Edge layers in thin elastic plates

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Abstract. *This paper deals with the asymptotics of the displacement of a thin elastic* 3D *plate when it is submitted to various boundary conditions on its lateral face: namely, hard and soft clamped conditions, and hard support. Of particular interest is the influence of the edges of the plate where boundary conditions of different types meet. Relying on general results of* [11, 12] *for the hard clamped case, we see that the clamped plate (hard and soft) admit strong boundary layers, in which are concentrated the edge layers, while the hard supported plate has no edge layer and even no boundary layer at all in certain situations. We conclude with hints about corner layers, in the case when the mean surface of the plate itself is polygonal.*

1 INTRODUCTION

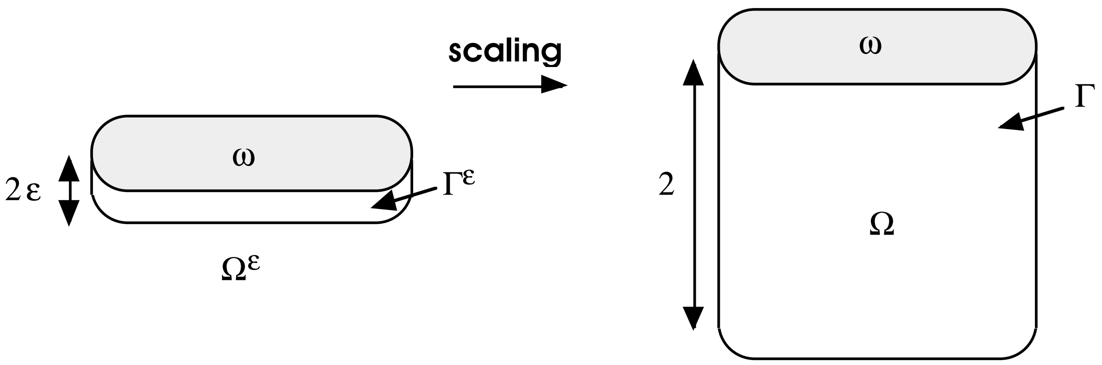

Figure 1. Thin Plate and Scaling

In the physical word, a plate is a three dimensional object characterized by its plane *mean surface* ω and its *thickness* d, the latter being small with respect to the other dimensions. We will denote by $\Omega^\varepsilon = \omega \times (-\varepsilon, +\varepsilon)$ such a plate.

1.a Models

Many physical quantities are modelized by boundary value problems in a plate: let us mention the heat equilibrium, the vibration modes, the displacement field when a traction or a body force is imposed... Here, we concentrate on the latter. Other coupled quantities are the strain and the stress tensors. Let us denote by $\boldsymbol{u}^\varepsilon$ the displacement field in the plate Ω^ε and e^ε_{ij} the corresponding strain tensor. In the theory of linear elasticity corresponding to small displacements, e^ε_{ij} is given by $\frac{1}{2}(\partial_i u_j + \partial_j u_i)$. The stress σ^ε obeys Hooke's law

$$Ae^\varepsilon = \sigma^\varepsilon$$

where A is the rigidity matrix of the constitutive material of the elastic plate Ω^ε (here we assume that the material is isotropic). The equations of equilibrium are

$$\operatorname{div} \sigma^\varepsilon = \boldsymbol{f}^\varepsilon,$$

where $\boldsymbol{f}^\varepsilon$ are the body forces inside Ω^ε. The complementing boundary conditions on the horizontal boundaries $\omega \times \{\pm\varepsilon\}$ are supposed to be the zero traction. On the lateral boundary $\Gamma^\varepsilon = \partial\omega \times (-\varepsilon, +\varepsilon)$, we can distinguish 3 natural components in the displacements or the tractions : normal, horizontal tangential, vertical, and we obtain 8 "canonical" lateral boundary conditions, according to how we choose to prescribe the displacement or the traction for each component.

All these equations result in a boundary value system in the 3D domain Ω^ε. It has been known for a long time that it is possible to replace such a problem by a 2D boundary value system in the mean surface ω. Let us quote the famous Kirchhoff-Love model and the Reissner-Mindlin plate. These models were derived by mechanical considerations. More recently, a larger class of 2D models, the *hierarchical models* was introduced, based on variational considerations.

1.b Comparison

The comparison between 3D and 2D models was first performed by the construction of infinite *formal* asymptotic expansions (i.e. without error estimates) [14, 15, 16]. Next, rigorous error estimates between the 3D solution and the Kirchhoff-Love solution $\boldsymbol{u}^0_{\mathrm{KL}}$ were proved in [5, 13, 4]: here $\boldsymbol{u}^0_{\mathrm{KL}}$ appears as the limit of the 3D solution as $\varepsilon \to 0$. Indeed $\boldsymbol{u}^0_{\mathrm{KL}}$ is the first term of an asymptotic development with respect to ε of $\boldsymbol{u}^\varepsilon$. Further terms were exhibited in [20], and the whole asymptotic expansion was constructed in [11, 12] for the hard clamped condition (which we refer to as the number one ①).

The main feature of these asymptotic expansions (both formal and rigorous) is the presence of two sorts of contributions. In order to explain this important fact, let us introduce coordinates in the plate: $x_* = (x_1, x_2)$ in ω and $z \in (-\varepsilon, \varepsilon)$. The scaled vertical variable is $x_3 = z/\varepsilon$. We also need local coordinates near the lateral boundary Γ^ε (or equivalently near the boundary $\partial\omega$ of ω): r the distance to $\partial\omega$ and s the arc length around $\partial\omega$. The first sort of contribution consists of displacements of the form $\varepsilon^k \underline{\boldsymbol{u}}^k(x_*, x_3)$ (*outer expansion*) and the second sort consists of *boundary layer* terms of the form $\varepsilon^k \boldsymbol{w}^k(\frac{r}{\varepsilon}, s, x_3)$ (*inner expansion*). The reason for the presence of a boundary layer is the following: the 3D model is a "singular perturbation problem" as the thickness parameter ε tends to 0, therefore a standard power series in ε is unable to describe correctly displacements, strains and stresses. Among works exhibiting boundary layers in neighboring situations, let us cite [21], [1], [24, 23].

1.c Boundary and edge layers

The first boundary layer term $\boldsymbol{w}^1$ plays a particular role: it is responsible for the main part of the difference between the 3D solution $\boldsymbol{u}^\varepsilon$ and the 2D solution $\boldsymbol{u}^0_{\mathrm{KL}}$, and this error depends on the norm in which it is evaluated. Moreover, we proved in [11, 12] for the clamped plate that the expected non-smoothness of the displacements (due to the presence of the edges of the plate) is concentrated in the boundary layer terms, as *edge layers*. Thus the precise description of $\boldsymbol{w}^1$ with its singularities allows to prove estimates in various norms and to be sure that they are sharp.

Just like in [1] where an asymptotics for the Reissner-Mindlin plate was constructed and in [3] where error bounds between the 3D solution and $\boldsymbol{u}^0_{\mathrm{KL}}$ was investigated in the case of different lateral boundary conditions, we can suspect a strong influence of the nature of these lateral boundary conditions on the asymptotical behavior of $\boldsymbol{u}^\varepsilon$. In this paper we present 3 out of the 8 cases: we recall our results concerning the *hard clamped* plate ①, then describe the *soft clamped* plate ②, and the *hard simply supported* plate ③. The whole results will be proved in [9], where the 8 conditions are systematically investigated, including the delicate case of the free plate. We choose these 3 cases as samples: they are representative of different phenomena and are the least difficult to describe.

But what kind of influence can we expect? On what quantities? Mainly on the terms of order one in the asymptotics $\boldsymbol{w}^1$ and $\boldsymbol{u}^1_{\mathrm{KL}}$. We will see that $\boldsymbol{u}^1_{\mathrm{KL}}$ is closely linked to $\boldsymbol{w}^1$. Thus we expect a visible interaction between the nature of lateral boundary conditions and the boundary layer terms. We are going to describe them, and to exhibit their singular part along the edges of the plate, which concentrate the whole singular behavior of the displacements. Of course, if the mean surface is not smooth but polygonal, the structures have a further level of complexity. We will briefly mention this aspect of things.

1.d Overview of the results

We describe our results in the scaled plate $\Omega = \omega \times (-1,1)$ for the scaled displacement field $\boldsymbol{u}(\varepsilon) = (u_1^\varepsilon, u_2^\varepsilon, \varepsilon\, u_3^\varepsilon)$. It can be interesting to distinguish between two regions in the plate: the middle part M and the boundary layer part N neighboring the lateral boundary $\Gamma = \partial\omega \times (-1,1)$. These are arbitrarily chosen with the help of a parameter r_0 small enough so that in the region N defined by $\{x \in \Omega,\ \mathrm{dist}(x,\Gamma) < r_0\}$, the triple (r, s, x_3) is a smooth system of coordinates. Then M is defined as $\Omega \setminus \overline{N}$. In M we have the asymptotics

$$\boldsymbol{u}(\varepsilon) = \boldsymbol{u}^0_{\mathrm{KL}} + \varepsilon\, \boldsymbol{u}^1_{\mathrm{KL}} + \varepsilon^2 \Big(\boldsymbol{u}^2_{\mathrm{KL}} + \boldsymbol{v}^2 \Big) + \text{h.o.t.}$$

with $\boldsymbol{u}^k_{\mathrm{KL}}$ Kirchhoff-Love displacements and the estimates in any norm

$$\|\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^0_{\mathrm{KL}}\| = \quad (\varepsilon) \quad \text{and} \quad \|\boldsymbol{u}(\varepsilon) - (\boldsymbol{u}^0_{\mathrm{KL}} + \varepsilon\, \boldsymbol{u}^1_{\mathrm{KL}})\| = \quad (\varepsilon^2).$$

This holds for any of the three cases ①, ② of ③, but also for the other ones. The term $\boldsymbol{u}^1_{\mathrm{KL}}$ is a long range manifestation of the boundary layers (see the relations below in section 4).

In N , the results have a clearer structure if we consider the field $\boldsymbol{u}$ in the local system of coordinates: $u_n = \boldsymbol{u} \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is the inward unit normal along Γ , and u_s the in-plane tangential component of $\boldsymbol{u}$. Then, in the three cases ①, ② of ③, we have

$$\begin{aligned}
u_n(\varepsilon) &= u^0_{\mathrm{KL},n} + \varepsilon \Big(u^1_{\mathrm{KL},n} + w^1_t(\tfrac{r}{\varepsilon}, s, x_3) \Big) + \varepsilon^2 \Big(u^2_{\mathrm{KL},n} + v^2_n + w^2_t(\tfrac{r}{\varepsilon}, s, x_3) \Big) + \text{h.o.t.} \\
u_s(\varepsilon) &= u^0_{\mathrm{KL},s} + \varepsilon\, u^1_{\mathrm{KL},s} + \varepsilon^2 \Big(u^2_{\mathrm{KL},s} + v^2_s + w^2_s(\tfrac{r}{\varepsilon}, s, x_3) \Big) + \text{h.o.t.} \\
u_3(\varepsilon) &= u^0_{\mathrm{KL},3} + \varepsilon\, u^1_{\mathrm{KL},3} + \varepsilon^2 \Big(u^2_{\mathrm{KL},3} + v^2_3 + w^2_3(\tfrac{r}{\varepsilon}, s, x_3) \Big) + \text{h.o.t.}
\end{aligned}$$

In the clamped plate (hard and soft), the leading part of the boundary layer, $w^1_t(\frac{r}{\varepsilon}, s, x_3)$ is present and not regular (not in H^2) in a generic way, in contrast with the hard simply supported plate where w^1_t is regular, and even absent in any point where the curvature of $\partial\omega$ is zero.

For the free plate, the situation would be converse: $w^1_t = 0$ and the leading part of the boundary layer would be $w^1_s(\frac{r}{\varepsilon}, s, x_3)$. But it is more difficult to explain the construction of asymptotics in this case[1].

[1] The above asymptotics for clamped and free plate are being verified numerically with the help of the code STRESSCHECK. Work in progress with Z. Yosibash.

2 EQUATIONS OF THE DISPLACEMENT

To simplify, we limit ourselves to the case when the rigidity matrix $A = (A_{ijkl})$ is that of an isotropic material, with Lamé coefficients λ and μ, namely:

$$A_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu\, (\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}).$$

Then, $\boldsymbol{u}^\varepsilon$ solves a boundary value problem whose variational formulation involves a space $V(\Omega^\varepsilon)$ of admissible displacements $\boldsymbol{v}^\varepsilon$ that undergoes the *stable* boundary conditions associated to each boundary condition ⓘ. Then, the variational problem reads: Find

$$\boldsymbol{u}^\varepsilon \in V(\Omega^\varepsilon) \ \ \textit{such that} \ \ \forall\, \boldsymbol{v}^\varepsilon \in V(\Omega^\varepsilon), \ \ \int_{\Omega^\varepsilon} A\, e(\boldsymbol{u}^\varepsilon) : e(\boldsymbol{v}^\varepsilon) = \int_{\Omega^\varepsilon} \boldsymbol{f}^\varepsilon \cdot \boldsymbol{v}^\varepsilon, \tag{2.1}$$

where e is the strain tensor and $\boldsymbol{f}^\varepsilon$ represents volume forces.

Let us fix the following notations: for a point in Ω^ε, $x_* = (x_1, x_2)$ are its horizontal coordinates, living in ω, and z is its vertical coordinate in $(-\varepsilon, +\varepsilon)$. Similarly, $\boldsymbol{v}_*$ denotes the two horizontal components (v_1, v_2) of a field $\boldsymbol{v}$.

We assume that all forces $\boldsymbol{f}^\varepsilon$ are derived from the same smooth "vertical profile" $\boldsymbol{f} \in {}^{\infty}(\bar{\Omega})^3$:

$$\boldsymbol{f}^\varepsilon_*(x_*, z) = \boldsymbol{f}_*(x_*, \frac{z}{\varepsilon}), \qquad f^\varepsilon_3(x_*, z) = \varepsilon\, f_3(x_*, \frac{z}{\varepsilon}). \tag{2.2}$$

This means that, on the fixed reference configuration $\Omega := \omega \times (-1, +1)$ satisfying

$$x^\varepsilon := (x_*, z) \in \Omega^\varepsilon \quad \Longleftrightarrow \quad x = (x_*, x_3) := (x_*, \frac{z}{\varepsilon}) \in \Omega, \tag{2.3}$$

the volume forces have the form $(f_1, f_2, \varepsilon f_3)$.

It is then natural to try to compare the displacements $\boldsymbol{u}^\varepsilon$ with each other on the scaled configuration Ω. The new unknown is the scaled displacement $\boldsymbol{u}(\varepsilon)$ whose components are defined for purpose of homogeneity by:

$$\boldsymbol{u}_*(\varepsilon)(x) = \boldsymbol{u}^\varepsilon_*(x^\varepsilon), \qquad u_3(\varepsilon)(x) = \varepsilon\, u^\varepsilon_3(x^\varepsilon).$$

Inserting this change of functions into the primitive boundary value problem (2.1) set on Ω^ε, we arrive at a new scaled boundary value problem set on Ω that reads: Find

$$\boldsymbol{u}(\varepsilon) \in V(\Omega) \ \ \textit{such that} \ \ \forall \boldsymbol{v} \in V(\Omega), \ \ \int_{\Omega} A\, \kappa(\varepsilon)\boldsymbol{u}(\varepsilon) : \kappa(\varepsilon)\boldsymbol{v} = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v} \tag{2.4}$$

where we have introduced the scaled strain tensor denoted by $\kappa(\varepsilon)(\boldsymbol{v})$ for any function $\boldsymbol{v} \in H^1(\Omega)^3$ and defined componentwise by (with α, $\beta \in \{1,2\}$):

$$\kappa_{\alpha\beta}(\varepsilon)(\boldsymbol{v}) = e_{\alpha\beta}(\boldsymbol{v}), \quad \kappa_{\alpha 3}(\varepsilon)(\boldsymbol{v}) = \varepsilon^{-1}\, e_{\alpha 3}(\boldsymbol{v}), \quad \kappa_{33}(\varepsilon)(\boldsymbol{v}) = \varepsilon^{-2}\, e_{33}(\boldsymbol{v}).$$

The variational space $V(\Omega)$ is made of functions $\boldsymbol{v} \in H^1(\Omega)^3$ with stable boundary conditions ①, ② or ③ on $\Gamma = \partial\omega \times (-1,+1)$, namely:

$$\left.\begin{array}{ll} ① & \boldsymbol{v} = 0 \\ ② & \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad v_3 = 0 \\ ③ & \boldsymbol{v}\times\boldsymbol{n} = 0 \quad \text{and} \quad v_3 = 0 \end{array}\right\} \quad \text{with } \boldsymbol{n} \text{ the inner unit normal to } \Gamma\,.$$

3 INNER – OUTER EXPANSION ANSATZ

Following [11] (*cf* also [10] for a short account) where this is done in case ①, we make a proposition of asymptotics for the scaled displacement $\boldsymbol{u}(\varepsilon)$ in the form a power series — the *outer* Ansatz:

$$\underline{\boldsymbol{u}}(\varepsilon)(x) \simeq \underline{\boldsymbol{u}}^0(x) + \varepsilon\,\underline{\boldsymbol{u}}^1(x) + \varepsilon^2\,\underline{\boldsymbol{u}}^2(x) + \cdots + \varepsilon^k\,\underline{\boldsymbol{u}}^k(x) + \cdots \tag{3.1}$$

where the $\underline{\boldsymbol{u}}^k$ are independent of ε, <u>corrected</u> by a boundary layer expansion — the *inner* Ansatz. These inner and outer expansions are familiar notions in the theory of matching asymptotics [17], where the idea is somewhat different: it consists of trying to describe the asymptotics either in primitive variables, or in boundary layer variables in different zones and to match both in an intermediate zone. Here we search for a combined expansion which is valid everywhere. More precisely, we find that the ingredients of a correct Ansatz are the following.

- Kirchhoff-Love displacements $\boldsymbol{u}^k_{\mathrm{KL}}$ with "generating function" $\boldsymbol{\zeta}^k = (\boldsymbol{\zeta}^k_*, \zeta^k_3)$, namely:
$$\boldsymbol{u}^k_{\mathrm{KL},*} = \boldsymbol{\zeta}^k_*(x_*) - x_3\,\nabla_*\zeta^k_3(x_*), \qquad u^k_{\mathrm{KL},3} = \zeta^k_3(x_*).$$
It is well known that the limit of $\boldsymbol{u}(\varepsilon)$ is a Kirchhoff-Love displacement. Indeed we find that such a displacement appears at each level of the asymptotics, so that the sum (3.4) below solves the whole problem (2.4).
- Displacements with mean values zero in each vertical fiber
$$\int_{-1}^{+1} \boldsymbol{v}^k(x_*, x_3)\,dx_3 = 0, \quad \forall x_* \in \omega. \tag{3.2}$$
Added to the previous Kirchhoff-Love displacements they constitute the outer expansion part of the Ansatz (3.1). They actually solve a Neumann problem on the interval $(-1,+1)$ very similar to the Neumann problem for

the operator ∂_3^2 in the x_3 vertical variable. Therefore, their existence classically results from a compatibility condition obtained after an integration on $(-1,+1)$ which yields in particular the membrane and bending equations (4.2) and (4.3) below, whereas (3.2) insures uniqueness by fixing a remaining constant.

- Boundary layer terms

$$\boldsymbol{w}^k = \boldsymbol{w}^k(\varepsilon^{-1} r, s, x_3) \quad \text{with} \quad \begin{cases} r \text{ the distance to } \partial\omega, \\ s \text{ the arc length in } \partial\omega. \end{cases} \tag{3.3}$$

They compensate for discrepancies in imposed lateral boundary conditions ①, ② and ③ respectively: indeed, as the scaled lateral boundary Γ is characteristic for the problem solved by each $\underline{\boldsymbol{u}}^k$ — in similarity to the operator ∂_3^2, the power series (3.1) is only able to solve the interior (volume) equations inside Ω and the horizontal boundary conditions, but not the lateral boundary conditions in general. The boundary layer terms describe phenomena "rapidly" varying and decreasing near Γ, and their introduction allows for a complete resolution. They constitute the inner expansion part of the Ansatz. For every k, $\boldsymbol{w}^k(t, s, x_3)$ is *exponentially decreasing* as $t \to +\infty$. With χ denoting a cut-off function equal to 1 in a neighborhood of $\partial\omega$, we consider the localized function $\chi(r)\, \boldsymbol{w}^k(\varepsilon^{-1} r, s, x_3)$.

Collecting all these features, we get the following expansion

$$\begin{aligned} \boldsymbol{u}(\varepsilon) \simeq \boldsymbol{u}^0_{\mathrm{KL}} + \varepsilon\, \boldsymbol{u}^1_{\mathrm{KL}} + \varepsilon\, \chi(r)\, (\boldsymbol{w}^1_*(\frac{r}{\varepsilon}, s, x_3), 0) \\ + \sum_{k\geq 2} \varepsilon^k \Big(\boldsymbol{u}^k_{\mathrm{KL}} + \boldsymbol{v}^k + \chi(r)\, \boldsymbol{w}^k(\frac{r}{\varepsilon}, s, x_3) \Big). \end{aligned} \tag{3.4}$$

Notice that the leading boundary layer term has a zero vertical component.

4 FIRST TERMS IN THE ASYMPTOTICS

4.a Terms of order zero

We proceed to a description of the leading terms of the Ansatz (3.4). As already mentioned, the leading term is the Kirchhoff-Love displacement of order zero:

$$\boldsymbol{u}^0_{\mathrm{KL}} = (\boldsymbol{\zeta}^0_* - x_3\, \nabla_* \zeta^0_3\,,\, \zeta^0_3). \tag{4.1}$$

In (4.1), the two-dimensional generator $\boldsymbol{\zeta}^0 = (\boldsymbol{\zeta}^0_*, \zeta^0_3)$ is defined on the mean surface ω of the plates Ω^ε. The in-plane generator $\boldsymbol{\zeta}^0_*$ solves a membrane equation in ω:

$$\mu\, \boldsymbol{\Delta}_* \boldsymbol{\zeta}_* + (\tilde\lambda + \mu) \nabla_* \operatorname{div}_* \boldsymbol{\zeta}_* = \boldsymbol{R}^0_{\mathrm{m}}, \tag{4.2}$$

with the “homogenized” Lamé coefficient $\tilde\lambda = 2\lambda\mu(\lambda+2\mu)^{-1}$ and with the right hand side $\boldsymbol{R}^0_{\rm m}$ defined as

$$\boldsymbol{R}^0_{\rm m}(x_*) = -\frac{1}{2}\int_{-1}^{+1} \boldsymbol{f}_*(x)\,dx_3.$$

Its vertical generator ζ^0_3 solves the bending equation

$$(\tilde\lambda + 2\mu)\Delta_*^2\zeta_3 = R^0_{\rm b}, \tag{4.3}$$

with the right hand side $R^0_{\rm b}$ defined as

$$R^0_{\rm b}(x_*) = \frac{3}{2}\int_{-1}^{+1} (f_3 + x_3\ \mathrm{div}_*\,\boldsymbol{f}_*)\,dx_3.$$

Here Δ_* is the in-plane Laplacian $\partial_{11}+\partial_{22}$ and $\boldsymbol{\Delta}_*$ is the bloc diagonal in-plane Laplacian $\Delta_*\mathbf{I}_2$. For smooth $\boldsymbol{f}$, the associated right hand sides are smooth as they actually depend on mean values of $\boldsymbol{f}$. Both membrane and bending problems are completed with the following zero boundary conditions on $\partial\omega$

$$\begin{array}{rrrrr}
① & \zeta^0_n = 0, & \zeta^0_s = 0, & \zeta^0_3 = 0, & \partial_n\zeta^0_3 = 0,\\
② & \zeta^0_n = 0, & T_s(\boldsymbol{\zeta}^0_*) = 0, & \zeta^0_3 = 0, & \partial_n\zeta^0_3 = 0,\\
③ & T_n(\boldsymbol{\zeta}^0_*) = 0, & \boldsymbol{\zeta}^0_s = 0, & \zeta^0_3 = 0, & M_n(\zeta^0_3) = 0,
\end{array} \tag{4.4}$$

with ζ_n and ζ_s the normal and tangential components of $\boldsymbol{\zeta}_*$, T_n and T_s the normal and tangential tractions associated to the membrane equation on ω and M_n the natural operator of order 2 associated to the bending equation. Since the boundary of ω is assumed to be smooth, as a consequence of classical results of elliptic regularity $\boldsymbol{u}^0_{\rm KL}$ is $\ ^\infty(\overline{\omega})$.

Whereas all the previous results concerning $\boldsymbol{u}^0_{\rm KL}$ are rather classical and can be obtained by different strategies (not only our construction of asymptotics, but also by a suitable choice of test functions in equation (2.4)), the situation of the further terms in the asymptotics is, to our knowledge, less well known.

Anyway, concerning the estimates of the error $\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^0_{\rm KL}$, we can prove that it behaves like $(\varepsilon^{1/2})$ in the $H^1(\Omega)$ norm and like (ε) in the $L^2(\Omega)$ norm.

4.b Traces of the power series Ansatz

As was indicated in the previous section, the part $\boldsymbol{v}^k$ satisfying (3.2) of the displacements $\underline{\boldsymbol{u}}^k$ are determined by the solution of a (generalized) Neumann problem. We obtain that $\boldsymbol{v}^0$, $\boldsymbol{v}^1 = 0$ and the following formulas for $\boldsymbol{v}^2$

$$\begin{array}{rcl}
v^2_\alpha &=& \bar p_2\,\partial_\alpha\,\mathrm{div}_*\,\boldsymbol{\zeta}^0_* + \bar p_3\,\partial_\alpha\Delta_*\zeta^0_3 - \frac{1}{\mu}\oint^{x_3}\left(\fint^{y_3} f_\alpha\right)dy_3\\
v^2_3 &=& \bar p_1\ \mathrm{div}_*\,\boldsymbol{\zeta}^0_* + \bar p_2\,\Delta_*\zeta^0_3
\end{array} \tag{4.5}$$

where $\oint^{x_3}$ denotes the primitive with zero mean value on $(-1,+1)$ and

$$\oint^{y_3} \cdot \, dz_3 \; := \; \frac{1}{2}\Big(\int_{-1}^{y_3} \cdot \; dz_3 - \int_{y_3}^{+1} \cdot \; dz_3\Big),$$

whereas $\bar{p}_j$ for $j = 1,2,3$ are the following polynomials of degree j :

$$\begin{aligned} \bar{p}_1(x_3) &= -\frac{\tilde{\lambda}}{2\mu}\, x_3, \qquad \bar{p}_2(x_3) = \frac{\tilde{\lambda}}{4\mu}\,\Big(x_3^2 - \frac{1}{3}\Big), \\ \bar{p}_3(x_3) &= \frac{1}{12\mu}\,\Big((\tilde{\lambda}+4\mu)\, x_3^3 - (5\tilde{\lambda}+12\mu)\, x_3\Big). \end{aligned} \tag{4.6}$$

The absence or presence of a first boundary layer term $\boldsymbol{w}^1$ depends only on whether or not v_3^2 satisfies the lateral boundary conditions. For reasons of homogeneity, the in-plane components of $\boldsymbol{v}^2$ only influence higher order terms.

In all three situations ①, ② and ③ the vertical component of the displacement $\boldsymbol{u}(\varepsilon)$ vanishes on Γ. Thus, we have to check whether $v_3^2 = 0$ on Γ, i.e. whether

$$\operatorname{div}_* \boldsymbol{\zeta}_*^0 = 0 \quad \text{on} \quad \partial\omega \qquad \text{and} \qquad \Delta_* \zeta_3^0 = 0 \quad \text{on} \quad \partial\omega. \tag{4.7}$$

We can see that for the clamped plates ① and ②, the boundary conditions satisfied by $\boldsymbol{u}_{\mathrm{KL}}^0$ do not imply (4.7) in general, in contrast to the case of the hard simply supported plates ③ for which interesting relations hold

$$T_n(\boldsymbol{\zeta}_*^0) \;=\; (\tilde{\lambda}+2\mu)\operatorname{div}_* \boldsymbol{\zeta}_*^0 + \frac{2\mu}{R}\,\zeta_n^0 \tag{4.8}$$

$$M_n(\zeta_3^0) \;=\; (\tilde{\lambda}+2\mu)\Delta_* \boldsymbol{\zeta}_3^0 + \frac{2\mu}{R}\,\partial_n \zeta_3^0 \tag{4.9}$$

with $R = R(s)$ the curvature radius of $\partial\omega$ at s. This means that the boundary conditions of v_3^2 are a mere consequence of those of $\boldsymbol{u}_{\mathrm{KL}}^0$ in any flat part of the boundary.

This fact is not so mysterious: in this hard simply supported plate, all the tangential components of the displacement are set to 0, whereas the normal component of the traction is zero. Therefore, in the neighborhood $\mathcal{V}$ of any flat part of the lateral boundary Γ, we can extend the displacement across Ω by reflection: the odd reflection of the tangential components and the even reflection of the normal component extend $\boldsymbol{u}(\varepsilon)$ to a new displacement $\tilde{\boldsymbol{u}}(\varepsilon)$ associated to zero horizontal boundary conditions and to extended volume forces $\widetilde{\boldsymbol{f}}$, which are at least in L^2. Thus the extended displacement satisfies estimates as if $\mathcal{V}$ is inside the plate, i.e. far from the boundary layer.

If the support of the volume forces avoids the lateral boundary Γ in the neighborhood $\mathcal{V}$, then there are *no boundary layer terms at all* in $\mathcal{V}$: $\boldsymbol{u}(\varepsilon)$ can be expanded in a power series inside $\mathcal{V}$. The situation is exactly the same in the case of the "sliding edge condition" where the normal component of the displacement and the tangential components of the traction are prescribed to be 0. The relation between this sliding edge condition on a rectangular plate and the periodic conditions is emphasized in [22].

4.c Terms of order one

The trace of v_3^2 being irrecoverable by a power series Ansatz (except in the particular case above), we introduce a first profile $\boldsymbol{\varphi}^1$ which will define the in-plane components w_1^1 and w_2^1, and the first non vanishing transverse component w_3^2 in accordance with the new scaling

$$(w_1^1, w_2^1, w_3^2) =: \boldsymbol{\varphi}^1 \equiv (\varphi_t^1, \varphi_s^1, \varphi_3^1) \quad \text{in cylindrical components.} \tag{4.10}$$

With r denoting the distance to $\partial\omega$ and s an arc-length coordinate in $\partial\omega$, we set $t = r\,\varepsilon^{-1}$. Then, the first profile $\boldsymbol{\varphi}^1$ satisfies

$$_{(0)}\boldsymbol{\varphi}^1 = 0 \tag{4.11}$$

where the left hand side $\;_{(0)}(\varphi_t, \varphi_s, \varphi_3)$ describes an uncoupled elasticity - Laplacian system in the *two variables t and x_3*:

$$\mu(\partial_{tt} + \partial_{33}) \begin{pmatrix} \varphi_t \\ \varphi_3 \end{pmatrix} + (\lambda + \mu) \begin{pmatrix} \partial_t \\ \partial_3 \end{pmatrix} (\partial_t\varphi_t + \partial_3\varphi_3) =: \boldsymbol{E}(\boldsymbol{\varphi}_\sharp) \tag{4.12}$$

$$\mu(\partial_{tt} + \partial_{33})\, \varphi_s. \tag{4.13}$$

Notice that operators (4.12) and (4.13) have no tangential derivative ∂_s, so that although $\boldsymbol{\varphi}^1$ a priori depends on all variables t, s, x_3, we may consider that *s is a mere parameter*. Thus, (4.12) (resp. (4.13)) is the right hand side of a two-dimensional elasticity problem (resp. Laplacian problem) solved by $\boldsymbol{\varphi}_\sharp^1 := (\varphi_t^1, \varphi_3^1)$ (resp. φ_s^1) on the half-strip

$$\Sigma_+ = \Big\{(t, x_3) \in (0, +\infty) \times (-1, +1)\Big\}.$$

Interior equations (4.12)-(4.13) are completed with the zero Neumann condition on the horizontal sides $x_3 = \pm 1$, and conditions ⓘ for the whole Ansatz on $t = 0$. Equations on the components $\boldsymbol{\varphi}_\sharp$ and φ_s are uncoupled. The boundary condition that we want to compensate for is the only Dirichlet trace of v_3^2 at this stage. This involves $\boldsymbol{\varphi}_\sharp^1$ only and we can prove:

$$\text{In cases ①, ② and ③,} \quad \varphi_s^1 = 0. \tag{4.14}$$

Concerning $\boldsymbol{\varphi}_\sharp^1$, one should have $\varphi_3^1 + v_3^2 = 0$ on $\partial\omega$. But the solution of (4.12) with this Dirichlet condition (and the remaining zero condition ⓘ) does not provide exponentially decreasing solutions. One needs a correction on the boundary to compensate for the (polynomial) behavior as $t \to +\infty$ of such solutions. This generates the Kirchhoff-Love displacement $\boldsymbol{u}_{\mathrm{KL}}^1$ of order one, namely:

$$\boldsymbol{u}_{\mathrm{KL}}^1 = (\boldsymbol{\zeta}_*^1 - x_3\,\nabla_*\zeta_3^1\,,\,\zeta_3^1) \tag{4.15}$$

where the generators $\boldsymbol{\zeta}_*^1$ and ζ_3^1 solve membrane and bending equations respectively *cf* (4.2), (4.3) with interior data 0 , and non-zero boundary conditions.

In case ① these boundary conditions read:

$$\zeta_n^1 = c_1^{①}\ \mathrm{div}_*\,\boldsymbol{\zeta}_*^0, \qquad \zeta_s^1 = 0, \qquad \zeta_3^1 = 0, \qquad \partial_n\zeta_3^1 = c_4^{①}\,\Delta_*\zeta_3^0. \tag{4.16}$$

The condition $\zeta_s^1 = 0$ in linked to the fact that $\varphi_s^1 = 0$. The quantities $c_1^{①}$ and $c_4^{①}$ are universal coefficients depending on the Lamé coefficients (λ, μ) , which appear as coupling constants between the behaviors in $t = 0$ and $t \to \infty$ of certain model profiles $\boldsymbol{\psi}_{\sharp,1}$ and $\boldsymbol{\psi}_{\sharp,2}$ generating the boundary layer terms: recalling that $v_3^2 = \bar p_1\,\mathrm{div}_*\,\boldsymbol{\zeta}_*^0 + \bar p_2\,\Delta_*\zeta_3^0$, these model profiles are given by

$$\left\{\begin{array}{rcll} \boldsymbol{E}(\boldsymbol{\psi}_{\sharp,j}) & = & 0 & \text{in } \Sigma_+ \\ \boldsymbol{T}(\boldsymbol{\psi}_{\sharp,j}) & = & 0 & \text{on } x_3 = \pm 1 \\ \psi_{t,j} & = & 0 & \text{on } t = 0 \\ \psi_{3,j} & = & -\bar p_j & \text{on } t = 0, \end{array}\right\} \qquad j = 1,\ 2,$$

and the coupling constants are such that

$$\boldsymbol{\psi}_{\sharp,1} = \boldsymbol{\varphi}_{\sharp,1}^{①} + c_1^{①}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \text{with } \boldsymbol{\varphi}_{\sharp,1}^{①} \text{ exp. decreasing as } t \to \infty\,,$$

respectively

$$\boldsymbol{\psi}_{\sharp,2} = \boldsymbol{\varphi}_{\sharp,2}^{①} + c_3^{①}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_4^{①}\begin{pmatrix} -x_3 \\ t \end{pmatrix}, \qquad \text{with } \boldsymbol{\varphi}_{\sharp,2}^{①} \text{ exp. decreasing as } t \to \infty\,.$$

Since it can be proved [9] that

$$\forall\lambda, \mu > 0 \qquad c_4^{①}(\lambda, \mu) \neq 0,$$

the term $\boldsymbol{u}_{\mathrm{KL}}^1$ is present in general. Finally the first profile in the boundary layer terms is $\boldsymbol{\varphi}^1 = (\varphi_t^1, \varphi_t^1, \varphi_3^1)$ with $\varphi_s^1 = 0$ and φ_t^1 , φ_3^1 defined as

$$\begin{pmatrix} \varphi_t^1 \\ \varphi_3^1 \end{pmatrix} = \mathrm{div}_*\,\boldsymbol{\zeta}_*^0(s)\ \boldsymbol{\varphi}_{\sharp,1}^{①}(t, x_3) + \Delta_*\zeta_3^0(s)\ \boldsymbol{\varphi}_{\sharp,2}^{①}(t, x_3). \tag{4.17}$$

Similar boundary conditions on $\partial\omega$ hold for the generator $\boldsymbol{\zeta}^1$ of $\boldsymbol{u}^1_{\mathrm{KL}}$ in case ②, with one difference concentrating on the tangential condition, namely:

$$\zeta^1_n = c^{\textcircled{2}}_1 \operatorname{div}_* \boldsymbol{\zeta}^0_*, \quad T_s(\boldsymbol{\zeta}^1_*) = c^{\textcircled{2}}_2\, \partial_s \operatorname{div}_* \boldsymbol{\zeta}^0_*, \quad \zeta^1_3 = 0, \quad \partial_n\zeta^1_3 = c^{\textcircled{2}}_4\, \Delta_*\zeta^0_3 \tag{4.18}$$

where $c^{\textcircled{2}}_1$ and $c^{\textcircled{2}}_4$ are the same coefficients as in case ① and $c^{\textcircled{2}}_2$ is another universal coefficient. The formula giving $\boldsymbol{\varphi}^1$ is similar to (4.17).

As already mentioned, the situation of the hard simple support ③ is different: owing to relations (4.8)-(4.9) we obtain as boundary conditions on $\partial\omega$ for the generator $\boldsymbol{\zeta}^1$ of $\boldsymbol{u}^1_{\mathrm{KL}}$:

$$T_n(\boldsymbol{\zeta}^1) = c^{\textcircled{3}}_1\, \frac{\zeta^0_n}{R^2}\,, \qquad \zeta^1_s = 0, \qquad \zeta^1_3 = 0, \qquad M_n(\zeta^1_3) = c^{\textcircled{3}}_4\, \frac{\partial_n\zeta^0_3}{R^2}\,, \tag{4.19}$$

and concerning the first boundary layer term: $\boldsymbol{\varphi}^1 = (\varphi^1_t, 0, \varphi^1_3)$ with $\varphi^1_s = 0$ and φ^1_t, φ^1_3 defined as

$$\begin{pmatrix}\varphi^1_t\\ \varphi^1_3\end{pmatrix} = \frac{\zeta^0_n(s)}{R(s)}\, \boldsymbol{\varphi}^{\textcircled{3}}_{\sharp,1}(t,x_3) + \frac{\partial_n\zeta^0_3(s)}{R(s)}\, \boldsymbol{\varphi}^{\textcircled{3}}_{\sharp,2}(t,x_3). \tag{4.20}$$

So, we see that the first boundary layer term is *absent* at any point of the lateral boundary with zero curvature.

Estimates concerning the terms of order one can be written as

$$\|\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^0_{\mathrm{KL}} - \varepsilon\boldsymbol{u}^1_{\mathrm{KL}}\|_{L^2(\Omega)} = \quad (\varepsilon^{3/2})$$

and

$$\|\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^0_{\mathrm{KL}} - \varepsilon\boldsymbol{u}^1_{\mathrm{KL}} - \varepsilon\left(\varphi^1_t(\frac{r}{\varepsilon}, s, x_3), 0, 0\right)\|_{H^1(\Omega)} = \quad (\varepsilon^{3/2}).$$

4.d Terms of order two

We conclude this section by a few words about terms of order 2. Besides the part of $\boldsymbol{v}^2_*$ which depends on $\boldsymbol{f}$, see (4.5), we see that the polynomial part of order two has the degrees $(3,3,2)$. Thus these degrees are minimal ones for a hierarchical model if we want to reach the order 2.

On the other hand, in any of the three cases ①, ② and ③, the only contribution to the part of order one in the boundary layers is the normal component $w^1_t = \varphi^1_t$. Both other ones begin at the order two: $w^1_3 = 0$ for reasons of homogeneity and $w^1_s = 0$ because of (4.14). We already gave formulas for $w^2_3 = \varphi^1_3$. Concerning $w^2_s = \varphi^2_s$, we can check that in the clamped cases, it has the form

$$\varphi^2_s = \partial_s \operatorname{div}_* \boldsymbol{\zeta}^0_*(s)\, \varphi^{\textcircled{i}}_{5,s} + \partial_s\Delta_*\zeta^0_3(s)\, \varphi^{\textcircled{i}}_{6,s}, \qquad \mathbf{i} = 1,\ 2,$$

where the $\varphi^{\textcircled{i}}_{j,s}$ are model profiles for the Laplacian, just like the $\varphi^{\textcircled{i}}_{j,\sharp}$ in (4.17) and (4.20) are model profiles for the elasticity $\boldsymbol{E}$.

5 SINGULARITIES ALONG THE EDGES OF THE PLATE

The plates Ω^ε have two *edges* $\gamma^\varepsilon_\pm$ defined as the intersection between the lateral face Γ^ε and the horizontal faces of the plate. They are transformed through the scaling (2.3) into the edges $\gamma_\pm = \partial\omega \times (-1, +1)$ of the scaled plate Ω where both lateral and horizontal boundary conditions meet: due to this change in type of boundary condition (in each of the cases ①, ② and ③), the scaled displacement $\boldsymbol{u}(\varepsilon)$ undergoes a change that alters the regularity, even if the right hand side $\boldsymbol{f}$ is smooth up to the boundary.

Notice that edges correspond to the corners $(0, \pm 1)$ of the half-strip Σ_+ as well, so that they alter the regularity of the profiles too. Finally, the regularity and the asymptotics of $\boldsymbol{u}(\varepsilon)$ in the neighborhood of the edges are governed by the singular exponents of the "reduced-normal problems" (4.12)-(4.13) on the half-strip Σ_+. It is worth noticing for the subsequent analysis that the opening angle of the domain Ω (resp. Σ_+) all along the edges is constant and equal to $\pi/2$. We illustrate all these features with cases ①, ② and ③.

5.a Clamped plates

Notice that the elasticity problem in both cases ① and ② is actually the same.

In the graph besides we plot

- On the lower curve: The first singular exponent (vertical axis) of the elasticity operator (4.12) with Neumann conditions on $x_3 = \pm 1$ and Dirichlet conditions on $t = 0$ versus the quantity $\log_{10}(\mu/\lambda)$ (horizontal axis)
- On the upper curve: The (real part) of the 2^{d} exponent of the same problem,
- The right line at 1 corresponds to the (first) singular exponent of the mixed Dirichlet-Neumann problem for the Laplace operator (4.13).

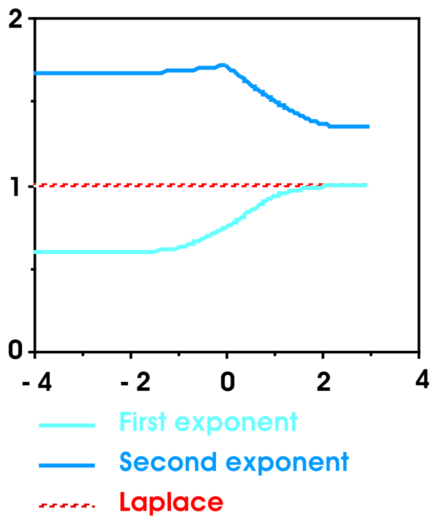

Figure 2. Singular exponents at the edges of the plate

We recall (see [18] for the general theory) that the singular exponents are the real (or complex) numbers ν such that the homogeneous problems (4.12) and (4.13) with the boundary conditions quoted above, have solutions of the form $\rho_{\pm}^{\nu}\phi(\theta_{\pm})$ on the quarter plane, with $(\rho_{\pm},\theta_{\pm})$ polar coordinates centered in $(0,\pm 1)$.

Note that the first exponent is less than 1, thus the displacements $\boldsymbol{u}(\varepsilon)$ have H^1 but not H^2 regularity. The second exponent being >1, we only have to subtract the first singular function in the expansion to attain H^2 regularity: in local coordinates (r,s,x_3) mentioned above

$$\boldsymbol{u}(\varepsilon)=\boldsymbol{u}_{\mathrm{reg}}(\varepsilon)+\sum_{+,-} c^{\pm}(\varepsilon)(s)\,\chi^{\pm}(\frac{r}{\varepsilon},x_3)\,\boldsymbol{\mathcal{S}}^{\pm}(\frac{r}{\varepsilon},x_3) \tag{5.1}$$

where $\boldsymbol{u}_{\mathrm{reg}}(\varepsilon)$ belongs to $H^2(\Omega)$, where $\chi^{\pm}(t,x_3)$ denotes a cut-off function equal to 1 in a neighborhood of $\partial\omega\times\{\pm 1\}$ and where the coefficients $c^{\pm}(\varepsilon)(s)$ have the regularity of $\ ^{\infty}(\partial\omega)$. The functions $\boldsymbol{\mathcal{S}}^{\pm}$ are singular solutions of the reduced-normal problems (4.12)-(4.13). This way (5.1) of writing the splitting of $\boldsymbol{u}(\varepsilon)$ is stable with respect to ε.

In the expansion (3.4) of $\boldsymbol{u}(\varepsilon)$, the non-regular terms concentrate on the boundary layer expansion $\sum_{k\geq 1}\varepsilon^k\,\chi(r)\,\boldsymbol{w}^k(\frac{r}{\varepsilon},s,x_3)$ as soon as the right hand side is regular up to the boundary. Indeed, the regularity result for $\boldsymbol{u}^0_{\mathrm{KL}}$ and $\boldsymbol{u}^1_{\mathrm{KL}}$ above mentioned in relation to the smoothness of the boundary $\partial\omega$ generalizes to Kirchhoff-Love displacements $\boldsymbol{u}^k_{\mathrm{KL}}$ which are generated by solutions of elliptic boundary value problems of membrane and bending types on ω. Moreover, recall that the functions $\boldsymbol{v}^k$ solve one-dimensional Neumann problems on the interval $(-1,+1)$. Besides, the boundary layer term $\boldsymbol{w}^k$ inherits the singular behavior of the profiles $\boldsymbol{\varphi}^k$ such that

$$(\boldsymbol{w}^k_1,\boldsymbol{w}^k_2,\boldsymbol{w}^{k+1}_3)=:\boldsymbol{\varphi}^k\equiv(\varphi^k_t,\varphi^k_s,\varphi^k_3)\quad\text{in cylindrical components.}$$

In general,

$$\boldsymbol{w}^k(t,s,x_3)=\boldsymbol{w}^k_{\mathrm{reg}}(t,s,x_3)+\sum_{+,-} c^{\pm,k}(s)\,\chi^{\pm}(t,x_3)\,\boldsymbol{\mathcal{S}}^{\pm}(t,x_3) \tag{5.2}$$

with $\boldsymbol{w}^k_{\mathrm{reg}}\in H^2(\Omega)$.

Results of [11, 12] apply to the regular part of $\boldsymbol{u}(\varepsilon)$ in the splitting (5.1), yielding the following asymptotics:

$$\|\boldsymbol{u}_{\mathrm{reg}}(\varepsilon)-\boldsymbol{u}^0_{\mathrm{KL}}-\varepsilon\,\chi\,\boldsymbol{w}^1_{\mathrm{reg}}(\frac{r}{\varepsilon},s,x_3)\|_{H^2(\Omega)}\leq C\,\varepsilon^{1/2}. \tag{5.3}$$

Note that the H^2 norm of $\boldsymbol{u}_{\mathrm{reg}}(\varepsilon)-\boldsymbol{u}^0_{\mathrm{KL}}$ does not tend to 0 with ε in general: taking account of the regular part of the first profile is necessary to obtain an estimate.

Asymptotics of the coefficients arising in (5.1) are linked to the coefficients appearing in (5.2):

$$c^{\pm}(\varepsilon) = \sum_{1 \le k \le N} \varepsilon^k \, c^{\pm,k} + \quad (\varepsilon^{N+1}).$$

Remark 5.1 For (monoclinic) anisotropic materials, we may rely on [6] and construct special linear combinations of the singularities $\mathcal{S}^{\pm}$ so that *Stable Singular Functions* are available. ■

5.b Hard simple support

The first singular exponent at the edges is 1. But since the boundary conditions are homogeneous, the reflection principle allows for proving that the coefficient of this singularity is zero. Thus the displacements $\boldsymbol{u}(\varepsilon)$ belong to $H^2(\Omega)$, and the profiles $\boldsymbol{w}^k$ too.

But as already said, we do not have any convergent estimate for $\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^0_{\mathrm{KL}}$ in $H^2(\Omega)$ norm, but similarly to (5.3)

$$\|\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^0_{\mathrm{KL}} - \varepsilon\,\chi\,\boldsymbol{w}^1(\frac{r}{\varepsilon}, s, x_3)\|_{H^2(\Omega)} \le C\,\varepsilon^{1/2}. \tag{5.4}$$

6 AND IF THE MEAN SURFACE HAS CORNERS...

We present now some facts when the mean surface ω is no longer smooth but polygonal. Then the plates Ω^ε are polyhedral and the displacements admit combined edges and corner singularities [8].

On the other hand, the limit Kirchhoff-Love displacement $\boldsymbol{u}^0_{\mathrm{KL}}$ has a singular behavior near the corners of ω. More precisely, its regularity is limited by the exponents of singularity of the membrane and bending operators with the boundary conditions corresponding to situations (i). These exponents at a vertex $\mathcal{O}$ of ω are the complex numbers ν such that the completely homogeneous boundary value problem under consideration admits solutions of the form $\rho^\nu \phi(\theta)$ on the sector tangent to ω at $\mathcal{O}$, with (ρ, θ) a system of polar coordinates defined in a neighborhood of the vertex.

Note that for an elastic material with Lamé constants $\tilde{\lambda}$ and μ, the singular exponents depend only on the ratio $\mu/\tilde{\lambda}$. Figure 3 displays this dependence for Dirichlet conditions (actually, $\log_{10}(\mu/\tilde{\lambda})$ is represented in the horizontal axis) when the opening of the angle is either $\pi/2$ or $3\pi/2$. Notice that since λ, $\mu > 0$, then $\frac{\mu}{\tilde{\lambda}} > \frac{1}{2}$ is the only physically admissible case, that is why we refer to the case $\frac{\mu}{\tilde{\lambda}} = \frac{1}{2}$ as a "physical limit".

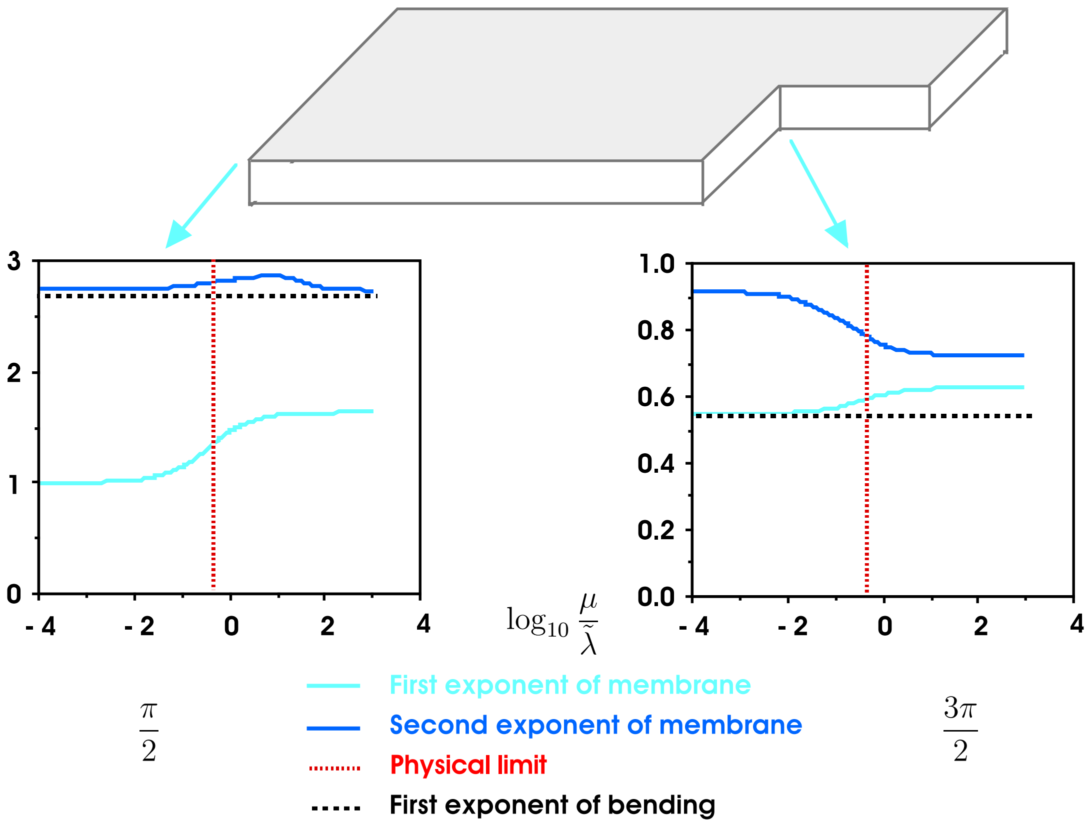

Figure 3. Dirichlet membrane and bending exponents

The regularity of ζ_*^0 is obtained by adding 1 to the value of the exponent. We see that for a convex polygon with a right angle, the first singular exponent is between 1 and 2, so the solution for the membrane problem with Dirichlet boundary conditions belongs to H^2 but not to H^3, unlike the case of a non convex polygon with right reentrant angle where the first singular exponent is less than 1 and thus the solution does not have the H^2 regularity.

The horizontal dotted lines represent the first exponent of the bending equation (bilaplacian) with Dirichlet boundary conditions. The regularity of ζ_3^0 is obtained by adding 2 to the value of this exponent.

As a consequence, in the vicinity of a right angle $\boldsymbol{u}_{\mathrm{KL}}^0$ still has the $H^2(\Omega)$ regularity but not the $H^3(\Omega)$ one. In the case of a reentrant angle, $\boldsymbol{u}_{\mathrm{KL}}^0$ does not belong to $H^2(\Omega)$. Note that the transverse component is more regular.

Figure 4 displays the variation of the first singular exponents versus the opening ϑ of the corner at the vertex $\mathcal{O}$ for the boundary value problems (4.4) of the membrane equation, namely those encountered in cases ①, ② and ③.

For mixed problems ② and ③, the situation deteriorates as the opening

In the graph besides we plot

- On the upper curve: For comparison, we give the variation of π/ω (Dirichlet or Neumann for Δ).
- On the middle curve: Singular exponents of the Dirichlet problem for case ① at the "physical limit" $\tilde{\lambda} = 2\mu$.
- On the lower curve: Singular exponents $|1 - \pi/\omega|$ of the mixed Dirichlet-Neumann problems: either the normal component of the displacement and the tangential component of the traction for ②, or the converse for ③.

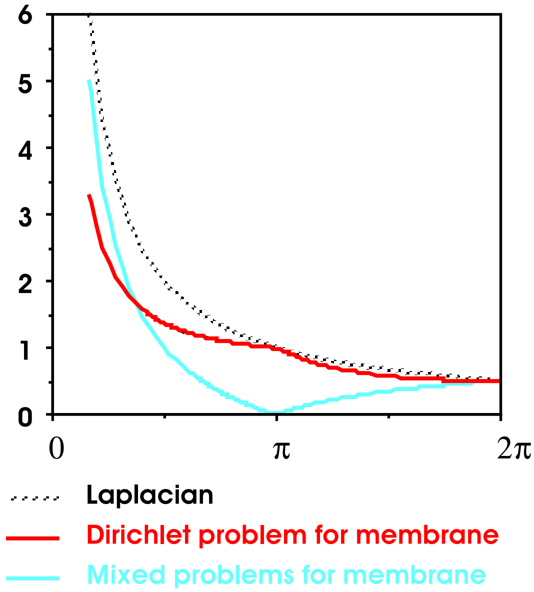

Figure 4. Exponents of singularity of the membrane equation

tends to π: this fact can be compared with the "Babuška paradox" [2]. We see that our construction algorithm of section 4 stops after the determination of $\boldsymbol{u}^0_{\mathrm{KL}}$. Indeed, if $2\pi/3 \leq \vartheta \leq 2\pi$ the right hand side of boundary conditions (4.18) is too singular to have a unique solution to the problem (due to the presence of a non trivial kernel formed by dual singular functions). Concerning (4.16) on a plate with reentrant corner, the algorithm would stop after the determination of $\boldsymbol{u}^1_{\mathrm{KL}}$ for a similar reason.

Getting around this difficulty requires the introduction of a new type of layer terms, namely *corner layers*, see [19], [7] and a modification of the boundary layer terms in the neighborhood of the corners. But this is another story... Another interesting relation is the problem of *geometric singular perturbation* in the hard simple supported plate: if ω is a polygon, the curvature of its boundary is (almost) everywhere zero, and according to (4.20) the boundary layer terms are absent. If instead of true corners, the mean surface ω has rounded corners with radius δ, then we have a first boundary layer term $\boldsymbol{w}^1$ supported in the curved zone, with a coefficient behaving like $1/\delta$. We can guess that in the limit $\delta \to 0$, we find again the corner layer terms.

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