathcal{N}^{\mathsf{lef}}_j$, and $\mathcal{N}^{\mathsf{rig}}_j$:

- $\partial_t\Psi(s,0) = 0$ and $\Psi(s,1) = 0$ $(s<0)$ for $\mathcal{L}_j$,
- $\partial_t\Phi(\sigma,0) = 0$ and $\Psi(\sigma,1) = 0$ $(\sigma<0)$ for $\mathcal{N}^{\mathsf{lef}}_j$,
- $\Phi(\sigma,0) = 0$ and $\Psi(\sigma,1) = 0$ $(\sigma>0)$ for $\mathcal{N}^{\mathsf{rig}}_j$.

Dirichlet-to-Neumann operators Here we introduce the Dirichlet-to-Neumann operators T^{rig} and T^{lef} which we use to solve the problems in the variables (σ,t). We denote by I the interface $\{0\}\times(0,1)$ between Hrig and Hlef.

On the right, and with Notation 6.2, we consider the problem:

$$\left(\mathcal{N}^{\mathsf{rig}}_0 - \frac{1}{8}\right)\Phi_{\mathsf{rig}} = 0 \ \text{ in } \ \mathsf{Hrig} \quad \text{and} \quad \Phi_{\mathsf{rig}}(0,t) = G(t)$$

where $G\in H^{1/2}_{00}(I)$. Since the first eigenvalue of the transverse part of $\mathcal{N}^{\mathsf{rig}}_0 - \frac{1}{8}$ is positive, this problem has a unique exponentially decreasing solution Φ_{rig}. Its exterior normal derivative $-\partial_\sigma\Phi_{\mathsf{rig}}$ on the line I is well defined in $H^{-1/2}(I)$. We define:

$$T^{\mathsf{rig}}G = \partial_n\Phi_{\mathsf{rig}} = -\partial_\sigma\Phi_{\mathsf{rig}}.$$

We have:

$$\langle T^{\mathsf{rig}}G, G\rangle = Q_{\mathsf{rig}}(\Phi_{\mathsf{rig}}) \geq C\|G\|^2_{H^{1/2}_{00}(I)}.$$

On the left, we consider the problem:

$$\left(\mathcal{N}^{\mathsf{lef}}_0 - \frac{1}{8}\right)\Phi_{\mathsf{lef}} = 0 \ \text{ in } \ \mathsf{Hlef} \quad \text{and} \quad \Phi_{\mathsf{lef}}(0,t) = G(t)$$

where $G\in H^{1/2}_{00}(I)$.

For all $G\in H^{1/2}_{00}(I)$ such that $\Pi_0 G = 0$, this problem has a unique exponentially decreasing solution Φ_{lef}. Its exterior normal derivative $\partial_\sigma\Phi_{\mathsf{lef}}$ on the line I is well defined in $H^{-1/2}(I)$. We define:

$$T^{\mathsf{lef}}G = \partial_n\Phi_{\mathsf{lef}} = \partial_\sigma\Phi_{\mathsf{lef}}.$$

We have:
$$\langle T^{\mathsf{lef}} G, G\rangle = Q_{\mathsf{lef}}(\Phi_{\mathsf{lef}}) \geq 0.$$

Proposition 6.3 *The operator $T^{\mathsf{rig}} + T^{\mathsf{lef}}\Pi_1$ is coercive on $H^{1/2}_{00}(I)$ with $\Pi_1 = \mathrm{Id} - \Pi_0$. In particular, it is invertible from $H^{1/2}_{00}(I)$ onto $H^{-1/2}(I)$.*

This proposition allows to prove the following lemma which is in the same spirit as Lemma 5.2, but now for transmission problems on Hlef $\cup$ Hrig:

Lemma 6.4 *Let $F_{\mathsf{lef}} = F_{\mathsf{lef}}(\sigma, t)$ and $F_{\mathsf{rig}} = F_{\mathsf{rig}}(\sigma, t)$ be real functions defined on* Hlef *and* Hrig*, respectively, with exponential decay with respect to σ. Let $G^0 \in H^{1/2}_{00}(I)$ and $H \in H^{-1/2}(I)$ be data on the interface $I = \partial\mathsf{Hlef} \cap \partial\mathsf{Hrig}$. Then there exists a unique $\zeta \in \mathbb{R}$ and a unique trace $G \in H^{1/2}_{00}(I)$ such that the transmission problem*

$$\begin{aligned}
\Big(\mathcal{N}_0^{\mathsf{lef}} - \frac{1}{8}\Big)\Phi_{\mathsf{lef}} &= F_{\mathsf{lef}} \quad \textit{in} \quad \mathsf{Hlef}, \quad \Phi_{\mathsf{lef}}(0,t) = G(t) + G^0(t) + \zeta c_0(t),\\
\Big(\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}\Big)\Phi_{\mathsf{rig}} &= F_{\mathsf{rig}} \quad \textit{in} \quad \mathsf{Hrig}, \quad \Phi_{\mathsf{rig}}(0,t) = G(t),\\
\partial_\sigma \Phi_{\mathsf{lef}}(0,t) - \partial_\sigma \Phi_{\mathsf{rig}}(0,t) &= H(t) \quad \textit{on} \quad I,
\end{aligned}$$

admits a (unique) solution $(\Phi_{\mathsf{lef}}, \Phi_{\mathsf{rig}})$ with exponential decay.

Proof: Let $(\Phi^0_{\mathsf{lef}}, \zeta_0)$ be the solution provided by Lemma 5.2 for the data $F = F_{\mathsf{lef}}$ and $G = 0$. Let Φ^0_{rig} be the unique exponentially decreasing solution of the problem

$$\Big(\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}\Big)\Phi^0_{\mathsf{rig}} = F_{\mathsf{rig}} \quad \text{in} \quad \mathsf{Hrig}, \quad \Phi^0_{\mathsf{rig}}(0,t) = 0.$$

Let H^0 be the jump $\partial_\sigma \Phi^0_{\mathsf{rig}}(0,t) - \partial_\sigma \Phi^0_{\mathsf{lef}}(0,t)$. If we define the new unknowns $\Phi^1_{\mathsf{rig}} = \Phi_{\mathsf{rig}} - \Phi^0_{\mathsf{rig}}$ and $\Phi^1_{\mathsf{lef}} = \Phi_{\mathsf{lef}} - \Phi^0_{\mathsf{lef}}$, the problem we want to solve becomes

$$\begin{aligned}
\Big(\mathcal{N}_0^{\mathsf{lef}} - \frac{1}{8}\Big)\Phi^1_{\mathsf{lef}} &= 0 \quad \text{in} \quad \mathsf{Hlef}, \quad \Phi^1_{\mathsf{lef}}(0,t) = G(t) + (\zeta - \zeta_0)c_0(t),\\
\Big(\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}\Big)\Phi^1_{\mathsf{rig}} &= 0 \quad \text{in} \quad \mathsf{Hrig}, \quad \Phi^1_{\mathsf{rig}}(0,t) = G(t),\\
\partial_\sigma \Phi^1_{\mathsf{rig}}(0,t) - \partial_\sigma \Phi^1_{\mathsf{lef}}(0,t) &= H(t) - H^0(t) \quad \text{on} \quad I.
\end{aligned}$$

Using Proposition 6.3 we can set $G = (T^{\mathsf{rig}} + T^{\mathsf{lef}}\Pi_1)^{-1}(H - H_0)$, which ensures the solvability of the above problem. $\square$

6.1.2 Construction of quasimodes

Terms of order h^0 Let us write the "interior" equations:

$$\begin{aligned}
\mathsf{lef}_s : \quad & \mathcal{L}_0\Psi_{\mathsf{lef},0} = \gamma_0\Psi_{\mathsf{lef},0}\\
\mathsf{lef}_\sigma : \quad & \mathcal{N}_0^{\mathsf{lef}}\Phi_{\mathsf{lef},0} = \gamma_0\Phi_{\mathsf{lef},0}\\
\mathsf{rig} : \quad & \mathcal{N}_0^{\mathsf{rig}}\Phi_{\mathsf{rig},0} = \gamma_0\Phi_{\mathsf{rig},0}\,.
\end{aligned}$$

The boundary conditions are:

$$\Psi_{\mathsf{lef},0}(0,t) + \Phi_{\mathsf{lef},0}(0,t) = \Phi_{\mathsf{rig},0}(0,t),$$
$$\partial_\sigma\Phi_{\mathsf{lef},0}(0,t) = \partial_\sigma\Phi_{\mathsf{rig},0}(0,t).$$

We get:

$$\gamma_0 = \frac{1}{8}, \quad \Psi_{\mathsf{lef},0} = g_0(s)c_0(t).$$

We now apply Lemma 6.4 with $F_{\mathsf{lef}} = 0$, $F_{\mathsf{rig}} = 0$, $G_0 = 0$, $H = 0$ to get

$$G = 0 \quad \text{and} \quad \zeta = 0.$$

We deduce: $\Phi_{\mathsf{lef},0} = 0$, $\Phi_{\mathsf{rig},0} = 0$ and, since $\zeta = -g_0(0)$, $g_0(0) = 0$. At this step, we do not have determined g_0 yet.

Terms of order $h^{1/3}$ The interior equations read:

$$\begin{aligned}
\mathsf{lef}_s : \quad & \mathcal{L}_0\Psi_{\mathsf{lef},1} = \gamma_0\Psi_{\mathsf{lef},1} + \gamma_1\Psi_{\mathsf{lef},0}\\
\mathsf{lef}_\sigma : \quad & \mathcal{N}_0^{\mathsf{lef}}\Phi_{\mathsf{lef},1} = \gamma_0\Phi_{\mathsf{lef},1} + \gamma_1\Phi_{\mathsf{lef},0}\\
\mathsf{rig} : \quad & \mathcal{N}_0^{\mathsf{rig}}\Phi_{\mathsf{rig},1} = \gamma_0\Phi_{\mathsf{rig},1} + \gamma_1\Phi_{\mathsf{rig},0}.
\end{aligned}$$

Using Lemma 5.3, the first equation implies:

$$\gamma_1 = 0, \quad \Psi_{\mathsf{lef},1}(s,t) = g_1(s)c_0(t).$$

The boundary conditions are:

$$g_1(0)c_0(t) + \Phi_{\mathsf{lef},1}(0,t) = \Phi_{\mathsf{rig},1}(0,t),$$
$$g_0'(0)c_0(t) + \partial_\sigma\Phi_{\mathsf{lef},1}(0,t) = \partial_\sigma\Phi_{\mathsf{rig},1}(0,t).$$

The system becomes:

$$\begin{aligned}
\mathsf{lef}_\sigma : \quad & \left(\mathcal{N}_0^{\mathsf{lef}} - \frac{1}{8}\right)\Phi_{\mathsf{lef},1} = 0,\\
\mathsf{rig} : \quad & \left(\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}\right)\Phi_{\mathsf{rig},1} = 0.
\end{aligned}$$

We apply Lemma 6.4 with $F_{\mathsf{lef}} = 0$, $F_{\mathsf{rig}} = 0$, $G_0 = 0$, $H = -g_0'(0)c_0(t)$ to get:

$$G = -g_0'(0)(T^{\mathsf{rig}} + T^{\mathsf{lef}}\Pi_1)^{-1}c_0.$$

Since $G = \Phi_{\mathsf{rig},1}$ and $\zeta = -g_1(0)$, this determines $\Phi_{\mathsf{lef},1}$, $\Phi_{\mathsf{rig},1}$ and $g_1(0)$.

Terms of order $h^{2/3}$ The interior equations write:

$$\begin{aligned}
\mathsf{lef}_s : \quad & \mathcal{L}_2\Psi_{\mathsf{lef},0} + \mathcal{L}_0\Psi_{\mathsf{lef},2} = \sum_{l+k=2} \gamma_l \Psi_{\mathsf{lef},k} \\
\mathsf{lef}_\sigma : \quad & \mathcal{N}_0^{\mathsf{lef}}\Phi_{\mathsf{lef},2} = \sum_{l+k=2} \gamma_l \Phi_{\mathsf{lef},k} \\
\mathsf{rig} : \quad & \mathcal{N}_0^{\mathsf{rig}}\Phi_{\mathsf{rig},2} = \frac{1}{8}\Phi_{\mathsf{rig},2},
\end{aligned}$$

with

$$\mathcal{L}_2\Psi_{\mathsf{lef},0} = -g_0''(s)c_0(t) + \frac{1}{\pi^3\sqrt{2}} s g_0(s)\partial_t^2(c_0).$$

Lemma 5.3 and then Lemma 5.4 imply:

$$-g_0'' - \frac{1}{4\pi\sqrt{2}} s g_0 = \gamma_2 g_0.$$

Thus, γ_2 is one of the eigenvalues of the Airy operator and g_0 an associated eigenfunction. In particular, this determines the unknown functions of the previous steps. We are led to take:

$$\Psi_{\mathsf{lef},2}(s,t) = \Psi_{\mathsf{lef},2}^{\perp} + g_2(s)c_0(t), \text{ with } \Psi_{\mathsf{lef},2}^{\perp} = 0$$

and to the system:

$$\begin{aligned}
\mathsf{lef}_\sigma : \quad & \left(\mathcal{N}_0^{\mathsf{lef}} - \frac{1}{8}\right)\Phi_{\mathsf{lef},2} = 0 \\
\mathsf{rig} : \quad & \left(\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}\right)\Phi_{\mathsf{rig},2} = 0.
\end{aligned}$$

Using Lemma 6.4, we find

$$G = -g_1'(0)(T^{\mathsf{rig}} + T^{\mathsf{lef}}\Pi_1)^{-1}c_0.$$

This determines $\Phi_{\mathsf{rig},2}$, $\Phi_{\mathsf{lef},2}$ and $g_2(0)$. The function g_1 is still unknown at this step.

Further terms Let us assume that we can write $\Psi_{\mathsf{lef},k} = \Psi_{\mathsf{lef},k}^{\perp} + g_k(s)c_0(t)$ for $0 \le k \le j$ and that $(g_k)_{0\le k\le j-2}$ and $(\Psi_{\mathsf{lef},k}^{\perp})_{0\le k\le j}$ are determined. Let us also assume that $g_{j-1}(0)$, $(\gamma_k)_{0\le k\le j}$, $(\Phi_{\mathsf{rig},k})_{0\le k\le j-1}$, $(\Phi_{\mathsf{lef},k})_{0\le k\le j-1}$ are already known. Finally, we assume that $g_j(0)$, $\Phi_{\mathsf{lef},j}$, $\Phi_{\mathsf{rig},j}$ are known once g_{j-1} is determined and that all the functions have an exponential decay.

Let us collect the terms of order $h^{(j+1)/3}$. The interior equations write:

$$\begin{aligned}
\mathsf{lef}_s : \quad & \sum_{k=0}^{j+1} \mathcal{L}_k \Psi_{\mathsf{lef},j+1-k} = \sum_{k=0}^{j+1} \gamma_k \Psi_{\mathsf{lef},j+1-k} \\
\mathsf{lef}_\sigma : \quad & \sum_{k=0}^{j+1} \mathcal{N}_k^{\mathsf{lef}} \Phi_{\mathsf{lef},j+1-k} = \sum_{k=0}^{j+1} \gamma_k \Phi_{\mathsf{lef},j+1-k} \\
\mathsf{rig} : \quad & \sum_{k=0}^{j+1} \mathcal{N}_k^{\mathsf{rig}} \Phi_{\mathsf{rig},j+1-k} = \sum_{k=0}^{j+1} \gamma_k \Phi_{\mathsf{rig},j+1-k},
\end{aligned}$$

We examine the first equation and notice that $\mathcal{L}_1 = 0$ and $\gamma_1 = 0$ so that $\Psi_{\mathsf{lef},j}$ does not appear. We can write this equation in the form:

$$\left(\mathcal{L}_0 - \frac{1}{8}\right)\Psi_{\mathsf{lef},j+1} = -\mathcal{L}_2\Psi_{\mathsf{lef},j-1} - \gamma_2\Psi_{\mathsf{lef},j-1} - \gamma_{j+1}\Psi_{\mathsf{lef},0} - \sum_{k=4}^{j+1}\mathcal{L}_k\Psi_{\mathsf{lef},j+1-k} - \sum_{k=3}^{j}\gamma_k\Psi_{\mathsf{lef},j+1-k}.$$

We apply Lemma 5.3 and we obtain an equation in the form:

$$-g''_{j-1} - \frac{1}{4\pi\sqrt{2}}sg_{j-1} - \gamma_2 g_{j-1} = f + \gamma_{j+1}g_0,$$

where f and $g_{j-1}(0)$ are known. Then, Lemma 5.4 applies and provides a unique value of γ_{j+1} such that g_{j-1} has an exponential decay. From the recursion assumption, we deduce that $g_j(0)$, $\Phi_{\mathsf{lef},j}$, $\Phi_{\mathsf{rig},j}$ are now determined. Lemma 5.3 uniquely determines $\Psi^{\perp}_{\mathsf{lef},j+1}$ such that:

$$\Psi_{\mathsf{lef},j+1} = \Psi^{\perp}_{\mathsf{lef},j+1} + g_{j+1}(s)c_0(t).$$

We can now write the system in the form:

$$\begin{aligned} &\mathsf{lef}_\sigma : && \left(\mathcal{N}_0^{\mathsf{lef}} - \frac{1}{8}\right)\Phi_{\mathsf{lef},j+1} = F_{\mathsf{lef}} \\ &\mathsf{rig} : && \left(\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}\right)\Phi_{\mathsf{rig},j+1} = F_{\mathsf{rig}}, \end{aligned}$$

where $F_{\mathsf{lef}}, F_{\mathsf{rig}}$ have an exponential decay. The transmission conditions are, cf. (6.4)–(6.5):

$$\begin{aligned} \Phi_{\mathsf{lef},j+1}(0,t) &= \Phi_{\mathsf{rig},j+1}(0,t) - \Psi_{\mathsf{lef},j+1}(0,t) \\ &= \Phi_{\mathsf{rig},j+1}(0,t) - \Psi^{\perp}_{\mathsf{lef},j+1}(0,t) - g_{j+1}(0)c_0(t) \end{aligned}$$

and

$$\partial_\sigma\Phi_{\mathsf{lef},j+1}(0,t) - \partial_\sigma\Phi_{\mathsf{rig},j+1}(0,t) = H(t) = -g'_j(0)c_0(t) + \tilde{H}(t),$$

where $\tilde{H}$ is known. We can apply Lemma 6.4 which determines $\Phi_{\mathsf{rig},j+1}$, $\Phi_{\mathsf{lef},j+1}$ (with an exponential decay) and $g_{j+1}(0)$ once g_j is known.

Quasimodes The previous construction leads to introduce:

$$\hat{\psi}_h^{[J]}(u,t) = \begin{cases} \displaystyle\sum_{j=0}^{J+2}\left(\Psi_{\mathsf{lef},j}\left(\frac{u}{h^{2/3}},t\right) + \Phi_{\mathsf{lef},j}\left(\frac{u}{h},t\right)\right)h^{j/3} & \text{when } u \leq 0 \\ \displaystyle\sum_{j=0}^{J+2}\Phi_{\mathsf{rig},j}\left(\frac{u}{h},\tau\right)h^{j/3} + u\,\hat{\chi}\left(\frac{u}{h}\right)R_{J,h}(\tau) & \text{when } u \geq 0\,, \end{cases} \tag{6.6}$$

where the correction term

$$R_{J,h}(\tau) = \partial_s \Psi_{\mathsf{lef},J+2}(0,\tau) h^{J/3}$$
$$- \sum_{j=J}^{J+2} \left(\frac{t\partial_t}{\pi\sqrt{2}} \Big(\Psi_{\mathsf{lef},j}(0,\tau) + \Phi_{\mathsf{lef},j}(0,\tau) \Big) \right) h^{j/3} + \sum_{j=J}^{J+2} \frac{\partial_\tau}{\pi\sqrt{2}} \Phi_{\mathsf{rig},j}(0,\tau) h^{j/3}$$

is added to make $\hat{\psi}_h^{[J]}$ satisfy the transmission condition (6.5). Here we have used a smooth cutoff function $\hat{\chi} = \hat{\chi}(u)$ supported in $(-\varepsilon_0, \varepsilon_0)$ with $\varepsilon_0 \in (0, \pi\sqrt{2})$. By construction, $\psi_h^{[J]}$ defined by the identity

$$\psi_h^{[J]}(x,y) = \hat{\chi}(u) \hat{\psi}_h^{[J]}(u,t)$$

belongs to the domain of $\mathcal{L}_{\mathsf{Gui}}(h)$. Using the exponential decays, for all $J \in \mathbb{N}$ we get the existence of $h_0 > 0$, $C(J,h_0) > 0$ such that for $h \in (0,h_0)$:

$$\left\| \left(\mathcal{L}_{\mathsf{Gui}}(h) - \sum_{j=0}^{J+2} \gamma_j h^{j/3} \right) \chi \psi_h^{[J]} \right\| \le C(J,h_0) h^{1+J/3}.$$

6.2 Agmon estimates and consequences

In this last subsection, we prove Theorem 1.9. For that purpose, we state Agmon estimates (the proof of which being a consequence of that $\mathcal{H}_{\mathsf{BO,Gui}}$ is a lower bound of $\mathcal{L}_{\mathsf{Gui}}(h)$ in the sense of quadratic forms) Complètement obscur :-(to show that the first eigenfunctions are essentially living in the triangle Tri so that we can compare the problem in the whole guide with the triangle.

Proposition 6.5 *Let (λ, ψ) be an eigenpair of $\mathcal{L}_{\mathsf{Gui}}(h)$ such that $|\lambda - \frac{1}{8}| \le Ch^{2/3}$. There exist $\alpha > 0$, $h_0 > 0$ and $C > 0$ such that for all $h \in (0,h_0)$, we have:*

$$\int_{u \ge 0} e^{\alpha h^{-1} x} \Big(|\psi|^2 + |h\partial_x \psi|^2 \Big) \, dx dy \le C \|\psi\|^2.$$

Proof of Theorem 1.9 Let ψ_n^h be an eigenfunction associated with $\lambda_{\mathsf{Gui},n}(h)$ and assume that the ψ_n^h are orthogonal in $L^2(\mathsf{Gui})$, and thus for the bilinear form $B_{\mathsf{Gui},h}$ associated with the operator $\mathcal{L}_{\mathsf{Gui}}(h)$. We introduce a smooth cutoff χ^h in x at the scale $h^{1-\varepsilon}$:

$$\chi^h(x) = \chi(x h^{\varepsilon - 1}) \quad \text{with} \quad \chi \equiv 1 \;\text{ if }\; |x| \le \tfrac{1}{2}, \quad \chi \equiv 0 \;\text{ if }\; |x| \ge 1$$

and we consider the functions $\chi^h \psi_n^h$. We denote:

$$\mathfrak{E}_{N_0}(h) = \operatorname{span}(\chi^h \psi_1^h, \ldots, \chi^h \psi_{N_0}^h).$$

We have:

$$Q_{\mathsf{Gui},h}(\psi_n^h) = \lambda_{\mathsf{Gui},n}(h) \|\psi_n^h\|^2$$

and deduce by the Agmon estimates of Proposition 6.5:

$$Q_{\mathsf{Gui},h}(\chi^h\psi_n^h) = \big(\lambda_{\mathsf{Gui},n}(h) + O(h^\infty)\big)\|\chi^h\psi_n^h\|^2.$$

In the same way, we get the "almost"-orthogonality, for $n \neq m$:

$$B_{\mathsf{Gui},h}(\chi^h\psi_n^h, \chi^h\psi_m^h) = O(h^\infty).$$

We deduce, for all $v \in \mathfrak{E}_{N_0}(h)$:

$$Q_{\mathsf{Gui},h}(v) \leq \big(\lambda_{\mathsf{Gui},N_0}(h) + O(h^\infty)\big)\|v\|^2.$$

We can extend the elements of $\mathfrak{E}_{N_0}(h)$ by zero so that $Q_{\mathsf{Gui},h}(v) = Q_{\mathsf{Tri}_{\varepsilon,h}}(v)$ for $v \in \mathfrak{E}_{N_0}(h)$ where $\mathsf{Tri}_{\varepsilon,h}$ is the triangle with vertices $(-\pi\sqrt{2}, 0)$, $(h^{1-\varepsilon}, 0)$ and $(h^{1-\varepsilon}, h^{1-\varepsilon} + \pi\sqrt{2})$. A dilation reduces us to:

$$\left(1 + \frac{h^{1-\varepsilon}}{\pi\sqrt{2}}\right)^{-2}(-h^2\partial_{\tilde{x}}^2 - \partial_{\tilde{y}}^2)$$

on the triangle Tri. The lowest eigenvalues of this new operator admits the lower bounds $\frac{1}{8} + z_{\mathsf{A}}(n)h^{2/3} - Ch^{1-\varepsilon}$; in particular, we deduce:

$$\lambda_{\mathsf{Gui},N_0}(h) \geq \frac{1}{8} + z_{\mathsf{A}}(N_0)h^{2/3} - Ch^{1-\varepsilon}.$$

This provides the separation of the eigenvalues and, joint with Proposition 6.1, this implies Theorem 1.9.

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