Asymptotics of Arbitrary Order for a Thin Elastic Clamped Plate, I. Optimal Error Estimates

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Abstract. *This paper is the first of a series of two, where we study the asymptotics of the displacement in a thin clamped plate made of a rigid "monoclinic" material, as the thickness of the plate tends to 0. The combination of a polynomial Ansatz (outer expansion) and of a boundary layer Ansatz (inner expansion) yields a complete multi-scale asymptotics of the displacement and leads to optimal error estimates in energy norm. We investigate the polynomial Ansatz in Part I, and the boundary layer Ansatz in Part II.*

If ε denotes the small parameter in the geometry, we first construct the algorithm for an infinite "even" Ansatz involving only even powers of ε, which is a natural extension of the usual Kirchhoff-Love Ansatz. The boundary conditions of the clamped plate being only satisfied at the order 0, we try to compensate for them by boundary layer terms: we rely on a result proved in Part II giving necessary and sufficient conditions for the exponential decay of such terms. In order to fulfill these conditions, the constructive algorithm for the boundary layer terms has to be combined with an "odd" polynomial Ansatz. The outcome is a two-scale asymptotics involving all nonnegative powers of ε, the in-plane space variables x_α, the transverse scaled variable x_3 and the quickly varying variable r/ε where r is the distance to the clamped part of the boundary.

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INTRODUCTION

The problem of the construction of two-dimensional models for thin plates was treated for a long time: the famous Kirchhoff model was stated around 1850, and the Reissner-Mindlin model, about 100 years later. The question of the validity of such models consists in the evaluation of the distance between the solution of the "true" three-dimensional problem and the solution deduced from the two-dimensional model: the faster the decay of this distance is as the plate thickness tends to 0, the better is the two-dimensional model.

The answer to such a question requires an asymptotic analysis. This is quite difficult, due to the fact that the small parameter ε intervenes as a parameter of *singular perturbation*: as ε tends to 0, the three-dimensional problem looses a part of its ellipticity (with respect to the in-plane space variables). As a consequence, a simple polynomial Ansatz with respect to ε cannot fulfill all boundary conditions, especially the Dirichlet condition of the hard clamped plate. Thus, we can expect the combination of an outer expansion (polynomial Ansatz) and of an inner expansion (boundary layer Ansatz).

The first approach is the construction of infinite formal asymptotic expansions (i.e. without error estimates) for both the inner and outer parts. We can quote the papers [18] by K. O. FRIEDRICHS & R. F. DRESSLER in 1961 and [19] by A. L. GOL'DENVEIZER in 1962, and more recently [20] by R. D. GREGORY & F. Y. WAN.

The other approach is to obtain rigorous bounds of the error, at least for the first terms. Such an approach is initiated by P. G. CIARLET and P. DESTUYNDER in [8], [14]. We can also quote the more recent work [32] by S. A. NAZAROV & I. S. ZORIN. See also the review by P. G. CIARLET [7, Ch.3].

Here, we provide a sort of synthesis of the two approaches: we construct the terms of arbitrary order in the inner and outer expansions, and we prove optimal estimates between the three-dimensional solution and any truncated series of the asymptotics: these estimates are optimal in the sense that the order of the error is exactly the norm of the first neglected terms. In this first part we insist on the construction algorithm of the expansions and the estimates in lower order norms (H^1, L^2 for the displacement field, L^2 for the strain tensor). In the second part, we prove the results about the exponential decay of the boundary layer terms that we need here, and by a complementary study of the behavior at the edges of the plate, we obtain estimates in higher order norms, and we deduce estimates in L^∞ norm and in special tensorized norms for the stress tensor.

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1 SETTING THE PROBLEM. STATEMENTS OF THE MAIN RESULTS.

1.1 The Plate Problem

We study the behavior of a thin elastic plate made of a homogeneous material which will be characterized through its rigidity matrix A. Practically, we consider a plate represented by an open set

$$\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon) \subset \mathbb{R}^3,$$

where $\omega \subset \mathbb{R}^2$ is a bounded plane domain with a smooth boundary (ω is the middle surface of the plate), and where the parameter $\varepsilon > 0$ stands for half the thickness. We consider the case when the plate Ω^ε is clamped along its lateral face

$$\Gamma_0^\varepsilon = \partial\omega \times (-\varepsilon, \varepsilon).$$

The unknown of the problem is the displacement u^ε which belongs to a space $V(\Omega^\varepsilon)$ of so-called "admissible displacements" having a finite elastic energy, varying in $\overline{\Omega}^\varepsilon$ and taking their values in $\mathbb{R}^3$; since we assume that the plate is clamped along its lateral face, admissible displacements are zero on Γ_0^ε. Hence:

$$V(\Omega^\varepsilon) = \{v = (v_1, v_2, v_3) \in H^1(\Omega^\varepsilon)^3 \mid v = 0 \quad \text{on} \quad \Gamma_0^\varepsilon\}.$$

For an admissible displacement u^ε, the stress tensor is defined by the formula

$$\sigma^\varepsilon = (\sigma_{ij}^\varepsilon) := A\, e(u^\varepsilon),$$

where A is the rigidity matrix and $e(u^\varepsilon) = (e_{ij}(u^\varepsilon))_{ij}$ is the linearized strain tensor defined by $e_{ij}(v) = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$. We will take a rigidity matrix of a quite general form, but we assume that its coefficients do not depend on ε (if not, we could obtain different asymptotic models).

We make the assumption [6] that we can take the linearized equations of elasticity. The displacement u^ε of the clamped plate corresponding to volume forces f^ε is the unique solution of the variational elasticity problem:

$$u^\varepsilon \in V(\Omega^\varepsilon), \tag{1.1a}$$

$$\forall v^\varepsilon \in V(\Omega^\varepsilon), \qquad \int_{\Omega^\varepsilon} A\, e(u^\varepsilon) : e(v^\varepsilon) = \int_{\Omega^\varepsilon} f^\varepsilon \cdot v^\varepsilon, \tag{1.1b}$$

where the dots : denote the usual scalar product for 3×3 matrices associated with the Frobenius norm $a : b = \sum_{ij} a_{ij} b_{ij}$. The displacement u^ε actually satisfies Neumann condition (zero normal traction) on the two "horizontal" faces of the plate $\omega \times \{\pm\varepsilon\}$. Our aim is to study the behavior of u^ε as $\varepsilon \to 0$ when f^ε depends on ε in a "natural" way. Indeed, we could obtain similar results with *prescribed normal tractions* on the two "horizontal" faces of the plate.

The 9×9 rigidity tensor A, with coefficients A_{ijkl}, satisfies the usual symmetry relations $A_{ijkl} = A_{jikl} = A_{ijlk} = A_{klij}$ and is supposed to be uniformly positive definite, namely:

$$\forall\, x^\varepsilon \in \overline{\Omega}^\varepsilon,\ \forall (t_{ij}) \in \mathbb{R}^9 \ \text{s.t.}\ t_{ij} = t_{ji}, \quad A_{ijkl}(x^\varepsilon)\, t_{kl} t_{ij} \geq c\, t_{ij}^2 \qquad (1.2)$$

where $c > 0$ is a positive constant. Moreover, as in [16], [38] we assume that

$$\begin{cases} A_{\alpha\beta\gamma 3} &=& 0 \qquad \forall \alpha, \beta, \gamma \in \{1, 2\} \\ A_{\alpha 333} &=& 0 \qquad \forall \alpha \in \{1, 2\}. \end{cases} \qquad (1.3)$$

The relations (1.3) are satisfied for any isotropic material, and also for an orthotropic material whose orthotropy axes are either normal or tangent to the plane containing ω. Property (1.3) is referred in [38] as the characterization of monoclinic materials, and it is noted there that this is the weakest assumption so that the system of elasticity can be uncoupled into a bending and a membrane problems.

To simplify our forthcoming analysis, we moreover assume at this stage that the coefficients of A do not depend on the "vertical" coordinate x_3. But, as in [39] for a linear heat conduction problem, the case of laminated materials might be worthwhile studying.

1.2 The scaling

Problem (1.1) was already studied by P. G. CIARLET and P. DESTUYNDER [8] [14] by means of an asymptotic expansion for the displacement-stress approach. See also [7] for a review on the subject. First, a scaling is worked out to get rid of the dependence of the reference set Ω^ε on the parameter $\varepsilon > 0$. So, the problem is set in an equivalent form on the reference set $\Omega = \omega \times (-1, 1)$, which is the image of Ω^ε through a dilation along the normal direction to the plane containing ω:

$$x^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) \in \Omega^\varepsilon \quad \longmapsto \quad x = (x_1^\varepsilon, x_2^\varepsilon, \varepsilon^{-1} x_3^\varepsilon) \in \Omega.$$

It is convenient to preserve the structure of the former problem which was associated with the tensor $e(u^\varepsilon)$. So we introduce a "scaled" linearized strain tensor denoted by $\kappa(\varepsilon)(v)$ for any function $v \in H^1(\Omega)^3$:

$$\kappa_{\alpha\beta}(\varepsilon)(v) = e_{\alpha\beta}(v), \quad \kappa_{\alpha 3}(\varepsilon)(v) = \varepsilon^{-1} e_{\alpha 3}(v), \quad \kappa_{33}(\varepsilon)(v) = \varepsilon^{-2} e_{33}(v), \qquad (1.4a)$$

where we use the convention that the Greek indices α and β span the set $\{1, 2\}$. Correspondingly, the convenient scaling of the components of u^ε writes

$$u_\alpha(\varepsilon)(x) = u_\alpha^\varepsilon(x^\varepsilon), \qquad u_3(\varepsilon)(x) = \varepsilon\, u_3^\varepsilon(x^\varepsilon), \qquad (1.4b)$$

for which the resulting scaled displacement $u(\varepsilon)$ satisfies

$$e(u^\varepsilon)(x^\varepsilon) = \kappa(\varepsilon)(u(\varepsilon))(x).$$

The corresponding canonical scaling for the body forces writes

$$f_\alpha(\varepsilon)(x) := f_\alpha^\varepsilon(x^\varepsilon) \quad \text{and} \quad f_3(\varepsilon)(x) := \varepsilon^{-1} f_3^\varepsilon(x^\varepsilon). \tag{1.4c}$$

Our analysis holds if the scaled body force $f(\varepsilon)$ satisfies an asymptotic property like

$$f(\varepsilon)(x) \simeq f^0(x) + \varepsilon\, f^1(x) + \varepsilon^2\, f^2(x) + \cdots + \varepsilon^k\, f^k(x) + \cdots \tag{1.5}$$

But, for simplicity we assume here that the only coefficient f_0 is non zero. For example, if f^ε is the force of gravity, this condition holds.

It may be directly seen that the scaled displacement $u(\varepsilon)$ solves the new problem:

$$u(\varepsilon) \in V(\Omega), \tag{1.6a}$$

$$\forall v \in V(\Omega), \qquad \int_\Omega A\,\kappa(\varepsilon)(u(\varepsilon)) : \kappa(\varepsilon)(v) = \int_\Omega f \cdot v, \tag{1.6b}$$

where

$$V(\Omega) = \{v \in H^1(\Omega)^3 \mid \quad v = 0 \quad \text{on} \quad \Gamma_0 = \partial\omega \times (-1,1)\} \tag{1.7}$$

and our purpose is to study the asymptotic behavior of $u(\varepsilon)$.

1.3 Related questions

There is a large amount of papers dealing with the asymptotic behavior of the displacement. We already mentioned [8] where the mixed variational formulation (involving both the stress tensor and the displacement) was used, rather than the "single" version that we use in this work, where the displacement remains the only unknown of the problem.

The model we develop here was extended to the case of coupled systems made of two different solids behaving in different ways at the limit as $\varepsilon \to 0$: examples are a three-dimensional body in which a thin plate is inserted (*cf.* [10]), or a thin plate where a slit contains a rod (*cf.* [21]), and also the junction between two plates [25, 27] or two rods (*cf.* [26]). More recently, structures involving shells have been considered: then the geometry of the mean surface has to be taken into account; *cf.* [11, 12, 13], [29], [35, 36, 37].

1.4 Polynomial Ansätze and boundary layers

The "asymptotic expansion method" for thin structures relies on special Ansätze about the behavior of $u(\varepsilon)$.

(i) THE POLYNOMIAL ANSATZ. We call polynomial Ansatz for $u(\varepsilon)$ a proposition of asymptotics of the form

$$u(\varepsilon)(x) \simeq u^0(x) + \varepsilon\, u^1(x) + \varepsilon^2\, u^2(x) + \cdots + \varepsilon^k\, u^k(x) + \cdots \qquad (1.8a)$$

where the functions u^0, u^1, etc... do not depend on ε. Inserting the Ansatz into the variational formulation (1.6) and identifying formally the powers of ε in both sides of the equation, one obtains equations for u^0, u^1, etc... The hypothesis (1.3) leads to assume that the odd terms in (1.8a) do not appear. Thus, we arrive at the notion of *"even polynomial Ansatz"*:

$$u(\varepsilon)(x) \simeq u^0(x) + \varepsilon^2\, u^2(x) + \cdots + \varepsilon^{2k}\, u^{2k}(x) + \cdots \qquad (1.8b)$$

In this respect, [8, 14] (see also [32]), have formally identified the three first terms of the Ansatz, namely u^0, u^2 and u^4.

(ii) KIRCHHOFF-LOVE DISPLACEMENTS. The first term u^0, as determined in the above mentioned works, satisfies the Dirichlet boundary conditions of the clamped plate and moreover

$$e_{i3}(u^0) = 0 \quad \text{in} \quad \Omega, \quad i = 1, 2, 3. \qquad (1.9)$$

This is the Kirchhoff-Love hypothesis. Conditions (1.9) hold if and only if there exist 3 functions ζ_1^0, ζ_2^0, ζ_3^0 defined on the two-dimensional mean surface ω such that

$$u^0(x) = \Big(\zeta_1^0(x_1, x_2) - x_3\, \partial_1 \zeta_3^0(x_1, x_2),\ \zeta_2^0(x_1, x_2) - x_3\, \partial_2 \zeta_3^0(x_1, x_2),\ \zeta_3^0(x_1, x_2) \Big).$$

In other words u^0 is a *Kirchhoff-Love displacement.*

(iii) THE JUSTIFICATION OF THE POLYNOMIAL ANSATZ. This is essential for the validation of the asymptotics. But neither u^2, nor u^4 in general, satisfies the Dirichlet boundary conditions. Such a situation contrasts with the case of *periodic* boundary conditions as investigated by J.C. PAUMIER in [34] where the validity and the convergence of the corresponding Ansatz (1.8b) are proved.

Thus, in the clamped plate, a method to perform such a justification is the construction of "correctors" as initiated by J.-L. LIONS in [28]. By this strategy, P. DESTUYNDER [14] and P. DESTUYNDER & I. GRUAIS [17] proved that the Kirchhoff-Love term u^0 is actually the limit of $u(\varepsilon)$ in the energy norm and that the following H^1- estimates hold:

$$\|u_i(\varepsilon) - u_i^0\|_{H^1(\Omega)} \leq C\, \varepsilon^{1/2} \qquad i = 1, 2, 3. \qquad (1.10)$$

(iv) BOUNDARY LAYERS. These estimates are limited to the first term u^0; this is linked to the fact that the other terms u^2, u^4, etc... do not satisfy the Dirichlet boundary conditions of the clamped plate. Even in this simplest case of the Dirichlet condition, the clamped part of the plate generates a "boundary layer" corresponding to an ill-posed problem. Typically, the elliptic problem (1.6) degenerates in the limit into a new problem which, roughly speaking, behaves like the Neumann problem for the operator ∂_3^2 and the surface Γ^0 is characteristic for this last operator.

Generally speaking, such a phenomenon is classical for *singular perturbation problems*: see the book [24] by A. M. IL'IN, where the construction of matching outer and inner expansions (corresponding to the polynomial Ansatz and the boundary layer terms respectively) is explained on many examples.

(v) MULTI-SCALE ASYMPTOTICS. The idea is to find *profiles* (see for instance the related works [1, 2] by G. ALLAIRE and [22, 23] by O. GUÈS where this notion is currently used) Ψ^0, Ψ^1 ... which are functions of the space variable x and also of other variables T such that for a certain relation of dependence of T with respect to x and ε, $T = T(x,\varepsilon)$, we have

$$u(\varepsilon)(x) \simeq \Psi^0(x,T) + \varepsilon\,\Psi^1(x,T) + \varepsilon^2\,\Psi^2(x,T) + \cdots + \varepsilon^k\,\Psi^k(x,T) + \cdots \qquad (1.11)$$

Such an approach is classical in homogeneization of periodic elastic structures: *cf.* the above references and the book by O.A. OLEINIK, A.S. SHAMAEV & G.A. YOSIFIAN [33] and also the references therein.

For related problems in thin plates, we can cite D. N. ARNOLD & R. S. FALK [3] for the two-dimensional Reissner-Mindlin model and C. SCHWAB [39] for the heat equilibrium scalar equation. Finally, while writing this paper we found out that S. A. NAZAROV & I. S. ZORIN [32] treated the same problem of thin plates for an isotropic material by this method of multi-scale asymptotics and determined the first terms up to the power ε^2.

In these two papers, our aim is threefold:

1. With the correct scaled variable(s) T, we construct the Ansatz (1.11) by calculating terms of arbitrarily high orders. This involves recurrence relations under the form of an algorithm providing the successive terms. The general setting of monoclinic materials allows a better understanding of the operators occurring in the algorithm.

2. We obtain optimal error estimates in the energy norm between the scaled displacement $u(\varepsilon)$ and a finite expansion where we keep only N terms of the Ansatz (1.11). Such a strategy is original when compared with earlier work on this subject (at the exception of [32]).

3. We construct edge layer displacements along the edges $\partial\omega \times \{\pm 1\}$ of Ω in order to obtain estimates in higher order norms and information about the structure of the solution.

1.5 Outline of the paper and main results

(i) HYPOTHESES. All our results are valid for the above assumptions about the rigidity matrix A, see (1.3), and the following smoothness hypotheses:

$$(\mathscr{H}) \qquad \begin{cases} \omega \text{ has a } \mathscr{C}^\infty \text{ boundary } \partial\omega \\ A = A(x_1, x_2) \in \mathscr{C}^\infty(\overline{\omega}, \mathscr{M}^{9\times 9}) \\ f \in \mathscr{C}^\infty(\overline{\Omega}, \mathbb{R}^3). \end{cases}$$

These assumptions are *necessary* if one wants to push the asymptotics up to any order N (see (1.13) below). But for any fixed N, it suffices that f belongs to $H^{N+4}(\Omega)$ to obtain the estimates (1.14) below, see also in §4.3 the regularity of the terms in the asymptotics if f is not supposed to be $\mathscr{C}^\infty$. On the other hand, the situation where the mean surface itself has corners (*cf.* [31]), combined with jumps in the lateral boundary conditions, gives rise to different behaviors in the asymptotics.

(*ii*) INFINITE ASYMPTOTIC EXPANSION. We start with an even polynomial Ansatz of the type (1.8b). We prove in §2 (Theorem 2.4) that there exist unique terms $u^0, \ldots u^{2k}, \ldots$ each one splitting into a function v^{2k} with zero mean values with respect to x_3 and into a Kirchhoff-Love displacement u^{2k}_{KL} satisfying Dirichlet conditions on $\partial\omega$, such that the sum $\sum_k \varepsilon^{2k} u^{2k}$ is formally a solution of (1.6b). This construction relies on the solution of Neumann problems with respect to the variable x_3 on the interval $(-1, +1)$ and the difficulty is to fulfill the compatibility conditions on the right hand sides in each point of the mean surface ω. With the exception of the first (Kirchhoff-Love) term u^0, the other terms do not satisfy the Dirichlet condition (1.6a).

Then we try to compensate for the Dirichlet boundary values of our even Ansatz by a boundary layer Ansatz of the form $\sum_{k\geq 1} \varepsilon^k w^k(\frac{r}{\varepsilon}, s, x_3)$ where (r, s) is a system of coordinates in ω such that the equation of the boundary $\partial\omega$ is $r = 0$. In order to reach this aim, considering the variable s as a parameter, it suffices to work with operators in the normal variables $(t, x_3) = (\frac{r}{\varepsilon}, x_3)$: in §3 we exhibit the corresponding reduction which consists of mixed Dirichlet-Neumann problems for a family of elasticity operators depending on the parameter s, called *reduced-normal problems* and defined on the half-strip

$$\Sigma^+ = \{(t, x_3) \in \mathbb{R}^+ \times (-1, 1)\}$$

with non-homogeneous Dirichlet conditions on $\{0\} \times (-1, 1)$ and zero Neumann conditions on the other sides $\mathbb{R}^+ \times \{-1, 1\}$. In this Part I, we only state the results which are needed here, and give the proofs in Part II. Roughly speaking, the solutions of these Dirichlet-Neumann problems are the sum of a rigid displacement (living in a space of dimension 4) and of a term exponentially decreasing at infinity.

Thus, at this stage, we have to correct the Dirichlet boundary values of our even Ansatz, in order that for each value of the parameter s, the "rigid displacement" is canceled. We perform in §4 such a correction by the help of an odd Ansatz. Then we achieve the construction of the whole asymptotics for $u(\varepsilon)$ by a two-step algorithm. The outcome of this algorithm (Theorem 4.1) is an asymptotics of the form (1.11)

$$\Psi^0(x, t) = u^0_{\mathrm{KL}}(x), \qquad \Psi^1(x, t) = u^1_{\mathrm{KL}}(x) - \chi(r)\, w^1(t, s, x_3) \tag{1.12a}$$

$$\Psi^k(x, t) = u^k_{\mathrm{KL}}(x) + v^k(x) - \chi(r)\, w^k(t, s, x_3) \quad \text{for} \quad k \geq 2 \tag{1.12b}$$

$$\text{with} \quad \begin{cases} u^k_{\mathrm{KL}} \text{ a Kirchhoff-Love term,} \\ v^k \text{ satisfies } \int_{-1}^1 v^k(x_1, x_2, x_3)\, dx_3 = 0, \ \forall (x_1, x_2) \in \omega, \\ w^k(t, s, x_3) \text{ is uniformly exponentially decreasing as } t \to +\infty, \\ \chi \text{ is a cut-off function equal to 1 in a neighborhood of } r = 0, \end{cases} \tag{1.12c}$$

which is (formally) solution of the problem (1.6a) & (1.6b). In this expansion, the "polynomial" part $\sum_{k \geq 0} \varepsilon^k u^k = \sum_{k \geq 0} \varepsilon^k (u_{\mathrm{KL}}^k + v^k)$ forms the *polynomial Ansatz*, whereas the part $\sum_{k \geq 1} \varepsilon^k w^k(\frac{r}{\varepsilon}, s, x_3)$ is the *boundary layer Ansatz*. Note that the boundary layer Ansatz starts with ε. It is interesting to compare the complexity of this algorithm, and in particular the occurrence of the odd polynomial terms due to the structure at infinity of solutions on the half strip Σ^+, with the situation of *scalar second-order operators* studied by C. SCHWAB [39]: in the scalar case, the interaction between polynomial Ansatz and boundary layer Ansatz is minimal (zero mean value for the Dirichlet condition in $t = 0$ is sufficient to ensure the exponential decay for the boundary layer terms), all terms in the asymptotics are of even order and the terms in the polynomial Ansatz are explicit.

(iii) ERROR ESTIMATES FOR THE DISPLACEMENT FIELD. Denoting for any $N \in \mathbb{N}$ by $U^N(\varepsilon)$ the truncated expansion (1.11):

$$U^N(\varepsilon)(x) = \Psi^0(x, \frac{r}{\varepsilon}) + \varepsilon\, \Psi^1(x, \frac{r}{\varepsilon}) + \varepsilon^2\, \Psi^2(x, \frac{r}{\varepsilon}) + \cdots + \varepsilon^N\, \Psi^N(x, \frac{r}{\varepsilon}), \qquad (1.13)$$

we obtain in §5 optimal estimates of the error between $u(\varepsilon)$ and $U^N(\varepsilon)$ in energy norm. We use the classical method in asymptotic analysis consisting in pushing forward the asymptotics a few terms further (here $(N+4)$), deducing rough estimates by Korn's inequality and concluding by an estimate of the energy norm of the neglected terms $(\Psi^{N+1}, \Psi^{N+2}, \Psi^{N+3}, \Psi^{N+4})$. Thus we prove in §5 that the error is of the same order as the energy norm of the first neglected term Ψ^{N+1}:

$$\|u(\varepsilon) - \sum_{k=0}^{N} \varepsilon^k u^k + \chi(r) \sum_{k=1}^{N} \varepsilon^k w^k(\frac{r}{\varepsilon}, s, x_3)\|_{H^1(\Omega)^3} \leq C\, \varepsilon^{N+1/2}, \qquad (1.14\mathrm{a})$$

and if one adds the boundary layer term of order $N+1$

$$\|u(\varepsilon) - \sum_{k=0}^{N} \varepsilon^k u^k + \chi(r) \sum_{k=1}^{N+1} \varepsilon^k w^k(\frac{r}{\varepsilon}, s, x_3)\|_{H^1(\Omega)^3} \leq C\, \varepsilon^{N+1}. \qquad (1.14\mathrm{b})$$

We moreover give estimates in L^2-norm:

$$\|u(\varepsilon) - \sum_{k=0}^{N} \varepsilon^k u^k + \chi(r) \sum_{k=1}^{N} \varepsilon^k w^k(\frac{r}{\varepsilon}, s, x_3)\|_{L^2(\Omega)^3} \leq C\, \varepsilon^{N+1}, \qquad (1.14\mathrm{c})$$

and if one adds the polynomial term of order $N+1$

$$\|u(\varepsilon) - \sum_{k=0}^{N+1} \varepsilon^k u^k + \chi(r) \sum_{k=1}^{N} \varepsilon^k w^k(\frac{r}{\varepsilon}, s, x_3)\|_{H^1(\Omega)^3} \leq C\, \varepsilon^{N+3/2}. \qquad (1.14\mathrm{d})$$

(iv) RELATION WITH THE KIRCHHOFF-LOVE MODEL AND OTHER MODELS OF PLATES. The first terms in the asymptotics have a special form. At the zero order, we recover the well-known fact that the first approximation for $u(\varepsilon)$ is $U^0 = u_{\mathrm{KL}}^0$:

$$u_{\mathrm{KL}}^0(x) = \left(\zeta_1^0 - x_3\, \partial_1 \zeta_3^0,\ \zeta_2^0 - x_3\, \partial_2 \zeta_3^0,\ \zeta_3^0 \right),$$

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with $(\zeta_1^0, \zeta_2^0) = \zeta_*^0$ the unique solution on the mean surface ω of the membrane equation

$$\zeta_*^0 \in \mathring{H}^1(\omega)^2 \quad \text{and} \quad -\partial_\beta M_{\alpha\beta\gamma\delta}\, e_{\gamma\delta}(\zeta_*^0) = \frac{1}{2} \int_{-1}^{+1} f_\alpha^0 \, dx_3 \,,$$

associated to the resultant rigidity matrix (which is positive definite)

$$M_{\alpha\beta\gamma\delta} = A_{\alpha\beta\gamma\delta} - A_{\alpha\beta33}\, A_{3333}^{-1}\, A_{33\gamma\delta},$$

and ζ_3^0 is the solution of the bending equation

$$\zeta_3^0 \in \mathring{H}^2(\omega) \quad \text{and} \quad \partial_{\alpha\beta}\, M_{\alpha\beta\gamma\delta}\, \partial_{\gamma\delta}(\zeta_3^0) = \frac{3}{2} \int_{-1}^{+1} (f_3^0 + x_3\, \partial_\alpha f_\alpha^0) \, dx_3 \,,$$

associated with the same rigidity matrix M. In the case of an isotropic material of Lamé constants λ and μ,

$$A_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu\, (\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}),$$

and:

$$M_{\alpha\beta\gamma\delta} = \frac{2\lambda\mu}{\lambda + 2\mu}\, \delta_{\alpha\beta}\, \delta_{\gamma\delta} + \mu\, (\delta_{\alpha\gamma}\, \delta_{\beta\delta} + \delta_{\alpha\delta}\, \delta_{\beta\gamma}).$$

At the order one, u^1 coincides with the Kirchhoff-Love displacement u_{KL}^1, whose "generating functions" (ζ_1^1, ζ_2^1) and ζ_3^1 are solutions of a membrane equation and a bending equation respectively, with zero right-hand side and a Dirichlet condition equal to the traces of the rigid displacements appearing for each s in the behavior at infinity of the solution of the "reduced-normal problem" with Dirichlet data equal to the traces of v^2 *cf.* (4.4) and (4.5). We will prove in Part II that the function u_{KL}^1 is not zero in general. The first term in the boundary layer series correlatively appears: this is w^1 and its third component w_3^1 is zero.

Taking into account the form of the first terms of the asymptotics we obtain the following error estimates between $u(\varepsilon)$ and u_{KL}^0:

$$\begin{cases} \|u_\alpha(\varepsilon) - u_{\mathrm{KL},\alpha}^0(\varepsilon)\|_{H^1(\Omega)} \le C\, \varepsilon^{1/2} & \|u_3(\varepsilon) - u_{\mathrm{KL},3}^0(\varepsilon)\|_{H^1(\Omega)} \le C\, \varepsilon, \\[2mm] \|u_\alpha(\varepsilon) - u_{\mathrm{KL},\alpha}^0(\varepsilon)\|_{L^2(\Omega)} \le C\, \varepsilon & \|u_3(\varepsilon) - u_{\mathrm{KL},3}^0(\varepsilon)\|_{L^2(\Omega)} \le C\, \varepsilon. \end{cases} \tag{1.15}$$

Moreover, the estimates (1.14) allow a precise evaluation of the approximation of the true displacement $u(\varepsilon)$ by a Kirchhoff-Love displacement $u_{\mathrm{KL}}(\varepsilon)$: let us take

$$u_{\mathrm{KL}}(\varepsilon) = u_{\mathrm{KL}}^0 + \varepsilon\, u_{\mathrm{KL}}^1. \tag{1.16}$$

Then (1.14a)-(1.14d) yield the estimates

$$\begin{cases} \|u_\alpha(\varepsilon) - u_{\mathrm{KL},\alpha}(\varepsilon)\|_{H^1(\Omega)} \le C\, \varepsilon^{1/2} & \|u_3(\varepsilon) - u_{\mathrm{KL},3}(\varepsilon)\|_{H^1(\Omega)} \le C\, \varepsilon^{3/2}, \\[2mm] \|u_\alpha(\varepsilon) - u_{\mathrm{KL},\alpha}(\varepsilon)\|_{L^2(\Omega)} \le C\, \varepsilon^{3/2} & \|u_3(\varepsilon) - u_{\mathrm{KL},3}(\varepsilon)\|_{L^2(\Omega)} \le C\, \varepsilon^2. \end{cases} \tag{1.17}$$

Only the estimate on the first two components in H^1-norm is not improved by the adjunction of $\varepsilon\,u^1_{\mathrm{KL}}$. We will not obtain any improvement of the three other estimates if we add $\varepsilon^2\,u^2_{\mathrm{KL}}$ to $u_{\mathrm{KL}}(\varepsilon)$. But, if we consider now the scaled linearized strain tensor, we have to include the polynomial terms up to the order 2 to obtain a convergence:

$$\|\kappa(\varepsilon)\big(u(\varepsilon) - (u^0_{\mathrm{KL}} + \varepsilon\,u^1_{\mathrm{KL}} + \varepsilon^2\,u^2_{\mathrm{KL}} + v^2)\big)\|_{L^2(\Omega)^9} \le C\,\varepsilon^{1/2}. \qquad (1.18)$$

We note that, with only one exception, all the above estimates are limited by the boundary layer terms. Since the boundary layer terms are not polynomial in the variable x_3 in general, other bi-dimensional models of plates (for example Reissner-Mindlin, or more generally Naghdi plates theories [30] or standard polynomial hierarchical models of plates) seem to be unable to provide a better approximation in energy norm as $\varepsilon \to 0$ than the Kirchhoff-Love model.

On the other hand, the third component $u_3(\varepsilon)$ of the displacement would be better described in the norm $L^2(\Omega)$ by a model which could be able to include v^2_3. In the same order of idea, if one evaluates of the error in other norms such as $H^1((-1,+1),\,H^{-1}(\omega))$ or $H^2((-1,+1),\,H^{-2}(\omega))$, or considers some average norms, higher order models can be more performant. An interesting application of this work could be a more precise investigation of higher order models of plates: *cf.* [4] for the analysis of shear correction factors and [38] for a-posteriori estimates.

2 THE "EVEN" ANSATZ

In this section, we are going to present an algorithm for the construction of an infinite even polynomial Ansatz of the type (1.8b) for the solution of problem (1.6). Our algorithm is an extension of the calculations of [8, 14], [32] determining the first two or three terms of this Ansatz.

2.1 The sequence of boundary value problems

Let us recall that problem (1.6) is a mixed boundary value problem with homogeneous Dirichlet conditions on $\Gamma_0 = \partial\omega \times (-1,1)$. For the presentation of our algorithm, it is easier to consider non-homogeneous Dirichlet conditions and volume forces set to 0. We return to the original problem (1.6) at the end of this section (§2.4). Let there be given h a triple of traces $h = (h_1, h_2, h_3)$ in $\mathscr{C}^\infty(\overline{\Gamma}_0)$. The problem we now consider is:

$$u(\varepsilon) \in H^1(\Omega)^3 \quad \text{and} \quad u(\varepsilon) = h \ \text{ on } \ \Gamma_0, \qquad (2.1\mathrm{a})$$

$$\forall v \in V(\Omega), \qquad \int_\Omega A\,\kappa(\varepsilon)(u(\varepsilon)) : \kappa(\varepsilon)(v) = 0. \qquad (2.1\mathrm{b})$$

Moreover, problem (2.1) will be useful by itself in further developments (§4).

Using the definition (1.4a) of $\kappa(\varepsilon)$ and integrating by parts, we obtain the following equivalent form for problem (2.1), which we split into the two groups of equations (2.2) and (2.3):

$$u_3(\varepsilon) = h_3 \ \text{ on } \ \Gamma_0, \tag{2.2a}$$

$$A_{3333}\, \partial_{33}\, u_3(\varepsilon) + \varepsilon^2 \Big(A_{33\gamma\delta}\, \partial_3\, e_{\gamma\delta}(u(\varepsilon)) + 2\, \partial_\alpha\, A_{\alpha3\gamma3}\, e_{\gamma3}(u(\varepsilon)) \Big) = 0 \ \text{ in } \ \Omega, \tag{2.2b}$$

$$A_{3333}\, \partial_3\, u_3(\varepsilon) + \varepsilon^2\, A_{33\gamma\delta}\, e_{\gamma\delta}(u(\varepsilon)) = 0 \ \text{ on } \ \{x_3 = \pm 1\} \cap \partial\Omega, \tag{2.2c}$$

$$u_\alpha(\varepsilon) = h_\alpha \ \text{ on } \ \Gamma_0, \tag{2.3a}$$

$$2\, A_{\alpha3\gamma3}\, \partial_3\, e_{\gamma3}(u(\varepsilon)) + \partial_\beta\, A_{\alpha\beta33}\, \partial_3\, u_3(\varepsilon) + \varepsilon^2\, \partial_\beta\, A_{\alpha\beta\gamma\delta}\, e_{\gamma\delta}(u(\varepsilon)) = 0 \ \text{ in } \ \Omega, \tag{2.3b}$$

$$A_{\alpha3\gamma3}\, e_{\gamma3}(u(\varepsilon)) = 0 \ \text{ on } \ \{x_3 = \pm 1\} \cap \partial\Omega. \tag{2.3c}$$

We use the repeated index convention along with the notation which gives to the Greek indices α, β, γ,... the values in $\{1,2\}$ corresponding to the in-plane variables. In this calculation, we took into account the fact that the coefficients of the rigidity matrix may depend on (x_1, x_2) but not on x_3. Equations (2.2a) and (2.3a) are the Dirichlet conditions on u_3 and (u_1, u_2) respectively, (2.2c) and (2.3c) the Neumann conditions, and (2.2b) and (2.3b) the interior equations.

As a consequence of hypothesis (1.3) about the coefficients of the rigidity matrix A, the only powers of ε present in (2.2)-(2.3) are ε^0 and ε^2. This is the reason for the introduction of the even Ansatz $u(\varepsilon)(x) \simeq u^0(x) + \varepsilon^2\, u^2(x) + \cdots + \varepsilon^{2k}\, u^{2k}(x) + \cdots$ Inserting this Ansatz in equations of (2.2) and (2.3) and identifying to 0 the factors of all the powers of ε, we find for any $k \in \mathbb{N}$:

$$u_3^{2k} = h_3^{2k} \ \text{ on } \ \Gamma_0, \tag{2.4a}$$

$$A_{3333}\, \partial_{33}\, u_3^{2k} + A_{33\gamma\delta}\, \partial_3\, e_{\gamma\delta}(u^{2k-2}) + 2\, \partial_\alpha\, A_{\alpha3\gamma3}\, e_{\gamma3}(u^{2k-2}) = 0 \ \text{ in } \ \Omega, \tag{2.4b}$$

$$A_{3333}\, \partial_3\, u_3^{2k} + A_{33\gamma\delta}\, e_{\gamma\delta}(u^{2k-2}) = 0 \ \text{ on } \ \{x_3 = \pm 1\} \cap \partial\Omega, \tag{2.4c}$$

$$u_\alpha^{2k} = h_\alpha^{2k} \ \text{ on } \ \Gamma_0, \tag{2.5a}$$

$$2\, A_{\alpha3\gamma3}\, \partial_3\, e_{\gamma3}(u^{2k}) + \partial_\beta\, A_{\alpha\beta33}\, \partial_3\, u_3^{2k} + \partial_\beta\, A_{\alpha\beta\gamma\delta}\, e_{\gamma\delta}(u^{2k-2}) = 0 \ \text{ in } \ \Omega, \tag{2.5b}$$

$$A_{\alpha3\gamma3}\, e_{\gamma3}(u^{2k}) = 0 \ \text{ on } \ \{x_3 = \pm 1\} \cap \partial\Omega, \tag{2.5c}$$

where we have set $h^{2k} = 0$ for $k > 0$ and $u^{2k-2} = 0$ for $k = 0$.

2.2 Kirchhoff-Love displacements

For $k = 0$, these two systems reduce to

$$u_3^0 = h_3^0 \qquad \text{on} \quad \Gamma_0, \tag{2.6a}$$

$$A_{3333}\, \partial_{33}\, u_3^0 = 0 \qquad \text{in} \quad \Omega, \tag{2.6b}$$

$$A_{3333}\, \partial_3\, u_3^0 = 0 \qquad \text{on} \quad \{x_3 = \pm 1\} \cap \partial\Omega, \tag{2.6c}$$

$$u_\alpha^0 = h_\alpha^0 \qquad \text{on} \quad \Gamma_0, \tag{2.7a}$$

$$2\, A_{\alpha 3 \gamma 3}\, \partial_3\, e_{\gamma 3}(u^0) + \partial_\beta\, A_{\alpha \beta 33}\, \partial_3\, u_3^0 = 0 \qquad \text{in} \quad \Omega, \tag{2.7b}$$

$$A_{\alpha 3 \gamma 3}\, e_{\gamma 3}(u^0) = 0 \qquad \text{on} \quad \{x_3 = \pm 1\} \cap \partial\Omega. \tag{2.7c}$$

To solve problems (2.6) and (2.7) (and more generally problems (2.4) and (2.5)), we note that, since A is positive definite, its diagonal elements are > 0 and its diagonal blocks are positive definite. Whence

$$A_{3333} > 0 \quad \text{and} \quad (A_{\alpha 3 \gamma 3}) = \begin{pmatrix} A_{1313} & A_{1323} \\ A_{2313} & A_{2323} \end{pmatrix} \text{ is positive definite.} \tag{2.8}$$

As a consequence of (2.8) we find that the only solutions of (2.6b)-(2.6c) are the functions $u_3^0(x_*, x_3) = \zeta_3^0(x_*)$ which do not depend on x_3. Here and elsewhere, x_* denotes the couple (x_1, x_2) of variables in the mean surface ω. Therefore, a necessary and sufficient condition to ensure the Dirichlet condition (2.6a) is that h_3^0 does not depend on x_3.

Taking advantage of (2.8) again, we find that problem (2.7) reduces to the following system whose unknown is $u_* = (u_1, u_2)$ (with the same convention of notation ∇_* denotes the gradient (∂_1, ∂_2) in ω):

$$u_*^0 = h_*^0 \qquad \text{on} \quad \Gamma_0, \tag{2.9a}$$

$$\partial_{33}\, u_*^0 = 0 \qquad \text{in} \quad \Omega, \tag{2.9b}$$

$$\partial_3\, u_*^0 + \nabla_*\, \zeta_3^0 = 0 \qquad \text{on} \quad \{x_3 = \pm 1\} \cap \partial\Omega. \tag{2.9c}$$

Thus the general solution of (2.9b)-(2.9c) is $u_*^0 = \zeta_*^0 - x_3\, \nabla_*\, \zeta_3^0$, where $\zeta_*^0 = (\zeta_1^0, \zeta_2^0)$ does not depend on x_3. In concordance with [7, §1.4], we may set:

Definition 2.1 We call *Kirchhoff-Love displacement* on Ω and denote by u_{KL} any displacement of the form

$$u_{\mathrm{KL}} = (u_{\mathrm{KL},*}, u_{\mathrm{KL},3}) \quad \text{with} \quad u_{\mathrm{KL},3} = \zeta_3 \quad \text{and} \quad u_{\mathrm{KL},*} = \zeta_* - x_3\, \nabla_*\, \zeta_3,$$

where ζ_3 and ζ_* are functions of x_* defined on $\overline{\omega}$, called the *generating functions* of u_{KL}. As a consequence of this definition $e_{i3}(u_{\mathrm{KL}}) = 0$. ∎

A necessary condition to ensure the Dirichlet condition (2.9a) is that h_*^0 be an affine function of x_3 for any $x_* \in \partial\omega$. Let us investigate the sufficiency.

As already hinted in §1, we introduce smooth local coordinates (r, s) in a tubular neighborhood $\mathcal{V}$ of $\partial\omega$. Let $L_1, \ldots, L_I$ be the lengths of the connected components of $\partial\omega$. We denote by S the disjoint union

$$S = L_1 \mathbb{S}^1 \,\dot{\cup}\, \cdots \,\dot{\cup}\, L_I \mathbb{S}^1. \tag{2.10}$$

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The variable r belongs to $(-\rho, \rho)$ and behaves like the distance to the boundary $\partial\omega$ inside ω and s belongs to S and behaves like the arc length along $\partial\omega$:

$$\text{for } x_* = X_*(r,s) \quad \begin{cases} x_* \in \mathcal{V} \cap \omega \iff 0 < r < \rho, \ s \in S \\ x_* \in \partial\omega \iff r = 0, \ s \in S, \\ x_* \in \partial\omega \implies DX_*(x_*) \in SO2. \end{cases} \tag{2.11}$$

Correspondingly, we define the contravariant change of coordinates on a displacement field u_* defined on $\overline{\omega}$, by

$$u_* = u_1 \, e_1 + u_2 \, e_2 = u_r \, n_r + u_s \, n_s \tag{2.12}$$

where (e_1, e_2) is the Cartesian basis, and

$$n_r(r,s) = DX_*(x_*) \, e_1, \qquad n_s(r,s) = DX_*(x_*) \, e_2, \tag{2.13}$$

whence

$$u_r(r,s) = DX_*(x_*)^{-1} u_*(x_*) \cdot e_1, \qquad u_s(r,s) = DX_*(x_*)^{-1} u_*(x_*) \cdot e_2. \tag{2.14}$$

Then we see that for any $x_* \in \partial\omega$:

$$\zeta_*^0(x_*) - x_3 \, \nabla_* \, \zeta_3^0(x_*) = \left(\zeta_r^0(0,s) - x_3 \, \partial_r \zeta_3^0(0,s) \right) n_r + \left(\zeta_s^0(0,s) - x_3 \, \partial_s \zeta_3^0(0,s) \right) n_s.$$

whence the Lemma:

Lemma 2.2 *Let h_r, h_n, h_s and h_3 be four smooth functions in $\mathscr{C}^\infty(\partial\omega)$. Then problem (2.6)-(2.7) with the Dirichlet data*

$$h_*^0 = (h_r - x_3 \, h_n) \, n_r + (h_s - x_3 \, \partial_s h_3) \, n_s \quad \text{and} \quad h_3^0 = h_3$$

has solutions which are Kirchhoff-Love displacements u_{KL}^0. The functions ζ_3^0 and ζ_*^0 such that $u_{\mathrm{KL},3}^0 = \zeta_3^0$ and $u_{\mathrm{KL},*}^0 = \zeta_*^0 - x_3 \, \nabla_* \, \zeta_3^0$ satisfy

$$\zeta_3^0(0,s) = h_3(s), \quad \partial_r \zeta_3^0(0,s) = h_n(s), \quad \zeta_r^0(0,s) = h_r(s), \quad \zeta_s^0(0,s) = h_s(s).$$

At this stage, the generating functions ζ_3^0 and ζ_*^0 are not uniquely determined. But, as we are going to show, if we solve problem (2.4) for $k = 1, 2$ and problem (2.5) for $k = 1$, we arrive at the equation of a *membrane* for ζ_*^0 and an equation of *bending* for ζ_3^0, and then these functions are uniquely determined.

We can also see that the conditions for solving a Dirichlet problem are very restrictive. But, in particular, if we consider homogeneous Dirichlet conditions, we have the solvability. Hence, there is no boundary layer at the rank of ε^0.

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2.3 Existence and uniqueness of an even Ansatz

We are ready to prove by induction the following theorem:

Theorem 2.3 *Let for any $k \in \mathbb{N}$, four functions h_r^{2k}, h_n^{2k}, h_s^{2k} and h_3^{2k} be given in $\mathscr{C}^\infty(\partial\omega)$. Moreover, we set $u^{-2} = 0$. Then, there exist a unique sequence of displacements $(v^{2k})_{k\in\mathbb{N}}$ and a unique sequence of Kirchhoff-Love displacements $(u_{\mathrm{KL}}^{2k})_{k\in\mathbb{N}}$ such that for any $k \in \mathbb{N}$*

$$
\begin{cases}
u^{2k} := v^{2k} + u_{\mathrm{KL}}^{2k} \text{ is solution of } (2.4b)\text{-}(2.4c) \;\&\; (2.5b)\text{-}(2.5c), \\[2mm]
\forall x_* \in \overline{\omega}, \quad \displaystyle\int_{-1}^{+1} v^{2k}(x_*, x_3)\, dx_3 = 0, \\[2mm]
u_{\mathrm{KL}}^{2k} = \left(\zeta_*^{2k} - x_3 \nabla_* \zeta_3^{2k},\; \zeta_3^{2k} \right) \text{ and} \\[2mm]
\quad \zeta_3^{2k}\big|_{\partial\omega} = h_3^{2k}, \quad \partial_r \zeta_3^{2k}\big|_{\partial\omega} = h_n^{2k}, \quad \zeta_r^{2k}\big|_{\partial\omega} = h_r^{2k}, \quad \zeta_s^{2k}\big|_{\partial\omega} = h_s^{2k}.
\end{cases}
$$

The functions v^{2k} and u_{KL}^{2k} are smooth on $\overline{\Omega}$.

Proof. If u^{2k-2} is given, problem (2.4b)-(2.4c) is simply a Neumann problem on the interval $(-1, +1)$ for each fixed $x_* \in \overline{\omega}$. Thus the solvability of this problem relies upon a *compatibility condition* obtained by integrating (2.4b) with respect to x_3 from -1 to $+1$ for any fixed $x_* \in \overline{\omega}$ and using the Neumann boundary conditions (2.4c). We obtain the system:

$$2\, \partial_\alpha\, A_{\alpha3\gamma3} \int_{-1}^{+1} e_{\gamma3}(u^{2k-2})\, dx_3 = 0, \tag{2.15a}$$

$$A_{3333}\, \partial_{33}\, u_3^{2k} = -A_{33\gamma\delta}\, \partial_3\, e_{\gamma\delta}(u^{2k-2}) - 2\, \partial_\alpha\, A_{\alpha3\gamma3}\, e_{\gamma3}(u^{2k-2}) \text{ in } \Omega, \tag{2.15b}$$

$$A_{3333}\, \partial_3\, u_3^{2k} = -A_{33\gamma\delta}\, e_{\gamma\delta}(u^{2k-2}) \text{ on } \{x_3 = \pm 1\} \cap \partial\Omega. \tag{2.15c}$$

Taking advantage of (2.8), if u_3^{2k} is given, problem (2.5b)-(2.5c) is also a Neumann problem and its solvability relies upon the compatibility condition obtained by integrating (2.5b) with respect to x_3 from -1 to $+1$ and using the Neumann boundary conditions (2.5c). We obtain the system:

$$\partial_\beta\, A_{\alpha\beta33} \int_{-1}^{+1} \partial_3\, u_3^{2k}\, dx_3 + \partial_\beta\, A_{\alpha\beta\gamma\delta} \int_{-1}^{+1} e_{\gamma\delta}(u^{2k-2})\, dx_3 = 0, \tag{2.16a}$$

$$A_{\alpha3\gamma3}\, \partial_{33}\, u_\gamma^{2k} = -A_{\alpha3\gamma3}\, \partial_{3\gamma}\, u_3^{2k} - \partial_\beta\, A_{\alpha\beta33}\, \partial_3\, u_3^{2k} - \partial_\beta\, A_{\alpha\beta\gamma\delta}\, e_{\gamma\delta}(u^{2k-2}) \text{ in } \Omega, \tag{2.16b}$$

$$A_{\alpha3\gamma3}\, \partial_3\, u_\gamma^{2k} = -A_{\alpha3\gamma3}\, \partial_\gamma\, u_3^{2k}, \text{ on } \{x_3 = \pm 1\} \cap \partial\Omega. \tag{2.16c}$$

Let us formulate now our induction hypothesis for any $\ell \in \mathbb{N}$:

$$
(\mathscr{I}^\ell) \quad
\begin{cases}
\forall k \leq \ell - 1, \quad u^{2k} \text{ is determined and pb } (2.15) \;\&\; (2.16) \text{ solved}, \\[2mm]
v^{2\ell} \text{ is s.t.} \quad \forall x_* \in \overline{\omega}, \quad \displaystyle\int_{-1}^{+1} v^{2\ell}(x_*, x_3)\, dx_3 = 0, \\[2mm]
\qquad\qquad (2.15) \;\&\; (2.16) \text{ hold for } k = \ell \text{ with } u^{2\ell} = v^{2\ell}, \\[2mm]
\qquad\qquad (2.15a) \text{ holds for } k = \ell + 1 \text{ with } u^{2\ell} = v^{2\ell}.
\end{cases}
$$

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It is clear that $(\mathscr{I}^0)$ holds with $u^{2k} = 0$ for $k < 0$ and $v^0 = 0$.

Let us assume that $(\mathscr{I}^\ell)$ holds. Thus $v_3^{2\ell}$ is a solution of (2.15b)-(2.15c). Therefore, for any function $\zeta_3^{2\ell} \in \mathscr{C}^\infty(\overline{\omega})$,

$$v_3^{2\ell} + \zeta_3^{2\ell} \text{ is a solution of (2.15b)-(2.15c).}$$

Since $v_*^{2\ell}$ is a solution of (2.16b)-(2.16c) with $u_3^{2\ell} = v_3^{2\ell}$, we easily see that for any function $\zeta_*^{2\ell} \in \mathscr{C}^\infty(\overline{\omega})^2$,

$$v_*^{2\ell} + \zeta_*^{2\ell} - x_3 \nabla_* \zeta_3^{2\ell} \text{ is a solution of (2.16b)-(2.16c) with } u_3^{2\ell} = v_3^{2\ell} + \zeta_3^{2\ell}.$$

In other words, for any Kirchhoff-Love displacement $u_{\mathrm{KL}}^{2\ell}$, the function $u^{2\ell} = v^{2\ell} + u_{\mathrm{KL}}^{2\ell}$ solves (2.15) and (2.16) for $k = \ell$. Moreover, since (2.15a) holds for $k = \ell + 1$ with $u^{2\ell} = v^{2\ell}$, it is straightforward that (2.15a) still holds for $k = \ell + 1$ with $u^{2\ell} = v^{2\ell} + u_{\mathrm{KL}}^{2\ell}$. As $A_{3333} \neq 0$, for any Kirchhoff-Love displacement $u_{\mathrm{KL}}^{2\ell}$, there exists a unique function $v_3^{2\ell+2}$ with zero mean values over x_3 which solves problem (2.15) for $k = \ell + 1$ with $u^{2\ell} = v^{2\ell} + u_{\mathrm{KL}}^{2\ell}$.

Now we investigate condition (2.16a) for $k = \ell + 1$. We "eliminate" the term $\partial_\beta A_{\alpha\beta33} \int_{-1}^{+1} \partial_3 u_3^{2\ell+2} \, dx_3$ with the help of problem (2.15) at the step $k = \ell + 1$: integrating (2.15b) in x_3 from the limit -1 and taking advantage of (2.15c) we find that

$$A_{3333} \, \partial_3 \, u_3^{2\ell+2} = -A_{33\gamma\delta} \, e_{\gamma\delta}(u_*^{2\ell}) - 2 \, \partial_\alpha \, A_{\alpha3\gamma3} \int_{-1}^{x_3} e_{\gamma3}(u^{2\ell}).$$

We examine the contributions of $v^{2\ell}$ and $u_{\mathrm{KL}}^{2\ell}$. Since $e_{\gamma3}(u_{\mathrm{KL}}^{2\ell}) = 0$, we obtain:

$$A_{3333} \, \partial_3 \, u_3^{2\ell+2} = -A_{33\gamma\delta} \, e_{\gamma\delta}(\zeta_*^{2\ell} - x_3 \nabla_* \zeta_3^{2\ell}) - A_{33\gamma\delta} \, e_{\gamma\delta}(v_*^{2\ell}) - 2 \, \partial_\alpha \, A_{\alpha3\gamma3} \int_{-1}^{x_3} e_{\gamma3}(v^{2\ell}).$$

Whence, taking into account that the mean value of $e_{\gamma\delta}(x_3 \nabla_* \zeta_3^{2\ell})$ is zero, the expression for condition (2.16a) when $k = \ell + 1$:

$$-\partial_\beta \, A_{\alpha\beta33} \, A_{3333}^{-1} \int_{-1}^{+1} \left(A_{33\gamma\delta} \, e_{\gamma\delta}(\zeta_*^{2\ell}) + A_{33\gamma\delta} \, e_{\gamma\delta}(v_*^{2\ell}) + 2 \, \partial_\alpha \, A_{\alpha3\gamma3} \int_{-1}^{x_3} e_{\gamma3}(v^{2\ell}) \right) dx_3$$
$$+ \partial_\beta \, A_{\alpha\beta\gamma\delta} \int_{-1}^{+1} e_{\gamma\delta}(\zeta_*^{2\ell}) \, dx_3 + \partial_\beta \, A_{\alpha\beta\gamma\delta} \int_{-1}^{+1} e_{\gamma\delta}(v_*^{2\ell}) \, dx_3 = 0.$$

We put together the terms which depend only on $v^{2\ell}$ and, taking advantage of the fact that $\int_{-1}^{+1} v^{2\ell} \, dx_3 = 0$ we set:

$$F_\alpha(v^{2\ell}) = \partial_\beta \, A_{\alpha\beta33} \, A_{3333}^{-1} \int_{-1}^{+1} \left(2 \, \partial_\alpha \, A_{\alpha3\gamma3} \int_{-1}^{x_3} e_{\gamma3}(v^{2\ell}) \right) dx_3.$$

Noting that the integrands involving $\zeta_*^{2\ell}$ do not depend on x_3, our condition (2.16a) when $k = \ell + 1$ takes the form:

$$-\partial_\beta \, A_{\alpha\beta33} \, A_{3333}^{-1} \, A_{33\gamma\delta} \, e_{\gamma\delta}(\zeta_*^{2\ell}) + \partial_\beta \, A_{\alpha\beta\gamma\delta} \, e_{\gamma\delta}(\zeta_*^{2\ell}) = \frac{1}{2} F_\alpha(v^{2\ell}). \tag{2.17}$$

But, as a consequence of the positivity of A, (*see* D. CAILLERIE [5]), the 4×4 matrix:

$$M_{\alpha\beta\gamma\delta} = A_{\alpha\beta\gamma\delta} - A_{\alpha\beta33} \, A_{3333}^{-1} \, A_{33\gamma\delta} \text{ is positive definite.} \tag{2.18}$$

Thus (2.17) is the equation of a *membrane* in the mean surface ω associated with the new two-dimensional "homogenized" rigidity matrix $M_{\alpha\beta\gamma\delta}$. We can complete it by Dirichlet boundary conditions on $\partial\omega$: there exists a unique solution $\zeta_*^{2\ell}$ to the problem

$$
\begin{cases}
\zeta_* \in H^1(\omega)^2 \quad \text{and} \quad \zeta_*\big|_{\partial\omega} = h_*^{2\ell}, \\[2mm]
\forall v_* \in \mathring{H}^1(\omega)^2, \qquad \displaystyle\int_\omega M_{\alpha\beta\gamma\delta}\, e_{\gamma\delta}(\zeta_*)\, e_{\alpha\beta}(v_*) = -\frac{1}{2}\int_\omega F_\alpha(v^{2\ell})\, v_\alpha\,.
\end{cases}
\tag{2.19}
$$

For this choice of $\zeta_*^{2\ell}$, the compatibility condition (2.16a) holds for $k = \ell + 1$. As the matrix $(A_{\alpha 3\gamma 3})$ is invertible (see (2.8)) problem (2.16b)-(2.16c) is solvable.

In order to investigate the compatibility condition (2.15a) for $k = \ell + 2$, we split $v_3^{2\ell+2}$ into two pieces

$$
v_3^{2\ell+2} = y_3^{2\ell+2} + z_3^{2\ell+2} \quad \text{s.t.} \int_{-1}^{+1} y_3^{2\ell+2}\, dx_3 = \int_{-1}^{+1} z_3^{2\ell+2}\, dx_3 = 0
$$

$$
\text{and} \qquad y_3^{2\ell+2} \text{ sol. of (2.15) with } u^{2\ell} = v^{2\ell} + (\zeta_*^{2\ell}, 0)
$$

$$
z_3^{2\ell+2} \text{ sol. of (2.15) with } u^{2\ell} = (-x_3\, \nabla_*\, \zeta_3^{2\ell}, \zeta_3^{2\ell})
$$

$y_3^{2\ell+2}$ is now completely determined whereas $z_3^{2\ell+2}$ has yet to be constructed. Correspondingly, we split $v_*^{2\ell+2}$ into two pieces

$$
v_*^{2\ell+2} = y_*^{2\ell+2} + z_*^{2\ell+2} \quad \text{s.t.} \int_{-1}^{+1} y_*^{2\ell+2}\, dx_3 = \int_{-1}^{+1} z_*^{2\ell+2}\, dx_3 = 0
$$

$$
\text{and} \qquad y_*^{2\ell+2} \text{ sol. of (2.16) with } u^{2\ell} = v^{2\ell} + (\zeta_*^{2\ell}, 0) \text{ and } u_3^{2\ell+2} = y_3^{2\ell+2}
$$

$$
z_*^{2\ell+2} \text{ sol. of (2.16) with } u^{2\ell} = (-x_3\, \nabla_*\, \zeta_3^{2\ell}, \zeta_3^{2\ell}) \text{ and } u_3^{2\ell+2} = z_3^{2\ell+2}.
$$

Then condition (2.15a) for $k = \ell + 2$ reduces to

$$
\partial_\alpha A_{\alpha 3\gamma 3} \int_{-1}^{+1} e_{\gamma 3}(z^{2\ell+2})\, dx_3 = -\partial_\alpha A_{\alpha 3\gamma 3} \int_{-1}^{+1} e_{\gamma 3}(y^{2\ell+2})\, dx_3.
\tag{2.20}
$$

The relation (2.16b) for $k = \ell + 1$ gives us, by definition of $z^{2\ell+2}$:

$$
2\, A_{\alpha 3\gamma 3}\, \partial_3\, e_{\gamma 3}(z^{2\ell+2}) = -\partial_\beta A_{\alpha\beta 33}\, \partial_3\, z_3^{2\ell+2} - \partial_\beta A_{\alpha\beta\gamma\delta}\, e_{\gamma\delta}(-x_3\, \nabla_*\, \zeta_3^{2\ell}).
$$

But we easily compute that

$$
z_3^{2\ell+2} = A_{3333}^{-1}\, A_{33\gamma\delta}\left(\frac{x_3^2}{2} - \frac{1}{6}\right)\partial_{\gamma\delta}\zeta_3^{2\ell}.
$$

Whence the relation

$$
2\, A_{\alpha 3\gamma 3}\, \partial_3\, e_{\gamma 3}(z^{2\ell+2}) = -x_3\, \partial_\beta A_{\alpha\beta 33}\, A_{3333}^{-1}\, A_{33\gamma\delta}\, \partial_{\gamma\delta}\, \zeta_3^{2\ell} + x_3\, \partial_\beta A_{\alpha\beta\gamma\delta}\, \partial_{\gamma\delta}\, \zeta_3^{2\ell}.
$$

Integrating from -1 to x_3 and taking advantage of the relation (2.16c) for $k = \ell+1$ with $u^{2\ell+2} = z^{2\ell+2}$ and $u^{2\ell} = (-x_3\, \nabla_*\, \zeta_3^{2\ell}, \zeta_3^{2\ell})$ we obtain

$$
2\, A_{\alpha 3\gamma 3}\, e_{\gamma 3}(z^{2\ell+2}) = \left(\frac{x_3^2}{2} - \frac{1}{2}\right)\left(-\partial_\beta A_{\alpha\beta 33}\, A_{3333}^{-1}\, A_{33\gamma\delta}\, \partial_{\gamma\delta}\, \zeta_3^{2\ell} + \partial_\beta A_{\alpha\beta\gamma\delta}\, \partial_{\gamma\delta}\, \zeta_3^{2\ell}\right).
$$

Integrating once more in x_3 from -1 to $+1$ and derivating with respect to x_α, we reduce condition (2.20) to

$$-\partial_{\alpha\beta} A_{\alpha\beta 33} A_{3333}^{-1} A_{33\gamma\delta} \partial_{\gamma\delta} \zeta_3^{2\ell} + \partial_{\alpha\beta} A_{\alpha\beta\gamma\delta} \partial_{\gamma\delta} \zeta_3^{2\ell} = \frac{3}{2} F_3(y^{2\ell+2}) \tag{2.21}$$

where

$$F_3(y^{2\ell+2}) = 2\, \partial_\alpha A_{\alpha 3\gamma 3} \int_{-1}^{+1} e_{\gamma 3}(y^{2\ell+2})\, dx_3.$$

Thus (2.21) is the equation of *bending* in ω associated with the rigidity matrix $M_{\alpha\beta\gamma\delta}$ defined in (2.18). We can complete it by Dirichlet boundary conditions on $\partial\omega$: there exists a unique solution $\zeta_3^{2\ell}$ to the problem

$$\begin{cases} \zeta_3 \in H^2(\omega) \quad \text{and} \quad \zeta_3\big|_{\partial\omega} = h_3^{2\ell}, \quad \partial_r\zeta_3\big|_{\partial\omega} = h_n^{2\ell}, \\[2mm] \forall v_3 \in \mathring{H}^2(\omega), \qquad \int_\omega M_{\alpha\beta\gamma\delta}\, \partial_{\gamma\delta}(\zeta_3)\, \partial_{\alpha\beta}(v_3) = \frac{3}{2} \int_\omega F_3(y^{2\ell+2})\, v_3\,. \end{cases} \tag{2.22}$$

Finally, the Kirchhoff-Love displacement $u_{\mathrm{KL}}^{2\ell}$ is completely determined and the compatibility conditions (2.16a), resp. (2.15a), hold for $k = \ell+1$, resp. $k = \ell+2$, which ensures the existence and the uniqueness of $v^{2\ell+2}$ satisfying the zero mean condition over x_3. The induction condition $(\mathscr{I}^{\ell+1})$ is proved. ∎

2.4   An even Ansatz for problem (1.6)

The above algorithm can also be applied to the situation of problem (1.6), with the right-hand side $f(\varepsilon) = f^0$. In this case, we set $f^k = 0$ for any non zero integer k. But Theorem 2.4 below applies in the case when $f(\varepsilon)$ has an expansion like (1.5) with only even order terms. Then, instead of problems (2.4) and (2.5) we obtain

$$u_3^{2k} = 0 \ \ \text{on} \ \ \Gamma_0, \tag{2.23a}$$
$$A_{3333}\, \partial_{33}\, u_3^{2k} + A_{33\gamma\delta}\, \partial_3\, e_{\gamma\delta}(u^{2k-2}) + 2\, \partial_\alpha A_{\alpha 3\gamma 3}\, e_{\gamma 3}(u^{2k-2}) = -f_3^{2k-4} \ \ \text{in} \ \ \Omega, \tag{2.23b}$$
$$A_{3333}\, \partial_3\, u_3^{2k} + A_{33\gamma\delta}\, e_{\gamma\delta}(u^{2k-2}) = 0 \ \ \text{on} \ \ \{x_3 = \pm 1\} \cap \partial\Omega, \tag{2.23c}$$

$$u_\alpha^{2k} = 0 \ \ \text{on} \ \ \Gamma_0, \tag{2.24a}$$
$$2\, A_{\alpha 3\gamma 3}\, \partial_3\, e_{\gamma 3}(u^{2k}) + \partial_\beta A_{\alpha\beta 33}\, \partial_3\, u_3^{2k} + \partial_\beta A_{\alpha\beta\gamma\delta}\, e_{\gamma\delta}(u^{2k-2}) = -f_\alpha^{2k-2} \ \ \text{in} \ \ \Omega, \tag{2.24b}$$
$$A_{\alpha 3\gamma 3}\, e_{\gamma 3}(u^{2k}) = 0 \ \ \text{on} \ \ \{x_3 = \pm 1\} \cap \partial\Omega, \tag{2.24c}$$

Theorem 2.4 *Let for any $k \in \mathbb{N}$, a function f^{2k} be given in $\mathscr{C}^\infty(\overline{\Omega})^3$. Moreover, we set $u^{-2} = 0$. Then, for any $k \in \mathbb{N}$, there exist a unique displacement v^{2k} and a unique Kirchhoff-Love displacement u_{KL}^{2k} such that*

$$\begin{cases} u^{2k} := v^{2k} + u_{\mathrm{KL}}^{2k} \ \text{is solution of (2.23b)-(2.23c) \& (2.24b)-(2.24c)}, \\[2mm] \forall x_* \in \overline{\omega}, \quad \int_{-1}^{+1} v^{2k}(x_*, x_3)\, dx_3 = 0, \\[2mm] u_{\mathrm{KL}}^{2k} = \left(\zeta_*^{2k} - x_3 \nabla_* \zeta_3^{2k},\, \zeta_3^{2k}\right) \quad \text{and} \quad \zeta_3^{2k}\big|_{\partial\omega} = 0, \quad \partial_r\zeta_3^{2k}\big|_{\partial\omega} = 0, \quad \zeta_*^{2k}\big|_{\partial\omega} = 0. \end{cases}$$

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The functions v^{2k} and u_{KL}^{2k} are smooth on $\overline{\Omega}$.

The proof is similar to that of Theorem 2.3 and our calculations imply that the generating functions ζ_*^0 and ζ_3^0 of u_{KL}^0 are respectively the solutions of the membrane equations

$$\zeta_*^0 \in \mathring{H}^1(\omega)^2 \quad \text{and} \quad -\partial_\beta \, M_{\alpha\beta\gamma\delta} \, e_{\gamma\delta}(\zeta_*^0) = \frac{1}{2} \int_{-1}^{+1} f_\alpha^0 \, dx_3, \qquad (2.25)$$

and the bending equation

$$\zeta_3^0 \in \mathring{H}^2(\omega) \quad \text{and} \quad \partial_{\alpha\beta} \, M_{\alpha\beta\gamma\delta} \, \partial_{\gamma\delta}(\zeta_3^0) = \frac{3}{2} \int_{-1}^{+1} (f_3^0 + x_3 \, \partial_\alpha f_\alpha^0) \, dx_3 \,. \qquad (2.26)$$

Despite the fact that we may impose zero Dirichlet boundary conditions on the Kirchhoff-Love parts u_{KL}^{2k} of the Ansatz, it is impossible to prescribe the Dirichlet traces of the parts v^{2k} with zero mean values. Since $v^0 = 0$, the traces of $u^0(x) + \varepsilon^2 \, u^2(x) + \cdots$ on Γ_0 behave like $\mathcal{O}(\varepsilon^2)$ but they are not zero in general. These traces indicate the existence of boundary layers in the neighborhood of Γ_0.

3 THE REDUCED-NORMAL PROBLEM ON A SEMI-INFINITE STRIP FOR THE DETERMINATION OF THE BOUNDARY LAYER ANSATZ

3.1 Guess of the boundary layers

As we have just seen, the even Ansatz formally associated with $u(\varepsilon)$, namely

$$u^+(\varepsilon) := u^0(x) + \varepsilon^2 \, u^2(x) + \cdots + \varepsilon^{2k} \, u^{2k}(x) + \cdots \qquad (3.1)$$

yields a sort of approximation of $u(\varepsilon)$ which, in the sense of asymptotic expansions, satisfies

$$u^+(\varepsilon) \in H^1(\Omega)^3, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.2\mathrm{a})$$

$$\forall v \in V(\Omega), \qquad \int_\Omega A \, \kappa(\varepsilon)(u^+(\varepsilon)) : \kappa(\varepsilon)(v) = \int_\Omega f \cdot v, \qquad (3.2\mathrm{b})$$

but $u^+(\varepsilon)$ does not belong to $V(\Omega)$. Only the leading Kirchhoff-Love term u^0 belongs to $V(\Omega)$ and, indeed, it coincides with the limit of $u(\varepsilon)$ for the topology of $H^1(\Omega)^3$ as $\varepsilon \to 0$, *cf.* [8], [14], [15], [9] and [7].

Since $u(\varepsilon) - u^0$ belongs to $V(\Omega)$, we may hope that the even Ansatz can be completed by a *boundary layer series* so that we obtain the true complete asymptotics of $u(\varepsilon)$. By *boundary layer term* we mean a function defined on $\overline{\Omega}$ with its support concentrated in the neighborhood of Γ_0, the "width" of its support behaving like ε.

Such a sort of function would be conveniently described in the local coordinates (r, s, x_3) where r denotes the variable counted along the reentrant unit normal n_r

to $\partial\omega$ and $s \in S$ is a curvilinear coordinate along $\partial\omega$, see (2.10)-(2.11). Then a *boundary layer* term is given by a function of the type

$$w(\varepsilon) : (r, s, x_3) \longmapsto w(\frac{r}{\varepsilon}, s, x_3) \tag{3.3}$$

where w is a function of $(t, s, x_3) \in \mathbb{R}^+ \times S \times (-1, +1)$ and is called *profile*. We intend to search profiles w rapidly decreasing with respect to t which could compensate for the Dirichlet traces of $u^+(\varepsilon) - u^0 = \varepsilon^2 u^2(x) + \cdots +$. Ideally, $w(\varepsilon)$ has to fulfill the same Dirichlet conditions as $\varepsilon^2 u^2(x) + \cdots +$, the same Neumann conditions on $\{x_3 = \pm 1\} \cap \partial\Omega$ and also the homogeneous interior elasticity system inside Ω.

In this §3, we construct the equations governing the boundary layer terms and exhibit the behavior as $t \to +\infty$ of the solutions of these equations. In §4, we will adjust the different terms to obtain the complete asymptotics of u. As already hinted in the introduction, we will see that this is more complicated than the mere juxtaposition of the even Ansatz and a boundary layer series.

3.2 Position of the reduced-normal problem

We are interested in solving a problem with non-homogeneous Dirichlet traces like (2.1), in the new coordinates (r, s, x_3). The change of variables $x_* = (x_1, x_2) \mapsto (r, s)$, with $x_* = X_*(r, s)$ and the corresponding change in the displacements $u_* \mapsto (u_r, u_s)$ is such that, *cf* (2.11)-(2.14):

$$\begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix} = DX_* \begin{pmatrix} \partial_r \\ \partial_s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = DX_* \begin{pmatrix} u_r \\ u_s \end{pmatrix}.$$

As a consequence, if we define the new scaled tensor $\tilde{\kappa}(\varepsilon)$ similarly to $\kappa(\varepsilon)$ in (1.4a):

$$\tilde{\kappa}_{\alpha\beta}(\varepsilon)(v) = e_{\alpha\beta}(\partial_r, \partial_s, \partial_3)(v),$$
$$\tilde{\kappa}_{\alpha3}(\varepsilon)(v) = \varepsilon^{-1} e_{\alpha3}(\partial_r, \partial_s, \partial_3)(v),$$
$$\tilde{\kappa}_{33}(\varepsilon)(v) = \varepsilon^{-2} e_{33}(\partial_r, \partial_s, \partial_3)(v),$$

where $e_{11}(\partial_r, \partial_s, \partial_3)(v) = \partial_r u_r$, $e_{12}(\partial_r, \partial_s, \partial_3)(v) = \frac{1}{2}(\partial_r u_s + \partial_s u_r)$, etc... Problem (2.1) is transformed into a similar problem in the domain $\tilde{\Omega} = (0, \rho) \times S \times (-1, +1)$ with "lateral face" $\tilde{\Gamma}_0 = \{0\} \times S \times (-1, +1)$

$$w(\varepsilon) \in H^1(\tilde{\Omega})^3 \quad \text{and} \quad w(\varepsilon) = h(\varepsilon) \quad \text{on} \quad \tilde{\Gamma}_0, \tag{3.4a}$$

$$\forall v \in V(\tilde{\Omega}), \qquad \int_{\tilde{\Omega}} \tilde{A}\, \tilde{\kappa}(\varepsilon)(w(\varepsilon)) : \tilde{\kappa}(\varepsilon)(v) + \tilde{\mathcal{A}}'(w(\varepsilon), v) = 0, \tag{3.4b}$$

where the 9×9 matrix $\tilde{A} = \tilde{A}(r, s)$ has the same features as A (positivity, smoothness and property (1.3) on the coefficients) and $\tilde{\mathcal{A}}'$ is a first order integro-differential form; here $\boldsymbol{i}$ and $\boldsymbol{j}$ denote multi-indices in $\mathbb{N}^3$:

$$\tilde{\mathcal{A}}'(w, v) = \sum_{|\boldsymbol{i}|+|\boldsymbol{j}| \le 1} \int_{\tilde{\Omega}} c_{\boldsymbol{ij}}(r, s)\, \partial^{\boldsymbol{i}} w\, \partial^{\boldsymbol{j}} v.$$

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Keeping in mind that we want solutions of the form (3.3), we apply to problem (3.4) the change of variables

$$(r, s, x_3) \longmapsto (t, s, x_3) \quad \text{where} \quad t = \frac{r}{\varepsilon}$$

thus $\partial_t = \varepsilon \, \partial_r$ and for purposes of homogeneity we introduce the following change in the displacements

$$(w_r, w_s, w_3) \longmapsto (\varphi_t, \varphi_s, \varphi_3) = (\varepsilon \, w_r, \varepsilon \, w_s, w_3). \tag{3.5}$$

Now, problem (3.4) is transformed into the problem on $\widetilde{\Sigma} := \mathbb{R}^+ \times S \times (-1, +1)$:

$$\varphi \in H^1(\widetilde{\Sigma})^3 \quad \text{and} \quad \varphi = g \quad \text{on} \quad \widetilde{\Gamma}_0, \tag{3.6a}$$

$$\forall v \in V(\widetilde{\Sigma}), \quad \int_{\widetilde{\Sigma}} \widetilde{A}(\varepsilon t, s) \, e(\partial_t, 0, \partial_3)(\varphi) : e(\partial_t, 0, \partial_3)(v) + \sum_{k=1}^{4} \varepsilon^k \, \widetilde{\mathcal{A}}^k(\varphi, \, v) = 0, \tag{3.6b}$$

where $\widetilde{\mathcal{A}}^k$ is a second order integro-differential form:

$$\widetilde{\mathcal{A}}^k(\varphi, \, v) = \sum_{|i|, \, |j| \leq 1} \int_{\widetilde{\Sigma}} c_{ij}^k(\varepsilon t, s) \, \partial^i \varphi \, \partial^j v.$$

We have also to take into account the variable εt in the coefficients of the operators. With the help of the Taylor expansion of the coefficients c_{ij}^k in $t = 0$, we obtain for problem (3.6) the formal expansion

$$\varphi \in H^1(\widetilde{\Sigma})^3 \quad \text{and} \quad \varphi = g \quad \text{on} \quad \widetilde{\Gamma}_0, \tag{3.7a}$$

$$\forall v \in V(\widetilde{\Sigma}), \quad \int_{\widetilde{\Sigma}} \widetilde{A}(0, s) \, e(\partial_t, 0, \partial_3)(\varphi) : e(\partial_t, 0, \partial_3)(v) + \sum_{k \geq 1} \varepsilon^k \, \mathcal{B}^k(\varphi, \, v) = 0, \tag{3.7b}$$

where for any $k \geq 1$, $\mathcal{B}^k$ is a second order integro-differential form:

$$\mathcal{B}^k(\varphi, \, v) = \sum_{|i|, \, |j| \leq 1} \int_{\widetilde{\Sigma}} b_{ij}^k(t, s) \, \partial^i \varphi \, \partial^j v$$

whose coefficients b_{ij}^k are polynomials of t for each fixed s. Of course, we obtain finite expansions with remainders by Taylor formulas.

Thus, we see that the "principal part" (in the sense of the powers of ε) of this problem does not involve any more the tangential derivation ∂_s: thus the variable s is a mere parameter.

Definition 3.1 The matrix $\widetilde{A}(r, s)$ being the matrix transformed from A by the change of variables $(x_1, x_2) \mapsto (r, s)$, we denote by B the matrix

$$B = B(s) = (B_{ijkl}) = \widetilde{A}(0, s).$$

The matrix B is positive definite, depends smoothly on $s \in S$ and its coefficients satisfy the property (1.3). Let Σ^+ and γ_0 denote the semi-infinite strip and its lateral face:

$$\Sigma^+ = \{(t, x_3) \mid t > 0, \, -1 < x_3 < +1\} \quad \text{and} \quad \gamma_0 = \{(t, x_3) \mid t = 0, \, -1 < x_3 < +1\}.$$

The space of admissible displacements on Σ^+ is

$$V(\Sigma^+) = \{v \in H^1(\Sigma^+)^3 \mid v = 0 \ \text{on} \ \gamma_0\}. \tag{3.8}$$

For each s fixed in S, the *reduced-normal problem* is the mixed Dirichlet-Neumann problem in Σ^+ :

$$\varphi \in H^1(\Sigma^+)^3 \quad \text{and} \quad \varphi\big|_{\gamma_0} = g, \tag{3.9a}$$

$$\forall v \in V(\Sigma^+), \qquad \int_{\Sigma^+} B(s)\, e(\partial_t, 0, \partial_3)(\varphi) : e(\partial_t, 0, \partial_3)(v) = 0, \tag{3.9b}$$

with unknown $\varphi = (\varphi_t, \varphi_s, \varphi_3)$ and data $g = (g_t, g_s, g_3)$. ∎

3.3 Behavior at infinity of solutions of the reduced-normal problem

The above problem (3.9) is an elasticity problem associated with the linearized strain tensor $e(\partial_t, 0, \partial_3)$. So we can expect that the space $\mathcal{R}$ of the rigid displacements associated with the tensor $e(\partial_t, 0, \partial_3)$ will play an important role:

$$R \in \mathcal{R} \quad \Longleftrightarrow \quad e(\partial_t, 0, \partial_3)(R) = 0.$$

This means explicitly:

$$\partial_t R_t = 0, \ \ \partial_3 R_3 = 0; \quad \partial_t R_s = 0, \ \ \partial_3 R_s = 0; \quad \text{and} \quad \partial_t R_3 + \partial_3 R_t = 0.$$

This implies that the pair (R_t, R_3), as a function of the variables (t, x_3), is a rigid displacement whereas R_s is constant. So we find that

$$\mathcal{R} = \{R = (R_t, R_s, R_3) = (c_t, c_s, c_3) + c_n\,(-x_3, 0, t) \mid c_t, c_s, c_3, c_n \in \mathbb{R}\}. \tag{3.10}$$

The following result states that the solutions of problem (3.9) are exponentially decreasing towards rigid displacements as the distance to the clamped face γ_0 becomes "large".

Theorem 3.2 *Let s belong to S. For all $g \in H^{1/2}(\gamma_0)^3$ there exists a unique rigid displacement $R = R(s)(g) \in \mathcal{R}$ so that the problem*

$$\varphi \in H^1(\Sigma^+)^3 \quad \text{and} \quad \varphi\big|_{\gamma_0} = g + R\big|_{\gamma_0}, \tag{3.11a}$$

$$\forall v \in V(\Sigma^+), \qquad \int_{\Sigma^+} B(s)\, e(\partial_t, 0, \partial_3)(\varphi) : e(\partial_t, 0, \partial_3)(v) = 0, \tag{3.11b}$$

has a (unique) solution. Moreover, there is $\eta_0 > 0$, which does not depend on g, such that $e^{\eta_0 t}\,\varphi(t, x_3)$ is bounded on Σ^+.

Indeed, in order to take account of nonzero right hand-sides coming from the operators $\mathcal{B}^k$ of formula (3.7b), we are led to use the following more general statement. The natural spaces involved require an exponential decay at infinity:

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Definition 3.3 For $\eta \in \mathbb{R}$ let $H^1_\eta(\Sigma^+)$ be the space of functions v such that $e^{\eta t}v$ belongs to $H^1(\Sigma^+)$ and let $V'_\eta(\Sigma^+)$ be the space of distributions $w = (w_t, w_s, w_3)$ such that $e^{\eta t}w$ belongs to the dual space of $V(\Sigma^+)$. Let $\boldsymbol{K}^1_\eta(\Sigma^+)$ and $\boldsymbol{K}^{-1}_\eta(\Sigma^+)$ be the corresponding intersection spaces

$$\boldsymbol{K}^1_\eta(\Sigma^+) = \bigcap_{\bar\eta < \eta} H^1_{\bar\eta}(\Sigma^+)^3 \quad \text{and} \quad \boldsymbol{K}^{-1}_\eta(\Sigma^+) = \bigcap_{\bar\eta < \eta} V'_{\bar\eta}(\Sigma^+).$$

We define $\boldsymbol{K}^{\pm 1}_\eta(\Omega \times \mathbb{R}^+)$ in a similar way (such spaces describe the regularity of complete profiles $\Psi(x, t)$). ∎

Corollary 3.4 *For all $g \in H^{1/2}(\gamma_0)^3$ and for all $f \in \boldsymbol{K}^{-1}_{\eta_0}(\Sigma^+)$ there exists a unique rigid displacement $R = R(s)(g, f) \in \mathcal{R}$ so that the problem*

$$\varphi \in H^1(\Sigma^+)^3 \quad and \quad \varphi\Big|_{\gamma_0} = g + R\Big|_{\gamma_0}, \tag{3.12a}$$

$$\forall v \in V(\Sigma^+), \qquad \int_{\Sigma^+} B\, e(\partial_t, 0, \partial_3)(\varphi) : e(\partial_t, 0, \partial_3)(v) = \int_{\Sigma^+} f\, v, \tag{3.12b}$$

has a (unique) solution. Moreover, this solution φ belongs to $\boldsymbol{K}^1_{\eta_0}(\Sigma^+)$.

4 COMPLETE ASYMPTOTICS

4.1 The first terms in the boundary layer series

We consider once more "the" even Ansatz (3.1) that we have constructed in §2. Let us recall that each component u^{2k} is the sum of a term v^{2k} with null means with respect to x_3 and of a Kirchhoff-Love displacement u^{2k}_{KL}. Only two components are completely determined: viz. the first Kirchhoff-Love displacement u^0_{KL} whose generating functions ζ^0_* and ζ^0_3 are the unique solutions of problems (2.25) and (2.26) respectively, and the component v^2 which only depends on u^0.

The leading term in the Dirichlet trace of the even Ansatz $u^+(\varepsilon)$ is $\varepsilon^2\, u^2\big|_{\Gamma_0}$. We propose as first term of the boundary layer series

$$(\varepsilon\, w^1_r, \varepsilon\, w^1_s, \varepsilon^2\, w^2_3)\big(\frac{r}{\varepsilon}, s, x_3\big) \tag{4.1}$$

which, by the change of functions (3.5), corresponds to the new unknown

$$\varepsilon^2\, \varphi = (\varepsilon^2\, w^1_r, \varepsilon^2\, w^1_s, \varepsilon^2\, w^2_3).$$

This is the reason for this curious inhomogeneous form. This "first term" compensates only for the trace of $\varepsilon^2\, u^2_3$. So, for each fixed $s \in S$ we search for a solution $\varphi(s)(t, x_3)$ of problem (*cf.* (3.9))

$$\begin{cases} \varphi(s) \in H^1(\Sigma^+)^3 \quad \text{and} \quad \varphi(s)\Big|_{\gamma_0} = g(s), \\[2mm] \forall v \in V(\Sigma^+), \qquad \int_{\Sigma^+} B(s)\, e(\partial_t, 0, \partial_3)(\varphi(s)) : e(\partial_t, 0, \partial_3)(v) = 0, \end{cases} \tag{4.2}$$

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with data $g(s) = (0, 0, u_3^2(s)|_{\gamma_0})$. According to Theorem 3.2, we know that, for each fixed $s \in S$, there exists a rigid displacement $R = R(s) \in \mathcal{R}$ so that problem (4.2) with the new data $g(s) = (0, 0, u_3^2(s)|_{\gamma_0}) + R(s)|_{\gamma_0}$ has an exponentially decreasing solution, that we denote $\varphi^2(s)$. Then the Dirichlet traces of φ^2 are

$$\varphi_t^2(s)\Big|_{\gamma_0} = c_t(s) - c_n(s)\, x_3, \quad \varphi_s^2(s)\Big|_{\gamma_0} = c_s(s), \quad \varphi_3^2(s)\Big|_{\gamma_0} = c_3(s) + u_3^2(s)\Big|_{\gamma_0} \quad (4.3)$$

and returning to the boundary layer Ansatz (4.1):

$$\varepsilon\, w_r^1\Big|_{\gamma_0} = \varepsilon\Big(c_t - c_n\, x_3\Big), \quad \varepsilon\, w_s^1\Big|_{\gamma_0} = \varepsilon\, c_s, \quad \varepsilon^2\, w_3^2\Big|_{\gamma_0} = \varepsilon^2\Big(c_3 + u_3^2\Big|_{\gamma_0}\Big).$$

The additional trace c_3 in the last term causes no problem because we know that $u^2 = v^2 + u_{\mathrm{KL}}^2$ with v^2 uniquely determined, but the traces of the generating functions ζ_*^2 and ζ_3^2 remain to be chosen. In particular we can impose that $\zeta_3^2|_{\partial\omega} = c_3$.

The occurrence of the traces c_t, c_n and c_s at the level of ε is somewhat unexpected. Indeed, we may compensate for them by the help of a new Kirchhoff-Love displacement u_{KL}^1 which should satisfy the following properties: its traces $u_{\mathrm{KL},r}^1$ and $u_{\mathrm{KL},s}^1$ have to coincide with $c_t - c_n\, x_3$ and c_s respectively, whereas the trace of its third component $u_{\mathrm{KL},3}^1$ has to be 0. Moreover, its generating functions ζ_*^1 and ζ_3^1 have to satisfy the equations inside ω governing the first term of an (odd) Ansatz for a problem similar to (2.1), *cf.* (2.19) and (2.22).

Using the formula $u_{\mathrm{KL}}^1 = (\zeta_*^1 - x_3\, \nabla_*\, \zeta_3^1, \zeta_3^1)$, we obtain that ζ_*^1 and ζ_3^1 are the unique solutions of

$$\begin{cases} \zeta_*^1 \in H^1(\omega)^2 \quad \text{and} \quad \zeta_r^1\Big|_{\partial\omega} = c_t, \quad \zeta_s^1\Big|_{\partial\omega} = c_s, \\[2mm] \forall v_* \in \mathring{H}^1(\omega)^2, \qquad \displaystyle\int_\omega M_{\alpha\beta\gamma\delta}\, e_{\gamma\delta}(\zeta_*^1)\, e_{\alpha\beta}(v_*) = 0, \end{cases} \quad (4.4)$$

and

$$\begin{cases} \zeta_3^1 \in H^2(\omega) \quad \text{and} \quad \zeta_3^1\Big|_{\partial\omega} = 0, \quad \partial_r\zeta_3^1\Big|_{\partial\omega} = c_n, \\[2mm] \forall v_3 \in \mathring{H}^2(\omega), \qquad \displaystyle\int_\omega M_{\alpha\beta\gamma\delta}\, \partial_{\gamma\delta}(\zeta_3^1)\, \partial_{\alpha\beta}(v_3) = 0, \end{cases} \quad (4.5)$$

respectively (we recall that $M_{\alpha\beta\gamma\delta} = A_{\alpha\beta\gamma\delta} - A_{\alpha\beta33}\, A_{3333}^{-1}\, A_{33\gamma\delta}$ and that it is positive definite, *see* (2.18)).

Thus, in order to find one term of the boundary layer series, we have completely determined four traces, which corresponds exactly to the data at each step of the even Ansatz (Theorem 2.3).

4.2 Asymptotic expansion of arbitrary order

Here is our result concerning the formal solution of problem (1.6).

Theorem 4.1 *Let* $f \in \mathscr{C}^\infty(\Omega)^3$ *. For each* $k \in \mathbb{N}$ *there exist a unique Kirchhoff-Love displacement* u_{KL}^k *, a unique displacement* v^k *such that* $\int_{-1}^1 v^k(x_*, x_3)\, dx_3 = 0$ *for all* $x_* \in \omega$ *, and a unique profile* $w^k \in \boldsymbol{K}_{\eta_0}^1(\Sigma^+)$ *(see Corollary 3.4), such that the series*

$$U(\varepsilon)(x) = \sum_{k \in \mathbb{N}} \varepsilon^k \left(u_{\mathrm{KL}}^k(x) + v^k(x) - \chi(r)\, w^k(\frac{r}{\varepsilon}, s, x_3) \right) \qquad (4.6)$$

formally satisfies [*]

$$U(\varepsilon) \in V(\Omega), \qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.7\mathrm{a})$$

$$\forall v \in V(\Omega), \qquad \int_\Omega A\, \kappa(\varepsilon)(U(\varepsilon)) : \kappa(\varepsilon)(v) = \int_\Omega f \cdot v. \qquad (4.7\mathrm{b})$$

The smooth cut-off function χ *is equal to* 1 *in a neighborhood of* 0 *and equal to* 0 *for* $r \geq \rho$ *. The displacement* v^0 *is zero, and the profile* w^0 *is also zero. The third component* w_3^1 *of the profile* w^1 *is equal to* 0 *.*

Proof. We make use of an induction argument. We reformulate problems (2.15) and (2.16) as:

$$2\, \partial_\alpha\, A_{\alpha 3\gamma 3} \int_{-1}^{+1} e_{\gamma 3}(u^{k-2})\, dx_3 = 0, \qquad\qquad (4.8\mathrm{a})$$

$$\begin{cases} A_{3333}\, \partial_{33}\, u_3^k = -A_{33\gamma\delta}\, \partial_3\, e_{\gamma\delta}(u^{k-2}) - 2\, \partial_\alpha\, A_{\alpha 3\gamma 3}\, e_{\gamma 3}(u^{k-2}), & \text{in} \quad \Omega, \\[2mm] A_{3333}\, \partial_3\, u_3^k = -A_{33\gamma\delta}\, e_{\gamma\delta}(u^{k-2}) & \text{on} \quad \{x_3 = \pm 1\} \cap \partial\Omega, \end{cases} \qquad (4.8\mathrm{b})$$

and

$$\partial_\beta\, A_{\alpha\beta 33} \int_{-1}^{+1} \partial_3\, u_3^k\, dx_3 + \partial_\beta\, A_{\alpha\beta\gamma\delta} \int_{-1}^{+1} e_{\gamma\delta}(u^{k-2})\, dx_3 = 0, \qquad (4.9\mathrm{a})$$

$$\begin{cases} A_{\alpha 3\gamma 3}\, \partial_{33}\, u_\gamma^k = -A_{\alpha 3\gamma 3}\, \partial_{3\gamma}\, u_3^k - \partial_\beta\, A_{\alpha\beta 33}\, \partial_3\, u_3^k - \partial_\beta\, A_{\alpha\beta\gamma\delta}\, e_{\gamma\delta}(u^{k-2}) \text{ in } \Omega, \\[2mm] A_{\alpha 3\gamma 3}\, \partial_3\, u_\gamma^k = -A_{\alpha 3\gamma 3}\, \partial_\gamma\, u_3^k, & \text{on} \quad \{x_3 = \pm 1\} \cap \partial\Omega, \end{cases} \qquad (4.9\mathrm{b})$$

respectively.

Our induction hypothesis applies to the displacements u_{KL}^k and v^k, for $k \geq 0$, the traces h_r^k, h_n^k, h_s^k and h_3^k on $\partial\omega$, for $k \geq 0$, and the scaled profiles φ^k, for $k \geq 2$, which determine the profiles w^k by

$$\varphi_t^k = w_t^{k-1}, \quad \varphi_s^k = w_s^{k-1}, \quad \varphi_3^k = w_3^k. \qquad (4.10)$$

Let $\ell \geq 1$. Our induction hypothesis consists of the assertions (4.11)-(4.15) below and will be denoted $(\widetilde{\mathscr{I}}^\ell)$ in the sequel.

---

[*] This means that for any $N \in \mathbb{N}$, if one cuts the series by $k \leq N$, defining U^N, then the trace $U^N|_{\Gamma_0}$ is a $\mathcal{O}(\varepsilon^{N_1})$ with $N_1 \to \infty$ as $N \to \infty$, and the expansion with respect to ε of $v \mapsto \int_\Omega \left(A\,\kappa(\varepsilon)(U(\varepsilon)) : \kappa(\varepsilon)(v) - f \cdot v \right)$ in terms $\varepsilon^k\, \Phi^k(x, r/\varepsilon)$ with $\Phi^k(x, t) \in \boldsymbol{K}^{-1}(\Omega \times \mathbb{R}^+)$, yields a $\mathcal{O}(\varepsilon^{N_2})$ with $N_2 \to \infty$ as $N \to \infty$. This is different from the formal approach of [7] which, just like our algorithm for the "even Ansatz", does not include boundary layer terms.

We suppose that the following are determined:

$$
\begin{aligned}
u^k_{\mathrm{KL}} &\quad \text{for } k = 0, \ldots, 2\ell - 2, \\
v^k &\quad \text{for } k = 0, \ldots, 2\ell, \\
h^k_r,\ h^k_n,\ h^k_s &\quad \text{for } k = 0, \ldots, 2\ell - 2, \\
h^k_3 &\quad \text{for } k = 0, \ldots, 2\ell - 1, \\
\varphi^k &\quad \text{for } k = 1, \ldots, 2\ell - 1.
\end{aligned}
\tag{4.11}
$$

Letting $u^k = u^k_{\mathrm{KL}} + v^k$ *and* $u^k_{\mathrm{KL}} = (\zeta^k_* - x_3 \nabla_* \zeta^k_3, \zeta^k_3)$ *, we have the following properties:*

$$
\zeta^k_r \big|_{\partial\omega} = h^k_r, \quad \zeta^k_s \big|_{\partial\omega} = h^k_s, \quad \zeta^k_3 \big|_{\partial\omega} = h^k_3, \quad \partial_r \zeta^k_3 \big|_{\partial\omega} = h^k_n,
\tag{4.12}
$$

for $k = 0, \ldots, 2\ell - 2$ *, and*

$$
\begin{aligned}
u^k &\text{ solution of (4.8b)-(4.9b)} &\text{for } k = 0, \ldots, 2\ell - 2, \\
v^k &\text{ solution of (4.8b)-(4.9b)} &\text{for } k = 2\ell - 1,\ 2\ell, \\
v^k &\text{ satisfies (4.8a)} &\text{for } k = 2\ell + 1,\ 2\ell + 2.
\end{aligned}
\tag{4.13}
$$

Moreover, for all $s \in S$ *, set:*

$$
\begin{aligned}
g^k_t &= v^{k-1}_r \big|_{\Gamma_0} + h^{k-1}_r - x_3 \, h^{k-1}_n, \\
g^k_s &= v^{k-1}_s \big|_{\Gamma_0} + h^{k-1}_s - x_3 \, \partial_s h^{k-1}_3, \\
g^k_3 &= v^k_3 \big|_{\Gamma_0} + h^k_3,
\end{aligned}
\tag{4.14}
$$

Then, $\varphi^k(s)$ *belongs to* $\boldsymbol{K}^1_{\eta_0}(\Sigma^+)$ *and is solution of (cf. problem (3.7)):*

$$
\begin{cases}
\varphi^k(s) \big|_{\gamma_0} = g^k(s) \\[2mm]
\forall v \in V(\Sigma^+), \ \displaystyle\int_{\Sigma^+} B(s)\, e(\partial_t, 0, \partial_3)(\varphi^k) : e(\partial_t, 0, \partial_3)(v) = -\int_{\Sigma^+} \sum_{j=1}^{k-1} \mathcal{B}^j(\varphi^{k-j},\, v)
\end{cases}
\tag{4.15}
$$

for $k = 0, \ldots, 2\ell - 1$ *.*

We initialize the recurrence by setting $u^{-2} = u^{-1} = 0$, and we easily see that the condition $(\widetilde{\mathscr{I}}^1)$, i.e. (4.11)-(4.15) for $\ell = 1$, holds with u^0_{KL} the first term of the even Ansatz, v^0, $v^1 = 0$ and v^2 the corresponding term in the even Ansatz, h^0_r, h^0_n, h^0_s, h^0_3, h^1_3 all equal to 0, $\varphi^1 = 0$.

We assume that $(\widetilde{\mathscr{I}}^\ell)$ (namely (4.11)-(4.15)) holds for an $\ell \geq 1$ and we start proving $(\widetilde{\mathscr{I}}^{\ell+1})$.

Our first step is the determination of $\varphi^{2\ell}$. Since φ^k belongs to $\boldsymbol{K}^1_{\eta_0}(\Sigma^+)$ for $k = 1, \ldots, 2\ell - 1$, then the right hand side of (4.15), $\sum_{j=1}^{2\ell-1} \mathcal{B}^j(\varphi^{k-j})$, belongs to $\boldsymbol{K}^{-1}_{\eta_0}(\Sigma^+)$. Then we can apply Corollary 3.4 and conclude that there exist $c_r(s)$, $c_n(s)$, $c_s(s)$ and $c_3(s)$ such that problem (4.15) for $k = 2\ell$ and $g^{2\ell}(s)$ defined by:

$$
\begin{aligned}
g^{2\ell}_t &= v^{2\ell-1}_r \big|_{\Gamma_0} + c_r - x_3 \, c_n, \\
g^{2\ell}_s &= v^{2\ell-1}_s \big|_{\Gamma_0} - x_3 \, \partial_s h^{2\ell-1}_3 + c_s, \\
g^{2\ell}_3 &= v^{2\ell}_3 \big|_{\Gamma_0} + c_3,
\end{aligned}
$$

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admits a unique solution $\varphi^{2\ell}$ belonging to $\boldsymbol{K}^1_{\eta_0}(\Sigma^+)$. Now we set:

$$h_r^{2\ell-1} = c_r, \quad h_n^{2\ell-1} = c_n, \quad h_s^{2\ell-1} = c_s, \quad h_3^{2\ell} = c_3.$$

Relying on the existence of the odd Ansatz including $u^{2\ell-3}$ and $v^{2\ell-1}$ and on the same argument used in the proof of the even Ansatz in Theorem 2.3, we deduce the existence of $u_{\mathrm{KL}}^{2\ell-1}$ such that the boundary conditions (4.12) hold for $k = 2\ell-1$, and the existence of $v^{2\ell+1}$, both of them satisfying (*cf.* (4.13)):

$$\begin{aligned} &u^k \text{ solution of (4.8b)-(4.9b)} &&\text{for } k = 2\ell - 1, \\ &v^k \text{ solution of (4.8b)-(4.9b)} &&\text{for } k = 2\ell + 1, \\ &v^{2\ell+1} \text{ satisfies (4.8a)} &&\text{for } k = 2\ell + 3. \end{aligned}$$

Next we determine $\varphi^{2\ell+1}$ as before, which yields the values of $h_r^{2\ell}$, $h_n^{2\ell}$ and $h_s^{2\ell}$. As $h_3^{2\ell}$ was already determined in the previous step, we rely on the even Ansatz to deduce the existence of $u_{\mathrm{KL}}^{2\ell}$ and $v^{2\ell+2}$ fulfilling the conditions involved in $(\widetilde{\mathscr{I}}^{\ell+1})$. $\blacksquare$

4.3 Regularity of the terms of the polynomial Ansatz

Let us assume that for a positive integer m, the right-hand side f in problem (1.6) has the following regularity

$$\begin{cases} f_\alpha \in L^2(I, H^{m-1}(\omega)) \cap H^{m-1}(I, L^2(\omega)) \\ f_3 \in H^1(I, H^{m-2}(\omega)) \cap H^{m-1}(I, L^2(\omega)). \end{cases} \tag{4.16}$$

Here I denotes the interval $(-1, +1)$. We note that, in fact f_α belongs to $H^{m-1}(\Omega)$ for $\alpha = 1, 2$ and that f_3 is a little less regular: f_3 and $\partial_3 f_3$ both belong to $H^{m-2}(\Omega)$. Starting from (4.16), we are going to follow the algorithms of the proofs of Theorems 2.3 and 4.1 in order to determine the regularity of the different terms in the asymptotics of $u(\varepsilon)$.

As the midsurface ω is smooth, by the ellipticity of the membrane and bending problems, we obtain

$$\zeta_\alpha^0 \in H^{m+1}(\omega) \quad \text{and} \quad \zeta_3^0 \in H^{m+2}(\omega). \tag{4.17}$$

From equation (2.23b) we deduce that $\partial_{33} v_3^2$ belongs to $H^\infty(I, H^m(\omega))$, whence solving a Neumann problem in the variable x_3, v_3^2 also belongs to $H^\infty(I, H^m(\omega))$. Next, from equation (2.24b) we deduce that $\partial_{33} v_\alpha^2$ belongs to $L^2(I, H^{m-1}(\omega)) \cap H^{m-1}(I, L^2(\omega))$, whence $v_\alpha^2 \in H^2(I, H^{m-1}(\omega)) \cap H^{m+1}(I, L^2(\omega))$.

As a consequence the trace of v_3^2 on Γ_0 belongs to $H^{1/2}(I, H^{m-1/2}(\partial\omega))$. Thus the traces c_t, c_s, c_n *cf.* (4.4) and (4.5), of the generating functions of u_{KL}^1 satisfy

$$\zeta_\alpha^1\big|_{\partial\omega} \in H^{m-1/2}(\partial\omega) \quad \text{and} \quad \partial_n \zeta_3^1\big|_{\partial\omega} \in H^{m-1/2}(\partial\omega).$$

Whence by solution of the membrane and bending problems:

$$\zeta_\alpha^1 \in H^m(\omega) \quad \text{and} \quad \zeta_3^1 \in H^{m+1}(\omega). \tag{4.18}$$

Returning to the "even Ansatz", we also deduce from the regularity of v^2 that $e_{\gamma 3}(v^2)$ belongs to $H^1(I, H^{m-1}(\omega)) \cap H^m(I, L^2(\omega))$, whence the right hand side $F_\alpha(v^2)$ of problem (2.19) is in $H^{m-3}(\omega)$ and by elliptic regularity ζ_α^2 is in $H^{m-1}(\omega)$. Going on, we obtain that, by solution of the Neumann problems (2.15b)-(2.15c) and (2.16b)-(2.16c), y_3^4 belongs to $H^3(I, H^{m-2}(\omega)) \cap H^{m+1}(I, L^2(\omega))$ and y_α^4 to $H^4(I, H^{m-3}(\omega)) \cap H^{m+1}(I, L^2(\omega))$. Whence $e_{\gamma 3}(y^4)$ belongs to $H^3(I, H^{m-3}(\omega)) \cap H^m(I, L^2(\omega))$ and the right hand side $F_3(y^4)$ of problem (2.22) to $H^{m-4}(\omega)$. Therefore ζ_3^2 belongs to $H^m(\omega)$.

Going on, we prove

Lemma 4.2 *Under the assumption* (4.16), *we have for any* $0 \le k \le m$

$$\zeta_\alpha^k \in H^{m+1-k}(\omega) \quad and \quad \zeta_3^k \in H^{m+2-k}(\omega), \tag{4.19}$$

and for $2 \le k \le m$

$$\begin{cases} v_\alpha^k \in H^k(I, H^{m+1-k}(\omega)) \cap H^{m+1}(I, L^2(\omega)) \\ v_3^k \in H^{k-1}(I, H^{m+2-k}(\omega)) \cap H^{m+1}(I, L^2(\omega)), \end{cases} \tag{4.20}$$

or, equivalently

$$\begin{cases} \partial_3^\ell v_\alpha^k \in H^{m+1-k}(\Omega), \quad 0 \le \ell \le k \\ \partial_3^\ell v_3^k \in H^{m+2-k}(\Omega), \quad 0 \le \ell \le k-1. \end{cases} \tag{4.21}$$

Of course, if f is $\mathscr{C}^\infty(\overline{\Omega})$, all the terms u_{KL}^k and v_k of the polynomial Ansatz are also $\mathscr{C}^\infty(\overline{\Omega})$. On the other hand, the regularity of the profiles w^k is severely limited by the singularities along the edges of the plate (*see* Part II).

5 ERROR ESTIMATES IN ENERGY NORM

5.1 The main error estimate.

For the justification of an asymptotic expansion, two methods may be considered according as to whether one knows only the leading term or one can compute terms of arbitrarily high orders. The first one has given rise to numerous articles on the subject (*cf.* [8], [7] and the references that we quoted in §1.3) and results are by now classical: In [7], it is shown that the three-dimensional solution $u(\varepsilon)$ of the scaled problem converges for the topology of $H^1(\Omega)$ towards the leading term u^0 of the Ansatz. There, no mention is made about the way it converges, except in [14, 15, 17] where an estimation of the error is obtained.

The other method which is actually considered here, uses the formal asymptotic expansion (4.6) which is valid up to an arbitrarily high order N. It yields an *optimal estimation of the error between the scaled displacement $u(\varepsilon)$ and the Ansatz of order N*. With the notations already used in Theorem 4.1, we state the main result:

Theorem 5.1 *Let* $u(\varepsilon)$ *be the solution of problem* (1.6). *There holds* $\forall N \geq 0$:

$$\left\| u(\varepsilon) - \sum_{k=0}^{N} \varepsilon^k \, u^k + \chi(r) \sum_{k=1}^{N} \varepsilon^k \, w^k(\frac{r}{\varepsilon}, s, x_3) \right\|_{H^1(\Omega)^3} \leq C \, \varepsilon^{N+1/2} \qquad (5.1)$$

where $u^k = u^k_{\mathrm{KL}} + v^k$ *and with* u^k_{KL}, v^k, w^k *and* χ *as defined in Theorem* 4.1.

As developed in the next section, the proof of this theorem relies on energy estimates, for which we do not try to reach optimality, and on a very simple argument consisting in pushing the development a few terms further and relying on the following estimates of each term in the asymptotics:

$$\|u^k\|_{H^1(\Omega)} \leq C \quad \text{and} \quad \|\chi(r) \, w^k(\frac{r}{\varepsilon}, s, x_3)\|_{H^1(\Omega)} \leq C \, \varepsilon^{-1/2}. \qquad (5.2)$$

The corresponding L^2-estimates are

$$\|u^k\|_{L^2(\Omega)} \leq C \quad \text{and} \quad \|\chi(r) \, w^k(\frac{r}{\varepsilon}, s, x_3)\|_{L^2(\Omega)} \leq C \, \varepsilon^{1/2}. \qquad (5.3)$$

So, we can deduce in a straightforward way the L^2-estimates (1.14c) and (1.14d) for the displacement and (1.18) for the strain tensor from Theorem 5.1 and from (5.2)-(5.3).

5.2 Proof of Theorem (5.1)

Setting:

$$\overline{U}^N(\varepsilon) = u(\varepsilon) - U^N(\varepsilon), \qquad (5.4)$$

where $U^N(\varepsilon)$ denotes the asymptotic expansion of order N, namely:

$$U^N(\varepsilon) = \sum_{k=0}^{N} \varepsilon^k \, u^k - \chi(r) \sum_{k=1}^{N} \varepsilon^k \, w^k(\frac{r}{\varepsilon}, s, x_3) \qquad (5.5)$$

we see that it is sufficient to establish an a priori estimate for the remainder $\overline{U}^N(\varepsilon)$ in the norm of the space $H^1(\Omega)^3$. Thanks to the coercivity of the elasticity operator and to Korn's inequality which holds for the elements of the space $V(\Omega)$, this will follow from an energy method which consists in an estimation of the energy associated with the remainder, namely:

$$\int_\Omega A\kappa(\varepsilon)(\overline{U}^N(\varepsilon)) : \kappa(\varepsilon)(\overline{U}^N(\varepsilon)).$$

(i) THE ENERGY METHOD. We split $U^N(\varepsilon)$ into its two natural parts:

$$U^N(\varepsilon) = V^N(\varepsilon) - \chi(r) \, W^N(\varepsilon) \qquad (5.6)$$

where

$$V^N(\varepsilon) = \sum_{k=0}^{N} \varepsilon^k \, u^k \quad \text{and} \quad W^N(\varepsilon) = \sum_{k=1}^{N} \varepsilon^k \, w^k(\frac{r}{\varepsilon}, s, x_3). \qquad (5.7)$$

A close look at the construction of the complete asymptotics, in particular to (4.10), (4.14) and (4.15), allows to prove:

Lemma 5.2 *For any $N \in \mathbb{N}$, $U^N(\varepsilon)$ belongs to the space $V(\Omega)$ of admissible displacements satisfying the Dirichlet condition on the clamped part of the boundary.*

Thus, we have

$$\forall N \in \mathbb{N}, \qquad \overline{U}^N(\varepsilon) \in V(\Omega), \tag{5.8}$$

and $\overline{U}^N(\varepsilon)$ is the solution of the problem:

$$\overline{U}^N(\varepsilon) \in V(\Omega), \tag{5.9a}$$

$$\forall v \in V(\Omega), \quad \begin{cases} \displaystyle\int_\Omega A\,\kappa(\varepsilon)\big(\overline{U}^N(\varepsilon)\big) : \kappa(\varepsilon)(v) = \int_\Omega \big(F^N(\varepsilon) - G^N(\varepsilon)\big) \cdot v \\ \qquad\qquad + \displaystyle\int_{\partial\Omega \cap \{x_3 = \pm 1\}} \big(f^N(\varepsilon) - g^N(\varepsilon)\big) \cdot v \end{cases} \tag{5.9b}$$

where we have decomposed the error into the two following contributions: for all $v \in V(\Omega)$,

$$\int_\Omega A\,\kappa(\varepsilon)\big(u(\varepsilon) - V^N(\varepsilon)\big) : \kappa(\varepsilon)(v) = \int_\Omega F^N(\varepsilon) \cdot v + \int_{\partial\Omega \cap \{x_3 = \pm 1\}} f^N(\varepsilon) \cdot v, \tag{5.10}$$

and

$$\int_\Omega A\,\kappa(\varepsilon)\big(\chi(r)\,W^N(\varepsilon)\big) : \kappa(\varepsilon)(v) = \int_\Omega G^N(\varepsilon) \cdot v + \int_{\partial\Omega \cap \{x_3 = \pm 1\}} g^N(\varepsilon) \cdot v. \tag{5.11}$$

In particular, for the choice $v = \overline{U}^N(\varepsilon)$ of the test function, we obtain the energy of the elasticity problem as the left-hand side of the variational equation (5.9b). Korn's inequality and the coercivity of the operator of elasticity will allow the method of energy to provide the following rough estimates.

Lemma 5.3

$$\|\overline{U}^N(\varepsilon)\|_{H^1(\Omega)^3} \leq C\,\varepsilon^{N-3}. \tag{5.12}$$

Proof. We have to estimate the right hand sides of (5.10) and (5.11).

Considering the equations (4.8) and (4.9) satisfied by u^k, and noting that the power of ε multiplying (4.8) is ε^{k-4} and the one multiplying (4.9) is ε^{k-2}, we obtain that:

$$F^N(\varepsilon) = \mathcal{O}(\varepsilon^{N-3}) \quad \text{and} \quad f^N(\varepsilon) = \mathcal{O}(\varepsilon^{N-3}). \tag{5.13}$$

Concerning (5.11), we first note

$$\left| \int_\Omega A\,\kappa(\varepsilon)\big(\chi\,W^N(\varepsilon)\big) : \kappa(\varepsilon)(v) - \int_\Omega A\,\kappa(\varepsilon)\big(W^N(\varepsilon)\big) : \kappa(\varepsilon)(\chi\,v)\right| \leq$$

$$\leq C\,\varepsilon^{-1}\Big(\|\chi'\,W^N\|_{L^2}\,\|\kappa(\varepsilon)(v)\|_{L^2} + \|v\|_{L^2}\,\|\chi'\,\kappa(\varepsilon)(W^N)\|_{L^2}\Big).$$

Taking advantage of the fact that w^k, or equivalently φ^k, is an element of the space $\boldsymbol{K}^1_{\eta_0}(\Sigma^+)$, (see Definition 3.3), we obtain:

$$\exists\, c > 0, \gamma > 0: \qquad \|\chi'\,W^N\|_{L^2} + \|\chi'\,\kappa(\varepsilon)(W^N)\|_{L^2} \leq c\,e^{-\gamma/\varepsilon}.$$

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Thus, there exist $c' > 0$ and $\gamma' > 0$, such that:

$$\left| \int_\Omega A\,\kappa(\varepsilon)(\chi\,W^N) : \kappa(\varepsilon)(v) - \int_\Omega A\,\kappa(\varepsilon)(W^N) : \kappa(\varepsilon)(\chi\,v) \right| \le c'\, e^{-\gamma'/\varepsilon} \|v\|_{H^1}. \qquad (5.14)$$

Setting

$$\tilde{v}(t, s, x_3) = \varepsilon^{-1}\,(\varepsilon\,\chi\,v_*,\ \chi\,v_3)(r, s, x_3) \quad \text{and} \quad \Phi^N(t, s, x_3) = (\varepsilon\,W_*^N,\ W_3^N)(r, s, x_3)$$

and relying on (3.6b), (3.7b) and (4.15), we obtain

$$\left| \int_{\widetilde{\Sigma}} \widetilde{A}(\varepsilon t, s)\, e(\partial_t, 0, \partial_3)(\Phi^N) : e(\partial_t, 0, \partial_3)(\tilde{v}) + \sum_{k=1}^4 \varepsilon^k\, \widetilde{\mathcal{A}}^k(\Phi^N,\ \tilde{v}) \right| = \mathcal{O}(\varepsilon^{N+1})\, \|\tilde{v}\|_{H^1(\widetilde{\Sigma})},$$

whence, with (3.4b):

$$\left| \int_\Omega A\,\kappa(\varepsilon)\big(W^N(\varepsilon)\big) : \kappa(\varepsilon)(\chi\,v) \right| = \mathcal{O}(\varepsilon^{N-1/2})\, \|v\|_{H^1(\Omega)}. \qquad (5.15)$$

Putting (5.14) and (5.15) together, we have:

$$\left| \int_\Omega A\,\kappa(\varepsilon)\big(\chi\,W^N(\varepsilon)\big) : \kappa(\varepsilon)(v) \right| = \mathcal{O}(\varepsilon^{N-1/2})\, \|v\|_{H^1(\Omega)}. \qquad (5.16)$$

Now, from (5.9) with (5.10), (5.11), (5.13) and (5.16), we may deduce an upper bound for the elastic energy associated with the remainder $\overline{U}^N(\varepsilon)$:

$$\int_\Omega A\,\kappa(\varepsilon)(\overline{U}^N(\varepsilon)) : \kappa(\varepsilon)(\overline{U}^N(\varepsilon)) \le C\,\varepsilon^{N-3} \|\overline{U}^N(\varepsilon)\|_{H^1(\Omega)} \qquad (5.17)$$

and we get from Korn's inequality, as $\overline{U}^N(\varepsilon)$ is an element of the space $V(\Omega)$:

$$\int_\Omega A\,\kappa(\varepsilon)(\overline{U}^N(\varepsilon)) : \kappa(\varepsilon)(\overline{U}^N(\varepsilon)) \le C\,\varepsilon^{N-3} \|\kappa(\varepsilon)(\overline{U}^N(\varepsilon))\|_{L^2(\Omega)}. \qquad (5.18)$$

We use Korn's inequality again and the positivity of the matrix A to conclude. ∎

(ii) IMPROVEMENT OF THE ESTIMATE. Lemma 5.3 yields the following estimate for $\overline{U}^{N+4}(\varepsilon)$:

$$\|\overline{U}^{N+4}(\varepsilon)\|_{H^1(\Omega)} \le C\,\varepsilon^{N+1}, \qquad (5.19)$$

whence:

$$\|u(\varepsilon) - \sum_{k=0}^N \varepsilon^k\, u^k + \chi(r) \sum_{k=1}^N \varepsilon^k\, w^k(\frac{r}{\varepsilon}, s, x_3)\|_{H^1(\Omega)^3} \le$$

$$\le C\,\varepsilon^{N+1} + \sum_{k=N+1}^{N+4} \varepsilon^k \left(\|u^k\|_{H^1(\Omega)^3} + \|\chi(r)\,w^k(\frac{r}{\varepsilon}, s, x_3)\|_{H^1(\Omega)^3} \right). \qquad (5.20)$$

Relying on the estimates (5.2) of each term u^k and w^k for $N+1 \le k \le N+4$, the estimate (5.1) directly follows from (5.20).

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