The Influence of Lateral Boundary Conditions on the Asymptotics in Thin Elastic Plates II: Frictional, Sliding Edge and Free Plates

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Abstract. *This paper is the second in a series of two in which we care about the asymptotics of the three-dimensional displacement field in thin elastic plates as the thickness tends to zero. In Part I we have investigated four types of lateral boundary conditions when the transverse component is clamped. In this Part II we study four other types of lateral conditions when the transverse component is free. We prove that the displacement admits an infinite expansion which can be cut off at any order with optimal error estimates. We describe the first terms in this expansion, namely the first two Kirchhoff-Love displacements and the first boundary layer term. In contrast with the first four lateral conditions, the second Kirchhoff-Love displacement is purely bending and the first boundary layer term is also of bending type and has only one non-zero component (the in-plane tangential).*

INTRODUCTION

The problem of thin elastic plate bending in linearized elastostatics has been addressed for more than 150 years (the first correct model was presented in a paper by Kirchhoff [12] published in 1850). But, due to the singular perturbation nature of the problem as the thickness 2ε of the plate tends to zero, a rigorous mathematical analysis is not straightforward.

In the case of a plate which is hard clamped along its lateral side, the situation is now well-known. Starting with Friedrichs & Dressler [9] and Gol'denveizer [10] the construction of infinite *formal* asymptotic expansions was performed in order to describe the 3D solution and its behavior as the parameter ε approaches zero. But, in order to justify such asymptotic expansions, rigorous error estimates between the 3D solution and at least its limit had to be found. This was achieved by Shoikhet [23] and Ciarlet and Destuynder [4, 8, 2]. Further terms in the asymptotic expansion were exhibited by Nazarov & Zorin [17] for special loadings, and the whole asymptotic expansion was constructed by Dauge & Gruais [5, 6].

In this series of two papers, we investigate the influence of the lateral boundary conditions on the different terms of the asymptotics (which prove to have the same global structure in each case, with outer and inner expansion parts). We have chosen to study eight 'canonical' sorts of lateral boundary conditions obtained by prescribing in the three natural directions (normal, in-plane tangential, transverse) either the displacement or the traction. The splitting into two parts is related to the choice made for the transverse component: in each of the cases investigated in Part I, the transverse component of the displacement is supposed to be zero on the lateral side, and in Part II, it is the transverse component of the traction which is zero.

Of importance are the first boundary layer terms in each situation, since they bring the quantitative limitation of accuracy of bi-dimensional models. In the clamped and simple support cases investigated in Part I, we found a strong main boundary layer term with generically non-zero membrane and bending parts, whereas in the four cases of Part II, we find a first boundary layer term which has the bending type and only the in-plane tangential component non-zero. Moreover, the sub-principal term in the outer part of the expansion is a Kirchhoff-Love displacement as usual, but with zero membrane part. This means in particular that if the right hand side has the membrane type, then the solution of the 3D Lamé equations converges to the usual limit Kirchhoff-Love displacement with improved accuracy (see section 9).

This paper contains eight sections, organized as follows: in section 1 we introduce the problems on the thin plates, the eight sorts of lateral boundary conditions, and the scaled form of the problems; after that we present an outline of our results. In section 2 we recall some results from Part I, containing the algorithm of the construction rules for the outer and inner parts of the mixed Ansatz. In section 3 the conditions on the data ensuring the existence of exponentially decreasing solutions to the boundary layer problems are given. Sections 4 to 7 are devoted to each of the four remaining types of lateral boundary conditions : frictional I, sliding edge, frictional II and free plates. In section 8, we prove error estimates between the 3D solution and any truncated series from the infinite asymptotic expansion, and we analyze the regularity of the different terms in the asymptotics: whereas the outer expansion terms are smooth if the data are so, the profiles have singularities along the edges of the plate. We conclude this series of two papers in section 9 by considerations about relative errors between the 3D solution and a limit 2D solution, which has to be carefully chosen according to what we wish to approximate (the displacement in H^1 norm, or the strain in L^2 norm).

1 LATERAL BOUNDARY CONDITIONS

We consider a family of thin elastic plates Ω^ε given by

$$\Omega^\varepsilon = \omega \times (-\varepsilon, +\varepsilon) \quad \text{with } \omega \subset \mathbb{R}^2 \text{ a regular domain and } \varepsilon > 0$$

in the framework of three-dimensional linearized elastostatics. Our aim is to study the behavior of the displacement field $\boldsymbol{u}^\varepsilon$ in this family of plates Ω^ε as the thickness parameter ε tends to zero. As in Part I we will restrict ourselves to the consideration of plates which are constituted of a homogeneous, isotropic material with Lamé constants λ and μ. We assume that the boundary conditions on the upper and lower faces $\Gamma_\pm^\varepsilon := \omega \times \{-\varepsilon, +\varepsilon\}$ are of traction type. On the lateral side

$$\Gamma_0^\varepsilon := \partial\omega \times (-\varepsilon, +\varepsilon),$$

we consider (in this paper and its part I) the 8 'canonical' choices of boundary conditions which will be denoted by ⓘ where $i = 1, \cdots, 8$.

1.1 The primitive problem

As it is common, let $\boldsymbol{u} = (u_1, u_2, u_3)$ be the displacement field, and let $e(\boldsymbol{u})$ denote the associated linearized strain tensor $e_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. Then the stress tensor $\sigma(\boldsymbol{u})$ is given by Hooke's law

$$\sigma(\boldsymbol{u}) = A\, e(\boldsymbol{u}),$$

where the rigidity matrix $A = (A_{ijkl})$ is that of a homogeneous, isotropic material. The inward traction field at a point on the boundary is denoted by $\boldsymbol{T} = (T_1, T_2, T_3)$, defined as $\sigma(\boldsymbol{u})\,\boldsymbol{n}$ with $\boldsymbol{n}$ the unit interior normal to the boundary. To $\boldsymbol{u}$ we associate its normal component $u_n := \boldsymbol{u}\cdot\boldsymbol{n} = u_1 n_1 + u_2 n_2$ on Γ_0^ε, its horizontal tangential component $u_s := u_1 n_2 - u_2 n_1$ and the vertical component u_3, and the same holds for $\boldsymbol{T}$. To each boundary condition ⓘ corresponds two complementary sets of indices $A_{ⓘ}$ and $B_{ⓘ}$, the reunion of which is $\{n, s, 3\}$, where $A_{ⓘ}$ corresponds to the Dirichlet conditions of ⓘ ($\forall a \in A_{ⓘ},\ u_a = 0$) and $B_{ⓘ}$ to the Neumann conditions ($\forall b \in B_{ⓘ},\ T_b = 0$). Here is the table of the eight lateral boundary conditions (in boldface, those we treat in this second part).

ⓘ	Type	Dirichlet	Neumann	$A_{ⓘ}$	$B_{ⓘ}$
①	hard clamped	$\boldsymbol{u} = 0,$		$\{n, s, 3\}$	
②	soft clamped	$u_n,\ u_3 = 0,$	$T_s = 0$	$\{n, 3\}$	$\{s\}$
③	hard simply supported	$u_s,\ u_3 = 0,$	$T_n = 0$	$\{s, 3\}$	$\{n\}$
④	soft simply supported	$u_3 = 0,$	$T_n,\ T_s = 0$	$\{3\}$	$\{n, s\}$
⑤	**frictional I**	$u_n,\ u_s = 0,$	$T_3 = 0$	$\{n, s\}$	$\{3\}$
⑥	**sliding edge**	$u_n = 0,$	$T_s,\ T_3 = 0$	$\{n\}$	$\{s, 3\}$
⑦	**frictional II**	$u_s = 0,$	$T_n,\ T_3 = 0$	$\{s\}$	$\{n, 3\}$
⑧	**free**		$\boldsymbol{T} = 0$		$\{n, s, 3\}$

Table 1. Lateral boundary conditions.

To each boundary condition ⓘ is associated the space of displacements $V_{ⓘ}(\Omega^\varepsilon)$

$$V_{ⓘ}(\Omega^\varepsilon) = \{\boldsymbol{v} \in H^1(\Omega^\varepsilon)^3 \mid \quad v_a = 0 \text{ on } \Gamma_0^\varepsilon \quad \forall a \in A_{ⓘ}\}$$

and the space $\mathcal{R}_{\textcircled{i}}$ of the rigid motions satisfying the Dirichlet conditions of $V_{\textcircled{i}}$. Then, the variational formulation of the problem consists in finding

$$\begin{cases} \boldsymbol{u}^\varepsilon \in V_{\textcircled{i}}(\Omega^\varepsilon) \\ \forall \boldsymbol{v} \in V_{\textcircled{i}}(\Omega^\varepsilon), \quad \displaystyle\int_{\Omega^\varepsilon} A\, e(\boldsymbol{u}^\varepsilon) : e(\boldsymbol{v}) = \int_{\Omega^\varepsilon} \boldsymbol{f}^\varepsilon \cdot \boldsymbol{v} \; + \int_{\Gamma_+^\varepsilon} \boldsymbol{g}^{\varepsilon,+} \cdot \boldsymbol{v} \; - \int_{\Gamma_-^\varepsilon} \boldsymbol{g}^{\varepsilon,-} \cdot \boldsymbol{v}, \end{cases} \tag{1.1}$$

where $\boldsymbol{f}^\varepsilon$ represents the volume force and $\boldsymbol{g}^{\varepsilon,\pm}$ the prescribed horizontal tractions. If the right hand side satisfies the compatibility condition

$$\forall \boldsymbol{v} \in \mathcal{R}_{\textcircled{i}}(\Omega^\varepsilon), \quad \int_{\Omega^\varepsilon} \boldsymbol{f}^\varepsilon \cdot \boldsymbol{v} \; + \int_{\Gamma_+^\varepsilon} \boldsymbol{g}^{\varepsilon,+} \cdot \boldsymbol{v} \; - \int_{\Gamma_-^\varepsilon} \boldsymbol{g}^{\varepsilon,-} \cdot \boldsymbol{v} = 0, \tag{1.2}$$

then there exists a unique solution to (1.1) satisfying

$$\forall \boldsymbol{v} \in \mathcal{R}_{\textcircled{i}}(\Omega^\varepsilon), \quad \int_{\Omega^\varepsilon} \boldsymbol{u}^\varepsilon \cdot \boldsymbol{v} = 0. \tag{1.3}$$

The equations inside Ω^ε are given by

$$(\lambda + \mu)\, \partial_i \operatorname{div} \boldsymbol{u}^\varepsilon + \mu\, \Delta u_i^\varepsilon = -f_i^\varepsilon, \quad i = 1, 2, 3. \tag{1.4}$$

Denoting by the Greek indices α, β, γ the values in $\{1, 2\}$ corresponding to the in-plane variables, we can write the boundary conditions on the horizontal sides $\Gamma_\pm^\varepsilon$ as

$$2\mu\, e_{\alpha 3}(\boldsymbol{u}^\varepsilon) = g_\alpha^{\varepsilon,\pm}, \quad \alpha = 1, 2, \qquad \text{and} \qquad 2\mu\, \partial_3 u_3^\varepsilon + \lambda \operatorname{div} \boldsymbol{u}^\varepsilon = g_3^{\varepsilon,\pm}. \tag{1.5}$$

The boundary conditions on the lateral side Γ_0^ε can be written as

$$u_a^\varepsilon = 0, \quad \forall a \in A_{\textcircled{i}} \qquad \text{and} \qquad T_b^\varepsilon = 0, \quad \forall b \in B_{\textcircled{i}}, \tag{1.6}$$

where the normal, tangential horizontal and (tangential) vertical components of the traction $\boldsymbol{T}^\varepsilon$ on Γ_0^ε can be given with the help of the local coordinates n, s by

$$T_n^\varepsilon = \lambda \operatorname{div} \boldsymbol{u}^\varepsilon + 2\mu\, \partial_n u_n^\varepsilon, \quad T_s^\varepsilon = \mu(\partial_s u_n^\varepsilon + \partial_n u_s^\varepsilon + \tfrac{2}{R}\, u_s^\varepsilon), \quad T_3^\varepsilon = \mu(\partial_n u_3^\varepsilon + \partial_3 u_n^\varepsilon), \tag{1.7}$$

where $\frac{1}{R} = \kappa$ is the curvature of $\partial\omega$. The local coordinates (see Part I, §1.2) are used in a tubular neighborhood of the lateral boundary Γ_0^ε, in particular for the derivation of the boundary layer terms.

1.2 The scaling

To study the behavior of the unknown $\boldsymbol{u}^\varepsilon$ as ε tends to zero, it is convenient to get rid of the small parameter ε in the domain Ω^ε by means of a dilatation along the vertical axis ($x_3 = \varepsilon^{-1}\, \tilde{x}_3$). This leads to a fixed reference configuration Ω:

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \Omega^\varepsilon \mapsto x = (x_1, x_2, x_3) \in \Omega, \quad \text{where} \quad \Omega = \omega \times (-1, +1). \tag{1.8}$$

Then, the new unknown displacement is a vector field denoted by $\boldsymbol{u}(\varepsilon)$ which we define for the sake of homogeneity by:

$$u_\alpha(\varepsilon)(x) = u^\varepsilon_\alpha(\tilde{x}),\ \alpha = 1,2, \qquad u_3(\varepsilon)(x) = \varepsilon\, u^\varepsilon_3(\tilde{x}). \tag{1.9}$$

We may assume without theoretical restriction that the asymptotic expansion of the unknown displacement will begin with a zero-th order term in ε: so the following additional assumption on the right hand side is necessary

$$f^\varepsilon_\alpha(\tilde{x}) = f_\alpha(x),\ \alpha = 1,2, \qquad \varepsilon^{-1} f^\varepsilon_3(\tilde{x}) = f_3(x), \tag{1.10a}$$

$$\varepsilon^{-1} g^{\varepsilon,\pm}_\alpha(\tilde{x}) = g^\pm_\alpha(x_*),\ \alpha = 1,2, \qquad \varepsilon^{-2} g^{\varepsilon,\pm}_3(\tilde{x}) = g^\pm_3(x_*). \tag{1.10b}$$

This scalings or similar ones are standard, see e.g. Ciarlet [2] or Miara [13]. For simplification, we assume that the data $\boldsymbol{f}$ and $\boldsymbol{g}^\pm$ are regular up to the boundary, i.e. $\boldsymbol{f} \in \mathscr{C}^\infty(\overline{\Omega})^3$ and $\boldsymbol{g}^\pm \in \mathscr{C}^\infty(\overline{\varpi})^3$. After the scaling (1.9), (1.10), problem (1.1) is transformed into a new boundary value problem on Ω, where now the operators depend on the small parameter ε: The variational formulation of the problem for the scaled displacement $\boldsymbol{u}(\varepsilon)$ consists in finding

$$\begin{cases} \boldsymbol{u}(\varepsilon) \in V_{(\mathrm{i})}(\Omega) \\ \forall\, \boldsymbol{v} \in V_{(\mathrm{i})}(\Omega), \quad \displaystyle\int_\Omega A\,\theta(\varepsilon)(\boldsymbol{u}(\varepsilon)) : \theta(\varepsilon)(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \;+ \int_{\Gamma_+} \boldsymbol{g}^+\cdot\boldsymbol{v} \;- \int_{\Gamma_-} \boldsymbol{g}^-\cdot\boldsymbol{v}, \end{cases} \tag{1.11}$$

where

$$V_{(\mathrm{i})}(\Omega) := \{\boldsymbol{v} \in H^1(\Omega)^3 \mid \quad v_a = 0 \text{ on } \Gamma_0 \quad \forall a \in A_{(\mathrm{i})}\} \tag{1.12}$$

is the space of geometrically admissible displacements associated to the problem with lateral boundary conditions (i) and $\theta(\varepsilon)(\boldsymbol{v})$ denotes the scaled linearized strain tensor defined by

$$\theta_{\alpha\beta}(\varepsilon)(\boldsymbol{v}) := e_{\alpha\beta}(\boldsymbol{v})\,, \quad \theta_{\alpha 3}(\varepsilon)(\boldsymbol{v}) := \varepsilon^{-1}\, e_{\alpha 3}(\boldsymbol{v})\,, \quad \theta_{33}(\varepsilon)(\boldsymbol{v}) := \varepsilon^{-2}\, e_{33}(\boldsymbol{v})\,, \tag{1.13}$$

for $\alpha,\beta = 1,2$ and it holds

$$\theta(\varepsilon)(\boldsymbol{u}(\varepsilon)) = e(\boldsymbol{u}^\varepsilon).$$

Denoting by $\mathcal{R}_{(\mathrm{i})}(\Omega)$ the space of rigid motions satisfying the Dirichlet conditions of $V_{(\mathrm{i})}(\Omega)$, the compatibility condition (1.2) becomes

$$\forall\, \boldsymbol{v} \in \mathcal{R}_{(\mathrm{i})}(\Omega), \quad \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \;+ \int_{\Gamma_+} \boldsymbol{g}^+\cdot\boldsymbol{v} \;- \int_{\Gamma_-} \boldsymbol{g}^-\cdot\boldsymbol{v} = 0, \tag{1.14}$$

and $\boldsymbol{u}(\varepsilon)$ satisfies the orthogonality condition

$$\forall\, \boldsymbol{v} \in \mathcal{R}_{(\mathrm{i})}(\Omega), \quad \int_\Omega \boldsymbol{u}(\varepsilon)\,\cdot\,\boldsymbol{v} = 0\,. \tag{1.15}$$

1.3 Outline of results

The displacement $\boldsymbol{u}^\varepsilon$ on the thin plate Ω^ε can be expanded in the following way in the sense of asymptotic expansions* (see Theorems 4.2, 5.1, 6.1, 7.1)

$$\boldsymbol{u}^\varepsilon \simeq \frac{1}{\varepsilon}\boldsymbol{u}^0_{\mathrm{KL,b}} + \boldsymbol{u}^0_{\mathrm{KL,m}} + \boldsymbol{u}^1_{\mathrm{KL,b}} + \varepsilon(\boldsymbol{u}^1_{\mathrm{KL,m}} + \boldsymbol{u}^2_{\mathrm{KL,b}} + \tilde{\boldsymbol{v}}^1 + \boldsymbol{\varphi}^1) + \dots \\ \dots + \varepsilon^k(\boldsymbol{u}^k_{\mathrm{KL,m}} + \boldsymbol{u}^{k+1}_{\mathrm{KL,b}} + \tilde{\boldsymbol{v}}^k + \boldsymbol{\varphi}^k) + \cdots \tag{1.16}$$

where

- for $k \geq 0$, $\boldsymbol{u}^k_{\mathrm{KL,b}}$ and $\boldsymbol{u}^k_{\mathrm{KL,m}}$ are the bending and membrane parts of a Kirchhoff-Love displacement (see Part I, Definition 1.1):

$$\boldsymbol{u}^k_{\mathrm{KL,b}} = (-\tilde{x}_3\partial_1\zeta^k_3, -\tilde{x}_3\partial_2\zeta^k_3, \zeta^k_3) \quad \text{and} \quad \boldsymbol{u}^k_{\mathrm{KL,m}} = (\zeta^k_1, \zeta^k_2, 0), \tag{1.17}$$

  where ζ^k_1, ζ^k_2 and ζ^k_3 are functions of the in-plane variables x_*;
- for $k \geq 1$, $\tilde{\boldsymbol{v}}^k = \tilde{\boldsymbol{v}}^k(x_*, \frac{\tilde{x}_3}{\varepsilon})$, i.e. does not depend on ε in the scaled domain and has a mean value zero on each fiber $x_* \times (-\varepsilon, \varepsilon)$;
- for $k \geq 1$, $\boldsymbol{\varphi}^k = \boldsymbol{\varphi}^k(\frac{r}{\varepsilon}, s, \frac{\tilde{x}_3}{\varepsilon})$ is a boundary layer profile.

The generators $(\zeta^k_1, \zeta^k_2) = \boldsymbol{\zeta}^k_*$ and ζ^k_3 of the above Kirchhoff displacements are solutions of membrane and bending equations respectively, with boundary conditions on $\partial\omega$ involving, in each of the four investigated cases:

	Membrane part		**Bending part**	
5	ζ_n	ζ_s	ζ_3	$\partial_n\zeta_3$
6	ζ_n	$T^{\mathrm{m}}_s(\boldsymbol{\zeta}_*)$	$\partial_n\zeta_3$	$N_n(\zeta_3)$
7	$T^{\mathrm{m}}_n(\boldsymbol{\zeta}_*)$	ζ_s	ζ_3	$M_n(\zeta_3)$
8	$T^{\mathrm{m}}_n(\boldsymbol{\zeta}_*)$	$T^{\mathrm{m}}_s(\boldsymbol{\zeta}_*)$	$M_n(\zeta_3)$	$N_n(\zeta_3)$

Table 2. Boundary conditions for the Kirchhoff-Love displacements.

Here ζ_n and ζ_s are the normal and tangential components of $\boldsymbol{\zeta}_*$, T^{m}_n and T^{m}_s are the normal and tangential components of the traction associated with the membrane operator. M_n is the second order boundary operator and N_n the third order boundary operator associated with the bending operator. The mechanical interpretation of this boundary operators is that M_n corresponds to the 'Kirchhoff bending moment' and N_n corresponds to the 'Kirchhoff shear force' on the lateral side of the plate (up to constants only depending on λ and μ).

Unlike the case of the four other lateral boundary conditions, the boundary data for $\boldsymbol{\zeta}^0$ are not all zero.

*This means that if a norm is fixed — for example the H^1 or the L^2 norm of the displacements or the energy — then there exists a shift d such that for any k the norm of the difference between $\boldsymbol{u}^\varepsilon$ and the truncated series of (1.16) at the order k is estimated by $c_k\varepsilon^{k-d}$, see §8 for more details.

In the cases ⑤ and ⑦, we assume for simplicity that ω is simply connected. Then the trace of ζ_3^0 on $\partial\omega$ is a prescribed constant (so that ζ_3^0 has a zero mean value in accordance with the orthogonality condition (1.15)) which can be obtained as the scalar product of the right hand side of the first bending limit problem versus the solution of typical problems for the bending operator, see (2.7).

In the cases ⑥ and ⑧ the boundary condition involving N_n is given by

$$N_n(\zeta_3^0) = \frac{3}{2}\left(\int_{-1}^{1} x_3\, f_n\, dx_3 + g_n^+ + g_n^-\right)\Big|_{\partial\omega}.$$

The mechanical interpretation of the right hand side in this relation reads that this expression has the dimension of a moment and can be understood as a prescribed moment on the lateral side of the plate, generated by f_n, g_n^+ and g_n^-. Obviously, this right hand side is zero, if the supports of the data f_n and $g_n^{\pm}$ avoid $\overline{\Gamma}_0$ and $\partial\omega$, respectively. The remaining boundary data for $\boldsymbol{\zeta}^0$ are all zero.

In contrast to the four 'clamped' lateral conditions, the boundary conditions relating to the membrane part $\boldsymbol{\zeta}_*^1$ are all zero, which combined with the fact that the interior right hand side is zero yields that $\boldsymbol{\zeta}_*^1$ is itself zero in each of these four 'free' lateral conditions ⑤ – ⑧.

The traces of ζ_3^1 are generically not zero: in cases ⑤ and ⑦ (and if ω is simply connected) all traces can be expressed with the help of the function

$$L(s) = \left[-\frac{2}{3}(\tilde{\lambda} + 2\mu)\partial_n\Delta_*\,\zeta_3^0 + \int_{-1}^{+1} x_3\, f_n\, dx_3 + g_n^+ + g_n^-\right]\Big|_{\partial\omega}.$$

In cases ⑥ and ⑧ the prescribed values of the traces involve more complicated operators. We write the boundary data for ζ_3^1 in a condensed form in the next table (we recall that κ is the curvature of $\partial\omega$).

⑤	$\Lambda^{⑤}$	0
⑥	0	$P^{⑥}(\zeta_3^0) + \kappa K^{⑥}(f_n, g_n^{\pm})$
⑦	$\Lambda^{⑦}$	L
⑧	$\partial_s(\partial_n + \kappa)\partial_s\zeta_3^0$	$P^{⑧}(\zeta_3^0) + \kappa K^{⑧}(f_n, g_n^{\pm})$

Table 3. Boundary data for ζ_3^1.

Here $\Lambda^{⑤}$ and $\Lambda^{⑦}$ are special double primitives of L on $\partial\omega$. $P^{⑥}$ is a linear combination of $\partial_s\kappa^2\partial_s$, $(\kappa\partial_s)^2$ and $\kappa\partial_n\Delta_*$, and $P^{⑧}$ of $\kappa\partial_n\Delta_*$, $\partial_s(\kappa(\partial_n+\kappa)\partial_s$ and $\kappa\partial_s(\partial_n+\kappa)\partial_s$. Finally, $K^{⑥}$ and $K^{⑧}$ are operators preserving the support with respect to the in-plane variables.

The first non-Kirchhoff displacement $\tilde{v}^1$ is completely determined by $\boldsymbol{\zeta}^0$, but in a way which does not depend on the lateral boundary conditions:

$$\tilde{\boldsymbol{v}}^1(x_*, x_3) = \frac{\lambda}{6(\lambda+2\mu)}\Big(0,\ 0,\ -6x_3\,\mathrm{div}_*\,\boldsymbol{\zeta}_*^0 + (3x_3^2-1)\,\Delta_*\zeta_3^0\Big). \tag{1.18}$$

Again in contrast to the four 'clamped' lateral conditions, the normal and transverse components of the first boundary layer profile $\boldsymbol{\varphi}^1$ are always zero in the cases ⑤ – ⑧. Only the in-plane tangential component φ^1_s is generically non-zero, and it is odd with respect to x_3. This means that $\boldsymbol{\varphi}^1$ is a bending displacement.

The component φ^1_s can be written in tensor product form $\ell^{\rm s}(s)\,\bar\varphi^{\rm s}(t,x_3)$ according to the following table

Case	$\ell^{\rm s}$	$\bar\varphi^{\rm s}$
⑤	$\partial_s\zeta^0_3$	$\bar\varphi^{\rm s}_{\rm Dir}$
⑥	$\kappa\partial_s\zeta^0_3$	$\bar\varphi^{\rm s}_{\rm Neu}$
⑦	$\partial_s\zeta^0_3$	$\bar\varphi^{\rm s}_{\rm Dir}$
⑧	$(\partial_n+\kappa)\partial_s\zeta^0_3$	$\bar\varphi^{\rm s}_{\rm Neu}$

Table 4. The first boundary layer profile.

Here $\bar\varphi^{\rm s}_{\rm Dir}$ and $\bar\varphi^{\rm s}_{\rm Neu}$ are solutions on $\mathbb{R}^+\times(-1,1)$ of a Dirichlet-Neumann and a Neumann problem respectively with x_3 as boundary condition on $t=0$, see Lemmas 3.6 and 3.7.

Note the presence of κ in front of the traces for the sliding edge case ⑥: the existence of boundary layer terms is linked to non-zero curvature.

2 INNER – OUTER EXPANSION ANSATZ

Here we recall some general principles and basic formulas from Part I, §2, 3.

2.1 The Ansatz

With r being the distance to $\partial\omega$, s an arc-length coordinate in $\partial\omega$ and t denoting $r\,\varepsilon^{-1}$, the scaled form of the expansion (1.16) can be written as:

$$\boldsymbol{u}(\varepsilon)(x)\simeq\boldsymbol{u}^0_{\rm KL}+\varepsilon\boldsymbol{u}^1(x,\frac{r}{\varepsilon})+\cdots+\varepsilon^k\boldsymbol{u}^k(x,\frac{r}{\varepsilon})+\cdots \tag{2.1}$$

where

$$\begin{aligned}\boldsymbol{u}^1(x,t)&=\boldsymbol{u}^1_{\rm KL}\qquad\;\;+\chi(r)\,\boldsymbol{w}^1(t,s,x_3)\qquad\text{with}\quad w^1_3=0,\\ \boldsymbol{u}^k(x,t)&=\boldsymbol{u}^k_{\rm KL}+\boldsymbol{v}^k+\chi(r)\,\boldsymbol{w}^k(t,s,x_3)\qquad\text{for}\quad k\geq 2.\end{aligned} \tag{2.2}$$

The links with expansion (1.16) are simply provided by the following relations

$$\boldsymbol{u}^k_{\rm KL}=\boldsymbol{u}^k_{\rm b}+\boldsymbol{u}^k_{\rm m},\quad \boldsymbol{v}^k=(\tilde{\boldsymbol{v}}^k_*,\tilde v^{k-1}_3),\quad \boldsymbol{w}^k=(\boldsymbol{\varphi}^k_*,\varphi^{k-1}_3). \tag{2.3}$$

Thus, with the splitting of coordinates, components and gradients in plane and vertical variables $x=(x_*,x_3)$, $\boldsymbol{\zeta}=(\boldsymbol{\zeta}_*,\zeta_3)$ and $\nabla=(\nabla_*,\partial_3)^\top$, the three types of terms in (2.1) are

- $\boldsymbol{u}^k_{\rm KL}$: Kirchhoff-Love displacements with 'generating functions' $\boldsymbol{\zeta}^k = (\boldsymbol{\zeta}^k_*, \zeta^k_3)$, i.e. $\boldsymbol{u}^k_{\rm KL}(x) = \Big(\boldsymbol{\zeta}^k_*(x_*) - x_3\nabla_*\zeta^k_3(x_*), \zeta^k_3(x_*)\Big)$,
- $\boldsymbol{v}^k$: displacements with zero mean value: $\forall x_* \in \overline{\omega}$, $\int_{-1}^{+1} \boldsymbol{v}^k(x_*, x_3)\, dx_3 = 0$,
- $\boldsymbol{w}^k$: exponentially decreasing profiles as $t \to +\infty$ and χ is a cut-off function equal to 1 in a neighborhood of $\partial\omega$,

The determination of the asymptotics (2.1) is split into two steps. The first one consists in finding *all* suitable power series

$$\underline{\boldsymbol{u}}(\varepsilon)(x) \simeq \underline{\boldsymbol{u}}^0(x) + \varepsilon\underline{\boldsymbol{u}}^1(x) + \cdots + \varepsilon^k\underline{\boldsymbol{u}}^k(x) + \cdots \tag{2.4}$$

which solve in the sense of asymptotic expansions the interior equations in Ω and conditions of traction on the horizontal sides $\Gamma_\pm$, see §2.2. The second step consists in finding the profiles $\boldsymbol{w}^k$ so that $\sum_k \varepsilon^k \boldsymbol{w}^k(r\varepsilon^{-1}, s, x_3)$ solves the inner equations in Ω with zero volume force, zero tractions on $\Gamma_\pm$ and compensate for the residual produced by the outer expansion series (2.4) in the lateral boundary conditions, see §2.3 to 2.5.

We emphasize that the Ansatz presented in (2.1) and (2.2) is unique in the following sense: If we take an Ansatz combining inner and outer expansion series, where the boundary layer series starts with smaller powers of ε in comparison with the power series as in (2.1), we will obtain the same result, i.e. the same series as in (2.1). Thus, the Ansatz described above is in this sense optimal.

2.2 The algorithms of the outer expansion part

The generator $\boldsymbol{\zeta}_0$ of the zero-th order term $\underline{\boldsymbol{u}}^0$ solves an uncoupled membrane-bending equation. Recall that Δ_* denotes the horizontal Laplacian $\partial_{11} + \partial_{22}$ and let $\boldsymbol{\Delta}_*$ denote the diagonal horizontal Laplacian. With the help of the 'homogenized' Lamé coefficient

$$\tilde\lambda = 2\lambda\mu(\lambda + 2\mu)^{-1}, \tag{2.5}$$

let us introduce the membrane operator $L^{\rm m}$ (plane stress model) by

$$L^{\rm m}\boldsymbol{\zeta}_* = \mu\,\boldsymbol{\Delta}_*\boldsymbol{\zeta}_* + (\tilde\lambda + \mu)\nabla_*\,{\rm div}_*\,\boldsymbol{\zeta}_* \tag{2.6}$$

and the bending operator $L^{\rm b}$ (Kirchhoff model) by

$$L^{\rm b}\zeta_3 = (\tilde\lambda + 2\mu)\Delta^2_*\zeta_3\,. \tag{2.7}$$

The horizontal generator $\boldsymbol{\zeta}^0_* = (\zeta^0_1, \zeta^0_2)$ solves the membrane equation $L^{\rm m}\boldsymbol{\zeta}^0_* = \boldsymbol{R}^0_{\rm m}$, with the right hand side $\boldsymbol{R}^0_{\rm m}$ defined as

$$\boldsymbol{R}^0_{\rm m}(x_*) = -\frac{1}{2}\left[\int_{-1}^{+1} \boldsymbol{f}_*(x_*, x_3)\, dx_3 + \boldsymbol{g}^+_*(x_*) - \boldsymbol{g}^-_*(x_*)\right], \tag{2.8}$$

and the vertical generator ζ_3^0 solves the bending equation $L^{\rm b}\zeta_3^0 = R_{\rm b}^0$, with the right hand side $R_{\rm b}^0$ defined as

$$R_{\rm b}^0(x_*) = \frac{3}{2}\left[\int_{-1}^{+1} f_3\,dx_3 + g_3^+ - g_3^- + \operatorname{div}_*\Big(\int_{-1}^{+1} x_3\,\boldsymbol{f}_*\,dx_3 + \boldsymbol{g}_*^+ + \boldsymbol{g}_*^-\Big)\right]. \tag{2.9}$$

The introduction of the following linear operators allows a description of the outer expansion algorithm, namely

- $G\ :\ (\boldsymbol{f},\boldsymbol{g}^\pm)\mapsto G(\boldsymbol{f},\boldsymbol{g}^\pm)$ and $H\ :\ (\boldsymbol{f},\boldsymbol{g}^\pm)\mapsto H(\boldsymbol{f},\boldsymbol{g}^\pm)$ continuous from $\mathscr{C}^\infty(\overline{\Omega})^3\times\mathscr{C}^\infty(\overline{\omega})^6$ into $\mathscr{C}^\infty(\overline{\Omega})^3$,
- $V\ :\ \boldsymbol{\zeta}\mapsto V\boldsymbol{\zeta}$ continuous from $\mathscr{C}^\infty(\overline{\omega})^3$ into $\mathscr{C}^\infty(\overline{\Omega})^3$,
- $W\ :\ \boldsymbol{v}\mapsto W\boldsymbol{v}$ continuous from $\mathscr{C}^\infty(\overline{\Omega})^3$ into itself,
- $X\ :\ \boldsymbol{\zeta}_*\mapsto X\boldsymbol{\zeta}_*$ continuous from $\mathscr{C}^\infty(\overline{\omega})^2$ into $\mathscr{C}^\infty(\overline{\Omega})^3$,
- $F\ :\ \boldsymbol{v}\mapsto F\boldsymbol{v} = (F_*\boldsymbol{v}, F_3\boldsymbol{v})$ continuous from $\mathscr{C}^\infty(\overline{\Omega})^3$ into $\mathscr{C}^\infty(\overline{\omega})^3$,

so that every expansion (2.4) formally solves the inner equations in Ω and the conditions of traction on the horizontal sides $\Gamma_\pm$ if and only if,

k	$\underline{\boldsymbol{u}}^k$	$\boldsymbol{v}^k$	$\boldsymbol{x}^{k-2}$	$\boldsymbol{R}_{\rm m}^k$	$R_{\rm b}^k$
0	$\boldsymbol{u}_{\rm KL}^0$	—	—	$\boldsymbol{R}_{\rm m}^0$	$R_{\rm b}^0$
2	$\boldsymbol{u}_{\rm KL}^2+\boldsymbol{v}^2$	$V\boldsymbol{\zeta}^0+\boldsymbol{x}^0$	G	$\boldsymbol{R}_{\rm m}^2$	$3F_3(W\boldsymbol{v}^2+H)$
4	$\boldsymbol{u}_{\rm KL}^4+\boldsymbol{v}^4$	$V\boldsymbol{\zeta}^2+\boldsymbol{x}^2$	$W\boldsymbol{v}^2+H$	$F_*\boldsymbol{v}^4$	$3F_3(W\boldsymbol{v}^4+X\boldsymbol{\zeta}_*^4)$
$2\ell+2$	$\boldsymbol{u}_{\rm KL}^{2\ell+2}+\boldsymbol{v}^{2\ell+2}$	$V\boldsymbol{\zeta}^{2\ell}+\boldsymbol{x}^{2\ell}$	$W\boldsymbol{v}^{2\ell}+X\boldsymbol{\zeta}_*^{2\ell}$	$F_*\boldsymbol{v}^{2\ell+2}$	$3F_3(W\boldsymbol{v}^{2\ell+2}+X\boldsymbol{\zeta}_*^{2\ell+2})$
1	$\boldsymbol{u}_{\rm KL}^1$	—	—	0	0
$2\ell+1$	$\boldsymbol{u}_{\rm KL}^{2\ell+1}+\boldsymbol{v}^{2\ell+1}$	$V\boldsymbol{\zeta}^{2\ell-1}$	—	0	0

Table 5. Algorithm formulas.

The most important and most characteristic of these operators is V which is defined for $\boldsymbol{\zeta}\in\mathscr{C}^\infty(\overline{\omega})^3$ as

$$\begin{aligned}(V\boldsymbol{\zeta})_\alpha &= \bar p_2\,\partial_\alpha\operatorname{div}_*\boldsymbol{\zeta}_* + \bar p_3\,\partial_\alpha\Delta_*\zeta_3\\ (V\boldsymbol{\zeta})_3 &= \bar p_1\,\operatorname{div}_*\boldsymbol{\zeta}_* + \bar p_2\,\Delta_*\zeta_3\end{aligned} \tag{2.10}$$

with $\bar p_j$ for $j=1,2,3$ the polynomials in the variable x_3 of degrees j defined as

$$\begin{aligned}\bar p_1(x_3) = -\frac{\tilde\lambda}{2\mu}\,x_3,\qquad \bar p_2(x_3) = \frac{\tilde\lambda}{4\mu}\Big(x_3^2-\frac{1}{3}\Big),\\ \bar p_3(x_3) = \frac{1}{12\mu}\Big((\tilde\lambda+4\mu)\,x_3^3-(5\tilde\lambda+12\mu)\,x_3\Big).\end{aligned} \tag{2.11}$$

Indeed this operator V is present in each term $\underline{\boldsymbol{u}}^k$ and it generates the kernel $\mathscr{K}$ of the elasticity operator on the infinite plate $\mathbb{R}^2 \times (-1,1)$ with prescribed horizontal tractions by

$$\boldsymbol{u} \in \mathscr{K} \iff \exists \boldsymbol{\zeta} \text{ with } L^{\mathrm{m}}\boldsymbol{\zeta}_* = 0\,,\ L^{\mathrm{b}}\zeta_3 = 0\,, \text{such that } \boldsymbol{u} = \boldsymbol{u}_{\mathrm{KL}} + V\boldsymbol{\zeta}\,, \quad (2.12)$$

where $\boldsymbol{u}_{\mathrm{KL}}$ is the Kirchhoff-Love displacement generated by $\boldsymbol{\zeta}$, compare also [14].

Since we will have to use G explicitly, we give its formula here: for $(\boldsymbol{f}, \boldsymbol{g}^{\pm}) \in \mathscr{C}^\infty(\overline{\Omega})^3 \times \mathscr{C}^\infty(\overline{\omega})^6$, $(G(\boldsymbol{f}, \boldsymbol{g}^{\pm}))_3$ is zero and for $\alpha = 1,2$

$$(G(\boldsymbol{f}, \boldsymbol{g}^{\pm}))_\alpha = \frac{1}{2\mu} \oint^{x_3} \Big[\Big(-2 \,⨍^{y_3} f_\alpha \Big) + (g^+_\alpha - g^-_\alpha + \int_{-1}^{+1} f_\alpha)\, y_3 + g^+_\alpha + g^-_\alpha \Big] dy_3 \quad (2.13)$$

with $\oint$ the primitive with zero mean value and $⨍^{y_3}$ the primitive which shifts the parities.

The formulas for the other solution operators W, X, H and F can be found in §2.2 of Part I. All of these operators have a property of preserving the support with respect to the in-plane variables: we say that they are *'in-plane local'* according to the following definition.

Definition 2.1 An operator A acting from a space E of functions defined on Ω into a space of functions defined on *(i)* Ω, or *(ii)* ω, is said to be *'in-plane local'* if for all $e \in E$ and $x_* \in \omega$ there holds

$$\forall x_3 \in (-1,1),\ (x_*, x_3) \notin \operatorname{supp}(e) \implies \begin{cases} \textit{(i)} & \forall x_3 \in (-1,1),\ (x_*, x_3) \notin \operatorname{supp}(Ae), \\ \textit{(ii)} & x_* \notin \operatorname{supp}(Ae). \end{cases}$$

■

The recursive formulas for the $\boldsymbol{v}^k$ are also given in Part I, §2.3.

2.3 The interior and horizontal equations of the inner expansion part

Assuming that $\sum_k \varepsilon^k \underline{\boldsymbol{u}}^k$ already fulfills the relations in Table 5, we determine now the equations satisfied by the profiles $\boldsymbol{\varphi}^k$ so that

$$\sum_{k \geq 0} \varepsilon^k \underline{\boldsymbol{u}}^k + \sum_{k \geq 1} \varepsilon^k (\boldsymbol{\varphi}^k_*, \varepsilon \varphi^k_3) \quad (2.14)$$

satisfies the inner equations in Ω and the horizontal traction conditions of problem (1.11). Thus $\sum_{k \geq 1} \varepsilon^k (\boldsymbol{\varphi}^k_*, \varepsilon\varphi^k_3)$ has to satisfy the inner equations with zero volume force in Ω, and zero tractions on $\Gamma^{\pm}$.

(i) Interior equations. In variables (t, s, x_3), and unknown $\boldsymbol{\varphi} \sim \sum_k \varepsilon^k \boldsymbol{\varphi}^k$ the interior equations become

$$\mathscr{B}(\varepsilon\,;\, t, s\,;\, \partial_t, \partial_s, \partial_3)\, \boldsymbol{\varphi} = 0.$$

The expressions for the three components $(\mathscr{B}\boldsymbol{\varphi})_t$, $(\mathscr{B}\boldsymbol{\varphi})_s$ and $(\mathscr{B}\boldsymbol{\varphi})_3$ of $\mathscr{B}\boldsymbol{\varphi}$ can be found in Part I, formula (3.7), where we introduced the notation

$$\rho = R(s) - r = R(s) - \varepsilon t$$

representing the curvature radius in s of the curve $\{x_* \in \omega,\ \mathrm{dist}(x_*, \partial\omega) = r\}$.

A Taylor expansion in $t = 0$ of $\rho^{-1} = (R - \varepsilon t)^{-1}$ yields an asymptotic expansion of $\mathscr{B}$ in a power series of ε:

$$\mathscr{B} \sim \mathscr{B}^{(0)} + \varepsilon\mathscr{B}^{(1)} + \cdots + \varepsilon^k\mathscr{B}^{(k)} + \cdots$$

where the $\mathscr{B}^{(k)}(t, s\,; \partial_t, \partial_s, \partial_3)$ are partial differential systems of order 2 with polynomial coefficients in t independent from ε. The first terms $\mathscr{B}^{(0)}$ and $\mathscr{B}^{(1)}$ are given by the expressions:

$$\begin{aligned}
(\mathscr{B}^{(0)}\boldsymbol{\varphi})_t &= \mu\Big(\partial_{tt}\varphi_t + \partial_{33}\varphi_t\Big) + (\lambda+\mu)\,\partial_t\Big(\partial_t\varphi_t + \partial_3\varphi_3\Big)\\
(\mathscr{B}^{(0)}\boldsymbol{\varphi})_s &= \mu\Big(\partial_{tt}\varphi_s + \partial_{33}\varphi_s\Big)\\
(\mathscr{B}^{(0)}\boldsymbol{\varphi})_3 &= \mu\,\Big(\partial_{tt}\varphi_3 + \partial_{33}\varphi_3\Big) + (\lambda+\mu)\,\partial_3\Big(\partial_t\varphi_t + \partial_3\varphi_3\Big)
\end{aligned} \tag{2.15}$$

and, with the curvature $\kappa = \frac{1}{R}$:

$$\begin{aligned}
(\mathscr{B}^{(1)}\boldsymbol{\varphi})_t &= -\mu\,\kappa\,\partial_t\varphi_t + (\lambda+\mu)\,\partial_t\Big(-\kappa\,\varphi_t + \partial_s\varphi_s\Big)\\
(\mathscr{B}^{(1)}\boldsymbol{\varphi})_s &= \mu\,\kappa\Big(\partial_{tt}(t\varphi_s) + \partial_{33}(t\varphi_s)\Big) - \mu\,\kappa\,\partial_t\varphi_s + (\lambda+\mu)\,\partial_s\Big(\partial_t\varphi_t + \partial_3\varphi_3\Big)\\
(\mathscr{B}^{(1)}\boldsymbol{\varphi})_3 &= -\mu\,\kappa\,\partial_t\varphi_3 + (\lambda+\mu)\,\partial_3\Big(-\kappa\,\varphi_t + \partial_s\varphi_s\Big).
\end{aligned} \tag{2.16}$$

Thus, the interior equation $\mathscr{B}(\varepsilon)\boldsymbol{\varphi} = 0$ can be written as

$$\mathscr{B}^{(0)}\boldsymbol{\varphi} + \varepsilon\mathscr{B}^{(1)}\boldsymbol{\varphi} + \cdots \varepsilon^k\mathscr{B}^{(k)}\boldsymbol{\varphi} + \cdots \sim 0,$$

which, going back to the terms of the series $\boldsymbol{\varphi} \sim \sum_k \varepsilon^k\boldsymbol{\varphi}^k$, yields

$$\forall k \geq 0, \qquad \sum_{\ell=0}^{k} \mathscr{B}^{(\ell)}\boldsymbol{\varphi}^{k-\ell} = 0. \tag{2.17}$$

(*ii*) Horizontal boundary conditions. The boundary conditions on the horizontal sides $x_3 = \pm 1$ are

$$\begin{aligned}
&\mu(\partial_3\varphi_t + \partial_t\varphi_3) = 0,\\
&\mu\partial_3\varphi_s + \varepsilon\,\mu\,\partial_s\varphi_3 = 0,\\
&(\lambda+2\mu)\partial_3\varphi_3 + \lambda\,\partial_t\varphi_t + \varepsilon\,\lambda\Big(-\tfrac{1}{\rho}\varphi_t + \tfrac{R}{\rho}\partial_s(\tfrac{R}{\rho}\varphi_s)\Big) = 0.
\end{aligned}$$

Similarly to the interior equations, we can develop these horizontal boundary conditions $\mathscr{G}$ in powers of ε:

$$\mathscr{G} \sim \mathscr{G}^{(0)} + \varepsilon\mathscr{G}^{(1)} + \cdots \varepsilon^k\mathscr{G}^{(k)} + \cdots \tag{2.18}$$

where the $\mathscr{G}^{(k)}(t,s\,;\partial_t,\partial_s,\partial_3)$ are partial differential systems of order 1 with polynomial coefficients in t. The expressions for $\mathscr{G}^{(0)}$ and $\mathscr{G}^{(1)}$ are:

$$\begin{array}{rclrcl}
(\mathscr{G}^{(0)}\boldsymbol{\varphi})_t &=& \mu(\partial_3\varphi_t+\partial_t\varphi_3), & (\mathscr{G}^{(1)}\boldsymbol{\varphi})_t &=& 0,\\
(\mathscr{G}^{(0)}\boldsymbol{\varphi})_s &=& \mu\partial_3\varphi_s, & (\mathscr{G}^{(1)}\boldsymbol{\varphi})_s &=& \mu\partial_s\varphi_3,\\
(\mathscr{G}^{(0)}\boldsymbol{\varphi})_3 &=& (\lambda+2\mu)\partial_3\varphi_3+\lambda\,\partial_t\varphi_t, & (\mathscr{G}^{(1)}\boldsymbol{\varphi})_3 &=& \lambda(-\kappa\,\varphi_t+\partial_s\varphi_s).
\end{array} \tag{2.19}$$

Thus, the horizontal boundary conditions $\mathscr{G}(\varepsilon)\boldsymbol{\varphi}=0$ can be written as

$$\mathscr{G}^{(0)}\boldsymbol{\varphi}+\varepsilon\mathscr{G}^{(1)}\boldsymbol{\varphi}+\cdots\varepsilon^k\mathscr{G}^{(k)}\boldsymbol{\varphi}+\cdots\sim 0,$$

which, going back to the terms of the series $\boldsymbol{\varphi}\sim\sum_k\varepsilon^k\boldsymbol{\varphi}^k$, yields

$$\forall k\geq 0,\qquad \sum_{\ell=0}^{k}\mathscr{G}^{(\ell)}\boldsymbol{\varphi}^{k-\ell}=0. \tag{2.20}$$

2.4 The matching of lateral boundary conditions

It remains to formulate the equations that have to hold in order that the series (2.14) satisfies the lateral boundary conditions of problem (1.11).

(*i*) Lateral Dirichlet boundary conditions. Let $\sum_k\varepsilon^kD_n^k$, $\sum_k\varepsilon^kD_s^k$ and $\sum_k\varepsilon^kD_3^k$ be the normal, tangential and vertical components of the lateral Dirichlet traces of the series (2.14). The lateral Dirichlet boundary conditions then read

$$\forall k\geq 0,\qquad D_n^k=0\ \text{ if }\ n\in A,\quad D_s^k=0\ \text{ if }\ s\in A,\quad D_3^k=0\ \text{ if }\ 3\in A, \tag{2.21}$$

which immediately yields the Dirichlet conditions for the whole expansion (2.14).

For the terms D^k, we have

$$D_n^0=\underline{u}_n^0,\quad D_s^0=\underline{u}_s^0,\quad D_3^0=\underline{u}_3^0,\quad D_3^1=\underline{u}_3^1, \tag{2.22}$$

and for $k\geq 1$

$$\begin{array}{rcl}
D_n^k &=& \varphi_t^k+\underline{u}_n^k,\\
D_s^k &=& \varphi_s^k+\underline{u}_s^k,\\
D_3^{k+1} &=& \varphi_3^k+\underline{u}_3^{k+1}\,.
\end{array} \tag{2.23}$$

(*ii*) Lateral Neumann boundary conditions. Let $\sum_k\varepsilon^kT_n^k$, $\sum_k\varepsilon^kT_s^k$ and $\sum_k\varepsilon^kT_3^k$ be the normal, tangential and vertical components of the lateral Neumann traces of the series (2.14). The lateral Neumann boundary conditions then read

$$\forall k\geq 0,\qquad T_n^k=0\ \text{ if }\ n\in B,\quad T_s^k=0\ \text{ if }\ s\in B,\quad T_3^k=0\ \text{ if }\ 3\in B, \tag{2.24}$$

which immediately yields the Neumann conditions for the whole expansion (2.14).

The evaluation of the terms T^k involves the traction $\boldsymbol{T}(\boldsymbol{\varphi})=(T_n,T_s,T_3)\boldsymbol{\varphi}$ associated to the interior operator $\mathscr{B}$, the traction $\boldsymbol{T}_*^{\mathrm{m}}=(T_n^{\mathrm{m}},T_s^{\mathrm{m}})$ associated to the membrane operator L^{m} (2.6) and the natural boundary conditions M_n and N_n associated to the bending operator L^{b} (2.7).

- Setting

$$\begin{array}{rclrcl} T_t^{(0)}(\boldsymbol{\varphi}) &=& \lambda\,\partial_3\varphi_3 + (\lambda+2\mu)\partial_t\varphi_t, & T_t^{(1)}(\boldsymbol{\varphi}) &=& \lambda(\partial_s\varphi_s - \frac{1}{R}\,\varphi_t),\\ T_s^{(0)}(\boldsymbol{\varphi}) &=& \mu\,\partial_t\varphi_s, & T_s^{(1)}(\boldsymbol{\varphi}) &=& \mu(\partial_s\varphi_t + \frac{2}{R}\,\varphi_s),\\ T_3^{(0)}(\boldsymbol{\varphi}) &=& \mu(\partial_t\varphi_3 + \partial_3\varphi_t), \end{array} \tag{2.25}$$

we have $(\varepsilon^{-1}T_n, \varepsilon^{-1}T_s, T_3) = (T_t^{(0)}, T_s^{(0)}, T_3^{(0)}) + \varepsilon(T_t^{(1)}, T_s^{(1)}, T_3^{(1)})$.

- The bilinear form associated with the membrane operator L^{m} reads:

$$a(\boldsymbol{\zeta}_*, \boldsymbol{\eta}_*) = \int_\omega \tilde\lambda\, e_{\alpha\alpha}(\boldsymbol{\zeta}_*)\; e_{\beta\beta}(\boldsymbol{\eta}_*) + 2\mu\, e_{\alpha\beta}(\boldsymbol{\zeta}_*)\; e_{\alpha\beta}(\boldsymbol{\eta}_*)$$

and the associated tractions are

$$T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*) = \tilde\lambda\,\mathrm{div}_*\,\boldsymbol{\zeta}_* + 2\mu\,\partial_n\zeta_n, \tag{2.26a}$$

$$T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*) = \mu(\partial_s\zeta_n + \partial_n\zeta_s + \tfrac{2}{R}\,\zeta_s). \tag{2.26b}$$

- The bilinear form associated with the bending operator L^{b} is:

$$b(\zeta_3, \eta_3) = \int_\omega \tilde\lambda\,\partial_{\alpha\alpha}\zeta_3\;\partial_{\beta\beta}\eta_3 + 2\mu\,\partial_{\alpha\beta}\zeta_3\;\partial_{\alpha\beta}\eta_3 \tag{2.27}$$

and its natural traces are

$$M_n(\zeta_3) = \tilde\lambda\,\Delta_*\zeta_3 + 2\mu\,\partial_{nn}\zeta_3, \tag{2.28a}$$

$$N_n(\zeta_3) = (\tilde\lambda + 2\mu)\partial_n\Delta_*\zeta_3 + 2\mu\,\partial_s(\partial_n + \tfrac{1}{R})\partial_s\zeta_3. \tag{2.28b}$$

Then, for the normal, tangential and vertical components of the lateral Neumann traces of the series (2.14) the following formulas are valid:

$$T_n^0 = 0,\quad T_n^1 = 0,\quad T_s^0 = 0,\quad T_s^1 = 0,\quad T_3^0 = 0, \tag{2.29}$$

and for $k \geq 1$:

$$\begin{aligned} T_n^{k+1} &= T_t^{(0)}(\boldsymbol{\varphi}^k) + T_t^{(1)}(\boldsymbol{\varphi}^{k-1}) + T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^{k-1}) - x_3\,M_n(\zeta_3^{k-1})\\ &\qquad + \lambda\,\partial_3\,\boldsymbol{x}_3^{k-1} + \lambda\,\mathrm{div}_*\,\boldsymbol{v}_*^{k-1} + 2\mu\,\partial_n v_n^{k-1} \end{aligned} \tag{2.30a}$$

$$\begin{aligned} T_s^{k+1} &= T_s^{(0)}(\boldsymbol{\varphi}^k) + T_s^{(1)}(\boldsymbol{\varphi}^{k-1}) + T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^{k-1}) - 2\mu x_3\,(\partial_n + \tfrac{1}{R})\partial_s\zeta_3^{k-1}\\ &\qquad + \mu\Big(\partial_s v_n^{k-1} + \partial_n v_s^{k-1} + \tfrac{2}{R}\,v_s^{k-1}\Big) \end{aligned} \tag{2.30b}$$

$$\begin{aligned} T_3^k &= T_3^{(0)}(\boldsymbol{\varphi}^k) + \mu(\bar p_2 + \bar p_3')\,\partial_n\Delta_*\zeta_3^{k-2}\\ &\qquad + \mu(\partial_n\boldsymbol{x}_3^{k-2} + \partial_3\boldsymbol{x}_n^{k-2}), \end{aligned} \tag{2.30c}$$

where $\bar p_1$, $\bar p_2$, $\bar p_3$ are introduced in (2.11) and the formulas for $\boldsymbol{x}^k$ can be found in Table 5.

2.5 Solving the inner expansion

According to the statements of the previous subsection it remains to find a sequence of profiles $(\boldsymbol{\varphi}^k)_k$ and a sequence of Kirchhoff-Love generators $(\boldsymbol{\zeta}^k)_k$ such that (2.17), (2.20), (2.21) and (2.24) hold. Let us consider now the profiles $\boldsymbol{\varphi}^k$ for $k \geq 1$ as main unknowns. In view of (2.17), (2.20) and (2.23) and (2.30), we see that the sequence of problems satisfied by the $\boldsymbol{\varphi}^k$ can be written in a recursive way: for each $k \geq 1$ the profile $\boldsymbol{\varphi}^k$ has to solve the equation

$$\mathscr{B}_{\text{ⓘ}}(\boldsymbol{\varphi}^k) = (\mathfrak{f}^k; \mathfrak{g}^k; \mathfrak{h}^k), \tag{2.31}$$

where

- $\mathscr{B}_{\text{ⓘ}}$ is the operator $\mathscr{B}^{(0)}$ inside the domain, the traction operator $\mathscr{G}^{(0)}$ on the horizontal sides, the Dirichlet traces on the lateral side for $a \in A_{\text{ⓘ}}$ and the Neumann traces on the lateral side for $b \in B_{\text{ⓘ}}$,
- $\mathfrak{f}^k$ and $\mathfrak{g}^k$ are the following functions of the previous profiles

$$\mathfrak{f}^k = -\sum_{\ell=1}^{k} \mathscr{B}^{(\ell)} \boldsymbol{\varphi}^{k-\ell} \quad \text{and} \quad \mathfrak{g}^k = -\sum_{\ell=1}^{k} \mathscr{G}^{(\ell)} \boldsymbol{\varphi}^{k-\ell}, \tag{2.32}$$

  so that (2.17)-(2.20) is solved, and $\mathfrak{h}^k$ involves previous profiles as well and certain traces of the Kirchhoff-Love generators $\boldsymbol{\zeta}^\ell$ according to (2.21)-(2.30).

The operators of $\mathscr{B}_{\text{ⓘ}}$ contain no derivative with respect to the tangential variable s, thus the role of s is reduced to that of a parameter and the equations (2.31) can be solved in the variables $t \in \mathbb{R}^+$ and $x_3 \in (-1,1)$. So we introduce the half-strip $\Sigma^+ = \mathbb{R}^+ \times (-1,1)$. Its boundary has two horizontal parts $\gamma_\pm = \mathbb{R}^+ \times \{x_3 = \pm 1\}$ and a lateral part $\gamma_0 = \{0\} \times (-1,1)$. Thus, we have

$$\mathscr{B}_{\text{ⓘ}}(\boldsymbol{\varphi}) = (\mathfrak{f}; \mathfrak{g}; \mathfrak{h}) \quad \Longleftrightarrow \quad \left\{ \begin{array}{rcll} \mathscr{B}^{(0)}(\boldsymbol{\varphi}) & = & \mathfrak{f}, & \text{in } \Sigma^+, \\ \mathscr{G}^{(0)}(\boldsymbol{\varphi}) & = & \mathfrak{g}, & \text{on } \gamma_\pm, \\ \varphi_a & = & \mathfrak{h}_a, & \text{on } \gamma_0, \quad \forall a \in A_{\text{ⓘ}}, \\ T_b^{(0)}(\boldsymbol{\varphi}) & = & \mathfrak{h}_b, & \text{on } \gamma_0, \quad \forall b \in B_{\text{ⓘ}}. \end{array} \right. \tag{2.33}$$

Essential is the possibility of finding *exponentially decreasing* solutions when $\mathfrak{f}$ and $\mathfrak{g}$ have the same property. This question will be investigated in the next section.

3 EXPONENTIALLY DECAYING PROFILES IN A HALF-STRIP

3.1 General principles

The properties of the operators $\mathscr{B}_{\text{ⓘ}}$ are closely linked to those of the corresponding operator $\underline{\mathscr{B}}$ on the full strip $\Sigma := \mathbb{R} \times (-1,1)$, defined as $\underline{\mathscr{B}}(\boldsymbol{\varphi}) = (\mathfrak{f}; \mathfrak{g})$ with $\mathfrak{f} = \mathscr{B}^{(0)}(\boldsymbol{\varphi})$ in Σ and $\mathfrak{g} = \mathscr{G}^{(0)}(\boldsymbol{\varphi})$ on $\mathbb{R} \times \{x_3 = \pm 1\}$, see also [16, Ch.5] and [14]. Let $\mathcal{P}$ be the space of polynomial displacements $\boldsymbol{Z}$ satisfying $\underline{\mathscr{B}}(\boldsymbol{Z}) = 0$.

Then $\mathcal{P}$ has eight dimensions and a basis of $\mathcal{P}$ is given by the following polynomial displacements $\boldsymbol{Z}^{[1]}, \cdots, \boldsymbol{Z}^{[8]}$

$$\boldsymbol{Z}^{[1]} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{Z}^{[2]} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{Z}^{[3]} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \boldsymbol{Z}^{[4]} = \begin{pmatrix} -x_3 \\ 0 \\ t \end{pmatrix}$$

$$\boldsymbol{Z}^{[5]} = \begin{pmatrix} t \\ 0 \\ \bar{p}_1 \end{pmatrix} \qquad \boldsymbol{Z}^{[6]} = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} \qquad \boldsymbol{Z}^{[7]} = \begin{pmatrix} -2tx_3 \\ 0 \\ t^2 + 2\bar{p}_2 \end{pmatrix}$$

$$\boldsymbol{Z}^{[8]} = \begin{pmatrix} -3t^2x_3 + 6\bar{p}_3 \\ 0 \\ t^3 + 6t\bar{p}_2 \end{pmatrix}$$

where $\bar{p}_1(x_3)$, $\bar{p}_2(x_3)$, $\bar{p}_3(x_3)$ are the polynomials previously introduced in (2.11). Let us note that these generators can be obtained from (2.12) by dimension restriction.

Let us introduce exponential weighted spaces H^m_η on the half-strip Σ^+:

Definition 3.1 Let $\eta \in \mathbb{R}$. For $m \geq 0$ let $H^m_\eta(\Sigma^+)$ be the space of functions v such that $e^{\eta t}v$ belongs to $H^m(\Sigma^+)$. We also denote $H^0_\eta(\Sigma^+)$ by $L^2_\eta(\Sigma^+)$. Similar definitions hold for $\mathbb{R}^+$. ■

We have, with η_0 the smallest exponent arising from the Papkovich-Fadle eigenfunctions, *cf* [19], [11], [6]:

Lemma 3.2 *Let η, $0 < \eta < \eta_0$. Let $\mathfrak{f}$ belong to $L^2_\eta(\Sigma^+)^3$ and $\mathfrak{g}$ belong to $L^2_\eta(\mathbb{R}^+)^6$, let $\mathfrak{h}_a$ belong to $H^{1/2}(\gamma_0)$ for each $a \in A_{ⓘ}$ and $\mathfrak{h}_b$ belong to $H^{-1/2}(\gamma_0)$ for each $b \in B_{ⓘ}$. Then there exist $\boldsymbol{\varphi} \in H^1_\eta(\Sigma^+)^3$ and $\boldsymbol{Z} \in \mathcal{P}$ so that*

$$\mathscr{B}_{ⓘ}(\boldsymbol{\varphi} + \boldsymbol{Z}) = (\mathfrak{f}; \mathfrak{g}; \mathfrak{h}).$$

But the solution given by Lemma 3.2 is not unique. Let $\mathcal{T}_{ⓘ}$ denote the space of the polynomial displacements $\boldsymbol{Z}$ such that there exists $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\boldsymbol{Z}) \in H^1_\eta(\Sigma^+)^3$ satisfying $\mathscr{B}_{ⓘ}(\boldsymbol{Z} + \boldsymbol{\varphi}(\boldsymbol{Z})) = 0$. Like in [6, Proposition 4.12], we can prove that $\dim \mathcal{T}_{ⓘ} = 4$. Thus $\mathcal{P}$ can be split in the direct sum of two four-dimensional spaces $\mathcal{Z}_{ⓘ}$ and $\mathcal{T}_{ⓘ}$, and we have as corollary:

Lemma 3.3 *Let $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ be as in Lemma* 3.2. *Then there exist $\boldsymbol{\varphi}$* unique *in $H^1_\eta(\Sigma^+)^3$ and $\boldsymbol{Z}$* unique *in the four-dimensional space $\mathcal{Z}_{ⓘ}$ so that*

$$\mathscr{B}_{ⓘ}(\boldsymbol{\varphi} + \boldsymbol{Z}) = (\mathfrak{f}; \mathfrak{g}; \mathfrak{h}).$$

Thus, we have a defect number equal to four for the solution of the sequence of the above equations (2.31) by exponentially decreasing displacements $\boldsymbol{\varphi}^k$, for each $s \in \partial\omega$. But four traces on $\partial\omega$ are still available, allowing to modify $\mathfrak{h}^k$. We prove in the following sections that for each ⓘ, these traces can be matched with the conditions ensuring exponential decay.

3.2 The operators acting on profiles

The operators $\mathscr{B}_{\textcircled{i}}$ decouple into operators acting separately on the pair of components (φ_t, φ_3) that we denote $\boldsymbol{\varphi}_\natural$, and on φ_s; on $\boldsymbol{\varphi}_\natural$ acts an elasticity operator with the Lamé constants λ and μ, and on φ_s a Laplace operator.

The interior elasticity operator in Σ^+ is

$$\boldsymbol{\varphi}_\natural \quad\longmapsto\quad \mathfrak{f}_\natural \;=\; \mu(\partial_{tt}+\partial_{33})\begin{pmatrix}\varphi_t\\ \varphi_3\end{pmatrix} + (\lambda+\mu)\begin{pmatrix}\partial_t\\ \partial_3\end{pmatrix}(\partial_t\varphi_t+\partial_3\varphi_3)\,, \tag{3.1}$$

its horizontal boundary conditions $\mathscr{G}^{(0)}$ (2.19) on $\gamma_\pm$ are

$$\boldsymbol{\varphi}_\natural \quad\longmapsto\quad \mathfrak{g}_\natural \;=\; \begin{pmatrix}\mu(\partial_3\varphi_t+\partial_t\varphi_3)\\ (\lambda+2\mu)\partial_3\varphi_3+\lambda\,\partial_t\varphi_t\end{pmatrix} \tag{3.2}$$

and the lateral boundary conditions are either Dirichlet's or Neumann's acting on the traction $\boldsymbol{T}_\natural^{(0)} = (T_t^{(0)}, T_3^{(0)})$, *cf* (2.25).

Similar to Part I, §4.2, let us introduce now the elasticity operators:

- $E_{\mathrm{Mix2}}:\ \boldsymbol{\varphi}_\natural \mapsto (\mathfrak{f}_\natural;\mathfrak{g}_\natural;\mathfrak{h}_\natural)$ with $\mathfrak{f}_\natural$ defined in (3.1), $\mathfrak{g}_\natural$ defined in (3.2) and $\mathfrak{h}_\natural$ the trace on γ_0
$$\mathfrak{h}_\natural = \Big(\varphi_t, T_3^{(0)}(\boldsymbol{\varphi}_\natural)\Big)\Big|_{\gamma_0}\quad,$$
- $E_{\mathrm{Free}}:\ \boldsymbol{\varphi}_\natural \mapsto (\mathfrak{f}_\natural;\mathfrak{g}_\natural;\mathfrak{h}_\natural)$ with $\mathfrak{f}_\natural$ defined in (3.1), $\mathfrak{g}_\natural$ defined in (3.2) and $\mathfrak{h}_\natural$ the trace of $\boldsymbol{T}_\natural^{(0)}(\boldsymbol{\varphi}_\natural)$ on γ_0,

whereas the Laplace operators are defined as:

- $L_{\mathrm{Dir}}:\ \varphi_s \mapsto (\mathfrak{f}_s;\mathfrak{g}_s;\mathfrak{h}_s)$ with $\mathfrak{f}_s=\mu\Delta\varphi_s$, $\mathfrak{g}_s=\mu\partial_3\varphi_s$ and $\mathfrak{h}_s=\varphi_s$ on γ_0,
- $L_{\mathrm{Neu}}:\ \varphi_s \mapsto (\mathfrak{f}_s;\mathfrak{g}_s;\mathfrak{h}_s)$ with $\mathfrak{f}_s=\mu\Delta\varphi_s$, $\mathfrak{g}_s=\mu\partial_3\varphi_s$ and $\mathfrak{h}_s=\mu\partial_t\varphi_s$ on γ_0.

Then we have the splittings:

$$\begin{aligned}\mathscr{B}_{⑤} &= E_{\mathrm{Mix2}}\oplus L_{\mathrm{Dir}}, &\qquad \mathscr{B}_{⑥} &= E_{\mathrm{Mix2}}\oplus L_{\mathrm{Neu}},\\ \mathscr{B}_{⑦} &= E_{\mathrm{Free}}\oplus L_{\mathrm{Dir}}, &\qquad \mathscr{B}_{⑧} &= E_{\mathrm{Free}}\oplus L_{\mathrm{Neu}}.\end{aligned} \tag{3.3}$$

3.3 The Laplacian on the half-strip

First we quote two Propositions from Part I, §4.3, where we investigated the solvability of problems for the operators L_{Dir} and L_{Neu}.

Proposition 3.4 *For $\eta>0$, let $f\in L^2_\eta(\Sigma^+)$, $g^\pm\in L^2_\eta(\mathbb{R}^+)^2$ and $h\in H^{1/2}(\gamma_0)$. If moreover $\eta<\pi/2$, then the problem*

$$L_{\mathrm{Dir}}(\psi) = (f;g^\pm;h)$$

has a unique solution $\psi=\varphi+\delta$ in $H^1_\eta(\Sigma^+)\oplus\mathrm{span}\{1\}$ with $\varphi\in H^1_\eta(\Sigma^+)$ and

$$\delta = \frac{1}{2\mu}\left(-\int_{\Sigma^+} t\,f(t,x_3)\,dt\,dx_3 + \int_{\mathbb{R}^+} t\Big(g^+(t)-g^-(t)\Big)\,dt + \mu\int_{-1}^{+1} h(x_3)\,dx_3\right). \tag{3.4}$$

Proposition 3.5 *For* $\eta > 0$ *, let* $f \in L^2_\eta(\Sigma^+)$ *,* $g^\pm \in L^2_\eta(\mathbb{R}^+)^2$ *and* $h \in H^{-1/2}(\gamma_0)$ *. If moreover* $\eta < \pi/2$ *, then the problem*

$$L_{\mathrm{Neu}}(\psi) = (f; g^\pm; h)$$

has a unique solution $\psi = \varphi + \delta\, t$ *in* $H^1_\eta(\Sigma^+) \oplus \operatorname{span}\{t\}$ *with* $\varphi \in H^1_\eta(\Sigma^+)$ *and*

$$\delta = \frac{1}{2\mu}\left(\int_{\Sigma^+} f(t, x_3)\, dt\, dx_3 - \int_{\mathbb{R}^+}\Big(g^+(t) - g^-(t)\Big)\, dt + \int_{-1}^{+1} h(x_3)\, dx_3\right). \qquad (3.5)$$

Later on we will need properties of exponentially decaying solutions of special problems involving L_{Dir} and L_{Neu} : these solutions appear as model profiles in the boundary layer terms of the expansion of $\boldsymbol{u}(\varepsilon)$. This is the topic of the following two lemmas.

Lemma 3.6 *Let* $\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}} \in H^1_\eta(\Sigma^+)$ *be the uniquely determined exponentially decaying solution of the problem*

$$L_{\mathrm{Dir}}(\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}) = (0; 0; x_3)\ ,$$

then it holds

$$\int_0^\infty \bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}(t, 1)\, dt > 0.$$

Proof.
$\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}$ is an odd function with respect to x_3 , i.e.

$$\forall t \in \mathbb{R}^+\ \forall x_3 \in (-1, 1) \quad \bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}(t, x_3) = -\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}(t, -x_3)\ .$$

Hence $\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}(t, 0) = 0$ for $t \in \mathbb{R}^+$. Moreover, $\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}$ is a harmonic function in Σ^+ and thus $\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}$ can be reflected by parity at the line $x_3 = 1$ according to the reflection principle of Schwarz for harmonic functions. Thus, we obtain a function $\tilde\varphi$, which is still harmonic, but now in $\widetilde\Sigma^+ = \mathbb{R}^+ \times (0, 2)$, with $\tilde\varphi(t, 1 + x_3) = \tilde\varphi(t, 1 - x_3) = \bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}(t, 1 - x_3)$ for all $t \in \mathbb{R}^+$ and $x_3 \in (0, 1)$. Hence $\tilde\varphi$ satisfies the following problem

$$\begin{array}{rcll} \Delta\,\tilde\varphi & = & 0 & \text{in } \ \widetilde\Sigma^+ \\ \tilde\varphi(t, 0) = \tilde\varphi(t, 2) & = & 0 & \text{for } \ t \in \mathbb{R}^+ \\ \tilde\varphi(0, x_3) & = & x_3 & \text{for } \ 0 < x_3 < 1 \\ \tilde\varphi(0, x_3) & = & 2 - x_3 & \text{for } \ 1 < x_3 < 2\,. \end{array}$$

This is a Dirichlet problem for Laplace equation. From the maximum principle for harmonic functions it follows $\tilde\varphi > 0$ in $\widetilde\Sigma^+$, hence the assertion. ■

Lemma 3.7 *Let* $\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}} \in H^1_\eta(\Sigma^+)$ *be the uniquely determined exponentially decaying solution of the problem*

$$L_{\mathrm{Neu}}(\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}) = (0; 0; 2\mu x_3)\ ,$$

then it holds

$$\int_0^\infty \bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}(t, 1)\, dt = -\frac{2}{3}\ .$$

Proof.
Twice integrating by parts of $x_3\,\Delta\bar\varphi^{\rm s}_{\rm Neu}(t,x_3)$ on rectangles $\Sigma_L=(0,L)\times(-1,1)$ with $L\to\infty$ (second Green's formula for Laplace) leads to the formula

$$0=-\int_{-1}^{+1}x_3\,\partial_t\bar\varphi^{\rm s}_{\rm Neu}(0,x_3)\,dx_3-\int_0^\infty\bar\varphi^{\rm s}_{\rm Neu}(t,1)\,dt+\int_0^\infty\bar\varphi^{\rm s}_{\rm Neu}(t,-1)\,dt\,.$$

Inserting the boundary condition $\partial_t\bar\varphi^{\rm s}_{\rm Neu}(0,x_3)=2x_3$ and having in mind that $\bar\varphi^{\rm s}_{\rm Neu}$ is odd with respect to x_3 yields

$$\int_{-1}^{+1}2x_3^2\,dx_3=-2\int_0^\infty\bar\varphi^{\rm s}_{\rm Neu}(t,1)\,dt\,,$$

hence by evaluating the integral on the left hand side the assertion. ■

3.4 Elasticity on the half-strip

As we have seen in Part I, §4.4, the displacements $\boldsymbol Z^{[i]}$ $i=1,\ldots,8$, generating the polynomial kernel of the operator $\underline{\mathscr B}$, satisfy the following duality relations.

Lemma 3.8 *Let $\boldsymbol T^{(0)}$ denote the lateral inward traction operator $(T^{(0)}_t,T^{(0)}_s,T^{(0)}_3)$, see (2.4). With σ the permutation*

$$\sigma(1)=5,\quad\sigma(2)=6,\quad\sigma(3)=8,\quad\sigma(4)=7,$$

$$\sigma(5)=1,\quad\sigma(6)=2,\quad\sigma(7)=4,\quad\sigma(8)=3,$$

the anti-symmetrized flux, which can be defined for any $L\in\mathbb R$ by

$$\underline\Phi(\boldsymbol Z^{[i]},\boldsymbol Z^{[j]}):=\int_{-1}^{+1}\Big(\boldsymbol T^{(0)}(\boldsymbol Z^{[i]})\cdot\boldsymbol Z^{[j]}-\boldsymbol T^{(0)}(\boldsymbol Z^{[j]})\cdot\boldsymbol Z^{[i]}\Big)(L,x_3)\,dx_3 \tag{3.6}$$

is independent of L, see [6, Lemma 3.1], and satisfies, for $i,j\in\{1,\cdots,8\}$

$$\underline\Phi(\boldsymbol Z^{[i]},\boldsymbol Z^{[j]})=\bar\gamma_i\,\delta_{j\sigma(i)} \tag{3.7}$$

with $\bar\gamma_i$ a non-zero real number.

For $i=2,6$ we find again the simple relations on which Propositions 3.4 and 3.5 are based. For the remaining values of i, the relations (3.7) apply to the bi-dimensional displacements $\boldsymbol Z^{[i]}_\natural$. Relying on the duality relations (3.7) and integration by parts, we are able to present formulas for the coefficients in the asymptotics at infinity of the solutions to the problems concerning the operators $E_{\rm Mix2}$ and $E_{\rm Free}$. Let us start with the proposition for $E_{\rm Mix2}$.

Proposition 3.9 *For $\eta>0$, let $\mathfrak f_\natural\in L^2_\eta(\Sigma^+)^2$, $\mathfrak g^\pm_\natural\in L^2_\eta(\mathbb R^+)^4$, $\mathfrak h_t\in H^{1/2}(\gamma_0)$ and $\mathfrak h_3\in H^{-1/2}(\gamma_0)$. If moreover $\eta<\eta_0$, then the problem*

$$E_{\rm Mix2}(\boldsymbol\psi)=(\mathfrak f_\natural;\mathfrak g^\pm_\natural;\mathfrak h_\natural)$$

has a unique solution in $H^1_\eta(\Sigma^+)^2 \oplus \mathrm{span}\{\boldsymbol{Z}^{[1]}_\natural, \boldsymbol{Z}^{[4]}_\natural, \boldsymbol{Z}^{[8]}_\natural\}$ *. Moreover*

$$\boldsymbol{\psi} = \boldsymbol{\varphi} + \delta_1 \boldsymbol{Z}^{[1]}_\natural + \delta_4 \boldsymbol{Z}^{[4]}_\natural + \delta_8 \boldsymbol{Z}^{[8]}_\natural \quad \textit{with} \quad \boldsymbol{\varphi} \in H^1_\eta(\Sigma^+)^2$$

and for $j = 3, 5, 7$

$$\delta_{\sigma(j)} = \frac{1}{\bar{\gamma}_{\sigma(j)}}\bigg(\int_{\Sigma^+} \mathfrak{f}_\natural \cdot \boldsymbol{Z}^{[j]}_\natural \, dt\, dx_3 - \int_{\mathbb{R}^+} \Big(\mathfrak{g}^+ \cdot \boldsymbol{Z}^{[j]}_\natural\Big|_{\gamma^+} - \mathfrak{g}^- \cdot \boldsymbol{Z}^{[j]}_\natural\Big|_{\gamma^-}\Big)\, dt \\ + \int_{-1}^{+1} \mathfrak{h}_3\, \boldsymbol{Z}^{[j]}_3 - \mathfrak{h}_t\, T^{(0)}_t(\boldsymbol{Z}^{[j]}_\natural)\, dx_3\bigg),$$

namely

$$\bar{\gamma}_8 \delta_8 = \int_{\Sigma^+} f_3 - \int_{\mathbb{R}^+} (g_3^+ - g_3^-) + \int_{-1}^{+1} \mathfrak{h}_3\,, \tag{3.8a}$$

$$\bar{\gamma}_1 \delta_1 = \int_{\Sigma^+} t f_t - \int_{\mathbb{R}^+} t(g_t^+ - g_t^-) - \int_{-1}^{+1} (\tilde{\lambda} + 2\mu)\mathfrak{h}_t - \frac{\tilde{\lambda}}{2\mu}\bigg(\int_{\Sigma^+} x_3 f_3 - \int_{\mathbb{R}^+} (g_3^+ + g_3^-) + \int_{-1}^{+1} x_3 \mathfrak{h}_3\bigg), \tag{3.8b}$$

$$\bar{\gamma}_4 \delta_4 = \int_{\Sigma^+} \mathfrak{f}_\natural \cdot \boldsymbol{Z}^{[7]}_\natural - \int_{\mathbb{R}^+} \Big(\mathfrak{g}^+ \cdot \boldsymbol{Z}^{[7]}_\natural - \mathfrak{g}^- \cdot \boldsymbol{Z}^{[7]}_\natural\Big) + 2\int_{-1}^{+1} \Big(\bar{p}_2 \mathfrak{h}_3 + (\tilde{\lambda} + 2\mu) x_3 \mathfrak{h}_t\Big)\,. \tag{3.8c}$$

The analogous proposition for E_{Free} reads:

Proposition 3.10 *For* $\eta > 0$ *, let* $\mathfrak{f}_\natural \in L^2_\eta(\Sigma^+)^2$ *,* $\mathfrak{g}^\pm_\natural \in L^2_\eta(\mathbb{R}^+)^4$ *and* $\mathfrak{h}_\natural \in H^{-1/2}(\gamma_0)^2$ *. If moreover* $\eta < \eta_0$ *, then the problem*

$$E_{\mathrm{Free}}(\boldsymbol{\psi}) = (\mathfrak{f}_\natural; \mathfrak{g}^\pm_\natural; \mathfrak{h}_\natural)$$

has a unique solution in $H^1_\eta(\Sigma^+)^2 \oplus \mathrm{span}\{\boldsymbol{Z}^{[5]}_\natural, \boldsymbol{Z}^{[7]}_\natural, \boldsymbol{Z}^{[8]}_\natural\}$ *. Moreover*

$$\boldsymbol{\psi} = \boldsymbol{\varphi} + \delta_5 \boldsymbol{Z}^{[5]}_\natural + \delta_7 \boldsymbol{Z}^{[7]}_\natural + \delta_8 \boldsymbol{Z}^{[8]}_\natural \quad \textit{with} \quad \boldsymbol{\varphi} \in H^1_\eta(\Sigma^+)^2$$

and for $j = 1, 3, 4$

$$\delta_{\sigma(j)} = \frac{1}{\bar{\gamma}_{\sigma(j)}}\bigg(\int_{\Sigma^+} \mathfrak{f}_\natural \cdot \boldsymbol{Z}^{[j]}_\natural \, dt\, dx_3 - \int_{\mathbb{R}^+} \Big(\mathfrak{g}^+ \cdot \boldsymbol{Z}^{[j]}_\natural\Big|_{\gamma^+} - \mathfrak{g}^- \cdot \boldsymbol{Z}^{[j]}_\natural\Big|_{\gamma^-}\Big)\, dt + \int_{-1}^{+1} \mathfrak{h}_\natural \cdot \boldsymbol{Z}^{[j]}_\natural\, dx_3\bigg),$$

namely

$$\bar{\gamma}_5 \delta_5 = \int_{\Sigma^+} f_t - \int_{\mathbb{R}^+} (g_t^+ - g_t^-) + \int_{-1}^{+1} \mathfrak{h}_t\,, \tag{3.9a}$$

$$\bar{\gamma}_8 \delta_8 = \int_{\Sigma^+} f_3 - \int_{\mathbb{R}^+} (g_3^+ - g_3^-) + \int_{-1}^{+1} \mathfrak{h}_3\,, \tag{3.9b}$$

$$\bar{\gamma}_7 \delta_7 = \int_{\Sigma^+} (-x_3 f_t + t f_3) + \int_{\mathbb{R}^+} \Big(g_t^+ + g_t^- - t(g_3^+ - g_3^-)\Big) - \int_{-1}^{+1} x_3 \mathfrak{h}_t\,. \tag{3.9c}$$

The space of rigid motions $\mathcal{R}_{⑤}$ is one-dimensional and spanned by the vertical translation $(0,0,1)$. Whence we have one compatibility condition (1.2), compensated by one orthogonality condition (1.3). In particular, the first terms ζ_3^0 and ζ_3^1 in the expansion of $u_3(\varepsilon)$ have to satisfy the zero mean value condition on ω. As can be seen in the next theorem, they are solutions of Dirichlet problems with data such that they satisfy this zero mean value condition. In order to formulate these Dirichlet conditions, we need the introduction of an auxiliary function: the solution η_ω of the problem

$$\begin{cases} (\tilde{\lambda}+2\mu)\Delta_*^2\,\eta_\omega = \dfrac{1}{\mathrm{mes}(\omega)} & \text{in } \omega \\ \eta_\omega = 0\,, \quad \partial_n\eta_\omega = 0 & \text{on } \partial\omega. \end{cases} \tag{4.1}$$

Integrations by parts yield immediately that if ζ is the solution of

$$\begin{cases} (\tilde{\lambda}+2\mu)\Delta_*^2\,\zeta = f & \text{in } \omega \\ \zeta = g\,, \quad \partial_n\zeta = 0 & \text{on } \partial\omega, \end{cases} \tag{4.2}$$

then there holds

$$\frac{1}{\mathrm{mes}(\omega)}\int_\omega \zeta\; dx_* = \int_\omega f\,\eta_\omega\; dx_* + \int_{\partial\omega} g\,N_n(\eta_\omega)\; ds. \tag{4.3}$$

Obviously, $\zeta = 1$ is the unique solution of problem (4.2) with $f = 0$ and $g = 1$. Thus, in this case we obtain from (4.3) the relation

$$\int_{\partial\omega} N_n(\eta_\omega)\; ds = 1\,. \tag{4.4}$$

We also need the following notation:

Notation 4.1 If L is an integrable function on $\partial\omega$ such that $\int_{\partial\omega} L = 0$, then we denote by $\oint_{\partial\omega} L$ the unique primitive of L along $\partial\omega$ with zero mean value on $\partial\omega$ (i.e. $\int_{\partial\omega}\oint L\,ds = 0$). The second primitive $\oint_{\partial\omega}\oint_{\partial\omega} L$ then makes sense. ■

Theorem 4.2 *Let us assume that the boundary of ω is connected. The expansion* (2.1)-(2.2) *holds for the solution $\boldsymbol{u}(\varepsilon)$ of problem* (1.11) *with lateral condition ⑤, compatibility condition* (1.14) *and orthogonality condition* (1.15)*. The Kirchhoff-Love generators $\boldsymbol{\zeta}_*^k$ and ζ_3^k are solutions of membrane and bending problems with interior data $\boldsymbol{R}_{\mathrm{m}}(\boldsymbol{\zeta}_*^k)$ and $R_{\mathrm{b}}(\zeta_3^k)$ described in §2.2, and Dirichlet boundary conditions. More precisely, $\boldsymbol{\zeta}^0$ and $\boldsymbol{\zeta}^1$ satisfy the following boundary conditions on $\partial\omega$*

$$\begin{aligned} &\zeta_n^0 = 0, \qquad \zeta_s^0 = 0, \qquad \partial_n\zeta_3^0 = 0, \qquad \zeta_3^0 = -\int_\omega R_{\mathrm{b}}^0\,\eta_\omega, \\ &\zeta_n^1 = 0, \qquad \zeta_s^1 = 0, \qquad \partial_n\zeta_3^1 = 0, \qquad \zeta_3^1 = c_4^{⑤}\Big(\oint_{\partial\omega}\oint_{\partial\omega} L - \int_{\partial\omega}\Big(\oint_{\partial\omega}\oint_{\partial\omega} L\Big)\,N_n(\eta_\omega)\Big), \end{aligned} \tag{4.5}$$

where L *is the function*

$$L(s)=\left[-\frac{2}{3}(\tilde\lambda+2\mu)\partial_n\Delta_*\,\zeta_3^0+\int_{-1}^{+1}x_3\,f_n\,dx_3+g_n^++g_n^-\right]\Big|_{\partial\omega} \tag{4.6}$$

and $c_4^{⑤}$ *is a coefficient only depending on* λ *and* μ. *The first boundary layer profile fulfills* $\boldsymbol{\varphi}_\natural^1=0$ *and* $\varphi_s^1=\partial_s\zeta_3^0(s)\,\bar\varphi_{\mathrm{Dir}}^{\mathrm{s}}$, *where* $\bar\varphi_{\mathrm{Dir}}^{\mathrm{s}}(t,x_3)$ *satisfies* $L_{\mathrm{Dir}}(\bar\varphi_{\mathrm{Dir}}^{\mathrm{s}})=(0;\ 0;\ x_3)$.

4.1 The traces of $\boldsymbol{u}_{\mathrm{KL}}^0$

According to (2.21) the Dirichlet traces D_n^0 and D_s^0 are zero, i.e. $u_{\mathrm{KL},n}^0$ and $u_{\mathrm{KL},s}^0$ are zero on Γ_0, whence $\zeta_n^0-x_3\partial_n\zeta_3^0=0$ and $\zeta_s^0-x_3\partial_s\zeta_3^0=0$ on $\partial\omega$. T_3^0 is always zero. The expressions in (4.5) concerning the boundary conditions for ζ_n^0, ζ_s^0 and $\partial_n\zeta_3^0$ are then obtained immediately.

We have moreover that $\partial_s\zeta_3^0=0$, whence $\zeta_3^0=c$ with c a constant on $\partial\omega$ (here of course, we use the assumption that $\partial\omega$ is connected). Hence ζ_3^0 solves problem (4.2) with $f=R_{\mathrm{b}}^0$ and $g=c$. According to the orthogonality condition $\int_\Omega u_3(\varepsilon)\,dx=0$, ζ_3^0 has a zero mean value on ω, which allows for the determination of the constant c with the help of formulas (4.3) and (4.4).

4.2 The traces of $\boldsymbol{u}_{\mathrm{KL}}^1$

By considering D_n^1, D_s^1 and T_3^1, it follows firstly that $\boldsymbol{\varphi}_\natural^1$ has to satisfy the problem

$$E_{\mathrm{Mix2}}(\boldsymbol{\varphi}_\natural^1)=(0;0;-\zeta_n^1+x_3\partial_n\zeta_3^1,0)\,. \tag{4.7}$$

From the cancellation of the constants δ_1, δ_4, δ_8 in Proposition 3.9, ensuring the existence of an exponentially decaying profile, the conditions $\zeta_n^1(s)=0$ and $\partial_n\zeta_3^1(s)=0$ on $\partial\omega$ are deduced. Of course, the constant δ_8 vanishes without any additional condition. The only exponentially decreasing solution is given by $\boldsymbol{\varphi}_\natural^1=0$.

Secondly, the problem for φ_s^1 reads

$$L_{\mathrm{Dir}}(\varphi_s^1)=(0;0;-\zeta_s^1+x_3\partial_s\zeta_3^1)\,.$$

Proposition 3.4 yields that the existence of a unique exponentially decaying solution is guaranteed if and only if $\zeta_s^1(s)=0$ on $\partial\omega$ is valid. This solution is $\varphi_s^1=\partial_s\zeta_3^1(s)\,\bar\varphi_{\mathrm{Dir}}^{\mathrm{s}}$, where $\bar\varphi_{\mathrm{Dir}}^{\mathrm{s}}$ satisfies $L_{\mathrm{Dir}}(\bar\varphi_{\mathrm{Dir}}^{\mathrm{s}})=(0;\ 0;\ x_3)$ and hence is odd with respect to x_3.

The next relations are deduced from $D_n^2=0$ and $T_3^2=0$ which leads to the following problem for $\boldsymbol{\varphi}_\natural^2$

$$E_{\mathrm{Mix2}}(\boldsymbol{\varphi}_\natural^2)=\Big(-\big(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1\big)_\natural\,;\ -\big(\mathscr{G}^{(1)}\boldsymbol{\varphi}^1\big)_\natural\,;\ \mathfrak{h}_t\,,\ \mathfrak{h}_3\Big)\,, \tag{4.8}$$

where the terms in the right hand side of (4.8) are given by

$$\big(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1\big)_t=(\lambda+\mu)\,\partial_t\partial_s\varphi_s^1\,,\quad \big(\mathscr{G}^{(1)}\boldsymbol{\varphi}^1\big)_t=0,$$

$$\mathfrak{h}_t=-\Big(\zeta_n^2-x_3\partial_n\zeta_3^2+\bar p_2\,\partial_n\,\mathrm{div}_*\,\boldsymbol{\zeta}_*^0+\bar p_3\,\partial_n\Delta_*\zeta_3^0+(G(\boldsymbol{f},\boldsymbol{g}^\pm))_n\Big)$$

and

$$(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_3 = (\lambda+\mu)\,\partial_3\partial_s\varphi^1_s, \quad (\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_3 = \lambda\,\partial_s\varphi^1_s,$$

$$\mathfrak{h}_3 = -\mu\left((\bar p_2 + \bar p_3')\,\partial_n\Delta_*\zeta^0_3 + \partial_3(G(\boldsymbol{f},\boldsymbol{g}^\pm))_n\right) .$$

From Proposition 3.9 we have $\delta_8 = 0$ as a necessary condition for the existence of an exponentially decaying profile. This condition yields

$$2\mu\,\partial_{ss}\zeta^1_3 \int_{\mathbb{R}^+} \bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}(t,1)\,dt = \int_{-1}^{+1} \mathfrak{h}_3\,dx_3\,. \tag{4.9}$$

But we obtain immediately from (2.13) after once integrating by parts that

$$\mu\left(G(\boldsymbol{f},\boldsymbol{g}^\pm)\right)_n\Big|_{-1}^{+1} = \int_{-1}^{+1} x_3\,f_n\,dx_3 + g^+_n + g^-_n \tag{4.10}$$

is valid. Then we deduce that $-\int_{-1}^{+1}\mathfrak{h}_3\,dx_3$ coincide with the function L in (4.6). With Lemma 3.6, we have

$$c^{⑤}_4 := -\left(2\mu\int_{\mathbb{R}^+}\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}(t,1)\,dt\right)^{-1} < 0\,.$$

Hence condition (4.9) reads

$$\partial_{ss}\zeta^1_3(s) = c^{⑤}_4\,L(s) \qquad \text{on } \partial\omega\,.$$

Assuming for the moment that there holds

$$\int_{\partial\omega} L(s)\,ds = 0\,, \tag{4.11}$$

we can choose the primitive of L with mean value zero to obtain the expression for $\partial_s\zeta^1_3(s)$ on $\partial\omega$. Then, with the help of formulas (4.3) and (4.4), we choose the unique primitive g of $\oint L$ such that the solution ζ^1_3 of problem (4.2) with $f = R^1_{\mathrm{b}} = 0$ and g has a mean value zero on ω. It remains to prove (4.11).

Relation (4.11) is indeed a simple consequence of the identity

$$\frac{3}{2}\int_{\partial\omega}\left\{\int_{-1}^{+1} x_3 f_n\,dx_3 + g^+_n + g^-_n\right\} ds = (\tilde\lambda + 2\mu)\int_{\partial\omega}\partial_n\Delta_*\zeta^0_3(s)\,ds\,, \tag{4.12}$$

which follows from

$$\int_\omega R^0_{\mathrm{b}}\,dx_* = (\tilde\lambda+2\mu)\int_\omega \Delta^2_*\zeta^0_3\,dx_*$$

by applying second Green's formula for Laplace to the functions $(\tilde\lambda+2\mu)\Delta_*\zeta^0_3$ and 1, the divergence theorem to $\int_{-1}^{+1} x_3 f_*\,dx_3 + g^+_* + g^-_*$ as well as from the assumed compatibility condition for the three-dimensional problem

$$\int_\omega\left\{\int_{-1}^{+1} f_3\,dx_3 + g^+_3 - g^-_3\right\} dx_* = 0\,.$$

The cancellation of the constants δ_1 and δ_4 (3.8) results in conditions for ζ_n^2 and $\partial_n\zeta_3^2$, respectively. The existence of a uniquely determined exponentially decaying profile $\boldsymbol{\varphi}_\natural^2$ is thus guaranteed. At the end, we just want to emphasize that the problem for $\boldsymbol{\zeta}_*^1$ is a completely homogeneous one, thus $\boldsymbol{\zeta}_*^1 \equiv 0$ holds. The recursivity of the algorithm can be proved analogously to Part I, §6.3.

Remark 4.3 The assumption that $\partial\omega$ is connected allows for simplifications but is not necessary to obtain an expansion (2.1)-(2.2). If $\partial\omega$ has $N \geq 2$ connected components $\partial_\nu\omega$, $\nu = 1,\ldots,N$, the trace of ζ_3^0 has a constant value γ_ν on each $\partial_\nu\omega$ and it is still possible to choose (uniquely) these values so that ζ_3^0 has a mean value zero in ω and such that the function L (4.6) has a mean value zero on each $\partial_\nu\omega$. This is a consequence of

Lemma 4.4 *Let $\mathscr{S}$ be the application from $\mathbb{R}^N$ to $\mathbb{R}^N$ defined as $\mathscr{S}(\gamma_1,\ldots,\gamma_N) = (\beta_1,\ldots,\beta_N)$ with $\beta_\nu = \int_{\partial_\nu\omega}\partial_n\Delta_*\zeta$ where ζ is the solution of problem* (4.2) *with $f = 0$ and $g = \gamma_\nu$ on $\partial_\nu\omega$. Then the kernel of $\mathscr{S}$ is formed by constants $(\gamma,\ldots,\gamma)$.*

The proof of this lemma relies on the Green formula

$$\int_\omega \Delta_*^2\zeta\cdot\zeta = \int_{\partial\omega}(\Delta_*\zeta\cdot\partial_n\zeta - \partial_n\Delta_*\zeta\cdot\zeta) + \int_\omega \Delta_*\zeta\cdot\Delta_*\zeta\,.$$

Indeed, if $(\gamma_1,\ldots,\gamma_N)$ belongs to the kernel of $\mathscr{S}$, the associated ζ is such that $\int_{\partial\omega}\partial_n\Delta_*\zeta\cdot\zeta$ is zero and the above equality yields that $\Delta_*\zeta$ is zero on ω. As $\partial_n\zeta$ is zero, then ζ is a constant γ. ■

5 SLIDING EDGE

We first have to study the space of rigid motions $\mathcal{R}_{⑥}$. If the mid-plane of the plate ω is not a disk or an annulus, then this space is one-dimensional and spanned by the vertical translation $(0,0,1)$. But if ω is a disk or an annulus, that we may suppose centered in 0, then $\mathcal{R}_{⑥}$ is two-dimensional generated by the vertical translation $(0,0,1)$ and the in-plane rotation $(x_2,-x_1,0)$. This can be seen, combining the compatibility conditions of the cases soft clamped ② in Part I, §6 and frictional I ⑤.

Theorem 5.1 *The expansion* (2.1)-(2.2) *holds for the solution $\boldsymbol{u}(\varepsilon)$ of problem* (1.11)*, with lateral condition* ⑥*, compatibility condition* (1.14) *and orthogonality condition* (1.15)*. The Kirchhoff-Love generators $\boldsymbol{\zeta}_*^k$ and ζ_3^k are solutions of membrane and bending problems with interior data $\boldsymbol{R}_{\rm m}(\boldsymbol{\zeta}_*^k)$ and $R_{\rm b}(\zeta_3^k)$, and boundary conditions prescribing ζ_n^k, $T_s^{\rm m}(\boldsymbol{\zeta}_*^k)$, and $\partial_n\zeta_3^k$, $N_n(\zeta_3^k)$ respectively. Moreover the mean value of ζ_3^k on ω is prescribed. More precisely, $\boldsymbol{\zeta}^0$ and $\boldsymbol{\zeta}^1$ satisfy the following boundary conditions on $\partial\omega$*

$$\begin{aligned}
&\zeta_n^0 = 0,\quad T_s^{\rm m}(\boldsymbol{\zeta}_*^0) = 0,\quad \partial_n\zeta_3^0 = 0,\quad N_n(\zeta_3^0) = \tfrac{3}{2}\left(\textstyle\int_{-1}^{+1} x_3\, f_n\, dx_3 + g_n^+ + g_n^-\right)\Big|_{\partial\omega},\\
&\zeta_n^1 = 0,\quad T_s^{\rm m}(\boldsymbol{\zeta}_*^1) = 0,\quad \partial_n\zeta_3^1 = 0,\quad N_n(\zeta_3^1) = c_1^{⑥}O + c_2^{⑥}P + c_3^{⑥}Q + \kappa\, K^{⑥}(f_n, g_n^{\pm}),
\end{aligned}\tag{5.1}$$

where O, P and Q are the traces on $\partial\omega$ of different operators acting on ζ_3^0 according to $O = \partial_s(\kappa^2\partial_s\zeta_3^0)$, $P = (\kappa\partial_s)^2\zeta_3^0$ and $Q = \kappa\partial_n\Delta_*\zeta_3^0$. The three constants $c_i^{⑥}$, $i = 1,2,3$, depend only on λ and μ. The operator $K^{⑥}$ depends only on the Lamé coefficients as well and is in-plane local, cf Def. 2.1. The mean values of ζ_3^0 and ζ_3^1 on ω are zero. The first boundary layer profile fulfills $\boldsymbol{\varphi}_\natural^1 = 0$ and $\varphi_s^1 = \kappa\partial_s\zeta_3^0(s)\,\bar\varphi_{\mathrm{Neu}}^{\mathrm{s}}$, cf Lemma 3.7.

We immediately see that the curvature κ can be factorized in the expression of $N_n(\zeta_3^1)$, just as it is in the first boundary layer term $\boldsymbol{\varphi}^1$. This is similar to the hard simple support situation, for converse symmetry reasons: in any flat part $\mathcal{V}$ of the boundary, the normal and tangential part of the displacement can be extended by odd and even reflections respectively. If moreover the support of the data avoids $\mathcal{V}$, there are *no boundary layer* terms and $\boldsymbol{u}(\varepsilon)$ can be expanded in a power series in $\mathcal{V}$.

In the special case when ω is a rectangle (in principle forbidden here!) and if the support of the data avoids the lateral boundary, the solution can be extended outside Ω in both in-plane directions into a periodic solution in $\mathbb{R}^2 \times I$: this link is indicated by PAUMIER in [20] where the periodic boundary conditions are addressed.

5.1 The traces of $\boldsymbol{u}_{\mathrm{KL}}^0$

The Dirichlet trace D_n^0 is zero according to (2.21), i.e. $\zeta_n^0 - x_3\partial_n\zeta_3^0 = 0$. Hence we have $\zeta_n^0 = 0$ and $\partial_n\zeta_3^0 = 0$ on $\partial\omega$. T_s^0, T_3^0 and T_s^1 are always zero.

We deduce the problem for $\boldsymbol{\varphi}_\natural^1$ from $D_n^1 = 0$ and $T_3^1 = 0$. Of course it is the same problem as in §4.2 given by (4.7). Thus, we obtain the conditions $\zeta_n^1(s) = 0$ and $\partial_n\zeta_3^1(s) = 0$ on $\partial\omega$ and $\boldsymbol{\varphi}_\natural^1 = 0$.

The condition $T_s^2 = 0$ yields that φ_s^1 has to satisfy the following problem

$$L_{\mathrm{Neu}}(\varphi_s^1) = (0; 0; -T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^0) + 2\mu x_3(\partial_n + \kappa)\partial_s\zeta_3^0). \tag{5.2}$$

Proposition 3.5 yields that the existence of a unique exponentially decaying solution is guaranteed if and only if $T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^0)(s) = 0$ on $\partial\omega$ is valid. Taking into account the already known condition $\partial_n\zeta_3^0(s) = 0$ on $\partial\omega$, this solution is given by, *cf* Lemma 3.7

$$\varphi_s^1 = \kappa\,\partial_s\zeta_3^0(s)\,\bar\varphi_{\mathrm{Neu}}^{\mathrm{s}}(t, x_3). \tag{5.3}$$

Let us now consider the problem for φ_s^2. With $T_s^3 = 0$ we obtain that φ_s^2 has to satisfy

$$L_{\mathrm{Neu}}(\varphi_s^2) = \Big(-(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_s\,;\,-(\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_s\,;\,\mathfrak{h}_s\Big), \tag{5.4}$$

where the terms in the right hand side of (5.4) are given by

$$(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_s = \mu\kappa\Big(\partial_{tt}(t\,\varphi_s^1) + \partial_{33}(t\,\varphi_s^1) - \partial_t\varphi_s^1\Big), \quad (\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_s = 0,$$
$$\mathfrak{h}_s = -\Big(2\mu\kappa\varphi_s^1 + T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^1) - 2\mu x_3(\partial_n + \kappa)\partial_s\zeta_3^1\Big),$$

since $\boldsymbol{\varphi}_\natural^1 = 0$. With the help of Proposition 3.5 and the fact that φ_s^1 is odd with respect to x_3 we deduce that there exists a uniquely determined exponentially decaying solution of problem (5.4) if and only if $T_s^{\mathrm{m}}(\boldsymbol{\zeta}_*^1)(s) = 0$ on $\partial\omega$. Taking into account relation (5.3) and the already known condition $\partial_n \zeta_3^1 = 0$ on $\partial\omega$, this solution is given by

$$\varphi_s^2 = -\kappa^2 \partial_s \zeta_3^0 \, \bar{\psi}_{\mathrm{Neu}}^{\mathrm{s}} + \kappa \, \partial_s \zeta_3^1 \, \bar{\varphi}_{\mathrm{Neu}}^{\mathrm{s}} \,, \tag{5.5}$$

where $\bar{\psi}_{\mathrm{Neu}}^{\mathrm{s}}$ is the exponentially decreasing solution of

$$L_{\mathrm{Neu}}(\bar{\psi}_{\mathrm{Neu}}^{\mathrm{s}}) = \mu(\Delta(t \, \bar{\varphi}_{\mathrm{Neu}}^{\mathrm{s}}) - \partial_t \bar{\varphi}_{\mathrm{Neu}}^{\mathrm{s}} \, ; \; 0 \, ; \; 2\bar{\varphi}_{\mathrm{Neu}}^{\mathrm{s}}), \tag{5.6}$$

so it is odd with respect to x_3.

The conditions $D_n^2 = 0$ and $T_3^2 = 0$ lead to a problem of the form (4.8) with the same expressions for $(\mathscr{B}^{(1)} \boldsymbol{\varphi}^1)_t$, $(\mathscr{B}^{(1)} \boldsymbol{\varphi}^1)_3$, $(\mathscr{G}^{(1)} \boldsymbol{\varphi}^1)_t$, $(\mathscr{G}^{(1)} \boldsymbol{\varphi}^1)_3$, $\mathfrak{h}_t$ and $\mathfrak{h}_3$. The only difference now is that we have to take into account $\varphi_s^1 = \kappa \partial_s \zeta_3^0(s) \, \bar{\varphi}_{\mathrm{Neu}}^{\mathrm{s}}$ instead of $\varphi_s^1 = \partial_s \zeta_3^0(s) \, \bar{\varphi}_{\mathrm{Dir}}^{\mathrm{s}}$ as we had in (4.8). From Proposition 3.9 we have as a necessary condition for the existence of an exponentially decaying profile $\delta_8 = 0$:

$$2\mu \partial_s (\partial_n + \kappa) \partial_s \zeta_3^0 \int_{\mathbb{R}^+} \bar{\varphi}_{\mathrm{Neu}}^{\mathrm{s}}(t, 1) \, dt = \int_{-1}^{+1} \mathfrak{h}_3 \, dx_3 \,.$$

As already obtained in §4.2, the expression of the right hand side holds

$$\int_{-1}^{+1} \mathfrak{h}_3 \, dx_3 = - \left[-\frac{2}{3} (\tilde{\lambda} + 2\mu) \partial_n \Delta_* \, \zeta_3^0 + \int_{-1}^{+1} x_3 \, f_n \, dx_3 + g_n^+ + g_n^- \right] \bigg|_{\partial\omega} .$$

Then Lemma 3.7 yields

$$\frac{2}{3} \underbrace{\Big((\tilde{\lambda} + 2\mu) \partial_n \Delta_* \, \zeta_3^0 + 2\mu \partial_s (\partial_n + \kappa) \partial_s \zeta_3^0 \Big)}_{= N_n(\zeta_3^0)} = \left(\int_{-1}^{+1} x_3 \, f_n \, dx_3 + g_n^+ + g_n^- \right) \bigg|_{\partial\omega} ,$$

hence the condition $N_n(\zeta_3^0)(s) = \frac{3}{2} \int_{-1}^{+1} x_3 \, f_n \, dx_3 + g_n^+ + g_n^-$ on $\partial\omega$.

The cancellation of the constants δ_1 and δ_4 (3.8b)-(3.8c) results in conditions for ζ_n^2 and $\partial_n \zeta_3^2$, respectively. If we split the operator $G(\boldsymbol{f}, \boldsymbol{g}^\pm) = G^{\mathrm{b}}(\boldsymbol{f}, \boldsymbol{g}^\pm) + G^{\mathrm{m}}(\boldsymbol{f}, \boldsymbol{g}^\pm)$ into bending and membrane part with parities (odd/even) and (even/odd), respectively, these conditions take the form

$$\partial_n \zeta_3^2 = c_4^{⑥} \partial_s (\kappa \partial_s) \zeta_3^0 + c_5^{⑥} \partial_n \Delta_* \zeta_3^0 + K^{\mathrm{b}}(f_n, g_n^\pm) \quad \text{and} \quad \zeta_n^2 = K^{\mathrm{m}}(f_n, g_n^\pm) \quad \text{on } \partial\omega \,,$$

where K^{b} and K^{m} are in-plane local operators. The existence of a uniquely determined exponentially decaying profile $\boldsymbol{\varphi}_\natural^2$ is thus guaranteed. It has the form $\boldsymbol{\varphi}_\natural^2 = \boldsymbol{\varphi}_\natural^{2,\mathrm{b}} + \boldsymbol{\varphi}_\natural^{2,\mathrm{m}}$. Later, for the determination of the boundary conditions satisfied by ζ_3^1, we will need information about its bending part $\boldsymbol{\varphi}_\natural^{2,\mathrm{b}}$, which is solution of the following problem in the half-strip:

$$\begin{aligned} E_{\mathrm{Mix2}}(\boldsymbol{\varphi}_\natural^{2,\mathrm{b}}) = {} & \partial_s (\kappa \partial_s) \zeta_3^0 \Big(-(\lambda + \mu) \partial_t \bar{\varphi}_{\mathrm{Neu}}^{\mathrm{s}} \, , \, -(\lambda + \mu) \partial_3 \bar{\varphi}_{\mathrm{Neu}}^{\mathrm{s}} \, ; \, 0 \, , \, -\lambda \bar{\varphi}_{\mathrm{Neu}}^{\mathrm{s}} \, ; \, c_4^{⑥} x_3 \, , \, 0 \Big) \\ & + \, \partial_n \Delta_* \zeta_3^0 \Big(0 \, ; \, 0 \, ; \, c_5^{⑥} x_3 - \bar{p}_3 \, , \, -\mu (\bar{p}_2 + \bar{p}_3') \Big) \\ & + \Big(0 \, ; \, 0 \, ; \, K^{\mathrm{b}}(f_n, g_n^\pm) \, x_3 - G^{\mathrm{b}}(\boldsymbol{f}, \boldsymbol{g}^\pm)_n \, , \, -\mu \, \partial_3 G^{\mathrm{b}}(\boldsymbol{f}, \boldsymbol{g}^\pm)_n \Big) . \end{aligned} \tag{5.7}$$

Now let us check the compatibility conditions for the existence of the generator $\boldsymbol{\zeta}^0$ of $\boldsymbol{u}^0_{\mathrm{KL}}$. Concerning ζ^0_3, we consider the following Green's formula linked with the bending operator

$$b(\zeta_3,\eta_3)=\int_\omega(\tilde\lambda+2\mu)\Delta_*^2\zeta_3\,\eta_3\,dx_*-\int_{\partial\omega}N_n(\zeta_3)\,\eta_3\,ds+\int_{\partial\omega}M_n(\zeta_3)\,\partial_n\eta_3\,ds\,, \qquad (5.8)$$

where $b(\zeta_3,\eta_3)$ is the bending bilinear form given in (2.27), M_n and N_n are the bending Neumann operators given in (2.28). The kernel of the problem for ζ^0_3 consists of constant functions. Thus, by Fredholm's alternative, this problem is solvable if and only if the right hand side R^0_{b} is such that the Green's formula (5.8) is valid for $\zeta_3=\zeta^0_3$ and $\eta_3=1$, thus that the condition

$$\int_\omega R^0_{\mathrm{b}}(x_*)\,dx_*-\int_{\partial\omega}\frac{3}{2}\left(\int_{-1}^{+1}x_3\,f_n\,dx_3+g^+_n+g^-_n\right)(0,s)\,ds=0$$

is fulfilled. With the help of the divergence theorem we can rewrite the left hand side as

$$\frac{3}{2}\int_\omega\left\{\int_{-1}^{+1}f_3\,dx_3+g^+_3-g^-_3\right\}dx_*\,,$$

which is nothing else than the assumed three-dimensional compatibility condition (1.2), and thus zero. The additional compatibility condition in the case when ω is a disk, is checked in exactly the same way as in the soft clamped situation ②, Part I, §6.1.

5.2 The traces of $\boldsymbol{u}^1_{\mathrm{KL}}$

The only remaining boundary condition is that for $N_n(\zeta^1_3)$. Therefore we only consider the problem for $\boldsymbol{\varphi}^3_\natural$, which is deduced from $D^3_n=0$ and $T^3_3=0$ and reads

$$E_{\mathrm{Mix2}}(\boldsymbol{\varphi}^3_\natural)=\Big(-\big(\mathscr{B}^{(1)}\boldsymbol{\varphi}^2\big)_\natural-\big(\mathscr{B}^{(2)}\boldsymbol{\varphi}^1\big)_\natural\,;\,-\big(\mathscr{G}^{(1)}\boldsymbol{\varphi}^2\big)_\natural-\big(\mathscr{G}^{(2)}\boldsymbol{\varphi}^1\big)_\natural\,;\;\mathfrak{h}_t\,,\;\mathfrak{h}_3\Big)\,. \qquad (5.9)$$

The boundary condition prescribing $N_n(\zeta^1_3)$ is then found by the cancellation of the coefficient δ_8 (3.8a). To make this explicit, we use

$$\begin{gathered}\big(\mathscr{B}^{(1)}\boldsymbol{\varphi}^2\big)_3=-\mu\kappa\partial_t\varphi^2_3+(\lambda+\mu)\,\partial_3(-\kappa\varphi^2_t+\partial_s\varphi^2_s)\,,\\ \big(\mathscr{B}^{(2)}\boldsymbol{\varphi}^1\big)_3=(\lambda+\mu)\,\partial_3(\kappa t\partial_s\varphi^1_s+\partial_s(\kappa t\varphi^1_s)),\\ \big(\mathscr{G}^{(1)}\boldsymbol{\varphi}^2\big)_3=\lambda(-\kappa\varphi^2_t+\partial_s\varphi^2_s)\,,\quad\big(\mathscr{G}^{(2)}\boldsymbol{\varphi}^1\big)_3=\lambda(\kappa t\partial_s\varphi^1_s+\partial_s(\kappa t\varphi^1_s))\,,\\ \mathfrak{h}_3=-\mu(\bar p_2+\bar p'_3)\,\partial_n\Delta_*\zeta^1_3\,.\end{gathered} \qquad (5.10)$$

The boundary layer terms involved are φ^1_s, φ^2_s and $\boldsymbol{\varphi}^{2,\mathrm{b}}_\natural$. Inserting the expressions (5.3), (5.5) and (5.7), this condition yields for $N_n(\zeta^1_3)$ an expression of the form $c^{⑥}_1\partial_s(\kappa^2\partial_s\zeta^0_3)+c^{⑥}_2(\kappa\partial_s)^2\zeta^0_3+c^{⑥}_3\kappa\partial_n\Delta_*\zeta^0_3+\kappa K^{⑥}(f_n,g^\pm_n)$.

The compatibility condition for ζ_3^1 can be checked using the same kind of argument as for the recursivity of compatibility conditions in the soft simple support case ④, see Part I, §8.2. Setting $\boldsymbol{\varphi} = \boldsymbol{\varphi}^1 + \varepsilon\boldsymbol{\varphi}^2$, we have by construction

$$\begin{aligned} N_n(\zeta_3^0 + \varepsilon\zeta_3^1) = \frac{3}{2}\left(\int_{\Sigma^+} \mathfrak{f}_3(\varepsilon) - \int_{\mathbb{R}^+} \Big(\mathfrak{g}_3^+(\varepsilon) - \mathfrak{g}_3^-(\varepsilon)\Big) + \int_{-1}^{+1} \mathfrak{h}_3(\varepsilon)\right) \\ + 2\mu\, \partial_s(\partial_n + \kappa)\partial_s(\zeta_3^0 + \varepsilon\zeta_3^1), \end{aligned} \tag{5.11}$$

where

$$\mathfrak{f}(\varepsilon) = \mathscr{B}\boldsymbol{\varphi} + \mathcal{O}(\varepsilon^2), \qquad \mathfrak{g}(\varepsilon) = \mathscr{G}\boldsymbol{\varphi} + \mathcal{O}(\varepsilon^2), \qquad \mathfrak{h}(\varepsilon) = \boldsymbol{T}\boldsymbol{\varphi} + \mathcal{O}(\varepsilon^2).$$

With $\boldsymbol{w}(\tilde{x}) = \chi(r)\,\boldsymbol{\varphi}(\frac{r}{\varepsilon}, s, \frac{\tilde{x}_3}{\varepsilon})$ on Ω^ε and integrating (5.11) along $\partial\omega$ we obtain for any rigid motion $\boldsymbol{v} = (0,0,a)$ in $\mathcal{R}_{⑥}$

$$\int_{\partial\omega} N_n(\zeta_3^0 + \varepsilon\zeta_3^1)\, v_3 = -\frac{3}{2}\int_{\Omega^\varepsilon} Ae(\boldsymbol{w}) : e(\boldsymbol{v}) + \mathcal{O}(\varepsilon^2) = \mathcal{O}(\varepsilon^2),$$

where we have used $\int_{\partial\omega} \partial_s(\partial_n + \kappa)\partial_s(\zeta_3^0 + \varepsilon\zeta_3^1)\, ds = 0$. The desired compatibility condition then follows, and even the argument applies to the recursivity of compatibility conditions.

In order that the orthogonality condition (1.15) holds, we choose ζ_3^1 with zero mean value. This also extends to any order k by recursivity: we remark that even if it happens that a certain boundary layer $\boldsymbol{\varphi}^\ell$ does not satisfy the orthogonality condition (1.15), we can choose the generator $\boldsymbol{\zeta}^{\ell+3}$ in such a way that (1.15) is fulfilled. At the end the $\boldsymbol{\zeta}^k$ are uniquely determined.

6 FRICTION II

Combining the compatibility conditions for the cases hard simple support ③, Part I, §7 and frictional I ⑤, it is clear that the space of rigid motions $\mathcal{R}_{⑦}$ is one-dimensional and spanned by the vertical translation $(0,0,1)$. Whence we have one compatibility condition (1.14), compensated by one orthogonality condition (1.15). In particular, the first terms ζ_3^0 and ζ_3^1 in the expansion of $u_3(\varepsilon)$ have to satisfy the zero mean value condition on ω. In the same spirit as in the case frictional I ⑤, in order to formulate the boundary conditions for ζ_3^0 and ζ_3^0, we need the introduction of an auxiliary function: the solution ξ_ω of the problem

$$\begin{cases} (\tilde{\lambda} + 2\mu)\Delta_*^2\, \xi_\omega = \dfrac{1}{\mathrm{mes}(\omega)} & \text{in } \omega \\ \xi_\omega = 0\,, \quad M_n(\xi_\omega) = 0 & \text{on } \partial\omega. \end{cases} \tag{6.1}$$

Integrations by parts yield immediately that if ζ is the solution of

$$\begin{cases} (\tilde{\lambda} + 2\mu)\Delta_*^2\, \zeta = f & \text{in } \omega \\ \zeta = g\,, \quad M_n(\zeta) = h & \text{on } \partial\omega, \end{cases} \tag{6.2}$$

then there holds

$$\frac{1}{\operatorname{mes}(\omega)} \int_\omega \zeta \, dx_* = \int_\omega f \, \xi_\omega \, dx_* + \int_{\partial\omega} g \, N_n(\xi_\omega) \, ds + \int_{\partial\omega} h \, \partial_n \xi_\omega \, ds. \tag{6.3}$$

Obviously, $\zeta = 1$ is the unique solution of problem (6.2) with $f = 0$, $g = 1$ and $h = 0$. Thus, in this case we obtain from (6.3) the relation

$$\int_{\partial\omega} N_n(\xi_\omega) \, ds = 1 \,. \tag{6.4}$$

Theorem 6.1 *Let us assume that the boundary of* ω *is connected. The expansion* (2.1)-(2.2) *holds for the solution* $\boldsymbol{u}(\varepsilon)$ *of problem* (1.11) *with lateral condition* ⑦, *compatibility condition* (1.14) *and orthogonality condition* (1.15). *The Kirchhoff-Love generators* $\boldsymbol{\zeta}_*^k$ *and* ζ_3^k *are solutions of membrane and bending problems with interior data* $\boldsymbol{R}_{\mathrm{m}}(\boldsymbol{\zeta}_*^k)$ *and* $R_{\mathrm{b}}(\zeta_3^k)$, *and boundary conditions prescribing* $T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^k)$, ζ_s^k, *and* $M_n(\zeta_3^k)$, ζ_3^k *respectively. More precisely,* $\boldsymbol{\zeta}^0$ *and* $\boldsymbol{\zeta}^1$ *satisfy the following boundary conditions on* $\partial\omega$

$$\begin{aligned} &T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^0) = 0, \qquad \zeta_s^0 = 0, \qquad M_n(\zeta_3^0) = 0, \qquad && \zeta_3^0 = -\int_\omega R_{\mathrm{b}}^0 \, \xi_\omega \,, \\ &T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^1) = 0, \qquad \zeta_s^1 = 0, \qquad M_n(\zeta_3^1) = c_3^{⑦} L, \qquad && \zeta_3^1 = c_4^{⑦} \Lambda \,. \end{aligned} \tag{6.5}$$

Here L *was already defined in* (4.6) *(case* ⑤ *) and*

$$\Lambda(s) = \left(\oint_{\partial\omega} \oint_{\partial\omega} L + 2\mu \int_{\partial\omega} L \, \partial_n \xi_\omega \; - \int_{\partial\omega} \Big(\oint_{\partial\omega} \oint_{\partial\omega} L \Big) \, N_n(\xi_\omega) \right),$$

where $\oint L$ *is the primitive with zero mean value introduced in Notation* 4.1. *The coefficients* $c_4^{⑦} = c_4^{⑤}$ *and* $c_3^{⑦} = -2\mu c_4^{⑦}$ *depend only on* λ *and* μ. *The first boundary layer profile fulfills* $\boldsymbol{\varphi}_\natural^1 = 0$ *and* $\varphi_s^1 = \partial_s \zeta_3^0(s) \, \bar{\varphi}_{\mathrm{Dir}}^{\mathrm{s}}$, *cf Lemma* 3.6.

6.1 The traces of $\boldsymbol{u}_{\mathrm{KL}}^0$

According to (2.21) the Dirichlet trace D_s^0 is zero, i.e. $u_{\mathrm{KL},s}^0$ is zero on Γ_0, whence $\zeta_s^0 - x_3 \partial_s \zeta_3^0 = 0$ on $\partial\omega$. Thus, we have the conditions $\zeta_s^0(s) = 0$ and $\partial_s \zeta_3(s) = 0$ on $\partial\omega$. T_n^0, T_3^0 and T_n^1 are always zero.

From the conditions $T_3^1 = 0$ and $T_n^2 = 0$ we deduce the problem for $\boldsymbol{\varphi}_\natural^1$. It reads

$$E_{\mathrm{Free}}(\boldsymbol{\varphi}_\natural^1) = (0; 0; -T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^0) + x_3 M_n(\zeta_3^0), 0) \,. \tag{6.6}$$

From the cancellation of the constants δ_5, δ_7, δ_8 in Proposition 3.10, ensuring the existence of an exponentially decaying profile, the conditions $T_n^{\mathrm{m}}(\boldsymbol{\zeta}_*^0)(s) = 0$ and $M_n(\zeta_3^0)(s) = 0$ on $\partial\omega$ are obtained. Of course, the constant δ_8 vanishes without any additional condition. The only exponentially decreasing solution is given by $\boldsymbol{\varphi}_\natural^1 = 0$.

The condition $\partial_s \zeta_3(s) = 0$ means $\zeta_3^0 = c$ with c a constant on $\partial\omega$. Hence ζ_3^0 solves problem (6.2) with $f = R_{\mathrm{b}}^0$, $g = c$ and $h = 0$. According to the orthogonality condition $\int_\Omega u_3(\varepsilon) \, dx = 0$, ζ_3^0 has a zero mean value on ω, which allows for the determination of the constant c with the help of formulas (6.3) and (6.4).

6.2 The traces of $\boldsymbol{u}^1_{\mathrm{KL}}$

By considering $D^1_s = 0$, we obtain that φ^1_s has to satisfy

$$L_{\mathrm{Dir}}(\varphi^1_s) = (0; 0; -\zeta^1_s + x_3\partial_s\zeta^1_3)\,.$$

In the same manner as in §4.1 we deduce the condition $\zeta^1_s(s) = 0$ on $\partial\omega$ and $\varphi^1_s = \partial_s\zeta^1_3(s)\,\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}$.

The next relations are deduced from $T^2_3 = 0$ and $T^3_n = 0$ which lead to the following problem for $\boldsymbol{\varphi}^2_\natural$

$$E_{\mathrm{Free}}(\boldsymbol{\varphi}^2_\natural) = \Big(-\big(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1\big)_\natural\,;\ -\big(\mathscr{G}^{(1)}\boldsymbol{\varphi}^1\big)_\natural\,;\ \mathfrak{h}_t\,,\ \mathfrak{h}_3\Big)\,, \tag{6.7}$$

where the terms in the right hand side of (6.7) are given by

$$\big(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1\big)_t = (\lambda+\mu)\,\partial_t\partial_s\varphi^1_s\,,\ \big(\mathscr{G}^{(1)}\boldsymbol{\varphi}^1\big)_t = 0\,,\ \mathfrak{h}_t = -\Big(\lambda\partial_s\varphi^1_s + T^{\mathrm{m}}_n(\boldsymbol{\zeta}^1_*) - x_3M_n(\zeta^1_3)\Big)$$

and

$$\big(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1\big)_3 = (\lambda+\mu)\,\partial_3\partial_s\varphi^1_s\,,\quad \big(\mathscr{G}^{(1)}\boldsymbol{\varphi}^1\big)_3 = \lambda\,\partial_s\varphi^1_s\,,$$

$$\mathfrak{h}_3 = -\mu\Big((\bar p_2 + \bar p_3')\,\partial_n\Delta_*\zeta^0_3 + \partial_3(G(\boldsymbol{f},\boldsymbol{g}^\pm))_n\Big)\,.$$

From Proposition 3.10 we know that for the existence of an exponentially decaying profile, the cancellation of the constants δ_5, δ_7 and δ_8 is necessary and sufficient. For the evaluation of the expressions in these conditions, we have to take into account that $\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}$ is odd with respect to the variable x_3. The cancellation of the constant δ_5 then leads to the boundary condition $T^{\mathrm{m}}_n(\boldsymbol{\zeta}^1_*)(s) = 0$ on $\partial\omega$.

The evaluation of the condition $\delta_8 = 0$ has been already done in §4.2, which yields in exactly the same way $\partial_{ss}\zeta^1_3(s) = c^{⑦}_4 L(s)$ on $\partial\omega$ with

$$c^{⑦}_4 = c^{⑤}_4 = -\left(2\mu\int_{\mathbb{R}^+}\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}(t,1)\,dt\right)^{-1} < 0$$

and

$$L(s) = \left[-\frac{2}{3}(\tilde\lambda + 2\mu)\partial_n\Delta_*\,\zeta^0_3 + \int_{-1}^{+1} x_3\,f_n\,dx_3 + g^+_n + g^-_n\right]\Bigg|_{\partial\omega}\,.$$

Inserting the expressions involved, the condition $\delta_7 = 0$ reads

$$\Bigg[(\lambda+\mu)\int_{\Sigma^+}(x_3\,\partial_t\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}} - t\,\partial_3\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}})\,dt\,dx_3 + \lambda\int_0^\infty t\,(\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}(1,t) - \bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}(-1,t))\,dt$$
$$+\,\lambda\int_{-1}^{+1} x_3\,\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}(0,x_3)\,dx_3\Bigg]\,\partial_{ss}\zeta^1_3 - \int_{-1}^{+1} x_3^2\,M_n(\zeta^1_3)\,dx_3 = 0.$$

Taking into account the parity of the boundary layer term $\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}$ and evaluation of the integral $\int_{-1}^{+1} x_3^2\,dx_3 = \frac{2}{3}$, the above condition becomes

$$\frac{2}{3}\,M_n(\zeta^1_3) = \partial_{ss}\zeta^1_3\Bigg[-\mu\int_{-1}^{+1} x_3\,\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}(0,x_3)\,dx_3 - 2\mu\int_0^\infty t\,\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}(1,t)\,dt\Bigg].$$

Applying the second Green's formula for Laplace on rectangles $\Sigma_L = (0,L)\times(-1,1)$ to the functions $\bar\varphi^{\rm s}_{\rm Dir}(t,x_3)$ and $w(t,x_3) = t\,x_3$ and considering $L\to+\infty$, the relation

$$2\int_0^\infty t\,\bar\varphi^{\rm s}_{\rm Dir}(t,1)\,dt = \int_{-1}^{+1} x_3\,\bar\varphi^{\rm s}_{\rm Dir}(0,x_3)\,dx_3 = \int_{-1}^{+1} x_3^2\,dx_3 = \frac{2}{3}$$

is obtained, and thus we have $M_n(\zeta_3^1) = -2\mu\partial_{ss}\zeta_3^1$. Inserting the expression obtained above for $\partial_{ss}\zeta_3^1$ leads to the condition $M_n(\zeta_3^1)(s) = -2\mu c_4^{⑦}L(s)$ on $\partial\omega$.

In the same manner as in the frictional I case ⑤, *cf* (4.12) there holds

$$\int_{\partial\omega} L(s)\,ds = 0\,. \tag{6.8}$$

Thus, we can choose the primitive of L with mean value zero to obtain the expression for $\partial_s\zeta_3^1(s)$ on $\partial\omega$. Then, with the help of formulas (6.3) and (6.4), we choose the unique primitive g of $\oint L$ such that the solution ζ_3^1 of problem (6.2) with $f = R_{\rm b}^1 = 0$, $h = -2\mu c_4^{⑦}L$ and g has a mean value zero on ω.

It remains to remark that the problem for $\boldsymbol{\zeta}_*^1$ is a completely homogeneous one, thus $\boldsymbol{\zeta}_*^1 \equiv 0$ holds.

7 FREE

The space of rigid motions $\mathcal{R}_{⑧}$ is six-dimensional and spanned by all rigid motions, the translations $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and the rotations $(x_3,0,-x_1)$, $(x_2,-x_1,0)$, $(0,x_3,-x_2)$.

Theorem 7.1 *The expansion* (2.1)-(2.2) *holds for the solution* $\boldsymbol{u}(\varepsilon)$ *of problem* (1.11) *with lateral condition* ⑧*, compatibility condition* (1.14) *and orthogonality condition* (1.15)*. The Kirchhoff-Love generators* $\boldsymbol{\zeta}_*^k$ *and* ζ_3^k *are solutions of membrane and bending problems with interior data* $\boldsymbol{R}_{\rm m}(\boldsymbol{\zeta}_*^k)$ *and* $R_{\rm b}(\zeta_3^k)$*, and boundary conditions prescribing* $T_n(\boldsymbol{\zeta}_*^k)$*,* $T_s(\boldsymbol{\zeta}_*^k)$*, and* $M_n(\zeta_3^k)$*,* $N_n(\zeta_3^k)$ *respectively. Moreover* $\boldsymbol{\zeta}_*^k$ *and* ζ_3^k *satisfy orthogonality conditions accordingly with* (1.15)*. More precisely,* $\boldsymbol{\zeta}^0$ *and* $\boldsymbol{\zeta}^1$ *satisfy the following boundary conditions on* $\partial\omega$

$$\begin{aligned}
&T_n^{\rm m}(\boldsymbol{\zeta}_*^0) = 0, \qquad T_s^{\rm m}(\boldsymbol{\zeta}_*^0) = 0, \qquad M_n(\zeta_3^0) = 0,\\
&\qquad\qquad N_n(\zeta_3^0) = \tfrac{3}{2}\Big(\textstyle\int_{-1}^{+1} x_3\,f_n\,dx_3 + g_n^+ + g_n^-\Big)\Big|_{\partial\omega},\\
&T_n^{\rm m}(\boldsymbol{\zeta}_*^1) = 0, \qquad T_s^{\rm m}(\boldsymbol{\zeta}_*^1) = 0, \qquad M_n(\zeta_3^1) = c_1^{⑧}\partial_s(\partial_n+\kappa)\partial_s\zeta_3^0,\\
&\qquad\qquad N_n(\zeta_3^1) = c_2^{⑧}Q + c_3^{⑧}S + c_4^{⑧}T + \kappa K^{⑧}(f_n, g_n^\pm),
\end{aligned} \tag{7.1}$$

where Q*,* S *and* T *are the traces on* $\partial\omega$ *of different operators acting on* ζ_3^0 *according to* $Q = \kappa\partial_n\Delta_*\zeta_3^0$*,* $S = \partial_s\kappa(\partial_n+\kappa)\partial_s\zeta_3^0$ *and* $T = \kappa\partial_s(\partial_n+\kappa)\partial_s\zeta_3^0$*. The constant* $c_1^{⑧}$ *coincides with the constant* $c_5^{④}$ *intervening in the soft simple support* *case (Th. 8.1 of Part I) and the* $c_j^{⑧}$ *for* $j = 2,3,4$ *are constants only depending on* λ *and* μ. *The operator* $K^{⑧}$ *depends only on the Lamé coefficients as well and is in-plane local, cf Def.* 2.1. *The first boundary layer profile fulfills* $\boldsymbol{\varphi}_\natural^1 = 0$ *and* $\varphi_s^1 = (\partial_n + \kappa)\partial_s\zeta_3^0(s)\,\bar\varphi_{\rm Neu}^{\rm s}$, *cf Lemma* 3.7.

7.1 The traces of $\boldsymbol{u}_{\rm KL}^0$

T_n^0, T_s^0, T_3^0, T_n^1 and T_s^1 are always zero. From the conditions $T_3^1 = 0$ and $T_n^2 = 0$ we obtain problem (6.6) for $\boldsymbol{\varphi}_\natural^1$. In exactly the same manner as in §6.1, we deduce the conditions $T_n^{\rm m}(\boldsymbol{\zeta}_*^0)(s) = 0$ and $M_n(\zeta_3^0)(s) = 0$ on $\partial\omega$ as well as $\boldsymbol{\varphi}_\natural^1 = 0$.

The condition $T_s^2 = 0$ yields, that φ_s^1 has to satisfy problem (5.2). As in §5.1, we obtain that $T_s^{\rm m}(\boldsymbol{\zeta}_*^0)(s) = 0$ on $\partial\omega$ holds. The exponentially decaying solution is then given by (*cf* Lemma 3.7)

$$\varphi_s^1 = (\partial_n + \kappa)\partial_s\zeta_3^0(s)\,\bar\varphi_{\rm Neu}^{\rm s}\,. \tag{7.2}$$

Let us now consider the problem for φ_s^2. With $T_s^3 = 0$ we obtain that φ_s^2 has to satisfy problem (5.4), hence the condition $T_s^{\rm m}(\boldsymbol{\zeta}_*^1)(s) = 0$ on $\partial\omega$ ensures the existence of an exponentially decaying profile. Taking into account the relation (7.2), this solution is given by

$$\varphi_s^2 = -\kappa(\partial_n + \kappa)\partial_s\zeta_3^0\,\bar\psi_{\rm Neu}^{\rm s} + (\partial_n + \kappa)\partial_s\zeta_3^1\,\bar\varphi_{\rm Neu}^{\rm s}\,, \tag{7.3}$$

where $\bar\psi_{\rm Neu}^{\rm s}$ is the solution of problem (5.6).

The next relations are deduced from $T_3^2 = 0$ and $T_n^3 = 0$ which lead to a problem of the form (6.7) for $\boldsymbol{\varphi}_\natural^2$ with the same expressions for $(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_t$, $(\mathscr{B}^{(1)}\boldsymbol{\varphi}^1)_3$, $(\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_t$, $(\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_3$, $\mathfrak{h}_t$ and $\mathfrak{h}_3$. The only difference now is that we have to take into account $\varphi_s^1 = (\partial_n + \kappa)\partial_s\zeta_3^0(s)\,\bar\varphi_{\rm Neu}^{\rm s}$ instead of $\varphi_s^1 = \partial_s\zeta_3^1(s)\,\bar\varphi_{\rm Dir}^{\rm s}$ as we had in (6.7). From Proposition 3.10 we know that for the existence of an exponentially decaying profile, the cancellation of the constants δ_5, δ_7 and δ_8 is necessary and sufficient. For the evaluation of the expressions in these conditions, we have to take into account that $\bar\varphi_{\rm Neu}^{\rm s}$ is odd with respect to the variable x_3. The cancellation of the constant δ_5 then leads to the boundary condition $T_n^{\rm m}(\boldsymbol{\zeta}_*^1)(s) = 0$ on $\partial\omega$.

Inserting the expressions involved, the condition $\delta_7 = 0$ reads

$$\Big[(\lambda+\mu)\int_{\Sigma^+}(x_3\,\partial_t\bar\varphi_{\rm Neu}^{\rm s} - t\,\partial_3\bar\varphi_{\rm Neu}^{\rm s})\,dt\,dx_3 + \lambda\int_0^\infty t\,(\bar\varphi_{\rm Neu}^{\rm s}(1,t) - \bar\varphi_{\rm Neu}^{\rm s}(1,t))\,dt$$
$$+\,\lambda\int_{-1}^{+1} x_3\,\bar\varphi_{\rm Neu}^{\rm s}(0,x_3)\,dx_3\Big]\,\partial_s(\partial_n+\kappa)\partial_s\zeta_3^0 - \int_{-1}^{+1} x_3^2\,M_n(\zeta_3^1)\,dx_3 = 0\,.$$

As the boundary layer term $\bar\varphi_{\rm Neu}^{\rm s}$ is odd, the above condition becomes

$$\frac{2}{3}\,M_n(\zeta_3^1) = \partial_s(\partial_n+\kappa)\partial_s\zeta_3^0\Big[-\mu\int_{-1}^{+1} x_3\,\bar\varphi_{\rm Neu}^{\rm s}(0,x_3)\,dx_3 - 2\mu\int_0^\infty t\,\bar\varphi_{\rm Neu}^{\rm s}(1,t)\,dt\Big].$$

Applying the second Green's formula for Laplace to the functions $\bar\varphi^{\rm s}_{\rm Neu}(t,x_3)$ and $w(t,x_3)=t\,x_3$, yields the relation

$$2\int_0^\infty t\,\bar\varphi^{\rm s}_{\rm Neu}(t,1)\,dt=\int_{-1}^{+1}x_3\,\bar\varphi^{\rm s}_{\rm Neu}(0,x_3)\,dx_3\,.$$

Thus we have $M_n(\zeta^1_3)=c_1^{⑧}\partial_s(\partial_n+\kappa)\partial_s\zeta^0_3$ on $\partial\omega$ with (compare Part I, §8.2)

$$c_1^{⑧}=c_5^{④}=-3\mu\int_{-1}^{+1}x_3\,\bar\varphi^{\rm s}_{\rm Neu}(0,x_3)\,dx_3\quad.$$

The evaluation of the condition $\delta_8=0$ has been already done in §5.1, which yields in exactly the same way the condition

$$N_n(\zeta^0_3)(s)=\frac{3}{2}\left(\int_{-1}^{+1}x_3\,f_n\,dx_3+g_n^++g_n^-\right)(0,s)\ \text{ on }\partial\omega\quad.$$

Thus, the existence of an exponentially decaying profile $\boldsymbol\varphi^2_\natural$ is guaranteed. Analogously as in §5.1 we split the operator $G(\boldsymbol f,\boldsymbol g^\pm)=G^{\rm b}(\boldsymbol f,\boldsymbol g^\pm)+G^{\rm m}(\boldsymbol f,\boldsymbol g^\pm)$ into bending and membrane part with parities (odd/even) and (even/odd), respectively. The profile has then the form $\boldsymbol\varphi^2_\natural=\boldsymbol\varphi^{2,\rm b}_\natural+\boldsymbol\varphi^{2,\rm m}_\natural$. Later, for the determination of the boundary conditions satisfied by ζ^1_3, we will need information about its bending part $\boldsymbol\varphi^{2,\rm b}_\natural$, which is solution of the following problem in the half-strip:

$$\begin{aligned}E_{\rm Free}(\boldsymbol\varphi^{2,\rm b}_\natural)\ &=\ \partial_s(\partial_n+\kappa)\partial_s\zeta^0_3\Big(-(\lambda+\mu)\partial_t\bar\varphi^{\rm s}_{\rm Neu}\,,\,-(\lambda+\mu)\partial_3\bar\varphi^{\rm s}_{\rm Neu}\,;\\ &\qquad\qquad 0\,,\,-\lambda\bar\varphi^{\rm s}_{\rm Neu}\,;\,-\lambda\bar\varphi^{\rm s}_{\rm Neu}+c_1^{⑧}x_3\,,\,0\Big)\\ &\quad+\ \partial_n\Delta_*\zeta^0_3\Big(0\,;\,0\,;\,0\,,\,-\mu(\bar p_2+\bar p_3')\Big)\\ &\quad+\Big(0\,;\,0\,;\,0\,,\,-\mu\,\partial_3G^{\rm b}(\boldsymbol f,\boldsymbol g^\pm)_n\Big)\,.\end{aligned}\tag{7.4}$$

Now let us check the compatibility conditions ensuring the existence of the generator $\boldsymbol\zeta^0$ of $\boldsymbol u^0_{\rm KL}$. Concerning $\boldsymbol\zeta^0_*$, we have to show that the membrane right hand side $\boldsymbol R^0_{\rm m}$ of the limit problem is orthogonal to each of the two-dimensional rigid motions $(1,0)$, $(0,1)$ and $(x_2,-x_1)$, since we have homogeneous traction boundary conditions in the problem for $\boldsymbol\zeta^0_*$. These orthogonality condition is clearly a consequence of the expression of the right hand side $\boldsymbol R^0_{\rm m}$ and of the compatibility conditions (1.2) assumed for the data $\boldsymbol f$ and $\boldsymbol g^\pm$ of the three-dimensional problem, since the two-dimensional rigid motions above do not depend on the vertical variable x_3.

The compatibility conditions for ζ^0_3 remains to be checked. We have to show that the Green's formula (5.8) holds for $\zeta_3=\zeta^0_3$ and separately for $\eta_3=1$, $\eta_3=x_1$ and $\eta_3=x_2$ (kernel of the problem for ζ^0_3). It has been already shown in §5.1 that the condition

$$\int_\omega R^0_{\rm b}(x_*)\,dx_*-\frac{3}{2}\int_{\partial\omega}\left(\int_{-1}^{+1}x_3\,f_n\,dx_3+g_n^++g_n^-\right)(0,s)\,ds=0$$

is fulfilled. Now let us check the Green's formula for $\eta_3 = x_1$, namely the condition

$$\int_\omega x_1\, R^0_{\mathrm{b}}(x_*)\, dx_* - \frac{3}{2}\int_{\partial\omega} x_1 \left(\int_{-1}^{+1} x_3\, f_n\, dx_3 + g_n^+ + g_n^-\right)(0,s)\, ds = 0\,.$$

With the help of the divergence theorem we can rewrite the left hand side as

$$\frac{3}{2}\left\{\int_\Omega (x_1\, f_3 - x_3\, f_1)\, dx_3\, dx_* + \int_\omega \Big\{x_1\,(g_3^+ - g_3^-) - (g_1^+ + g_1^-)\Big\}\, dx_*\right\}\,,$$

which is clearly zero, because it represents a compatibility condition for the three-dimensional problem we assumed in (1.2). Of course, the condition for $\eta_3 = x_2$ can be proved analogously.

7.2 The traces of $\boldsymbol{u}^1_{\mathrm{KL}}$

The only remaining boundary condition is that for $N_n(\zeta^1_3)$. Therefore we only consider the problem for $\boldsymbol{\varphi}^3_\natural$, which is deduced from $T^3_3 = 0$ and $T^4_n = 0$ and reads

$$E_{\mathrm{Free}}(\boldsymbol{\varphi}^3_\natural) = \Big(-(\mathscr{B}^{(1)}\boldsymbol{\varphi}^2)_\natural - (\mathscr{B}^{(2)}\boldsymbol{\varphi}^1)_\natural\,;\, -(\mathscr{G}^{(1)}\boldsymbol{\varphi}^2)_\natural - (\mathscr{G}^{(1)}\boldsymbol{\varphi}^1)_\natural\,;\; \mathfrak{h}_t\,,\; \mathfrak{h}_3\Big)\,. \quad (7.5)$$

The boundary condition prescribing $N_n(\zeta^1_3)$ is then found by the cancellation of the coefficient δ_8 (3.8a). We use again (5.10) and that the boundary layer terms involved are φ^1_s, φ^2_s and $\boldsymbol{\varphi}^{2,\mathrm{b}}_\natural$. Inserting the expressions (7.2), (7.3) and (7.4), this condition yields for $N_n(\zeta^1_3)$ an expression of the form

$$c_2^{⑧}\kappa\partial_n\Delta_*\zeta^0_3 + c_3^{⑧}\partial_s\kappa(\partial_n+\kappa)\partial_s\zeta^0_3(s) + c_4^{⑧}\kappa\partial_s(\partial_n+\kappa)\partial_s\zeta^0_3(0,s) + \kappa K^{⑧}(f_n, g_n^\pm)\,.$$

The compatibility conditions for ζ^1_3 can be checked similarly to the compatibility condition in the sliding edge situation ⑥ using exactly the same kind of argument as there, compare also Part I, §8.2. And of course, the same remark applies we made there concerning the conservation of the orthogonality condition (1.15) for the successive terms in the expansion.

8 ERROR ESTIMATES

8.1 In H^1 norm

In this section we extend the results obtained in [5, §5] for the hard clamped situation to plates with one of the eight 'canonical' boundary conditions on the lateral side. Since the way of proving the error estimates is strictly similar to what is presented in [5, §5], here we only give a sketch of the proof including the main steps and statements.

The justification of the formal asymptotic expansion (2.1), (2.2) which is valid up to an arbitrarily high order N, yields an *optimal estimation of the error between the scaled displacement* $\boldsymbol{u}(\varepsilon)$ *and the Ansatz of order* N. In the sequel, C always denotes a constant independent of ε (not necessarily everywhere the same constant, although the same letter is used).

Theorem 8.1 *Let $\boldsymbol{u}(\varepsilon)$ be the unique solution of problem* (1.11) *satisfying the mean value conditions* (1.15). *Then there holds* $\forall N \geq 0$

$$\|\boldsymbol{u}(\varepsilon)(x) - \boldsymbol{u}^0_{\mathrm{KL}}(x) - \sum_{k=1}^{N} \varepsilon^k \boldsymbol{u}^k(x, \frac{r}{\varepsilon})\|_{H^1(\Omega)^3} \leq C\, \varepsilon^{N+1/2} \tag{8.1}$$

with $\boldsymbol{u}^k(x, \frac{r}{\varepsilon})$ *given in* (2.2).

First we will give a sketch of the proof of Theorem 8.1 and then some conclusions from the estimate (8.1) will be drawn in section 9. The proof relies on energy estimates and on a very simple argument consisting in pushing the development a few terms further.

We define the space

$$\mathcal{V}_{\textcircled{i}}(\Omega) := \left\{ \boldsymbol{u} \in V_{\textcircled{i}}(\Omega) \;\middle|\; \forall\, \boldsymbol{v} \in \mathcal{R}_{\textcircled{i}}(\Omega), \quad \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} = 0 \right\} ,$$

then $\boldsymbol{u}(\varepsilon) \in \mathcal{V}_{\textcircled{i}}(\Omega)$ is obvious. Combining Korn's inequality without boundary conditions and the infinitesimal rigid displacement lemma we obtain a Korn inequality with boundary conditions for arbitrary $\boldsymbol{u} \in \mathcal{V}_{\textcircled{i}}(\Omega)$, compare [18] and [3], which reads in terms of the scaled linearized strain tensor $\theta(\varepsilon)$

$$\left(\int_\Omega A\theta(\varepsilon)(\boldsymbol{u}) : \theta(\varepsilon)(\boldsymbol{u}) \right)^{1/2} \geq C^* \|\theta(\varepsilon)(\boldsymbol{u})\|_{L^2(\Omega)^9} \geq C \|\boldsymbol{u}\|_{H^1(\Omega)} . \tag{8.2}$$

Setting

$$\overline{U}^N(\varepsilon) := \boldsymbol{u}(\varepsilon) - U^N(\varepsilon) , \tag{8.3}$$

where $U^N(\varepsilon)$ denotes the asymptotic expansion of order N , namely

$$U^N(\varepsilon) \;=\; \underbrace{\sum_{k=0}^{N} \varepsilon^k\, \underline{\boldsymbol{u}}^k}_{=:\, V^N(\varepsilon)} \;+\; \chi(r) \underbrace{\sum_{k=1}^{N} \varepsilon^k\, \boldsymbol{w}^k(\frac{r}{\varepsilon}, s, x_3)}_{=:\, W^N(\varepsilon)} \tag{8.4}$$

with $\underline{\boldsymbol{u}}^k := \boldsymbol{u}^k_{\mathrm{KL}} + \boldsymbol{v}^k$, compare §2.1 for notations, we realize that it is sufficient to establish an a priori estimate for the remainder $\overline{U}^N(\varepsilon)$ in the norm of the space $H^1(\Omega)^3$. Therefore, we split $U^N(\varepsilon)$ into its two natural parts

$$U^N(\varepsilon) = V^N(\varepsilon) + \chi(r)\, W^N(\varepsilon) .$$

Considering carefully the construction algorithm, in particular the derivation of the boundary layer terms, we observe

Lemma 8.2 *For any* $N \in \mathbb{N}$, $U^N(\varepsilon)$ *belongs to the space* $\mathcal{V}_{\textcircled{i}}(\Omega)$ *of geometrically admissible displacements satisfying the mean value conditions* (1.15).

Thus, we have

$$\forall N \in \mathbb{N}\,, \quad \overline{U}^N(\varepsilon) \in \mathcal{V}_{\text{ⓘ}}(\Omega)$$

and the variational form of the problem for $\overline{U}^N(\varepsilon)$ can be written down, where we split the deviation to the true solution into an error generated by $V^N(\varepsilon)$ and an error coming from $W^N(\varepsilon)$, compare [5, (5.8) – (5.11)]. For the choice $\boldsymbol{v} = \overline{U}^N(\varepsilon)$ of the test function in the variational formulation of the problem for $\overline{U}^N(\varepsilon)$, we obtain as one side of the resulting equation the energy associated to the remainder, namely

$$\int_\Omega A\,\theta(\varepsilon)(\overline{U}^N(\varepsilon)) \,:\, \theta(\varepsilon)(\overline{U}^N(\varepsilon))\,.$$

Korn's inequality (8.2) and the coercivity of the operator of elasticity then provides the following rough estimate

$$\|\overline{U}^N(\varepsilon)\|_{H^1(\Omega)^3} \le C\varepsilon^{N-3}$$

exactly in the same manner as in the proof of Lemma 5.3 in [5]. This estimate reads for $\overline{U}^{N+4}(\varepsilon)$

$$\|\overline{U}^{N+4}(\varepsilon)\|_{H^1(\Omega)^3} \le C\varepsilon^{N+1}\,,$$

whence

$$\begin{aligned} &\|\boldsymbol{u}(\varepsilon)(x) - \boldsymbol{u}^0_{\mathrm{KL}}(x) - \sum_{k=1}^{N} \varepsilon^k \boldsymbol{u}^k(x, \frac{r}{\varepsilon})\|_{H^1(\Omega)^3} \\ &\qquad \le C\,\varepsilon^{N+1} + \sum_{k=N+1}^{N+4} \varepsilon^k \left(\|\underline{\boldsymbol{u}}^k\|_{H^1(\Omega)^3} + \|\chi(r)\boldsymbol{w}^k(\frac{r}{\varepsilon}, s, x_3)\|_{H^1(\Omega)^3} \right)\,. \end{aligned} \tag{8.5}$$

With the help of the following H^1-estimates of each term in the asymptotics

$$\|\underline{\boldsymbol{u}}^k\|_{H^1(\Omega)^3} \le C \quad \text{and} \quad \|\chi(r)\boldsymbol{w}^k(\frac{r}{\varepsilon}, s, x_3)\|_{H^1(\Omega)^3} \le C\varepsilon^{-1/2}\,, \tag{8.6}$$

the estimate (8.1) directly follows from (8.5).

8.2 In other norms

The L^2-estimates of each term corresponding to (8.6)

$$\|\underline{\boldsymbol{u}}^k\|_{L^2(\Omega)^3} \le C \quad \text{and} \quad \|\chi(r)\boldsymbol{w}^k(\frac{r}{\varepsilon}, s, x_3)\|_{L^2(\Omega)^3} \le C\varepsilon^{1/2} \tag{8.7}$$

lead in a straightforward way to the following estimates in L^2-norm

$$\|\boldsymbol{u}(\varepsilon) - \sum_{k=0}^{N} \varepsilon^k\,\underline{\boldsymbol{u}}^k - \chi(r)\,\sum_{k=1}^{N} \varepsilon^k\,\boldsymbol{w}^k(\frac{r}{\varepsilon}, s, x_3)\|_{L^2(\Omega)^3} \le C\,\varepsilon^{N+1}\,. \tag{8.8}$$

The question of estimates in higher norms, H^2 for instance, is also considered in [6] for the clamped case. Such estimates require a splitting of the solution and of terms in the asymptotics, since in general the H^2 regularity is not attained. The situation is similar for all lateral conditions. Let us just emphasize that all the terms in the outer expansion are smooth, but also that the singularities along the edges $\partial\omega \times \{\pm 1\}$ of the plate are concentrated in the inner expansion: the model profiles are all non-smooth, with a regularity between $H^{3/2}$ and H^3. For example $\bar\varphi^{\mathrm{s}}_{\mathrm{Dir}}$ is almost H^2 and $\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}$ is almost H^3 whereas the profiles $\bar{\boldsymbol\varphi}^{\mathrm{m}}_{\mathrm{Dir},\natural}$ and $\bar{\boldsymbol\varphi}^{\mathrm{b}}_{\mathrm{Dir},\natural}$ occurring in the clamped plates have less regularity, *cf* [7].

9 CONCLUSIONS

Coming back to the family of thin domains Ω^ε, we will briefly address the question of the determination of a limit solution, and of the evaluation of the relative error between this limit and the 3D solution. The correct answer depends on the norm in which the error is evaluated and of the type of the loading.

9.1 H^1 norm

We have first to evaluate the behavior of the $H^1(\Omega^\varepsilon)$ norm denoted $\|\cdot\|_{H1}$ of each of the four types of components of series (1.16), namely $\boldsymbol u^k_{\mathrm{KL,b}}$, $\boldsymbol u^k_{\mathrm{KL,m}}$, $\tilde{\boldsymbol v}^k$ and $\boldsymbol\varphi^k$. We find:

$$\|\boldsymbol u^k_{\mathrm{KL,b}}\|_{H1} = \mathcal O(\varepsilon^{1/2}),\ \ \|\boldsymbol u^k_{\mathrm{KL,m}}\|_{H1} = \mathcal O(\varepsilon^{1/2}),\ \ \|\tilde{\boldsymbol v}^k\|_{H1} = \mathcal O(\varepsilon^{-1/2}),\ \ \|\boldsymbol\varphi^k\|_{H1} = \mathcal O(1).$$

In the case of a bending load such that $R^0_{\mathrm b}$, *cf* (2.8), is non-zero, we have

$$\frac{\|\boldsymbol u^\varepsilon - \varepsilon^{-1}\boldsymbol u^0_{\mathrm{KL,b}}\|_{H1}}{\|\boldsymbol u^\varepsilon\|_{H1}} \le C\,\varepsilon, \tag{9.1}$$

and this estimate is sharp for any lateral boundary condition, since the main contribution to the error comes from $\tilde{\boldsymbol v}^1$ which is equal to $(0,0,\bar p_2(x_3)\,\Delta_*\zeta^0_3)$: indeed, since we assumed that $R^0_{\mathrm b}$ is non-zero, $\Delta^2_*\zeta^0_3$ is non-zero, and $\tilde{\boldsymbol v}^1 \not\equiv 0$.

In the case of a membrane load such that $\boldsymbol R^0_{\mathrm m}$, *cf* (2.9), is non-zero, we have to include $\tilde{\boldsymbol v}^1$ in the limit solution to have a convergence: we set

$$\boldsymbol u^{\mathrm{lim}}_{\mathrm m} = \boldsymbol u^0_{\mathrm{KL,m}} + \varepsilon\tilde{\boldsymbol v}^1 = (\zeta^0_*,\ \bar p_1(\tilde x_3)\,\mathrm{div}_*\,\zeta^0_*). \tag{9.2}$$

Then

$$\frac{\|\boldsymbol u^\varepsilon - \boldsymbol u^{\mathrm{lim}}_{\mathrm m}\|_{H1}}{\|\boldsymbol u^\varepsilon\|_{H1}} \le C\,\varepsilon^{1/2}, \quad \text{in cases } ① - ④, \tag{9.3}$$

this estimate being generically optimal, in the sense that it is sharp when $\boldsymbol\varphi^1$ is non-zero, i.e. when $\mathrm{div}_*\,\zeta^0_*$ is non-zero on $\partial\omega$ in cases ①, ② and ④, and when $\kappa\zeta^0_n$ is non-zero on $\partial\omega$ in cases ③. On the other hand

$$\frac{\|\boldsymbol u^\varepsilon - \boldsymbol u^{\mathrm{lim}}_{\mathrm m}\|_{H1}}{\|\boldsymbol u^\varepsilon\|_{H1}} \le C\,\varepsilon, \quad \text{in cases } ⑤ - ⑧, \tag{9.4}$$

this estimate being generically optimal too, in the sense that it is sharp when $\tilde{\boldsymbol{v}}^2$ is non-zero, i.e. when $\operatorname{div}_* \zeta_*^0 \not\equiv 0$, compare also with [15] for a special membrane loading on a free plate.

9.2 Energy norm

We now set $\|\boldsymbol{u}\|_E = \left(\int_{\Omega^\varepsilon} Ae(\boldsymbol{u}) : e(\boldsymbol{u})\right)^{1/2}$. The energy of the four types of terms in the series (1.16) has the same behavior as their H^1 norm *except* the one concerning $\boldsymbol{u}^k_{\mathrm{KL,b}}$ whose energy is one order smaller:

$$\|\boldsymbol{u}^k_{\mathrm{KL,b}}\|_E = \mathcal{O}(\varepsilon^{3/2}).$$

We obtain exactly the same conclusions if we use this energy, or the L^2 norm of the strain tensor, or the complementary energy. We have to include the polynomial terms up to the order 2 to obtain a convergence: we set $\boldsymbol{u}^{\lim}_{\mathrm{m}}$ as above in (9.2) and moreover

$$\boldsymbol{u}^{\lim}_{\mathrm{b}} = \boldsymbol{u}^0_{\mathrm{KL,b}} + \varepsilon\tilde{\boldsymbol{v}}^1 = (-\varepsilon x_3 \nabla_* \zeta_3^0,\ \zeta_3^0 + \varepsilon \bar{p}_2(x_3)\,\Delta_* \zeta_3^0), \tag{9.5}$$

see also [21] and [22] in this context.

In the case of a bending load such that R^0_{b} is non-zero, we have

$$\frac{\|\boldsymbol{u}^\varepsilon - \boldsymbol{u}^{\lim}_{\mathrm{b}}\|_E}{\|\boldsymbol{u}^\varepsilon\|_E} \le C\,\varepsilon^{1/2}, \tag{9.6}$$

this estimate being generically optimal, in the sense that it is sharp when $\boldsymbol{\varphi}^1$ is non-zero, i.e. when ℓ^{b} is non-zero on $\partial\omega$ in cases ① – ④, *cf* Table 5 in Part I and when ℓ^{s} is non-zero on $\partial\omega$ in cases ⑤ – ⑧, *cf* Table 4 in Part II.

In the case of a membrane load such that R^0_{b} is non-zero, we have exactly the same behavior as with the H^1 norm, see (9.3) and (9.4). In particular, the condition for the optimality of the estimates is visibly sharp, which brings a conclusion to the work [1].

The observation of the first terms in the asymptotics also sheds light on the order of magnitude of the answer of the plate under the loading. The maximal answer rate (of order ε^{-2}) is obtained with a bending load such that R^0_{b} is non-zero and corresponds to the flexural nature of plates. In contrast, the membrane (or stretching) answer is of order 1 when $\boldsymbol{R}^0_{\mathrm{m}}$ is non-zero. Moreover, there are very many other types of loading (bending or membrane) whose answer rate is much lower.

ACKNOWLEDGEMENT

The third author is supported by a grant of the German National Science Foundation (DFG, graduate collegium ‘GKKS – Modelling and discretization methods in continua and fluids’). This support is gratefully acknowledged.

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