Koiter Estimate Revisited

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Abstract. *We prove a universal energy estimate between the solution of the three-dimensional Lamé system on a thin* clamped *shell and a displacement reconstructed from the solution of the classical Koiter model. The mid-surface S of the shell is an arbitrary smooth manifold with boundary. The bound of our energy estimate only involves the thickness parameter ε, constants attached to S, the loading, the two-dimensional energy of the solution of the Koiter model and "wave-lengths" associated with this latter solution. This result is in the same spirit as Koiter's who gave a heuristic estimate in* [21]. *Taking boundary layers into account, we obtain rigorous estimates, which prove to be sharp in the cases of plates and elliptic shells.*

1 INTRODUCTION

This paper deals with *shell theory* whose aim is the approximation of the three-dimensional linear elastic shell problem by a two-dimensional problem posed on the mid-surface. This is an old and difficult question. As written by KOITER & SIMMONDS in 1972 [23] *"Shell theory attempts the impossible: to provide a two-dimensional representation of an intrinsically three-dimensional phenomenon."*

1.A FRAMEWORK

Let us recall that a shell is a three-dimensional object characterized by its mid-surface S and its (half-)thickness ε. The mid-surface is a two-dimensional manifold embedded in $\mathbb{R}^3$. We assume that S if a $\mathscr{C}^\infty$ smooth compact orientable manifold with boundary. Let $S \ni P \mapsto \boldsymbol{n}(P) \in \mathbb{R}^3$ be a continuous unit normal field on S. We denote the shell by Ω^ε in order to remind the value ε of the thickness parameter which is small enough, $0 < \varepsilon \leq \varepsilon_0$, so that the representation

$$S \times (-\varepsilon, \varepsilon) \ni (P, x_3) \mapsto P + x_3\, \boldsymbol{n}(P) \in \mathbb{R}^3, \tag{1.1}$$

1E1

is a $\mathscr{C}^\infty$ diffeomorphism onto Ω^ε. In simpler words, Ω^ε is the surface S thickened in its normal direction by the thickness ε. Of course, if S a plane domain, Ω^ε is a plate.

As material law for the body Ω^ε, the most standard assumption is to consider the case of an homogeneous and isotropic material like in the literature quoted below. Such a material is characterized by its Lamé constants λ and μ, or, alternatively by its Young modulus E and its Poisson coefficient ν. For a given load $\boldsymbol{f}$, let $\boldsymbol{u}$ be the displacement field, solution of the problem (P_{3D}) consisting of the three-dimensional Lamé system on Ω^ε with clamped boundary conditions on its lateral boundary. We consider this $\boldsymbol{u}$ as the "exact" solution and address the question of the approximation of $\boldsymbol{u}$ via the solution $\boldsymbol{z}$ of a problem (P_{2D}) posed on the mid-surface S.

Many papers deal with this question. Concerning the classical aspects of the derivation of shell models, let us quote KOITER [20, 21, 22], JOHN [18], NAGHDI [25], NOVOZHILOV [27]. Concerning plates, the derivation of the first two-dimensional model was done much earlier, see KIRCHHOFF [19].

Most of classical shell models rely on a 3×3 system of equations on S depending on ε, which can be written in the form

$$\mathsf{K}(\varepsilon) := \mathsf{M} + \varepsilon^2\mathsf{B} \tag{1.2}$$ 1E2

where M is the *membrane* operator on S and B a *bending* operator. The above authors all agree about the definition of the membrane operator M. On the contrary, different expressions for B can be found in the literature. The most natural in a geometrical and mechanical point of view, is the one given by W. T. KOITER (see [21]) but the question of determining the *best* model was very controversial (see in particular [4] and the discussion in [22, 25]). Without special mention, *we always take* $\mathsf{K}(\varepsilon)$ *as the Koiter operator*.

So the equation in the mid-surface S takes the form $\mathsf{K}(\varepsilon)\boldsymbol{z} = \boldsymbol{g}$, with the mean value $\boldsymbol{g}$ of the load $\boldsymbol{f}$ across each normal fiber to S. When considering laterally clamped shells, this equation has to be complemented by the Dirichlet boundary condition and defines problem (P_{2D}). The unique solvability of this problem was proved by BERNADOU & CIARLET [3]. Let $\boldsymbol{z}$ be the solution of problem (P_{2D}). Natural questions arise:

- Q1 Is $\boldsymbol{z}$ itself a "valid" approximation of $\boldsymbol{u}$? In what sense ?
- Q2 Is it possible to reconstruct with $\boldsymbol{z}$ only, a three-dimensional displacement $\mathsf{U} = \mathsf{U}\boldsymbol{z}$ which would be an approximation of $\boldsymbol{u}$ in (relative) energy norm?

To the authors' knowledge, the first question to be addressed was Q2, by KOITER himself. Indeed, the energy norm seems to be the most natural one and the easiest to deal with. But, in general, $\boldsymbol{z}$ *is not an approximation of* $\boldsymbol{u}$ *in energy norm*, but in weaker norms, as stated and proved by SANCHEZ-PALENCIA [29] and CIARLET, LODS, MIARA [6, 8, 7] who gave answers to question Q1. Let us go back to Q2, which is our main point of interest.

1.B KOITER ESTIMATE

KOITER proposed for $\mathsf{U}\boldsymbol{z}$ a *modified Kirchhoff-Love* three-dimensional displacement, which we may write as

$$\mathsf{U}\boldsymbol{z} := \mathsf{U}^{\mathsf{KL}}\boldsymbol{z} + \mathsf{U}^{\mathsf{cmp}}\boldsymbol{z}, \tag{1.3}$$ 1E3

where $\mathsf{U}^{\mathsf{KL}}\boldsymbol{z}$ is the Kirchhoff-Love displacement associated with $\boldsymbol{z}$ and the complementary term $\mathsf{U}^{\mathsf{cmp}}\boldsymbol{z}$ is a transverse displacement quadratic in the normal variable x_3. It is easy to provide formulas for U^{KL} and $\mathsf{U}^{\mathsf{cmp}}$ in the case of plates: In the situation with zero curvature, we choose as system of coordinates the Cartesian coordinates (x_1, x_2, x_3) with x_1, x_2 (also denoted (x_α)) coordinates in the plane containing S and x_3 the coordinate in normal direction. The corresponding components of U are U_α, $\alpha = 1, 2$ and U_3, and similarly for the components z_α and z_3 of $\boldsymbol{z}$. We have

$$\begin{aligned} &\mathsf{U}^{\mathsf{KL}}_\alpha\boldsymbol{z} = z_\alpha - x_3\partial_\alpha z_3, && \mathsf{U}^{\mathsf{cmp}}_\alpha\boldsymbol{z} = 0,\\ &\mathsf{U}^{\mathsf{KL}}_3\boldsymbol{z} = z_3, && \mathsf{U}^{\mathsf{cmp}}_3\boldsymbol{z} = -p\,x_3(\partial_1 z_1 + \partial_2 z_2) + p\,\tfrac{x_3^2}{2}(\partial_1^2 + \partial_2^2)z_3, \end{aligned} \tag{1.4}$$

1E4

where $p = \lambda(\lambda + 2\mu)^{-1}$. Formulas for general shells are a natural geometrical extension of these formulas, see (2.19) later.

In his main papers [21, 22], KOITER obtained the following tentative energy estimate:

$$\mathsf{E}^\varepsilon_{\mathsf{3D}}\big[\boldsymbol{u} - \mathsf{U}\boldsymbol{z}\big] \le C_S\left(\frac{\varepsilon}{R} + \frac{\varepsilon^2}{L^2}\right)\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}], \tag{1.5}$$

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where $\mathsf{E}^\varepsilon_{\mathsf{3D}}$ is the quadratic energy functional associated with problem (P_{3D}) and $\mathsf{E}^\varepsilon_{\mathsf{2D}}$ is the quadratic "physical" energy associated with problem (P_{2D}). Moreover $1/R$ denotes the maximum principal curvature of S and L a "wave length" associated with the solution $\boldsymbol{z}$. Indeed L is a constant appearing in *inverse estimates* concerning the membrane and bending tensors of $\boldsymbol{z}$, see §2.E later.

It turns out that estimate (1.5) is not true: The error between $\boldsymbol{u}$ and $\mathsf{U}\boldsymbol{z}$ in energy norm is not bounded by the right-hand side quantity, in general, see the following paragraph. Nevertheless this same quantity happens to be a bound for the difference between the three-dimensional energy of $\mathsf{U}\boldsymbol{z}$ and the two-dimensional energy associated with $\boldsymbol{z}$: In a first part of this work, we indeed show the following bound for the difference between the energies of $\boldsymbol{z}$ and $\mathsf{U}\boldsymbol{z}$:

$$\left|\mathsf{E}^\varepsilon_{\mathsf{3D}}[\mathsf{U}\boldsymbol{z}] - \mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]\right| \le c_S\left(\frac{\varepsilon}{R} + \frac{\varepsilon^2}{L^2}\right)\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}], \tag{1.6}$$

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where c_S is an adimensional constant depending only on S.

1.C Thin plate counter-example and elliptic shell example

A method for the validation of estimate (1.5) is to apply it to families $(\boldsymbol{u}^\varepsilon)$ and $(\boldsymbol{z}^\varepsilon)$ obtained as solutions of problems (P_{3D}) and (P_{2D}) for each $\varepsilon \in (0, \varepsilon_0]$ when the load $\boldsymbol{f}(P, x_3) = \boldsymbol{g}(P)$ is independent of the transverse variable x_3, with a surface load $\boldsymbol{g}$ independent of ε. Note that $\boldsymbol{g}$ also coincides with the mean value of $\boldsymbol{f}$ across the normal fibers to S. Let us stress that, though $\boldsymbol{g}$ is independent of ε, the solution $\boldsymbol{z}^\varepsilon$ does depend on ε, and that the wave length L may also depend on ε.

But, in the situation of plates, L *does not depend on* ε and, of course, $\frac{1}{R} = 0$. Two years after the publication of [21, 22], it was already known that estimate (1.5) does not hold as $\varepsilon \to 0$ for plates. We read in [23] *"The somewhat depressing conclusion for most shell problems is, similar to the earlier conclusions of* GOL'DENWEIZER*, that no better accuracy of the solutions can be expected than of order* $\frac{\varepsilon}{L} + \frac{\varepsilon}{R}$*, even if the equations of first-approximation shell theory would permit, in principle, an accuracy of order* $\frac{\varepsilon^2}{L^2} + \frac{\varepsilon}{R}$*."*

The reason for this is also explained by JOHN [18] in these terms *"Concentrating on the interior we sidestep all kinds of delicate questions, with an attendant gain in certainty and generality. The information about the interior behavior can be obtained much more cheaply (in the mathematical sense) than that required for the discussion of boundary value problems, which form a more "transcendental" stage."*.

The presence of boundary layer terms for thin plates in the vicinity of the lateral part of the boundary was already pointed out by GOL'DENWEIZER [17] but a multi-scale asymptotic expansion combining (for plates) inner (boundary layer) and outer (regular) parts was only available later, see Chapters 15 and 16 in [24] and its bibliographical comments. A more specific form adapted for clamped thin plates is provided by NAZAROV & ZORIN in [26] and DAUGE & GRUAIS in [10]. From these results we can deduce the sharp estimates for plates, valid for a "standard" load $\boldsymbol{f}(P, x_3) = \boldsymbol{g}(P)$, see [11, §12]

$$\mathsf{E}^{\varepsilon}_{\mathsf{3D}}\big[\boldsymbol{u}^{\varepsilon} - \mathsf{U}\boldsymbol{z}^{\varepsilon}\big] \le b_S\,\varepsilon\,\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}^{\varepsilon}], \quad \text{as} \quad \varepsilon \to 0. \tag{1.7}$$ 1E5

In (1.7), the factor ε in the bound comes from the contribution of the three-dimensional boundary layer term along the lateral part of the boundary, and b_S^{-1} has the dimension of a length – so that $b_S\varepsilon$ is adimensional.

For shells, the complexity of a multi-scale analysis (if possible) is much higher. There is at least one situation where such an analysis was successfully performed: the case of *clamped elliptic shells*. In [14, 15], FAOU proved that

1. The solution $\boldsymbol{z} = \boldsymbol{z}^{\varepsilon}$ of the Koiter problem (P_{2D}) has a boundary layer in the vicinity of ∂S with length-scale $\sqrt{\varepsilon}$, which yields that the wave length L is also a $\mathcal{O}(\sqrt{\varepsilon})$,
2. The solution $\boldsymbol{u} = \boldsymbol{u}^{\varepsilon}$ of the Lamé problem (P_{3D}) has a complete three-scale asymptotics combining regular terms and boundary layer terms with length-scale $\sqrt{\varepsilon}$ and ε.

Relying on these two results it is possible to prove that estimate (1.7) holds true, and that it is also sharp. But now, both terms in the sum $\frac{\varepsilon^2}{L^2} + \frac{\varepsilon}{R}$ are a $\mathcal{O}(\varepsilon)$ and this proves that *the first Koiter estimate* (1.5) *is asymptotically valid for clamped elliptic shells.*

1.D SUMMARY OF RESULTS

In this paper, we prove universal estimates in the spirit of (1.5) without a priori knowledge of multi-scale expansions for $\boldsymbol{u}$ and $\boldsymbol{z}$. The result is given in Theorem 2.9. In comparison with (1.6), our estimate now involves the three following constants:

a) A global wave length L associated with $\boldsymbol{z}$ similar to the one which Koiter used,

b) A lateral wave length ℓ for $\boldsymbol{z}$, allowing to take boundary layer effects into account,

c) A curvature constant r depending on the curvature of S and its derivatives.

Besides these three main quantities, two more lengths D and d attached to the mean surface S take part in our statement.

Let us briefly describe our result under a simplifying hypothesis.

The constant L typically describes the characteristic length of layers appearing in the shell. According to the formal result in [28], this wave length can typically be assumed of size $L \geq \sqrt{L_S \varepsilon}$ where L_S has the dimension of a length and is uniformly bounded in ε. Under this assumption, and in the specific case where the loading forces $\boldsymbol{f}$ are constant along each normal fiber of the shell, our general estimate (2.24) in Theorem 2.9 yields

$$\mathsf{E}^{\varepsilon}_{\mathsf{3D}}\big[\boldsymbol{u} - \mathsf{U}\boldsymbol{z}\big] \leq a_S \,\Big(\frac{\varepsilon}{\ell} + \frac{\varepsilon^2}{\mathsf{r}^2} + \frac{\varepsilon^2}{L^2}\Big)\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}] \tag{1.8}$$

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where a_S is an adimensional constant. Note that the difference with Koiter's original estimate (1.6) lies in the presence of a boundary term depending on the lateral wave-length ℓ. Under the same assumption on L, and as a consequence of the energy estimate (1.6), we deduce from (1.8) the *relative energy estimate* for ε small enough:

$$\frac{\mathsf{E}^{\varepsilon}_{\mathsf{3D}}\big[\boldsymbol{u} - \mathsf{U}\boldsymbol{z}\big]}{\mathsf{E}^{\varepsilon}_{\mathsf{3D}}[\mathsf{U}\boldsymbol{z}]} \leq 2a_S \,\Big(\frac{\varepsilon}{\ell} + \frac{\varepsilon^2}{\mathsf{r}^2} + \frac{\varepsilon^2}{L^2}\Big). \tag{1.9}$$

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In the cases of plates and elliptic shells, the behavior of the three characteristic lengths L, ℓ, and r with respect to the thickness ε can be made explicit for families of solutions corresponding to a standard load $\boldsymbol{f}(P, x_3) = \boldsymbol{g}(P)$:

- For *plates*, the two wave-lengths L and ℓ are $\mathcal{O}(1)$, while $\mathsf{r} = +\infty$.
- For *elliptic shells*, ℓ is $\mathcal{O}(1)$, whereas L is $\mathcal{O}(\sqrt{\varepsilon R_\partial})$ where R_∂ is the curvature radius along the boundary of S.

In both cases our general estimate (1.8) gives back the optimal estimate (1.7).

1.E Plan

In §2, we introduce the three- and two-dimensional problems, their solutions $\boldsymbol{u}$ and $\boldsymbol{z}$, the different characteristic lengths, the reconstruction operator $\boldsymbol{z} \mapsto \mathsf{U}\boldsymbol{z}$ and finally, we state our results. In §3, we prove a priori estimates for Sobolev norms of $\boldsymbol{z}$ by Sobolev norms of its membrane and bending strain tensors $\boldsymbol{\gamma}$ and $\boldsymbol{\rho}$. This will serve to convert any norm of $\boldsymbol{z}$ appearing in our estimates in norms of $\boldsymbol{\gamma}$ and $\boldsymbol{\rho}$. Using the wave-lengths, these norms can be compared to the energy of $\boldsymbol{z}$. In §4, in a preliminary step, we prove the estimate (1.6) between the three- and two-dimensional energies.

In §5, we start the proof of the main estimate (1.8) with the introduction of two main steps: *(i)* A variational type estimate of the energy scalar product of the difference $\boldsymbol{u}-\mathsf{U}\boldsymbol{z}$ against all displacements $\boldsymbol{v}$ satisfying the clamped boundary conditions on the lateral boundary, and *(ii)* an energy estimate of a correcting displacement $\boldsymbol{u}^{\mathsf{cor}}$ constructed so that $\mathsf{U}\boldsymbol{z}+\boldsymbol{u}^{\mathsf{cor}}$ also satisfies the clamped boundary conditions.

Step *(i)* is performed in §7 with the help of the operator formal series solution developed in [14] which we recall in §6. Step *(ii)* is performed in §8 by an explicit construction. In the last section §9, we show that our estimate (2.24) is optimal for plates, for shallow shells in the sense of [9] and for elliptic shells. For this, we rely on [10], [2] and [15] which provide sharp asymptotic expansions in each of these three cases, respectively.

2 Statement of results

In this section, we now formulate precisely our assumptions, the definitions of problems (P_{3D}) and (P_{2D}) and of the different lengths occurring in estimates (1.8) and (1.6), and we state our main results. We use everywhere the convention of repeated indices for the contraction of tensors.

2.a The three-dimensional problem

In all this work $\{\Omega^\varepsilon\}_{\varepsilon\leq\varepsilon_0}$ denotes a family of elastic shells defined for ε_0 sufficiently small, made with an isotropic and homogeneous material characterized by its two Lamé coefficients λ and μ. The mid-surface of the shell is represented by a smooth 2-manifold S embedded in $\mathbb{R}^3$, compact with non-empty boundary ∂S. We stress that no other assumption is made on the geometry of the surface S. In particular, its main curvatures may have different signs, or even be zero, in which case the shell is a plate. The domain Ω^ε is then the image of the manifold $S\times(-\varepsilon,\varepsilon)$ by the application :

$$S\times(-\varepsilon,\varepsilon)\ni(P,x_3)\mapsto P+x_3\,\boldsymbol{n}(P)\in\mathbb{R}^3, \tag{2.1}$$

diffeo

where $\boldsymbol{n}$ is a continuous unit normal field on S. The shell has two faces $\Gamma^\varepsilon_\pm$ corresponding to $S\times\{\pm\varepsilon\}$ and a lateral boundary Γ^ε_0 corresponding to $\partial S\times(-\varepsilon,\varepsilon)$. The boundary conditions applied to the shell are the free traction conditions on the two faces $\Gamma^\varepsilon_\pm$ and the clamped conditions on Γ^ε_0. The space of admissible displacements is then

$$V(\Omega^\varepsilon)=\big\{\boldsymbol{u}\in\mathrm{H}^1(\Omega^\varepsilon)^3\quad|\quad\boldsymbol{u}=0\quad\text{on}\quad\Gamma^\varepsilon_0\big\}\,. \tag{2.2}$$

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If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two displacements on Ω^ε, we define the energy scalar product

$$a^\varepsilon_{\mathsf{3D}}(\boldsymbol{u},\boldsymbol{v})=\int_{\Omega^\varepsilon}A^{ijk\ell}\,e_{ij}(\boldsymbol{u})\,e_{k\ell}(\boldsymbol{v})\,\mathrm{d}V, \tag{2.3}$$

2Ea

where $\mathrm{d}V=\mathrm{d}t^1\mathrm{d}t^2\mathrm{d}t^3$ with a system $\{t^i\}$ of Cartesian coordinates in $\mathbb{R}^3$ and where

$$A^{ijk\ell}=\lambda\delta^{ij}\delta^{k\ell}+\mu(\delta^{ik}\delta^{j\ell}+\delta^{i\ell}\delta^{jk})$$

is the rigidity tensor of the material, with the Kronecker tensor δ^{ij}. The tensor $e_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$ is the strain tensor in Cartesian coordinates, where ∂_i denotes the derivative with respect to t^i. The associated quadratic three-dimensional energy of a displacement $\boldsymbol{v}$ is then:

$$\mathsf{E}^{\varepsilon}_{\mathsf{3D}}[\boldsymbol{v}] \ := \ a^{\varepsilon}_{\mathsf{3D}}(\boldsymbol{v}, \boldsymbol{v}). \tag{2.4}$$

2E3D

Our "exact solution" $\boldsymbol{u}$ is the displacement solution of the variational problem :

$$(P_{\mathsf{3D}}) \qquad \text{Find } \boldsymbol{u} \in V(\Omega^{\varepsilon}) \text{ such that } \quad \forall\, \boldsymbol{v} \in V(\Omega^{\varepsilon}), \quad a^{\varepsilon}_{\mathsf{3D}}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega^{\varepsilon}} \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}V,$$

where $\boldsymbol{f} \in \mathrm{L}(\Omega^{\varepsilon})^3$ represents the loading force.

2.B NORMAL COORDINATES AND TENSORS

The shell Ω^{ε} is diffeomorphic to the manifold $S \times (-\varepsilon, \varepsilon)$ via the application (2.1). Any local coordinate system (x_σ) on S yields a coordinate system (x_σ, x_3) on $S \times (-\varepsilon, \varepsilon)$ and thus an atlas on S provides an atlas on Ω^{ε} whose local maps are $U \times (-\varepsilon, \varepsilon)$ where U are the maps of the atlas on S. Such a coordinate system is called *normal coordinate system*, and induces a basis for tensor fields on Ω^{ε}.

This implies that every tensor on Ω^{ε} can be decomposed into several two-dimensional tensors depending smoothly on x_3 and living on S. Typically, any displacement (i.e. a 1-form on Ω^{ε}) $\boldsymbol{v}$ splits into

(i) a surfacic displacement (v_σ), which means that $x_3 \mapsto (v_\sigma(x_3))$ takes its values in 1-forms on S.

(ii) a function v_3, in other words $x_3 \mapsto v_3(x_3)$ takes its values in functions on S.

On the same way, for each fixed x_3, the strain tensor e_{ij} splits into: e_{33}, which is a function on S, $(e_{\sigma 3})$ which is a covariant tensor of order 1 on S, and $(e_{\alpha\beta})$ which is a covariant tensor of order 2 on S. These three surfacic tensors depend smoothly on x_3.

We denote by $\boldsymbol{a} = (a_{\alpha\beta})$ the metric tensor on S induced by the ambient metric in $\mathbb{R}^3$, and by $\boldsymbol{b} = (b_{\alpha\beta})$ the curvature tensor on S (see e.g. [12, 30]). These tensors are symmetric covariant tensors of order 2. Moreover, the metric tensor induces an isomorphism between covariant and contravariant tensors. For instance, the tensor b^{α}_{σ} is defined by $b^{\alpha}_{\sigma} = a^{\alpha\beta} b_{\beta\sigma}$, with the inverse $a^{\alpha\beta}$ of the metric tensor. We also denote by D_α the covariant derivative induced by the Riemannian metric $a_{\alpha\beta}$ on S.

Let us recall the definition of the Sobolev norm of a tensor on a manifold. Consider a covariant tensor field $\boldsymbol{\tau}$ of order k on S. In a local coordinate system, we denote by $\tau_{\alpha_1\alpha_2\cdots\alpha_k}$ its components. The norm $|\boldsymbol{\tau}|$ of $\boldsymbol{\tau}$ at a fixed point $P \in S$ is defined as

$$|\boldsymbol{\tau}|^2 = \tau^{\alpha_1\alpha_2\cdots\alpha_k} \tau_{\alpha_1\alpha_2\cdots\alpha_k} \tag{2.5}$$

E765

where $\tau^{\alpha_1\alpha_2\cdots\alpha_k}$ is the contravariant tensor associated with $\boldsymbol{\tau}$ using the metric tensor, as explained above. The expression (2.5) is independent of local coordinate systems.

Note that (2.5) makes sense for tensors of any type, as it depends only on the order of the tensor and not of its representation as covariant or contravariant tensor. We have for example $|\boldsymbol{b}|^2 = b^{\alpha\beta}b_{\alpha\beta} = b^\alpha_\beta b^\beta_\alpha$ so that can write $|\boldsymbol{b}| = |b_{\alpha\beta}| = |b^\beta_\alpha|$.

The L^2 norm of $\boldsymbol{\tau}$ is defined as

$$|\boldsymbol{\tau}|^2_{0\,;S} := \int_S |\boldsymbol{\tau}|^2 \mathrm{d}S.$$

For $n \in \mathbb{N}$, we denote by $\mathrm{D}_{[n]}\boldsymbol{\tau}$ the tensor of order $k+n$ with components

$$\mathrm{D}_{\beta_1} \cdots \mathrm{D}_{\beta_n} \tau_{\alpha_1\alpha_2\cdots\alpha_k}$$

in a local coordinate system. The semi norm of order n of $\boldsymbol{\tau}$ is thus

$$|\boldsymbol{\tau}|_{n\,;S} = |\mathrm{D}_{[n]}\boldsymbol{\tau}|_{0\,;S}\,. \tag{2.6}$$

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As the surface S is smooth, this expression makes sense on S for all n, and does not depend on a choice of a local coordinate system. We define similarly the semi-norms $|\boldsymbol{\tau}|_{n\,;\partial S}$ on the lateral boundary ∂S.

In the following, we denote by $\mathbf{H}^k(S)$ the space of 1-form fields (z_σ) such that $|z_\sigma|_{n\,;S} < \infty$ for $n = 0, \dots, k$, and by $\mathrm{H}^k(S)$ the corresponding space for functions.

2.c The two-dimensional problem

The Koiter operator on S is defined as $\mathsf{K}(\varepsilon) = \mathsf{M} + \varepsilon^2\mathsf{B}$ where M is the *membrane* operator and B the *bending* operator. Both of them involve the rigidity tensor $M^{\alpha\beta\sigma\delta}$ corresponding to the modified Lamé constants $\tilde{\lambda} = 2\lambda\mu/(\lambda+2\mu)$ and μ:

$$M^{\alpha\beta\sigma\delta} = \frac{2\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\delta} + \mu(a^{\alpha\sigma}a^{\delta\beta} + a^{\alpha\delta}a^{\beta\sigma}).$$

Both operators M and B act on spaces of $\boldsymbol{z} = (z_\sigma, z_3)$ where (z_σ) is a 1-form on S and z_3 a function on S. The target space contains elements of the form $\boldsymbol{g} = (g_\sigma, g_3)$ where (g_σ) is a 1-form on S and g_3 a function on S. Typical spaces for $\boldsymbol{z}$ are $\mathbf{H}^1 \times \mathrm{L}^2(S)$ and $\mathbf{H}^1 \times \mathrm{H}^2(S)$.

The operator M is the operator associated with the bilinear form a_{M} defined for any $\boldsymbol{z} = (z_\sigma, z_3)$ and $\boldsymbol{\eta} = (\eta_\sigma, \eta_3)$ in $\mathbf{H}^1 \times \mathrm{L}^2(S)$ by

$$(\boldsymbol{z}, \boldsymbol{\eta}) \mapsto a_{\mathsf{M}}(\boldsymbol{z}, \boldsymbol{\eta}) = \int_S M^{\alpha\beta\sigma\delta}\,\gamma_{\alpha\beta}(\boldsymbol{z})\,\gamma_{\sigma\delta}(\boldsymbol{\eta})\,\mathrm{d}S,$$

where the membrane strain tensor field

$$\gamma_{\alpha\beta}(\boldsymbol{z}) = \tfrac{1}{2}(\mathrm{D}_\alpha z_\beta + \mathrm{D}_\beta z_\alpha) - b_{\alpha\beta}z_3$$

is the change of metric tensor.

The operator B is associated with the bilinear form a_{B} defined for any $\boldsymbol{z}$ and $\boldsymbol{\eta}$ in $\mathbf{H}^1 \times \mathrm{H}^2(S)$ by

$$(\boldsymbol{z}, \boldsymbol{\eta}) \mapsto a_{\mathsf{B}}(\boldsymbol{z}, \boldsymbol{\eta}) = \frac{1}{3} \int_S M^{\alpha\beta\sigma\delta} \, \rho_{\alpha\beta}(\boldsymbol{z}) \, \rho_{\sigma\delta}(\boldsymbol{\eta}) \, \mathrm{d}S$$

where

$$\rho_{\alpha\beta}(\boldsymbol{z}) = \mathrm{D}_\alpha \mathrm{D}_\beta z_3 - b_\alpha^\sigma b_{\sigma\beta} z_3 + b_\alpha^\sigma \mathrm{D}_\beta z_\sigma + \mathrm{D}_\alpha b_\beta^\sigma z_\sigma \tag{2.7}$$ Erho

is the change of curvature tensor.

The two-dimensional energy scalar product is defined for $\boldsymbol{z}$, $\boldsymbol{\eta} \in \mathbf{H}^1 \times \mathrm{H}^2(S)$ by

$$a_{\mathsf{2D}}^{\varepsilon}(\boldsymbol{z}, \boldsymbol{\eta}) = a_{\mathsf{M}}(\boldsymbol{z}, \boldsymbol{\eta}) + \varepsilon^2 a_{\mathsf{B}}(\boldsymbol{z}, \boldsymbol{\eta}). \tag{2.8}$$ EKoiter

This bilinear form is associated with the Koiter operator $\mathsf{K}(\varepsilon) = \mathsf{M} + \varepsilon^2 \mathsf{B}$. The physical quadratic energy associated with a displacement $\boldsymbol{z}$ is defined as:

$$\mathsf{E}_{\mathsf{2D}}^{\varepsilon}[\boldsymbol{z}] \ := \ 2\varepsilon \, a_{\mathsf{2D}}^{\varepsilon}(\boldsymbol{z}, \boldsymbol{z}). \tag{2.9}$$ 2E2D

The right-hand side $\boldsymbol{g} = (g_\sigma, g_3)$ of the two-dimensional problem (P_{2D}) is defined on S as

$$\boldsymbol{g} = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \boldsymbol{f}(x_3) \, \mathrm{d}x_3. \tag{2.10}$$ 2Eg

The admissible two-dimensional displacement space is $\mathbf{H}_0^1 \times \mathrm{H}_0^2(S)$. The two-dimensional problem then writes:

$$(P_{\mathsf{2D}}) \quad \begin{array}{l} \text{Find } \boldsymbol{z} \in \mathbf{H}_0^1 \times \mathrm{H}_0^2(S) \quad \text{such that} \\ \qquad \forall\, \boldsymbol{\eta} \in \mathbf{H}_0^1 \times \mathrm{H}_0^2(S), \quad a_{\mathsf{2D}}^{\varepsilon}(\boldsymbol{z}, \boldsymbol{\eta}) = \displaystyle\int_S (a^{\alpha\beta} g_\alpha \eta_\beta + g_3 \eta_3) \, \mathrm{d}S. \end{array}$$

We define the *residual load* as

$$\boldsymbol{f}^{\mathsf{rem}} := \boldsymbol{f} - \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \boldsymbol{f}(x_3) \, \mathrm{d}x_3 \tag{2.11}$$ Eresload

In the sequel, we also use the notation $\Sigma(S) := \Gamma(T_1\overline{S}) \times \mathscr{C}^\infty(\overline{S})$ where $\Gamma(T_1\overline{S})$ denotes the space of smooth 1-form fields on $\overline{S}$ (see [14] for details). The elements of $\Sigma(S)$ are written (z_σ, z_3).

2.D PHYSICAL DIMENSIONS

We recall here the physical dimensions of the different objects present in the problem. In Table 1 we give the dimensions of the 3D objects. Here, E denotes the Young modulus of the material. We recall the formulas

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad \text{and} \quad \nu = \frac{\lambda}{2(\lambda + \mu)},$$

where ν is the adimensional Poisson coefficient. Conversely

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}.$$

Physical object	***Notation***	***Dimension***
Displacement	$\boldsymbol{u}$	m
Volume force	$\boldsymbol{f}$	$N.m^{-3}$
Energy	$\mathsf{E}_{\mathsf{3D}}[\boldsymbol{u}]$	$N.m$ (Joule)
Deformation rate	$e_{ij}(\boldsymbol{u})$	Adimensional
Material coefficients	E, λ, μ	$N.m^{-2}$ (Pascal)

Table 1. Physical dimensions of the 3D objects

For surfacic objects, we give the corresponding information in Table 2.

Note that in a local coordinate system on S, a partial derivative has the dimension of the inverse of a length. The expression of the Christoffel symbols

$$\Gamma^{\sigma}_{\alpha\beta} = a^{\sigma\delta}(\partial_\alpha a_{\beta\delta} + \partial_\beta a_{\alpha\delta} - \partial_\delta a_{\alpha\beta}) \tag{2.12}$$ Echri

then shows that they also have the dimension of the inverse of a length. Thus the dimension of the covariant derivative is coherent.

2.E WAVE LENGTHS

Before defining wave lengths attached to the solution $\boldsymbol{z}$ of (P_{2D}), we introduce a sequence of characteristic quantities depending on the curvature tensor of S.

Definition 2.1 *(i) We set $\kappa_0 = 1$ and define recursively for $j \geq 1$ the numbers κ_j by:*

$$\kappa_1 = \max_{P\in\overline{S}} |\boldsymbol{b}| \quad \textit{and} \quad \kappa_j = \sup\left(\kappa_{j-1}, \max_{P\in\overline{S}} \left|\mathrm{D}_{[j-1]}\boldsymbol{b}\right|^{1/j}\right) \quad \textit{for} \quad j \geq 2, \tag{2.13}$$ Edefk

Physical object	*Notation*	*Dimension*
Displacement	$\boldsymbol{z}$	m
Volume force	$\boldsymbol{g}$	$N.m^{-3}$
Energy	$\mathsf{E}_{\mathsf{2D}}[\boldsymbol{z}]$	$N.m$ (Joule)
Curvature	b_α^β	m^{-1}
Covariant derivative	D_α	m^{-1}
Change of metric tensor	$\gamma_{\alpha\beta}(\boldsymbol{z})$	Adimensional
Change of curvature tensor	$\rho_{\alpha\beta}(\boldsymbol{z})$	m^{-1}

Table 2. Physical dimensions of the 2D objects

where $\boldsymbol{b}$ *is the curvature tensor. For any* $j \geq 1$, *the constants* κ_j *have the dimension of the inverse of a length.*

(ii) For any tensor field on S *let for all* $n \in \mathbb{N}$ *the semi-norm* $|\boldsymbol{\tau}|^{(\boldsymbol{b})}_{n\,;S}$ *be defined by the expression**

$$\Big(|\boldsymbol{\tau}|^{(\boldsymbol{b})}_{n\,;S}\Big)^2 = \sum_{j=0}^{n} \kappa_j^{2j} |\boldsymbol{\tau}|^2_{n-j\,;S}\,. \tag{2.14}$$ Enormk

In the case of plates, we have $\kappa_j = 0$ for $j \geq 1$, and hence $|\boldsymbol{\tau}|^{(\boldsymbol{b})}_{n\,;S} = |\boldsymbol{\tau}|_{n\,;S}$, in contrast with the case when $\boldsymbol{b} \neq 0$.

With the definition (2.13) we have $\kappa_1 = 1/R$ where $1/R$ is the maximum principal curvature of S. As the covariant derivative has the dimension of the inverse of a length, we see that all the terms in the sum of the right-hand side of (2.14) have the same dimension.

Definition 2.2 *An operator* L *acting on tensor spaces on* S *is said to be* $\boldsymbol{b}$*-homogeneous of degree* n *if it is a linear combination with adimensional coefficients of contractions of tensors of the form*

$$B_1 \cdots B_n$$

where each B_j *is either the covariant derivative* D_σ *or the curvature tensor* $b_{\alpha\beta}$.

*We could have introduce factorial normalization terms in the definitions (2.13) and (2.6). This could in principle lead to analytic estimates in the case where S is analytic. In this situation, κ_j would tend to the analytic radius of convergence of $\boldsymbol{b}$ when $j \to \infty$.

Note that the operators $\boldsymbol{z} \mapsto \gamma_{\alpha\beta}(\boldsymbol{z})$ and $\boldsymbol{z} \mapsto \rho_{\alpha\beta}(\boldsymbol{z})$ are $\boldsymbol{b}$-homogeneous of degree 1 and 2 respectively. Similarly the membrane is $\boldsymbol{b}$-homogeneous of degree 2, which means that both surfacic and transverse components are $\boldsymbol{b}$-homogeneous of degree 2, and the bending operator is $\boldsymbol{b}$-homogeneous of degree 4.

This definition is motivated by the following lemma:

Lemma 2.3 *Let* L *be a* $\boldsymbol{b}$*-homogeneous operator of degree* n *acting on tensors* $\boldsymbol{\tau}$ *of order* k*, ans let* $s \in \mathbb{N}$*. Then there exists an adimensional constant* A *such that,*

$$\forall \boldsymbol{\tau}, \quad |\mathsf{L}\boldsymbol{\tau}|^{(\boldsymbol{b})}_{s\,;\,S} \leq A|\boldsymbol{\tau}|^{(\boldsymbol{b})}_{s+n\,;\,S}\,.$$

Let $\boldsymbol{\gamma}$ and $\boldsymbol{\rho}$ denote the membrane and bending strain tensors of the solution $\boldsymbol{z}$ of problem (P_{2D}). With our notations, we can reformulate Koiter's definition of the quantity L in [20, 21] as *"the wave length of the deformation pattern of shell theory, defined by the order of magnitude relations* $\mathrm{D}_{[1]}\boldsymbol{\gamma} = \mathcal{O}(\boldsymbol{\gamma}/L)$ *and* $\mathrm{D}_{[1]}\boldsymbol{\rho} = \mathcal{O}(\boldsymbol{\rho}/L)$*."*

Without being exactly the same, our definitions retain the idea of *inverse inequalities* for the membrane and bending strain tensors $\boldsymbol{\gamma}$ and $\boldsymbol{\rho}$.

Definition 2.4 *For* $\boldsymbol{z} \in \Sigma(S)$ *we denote by* $\boldsymbol{\gamma} = \gamma_{\alpha\beta}(\boldsymbol{z})$ *and* $\boldsymbol{\rho} = \rho_{\alpha\beta}(\boldsymbol{z})$ *the membrane and bending strain tensors associated with* $\boldsymbol{z}$*. We set* $L_0 = 1$ *and for all* $k \geq 1$*, we define the global wave length* L_k *of* $\boldsymbol{z}$ *as the* largest *constant such that there holds, for* $j = 1, \ldots, k$

$$|\boldsymbol{\gamma}|_{j\,;\,S} \leq L_k^{-j}|\boldsymbol{\gamma}|_{0\,;\,S} \quad \textit{and} \quad |\boldsymbol{\rho}|_{j\,;\,S} \leq L_k^{-j}|\boldsymbol{\rho}|_{0\,;\,S}\,. \tag{2.15}$$

2E2

Note that $L_1 \geq L_2 \geq \cdots$, and that L_k can be equivalently defined by requiring (2.15) for $j = k$ only. We have, by definition:

$$\big(|\boldsymbol{\gamma}|^{(\boldsymbol{b})}_{n\,;\,S}\big)^2 \leq |\boldsymbol{\gamma}|^2_{0\,;\,S} \sum_{j+k=n} L_k^{-2k}\kappa_j^{2j} \quad \text{and} \quad \big(|\boldsymbol{\rho}|^{(\boldsymbol{b})}_{n\,;\,S}\big)^2 \leq |\boldsymbol{\rho}|^2_{0\,;\,S} \sum_{j+k=n} L_k^{-2k}\kappa_j^{2j}. \tag{2.16}$$

2EGR

We now define a similar wave length, now for the norms on the boundary ∂S.

Definition 2.5 *With* $\boldsymbol{\gamma}$ *and* $\boldsymbol{\rho}$ *as in Definition 2.4, we define* ℓ *as the* largest *constant such that there holds*

$$|\boldsymbol{\gamma}|_{0\,;\,\partial S} \leq \ell^{-1/2}|\boldsymbol{\gamma}|_{0\,;\,S} \quad \textit{and} \quad |\boldsymbol{\rho}|_{0\,;\,\partial S} \leq \ell^{-1/2}|\boldsymbol{\rho}|_{0\,;\,S}\,. \tag{2.17a}$$

2E2t

$$|\boldsymbol{\gamma}|_{1\,;\,\partial S} \leq \ell^{-3/2}|\boldsymbol{\gamma}|_{0\,;\,S} \quad \textit{and} \quad |\boldsymbol{\rho}|_{1\,;\,\partial S} \leq \ell^{-3/2}|\boldsymbol{\rho}|_{0\,;\,S}\,. \tag{2.17b}$$

2E2b

Note that the quantity ℓ has also the dimension of a length.

2.F Shifted displacement and reconstructed displacement

Let $\boldsymbol{u}$ be the displacement solution of (P_{3D}). We can express this displacement in the *shifted* normal coordinates introduced by Naghdi (see [25]) and commonly used in classical shell theory. As a matter of fact, computations are easier when considering the shifted components. The *shifter* is the tensor μ_σ^β (see [25]) defined by

$$\mu_\sigma^\beta(x_\alpha, x_3) = \delta_\sigma^\beta - x_3 b_\sigma^\beta(x_\alpha),$$

where δ_σ^β is the Kronecker tensor. If $\boldsymbol{v} = (v_\sigma, v_3)$ is a displacement, the *shifted* displacement $\tilde{\boldsymbol{v}} = (\tilde{v}_\sigma, \tilde{v}_3)$ is defined by the relations

$$\tilde{v}_3 = v_3 \quad \text{and} \quad \tilde{v}_\sigma = (\mu^{-1})_\sigma^\beta v_\beta,$$

where $(\mu^{-1})_\sigma^\beta$ is the inverse of the shifter.

As they will be of constant use, we will denote by (w_σ, w_3) the shifted components of the displacement $\boldsymbol{u}$ solution of (P_{3D}), instead of $(\tilde{u}_\sigma, \tilde{u}_3)$. We denote by $\boldsymbol{w}$ the corresponding shifted displacement.

Let $\boldsymbol{z} = (z_\sigma, z_3)$ be solution of (P_{2D}). With $\boldsymbol{z}$, we associate the three-dimensional shifted displacement $\mathsf{W}\boldsymbol{z}$ defined by the formula

$$\mathsf{W}\boldsymbol{z} = \begin{cases} z_\sigma - x_3\theta_\sigma(\boldsymbol{z}), \\ z_3 - p\, x_3\gamma_\alpha^\alpha(\boldsymbol{z}) + p\,\frac{x_3^2}{2}\rho_\alpha^\alpha(\boldsymbol{z}), \end{cases} \tag{2.18}$$

2EW

where $\theta_\sigma(\boldsymbol{z}) = \mathrm{D}_\sigma z_3 + b_\sigma^\alpha z_\alpha$ and $p = \lambda(\lambda + 2\mu)^{-1}$. To this displacement $\mathsf{W}\boldsymbol{z}$ corresponds the displacement $\mathsf{U}\boldsymbol{z}$ in "unshifted" normal coordinates:

$$\mathsf{U}\boldsymbol{z} = \begin{cases} z_\sigma - x_3(\mathrm{D}_\sigma z_3 + 2b_\sigma^\alpha z_\alpha) + x_3^2\, b_\sigma^\alpha\theta_\alpha(\boldsymbol{z}), \\ z_3 - p\, x_3\gamma_\alpha^\alpha(\boldsymbol{z}) + p\,\frac{x_3^2}{2}\rho_\alpha^\alpha(\boldsymbol{z}), \end{cases} \tag{2.19}$$

2EU

2.G Korn inequalities

We now define a length D through Korn inequalities Ω^ε involving the Young modulus E and the thickness ε of the shell.

Proposition 2.6 *There exists a constant D independent on ε, having the dimension of a length, such that for all $\boldsymbol{v} \in V(\Omega^\varepsilon)$ defined in (2.2), we have*

$$\begin{aligned} \|\boldsymbol{v}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} &\leq \mathsf{D}^4 E^{-1}\varepsilon^{-2}\,\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}] \\ \|\mathrm{D}_\alpha\boldsymbol{v}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} + \|\partial_3\boldsymbol{v}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} &\leq \mathsf{D}^2 E^{-1}\varepsilon^{-2}\,\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]. \end{aligned} \tag{2.20}$$

Korn3D

Proof. We make the scaling $X_3 = \varepsilon^{-1}x_3$ mapping the manifold $S \times (-\varepsilon, \varepsilon)$ to the manifold $\Omega := S \times (-1,1)$. In Ω, the expression of the deformation tensor $e_{ij}(\varepsilon)(\boldsymbol{v})$ is obtained by changing ∂_{x_3} to $\varepsilon^{-1}\partial_{X_3}$. This definition coincides with the one in [8]. Note that the variable X_3 and the derivative ∂_{X_3} are adimensional, and that the tensor $e_{ij}(\varepsilon)(\boldsymbol{v})$ is also adimensional.

The lateral boundary $\Gamma_0 = \partial S \times (-1,1)$ is the image of Γ_0^ε by the scaling. We define the space $V(\Omega)$ as the set of $\boldsymbol{v} \in \mathrm{H}(\Omega)^3$ such that $\boldsymbol{v}\big|_{\Gamma_0} = 0$. On the manifold Ω, the following inequalities hold (see [8]): For all $\boldsymbol{v} \in V(\Omega)$,

$$\begin{aligned} \|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} &\leq C_1\varepsilon^{-1}\|e_{ij}(\varepsilon)(\boldsymbol{v})\|_{\mathrm{L}^2(\Omega)} \\ \|\mathrm{D}_\alpha\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} &\leq C_2\varepsilon^{-1}\|e_{ij}(\varepsilon)(\boldsymbol{v})\|_{\mathrm{L}^2(\Omega)} \end{aligned} \tag{2.21}$$

Ekorn12

The shear deformation tensor is written as the convergent series (see [25, 14]):

$$e_{\alpha 3}(\varepsilon)(\boldsymbol{v}) = \tfrac{1}{2}(\mathrm{D}_\alpha v_3 + \varepsilon^{-1}\partial_{X_3}v_\alpha) + \sum_{k=0}^{\infty} \varepsilon^j X_3^j (b^{j+1})_\alpha^\beta v_\beta$$

where $(b^j)_\alpha^\beta$ denotes the product of j-times the curvature tensor $\boldsymbol{b}$. We thus have

$$\|\partial_{X_3}v_\alpha\|_{\mathrm{L}^2(\Omega)} \leq \varepsilon\|e_{i3}(\varepsilon)(\boldsymbol{v})\|_{\mathrm{L}^2(\Omega)} + \varepsilon\Big(\sum_{j=0}^{\infty}\varepsilon^j\kappa_1^{j+1}\Big)\|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} + \varepsilon\|\mathrm{D}_\alpha\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}\,.$$

Combining this estimate with (2.21), we obtain the existence of C_3, having the dimension of a length, such that

$$\|\partial_{X_3}\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} \leq C_3\|e_{ij}(\varepsilon)(\boldsymbol{v})\|_{\mathrm{L}^2(\Omega)}\,,$$

which improves the corresponding estimate in [8]. We scale back to Ω^ε and get the estimates for the squared norms:

$$\begin{aligned} \|\boldsymbol{v}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} &\leq C_4\varepsilon^{-2}\|e_{ij}(\boldsymbol{v})\|^2_{\mathrm{L}^2(\Omega^\varepsilon)}\,, \\ \|\mathrm{D}_\alpha\boldsymbol{v}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} + \|\partial_3\boldsymbol{v}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} &\leq C_5\varepsilon^{-2}\|e_{ij}(\boldsymbol{v})\|^2_{\mathrm{L}^2(\Omega^\varepsilon)}\,. \end{aligned}$$

The constants C_4 and C_5 have the dimensions m^4 and m^2 respectively.

Using the relation between λ, μ and the Young modulus E we see that there exist adimensional constants a and A such that

$$aE \leq \lambda \leq AE \quad \text{and} \quad aE \leq \mu \leq AE. \tag{2.22}$$

EEmu

As $E > 0$, there exists an adimensional constant A_1 such that

$$\|e_{ij}(\boldsymbol{v})\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} \leq A_1E^{-1}\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}].$$

Denoting by D the length $\max((A_1C_4)^{1/4}, (A_1C_5)^{1/2})$, we obtain the result. ∎

2.H Main energy estimates

Our first result gives an estimate between the energy of a two-dimensional displacement $\boldsymbol{z}$ and the three-dimensional energy of the reconstructed displacement $\mathsf{U}\boldsymbol{z}$:

Theorem 2.7 *For all* $\boldsymbol{z} \in (\mathbf{H}^2 \cap \mathbf{H}^1_0)(S) \times (\mathrm{H}^3 \cap \mathrm{H}^2_0)(S)$ *, we have the following estimate*

$$\left| \mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}] - \mathsf{E}^{\varepsilon}_{\mathsf{3D}}[\mathsf{U}\boldsymbol{z}] \right| \le A \Big(\frac{\varepsilon}{R} + \frac{\varepsilon^2}{L_1^2} \Big) \, \mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}], \tag{2.23}$$ $\boxed{\texttt{4E1}}$

for an adimensional constant A *, where* L_1 *is the first wave length for* $\boldsymbol{z}$ *defined according to Definition 2.4, and* $R = \kappa_1^{-1}$ *according to Definition 2.1.*

Our main result gives an estimate between the three-dimensional displacement field and the reconstructed displacement. Besides the notations defined in the previous sections, we need one more characteristic length d of the shell S .

Definition 2.8 Let r denote the geodesic distance in S to the boundary ∂S , and let s be the arc-length along ∂S . We denote by d the maximal width of the tubular neighborhood in which $(r, s) \in [0, \mathsf{d}] \times \partial S$ defines a smooth coordinate system.

It is clear that d has the dimension of a length and that d is proportional to the maximum radius of curvature of the boundary ∂S viewed as a submanifold of S .

Theorem 2.9 *Let* $\boldsymbol{u}$ *and* $\boldsymbol{z}$ *be the solutions of* (P_{3D}) *and* (P_{2D}) *respectively, and let* $\mathsf{U}\boldsymbol{z}$ *be the displacement (2.19). Let* $L = L_4$ *and* ℓ *be defined in Definitions 2.4-2.5,* $\mathsf{r} = 1/\kappa_5$ *given by Definition 2.1. Assume that* $\varepsilon \le \varepsilon_0 < \min\{\mathsf{r}, \mathsf{d}\}$ *and that*

$$\sup_{\varepsilon \le \varepsilon_0} \left| \frac{\varepsilon}{\ell} + \frac{\varepsilon^2}{L^2} \right| \le M < \infty.$$

Then the following estimate holds:

$$\mathsf{E}^{\varepsilon}_{\mathsf{3D}}\big[\boldsymbol{u} - \mathsf{U}\boldsymbol{z}\big] \le a_S \left(B_S(\varepsilon; \boldsymbol{z}) \, \mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}] + \mathsf{D}^2 E^{-1} \|\boldsymbol{f}^{\mathsf{rem}}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} \right)$$

$$\text{with} \quad B_S(\varepsilon; \boldsymbol{z}) = \frac{\varepsilon}{\ell} + \frac{\varepsilon^2}{\mathsf{r}^2} + \frac{\varepsilon^2}{L^2} + \frac{\varepsilon^4 \mathsf{D}^2}{L^6} \tag{2.24}$$ $\boxed{\texttt{Eest}}$

where E *is the Young modulus,* $\boldsymbol{f}^{\mathsf{rem}}$ *is the residual load (2.11) and* D *the constant appearing in Proposition 2.6. The constant* a_S *is an adimensional constant such that*

$$a_S \le b_S(1 + M^3) \tag{2.25}$$ $\boxed{\texttt{Eadimcst}}$

where b_S *is an adimensional constant depending only on* S *.*

Our result can be viewed as an a posteriori estimation of the modeling error by means of the 2D solution. It is universal and does not specify any special dependency (or independence) of the loading $\boldsymbol{f}$ with respect to ε . We can possibly deduce from estimate

(2.24) a convergence result as $\varepsilon \to 0$ if we have more information about the behaviour of the wave length L and ℓ with respect to ε.

Remark 2.10 Theorem 2.9 does not require more regularity than L^2 for the loading $\boldsymbol{f}$. However, for the quantity $L = L_4$ to be finite, we need that the mean value $\boldsymbol{g}$ of $\boldsymbol{f}$ across the shell is more regular. The ellipticity of the Koiter operator with Dirichlet boundary conditions implies that L_4 is finite if $\boldsymbol{g}$ belongs to $\mathrm{H}^3(S)^3$.

Remark 2.11 If the loading belongs to $\mathrm{H}^1(\Omega^\varepsilon)$, owing to the estimate

$$\|\boldsymbol{f}^{\mathsf{rem}}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} \leq A_S \varepsilon^2 \|\nabla \boldsymbol{f}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)}$$

with an adimensional constant A_S, the contribution of $\boldsymbol{f}^{\mathsf{rem}}$ to the bound of $\mathsf{E}^\varepsilon_{\mathsf{3D}}\big[\boldsymbol{u} - \mathsf{U}\boldsymbol{z}\big]$ is of higher order. If $\boldsymbol{g} = 0$, the 2D displacement $\boldsymbol{z}$ is also 0, and we are in a regime of higher order answers.

3 A PRIORI ESTIMATES

Let $\boldsymbol{z} = (z_\sigma, z_3)$ where (z_σ) is a 1-form field on S and z_3 a function on S. In this section we prove estimates for the Sobolev norms of $\boldsymbol{z}$, first by Sobolev norms of its strain tensors $\boldsymbol{\gamma} := \boldsymbol{\gamma}(\boldsymbol{z})$ and $\boldsymbol{\rho} := \boldsymbol{\rho}(\boldsymbol{z})$ and then, with the help of the wave lengths L_k, by its quadratic energy $\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]$, *cf* (2.9).

Lemma 3.1 *There exists a positive adimensional constant* A *such that*

$$\forall\, \boldsymbol{z} \in \mathbf{H}^2 \times \mathrm{H}^1(S), \qquad |z_\sigma|_{2\,;\,S} \;\leq\; A\big(|\boldsymbol{\gamma}|_{1\,;\,S} + \kappa_1 |\boldsymbol{z}|_{1\,;\,S} + \kappa_2^2 |\boldsymbol{z}|_{0\,;\,S}\big), \quad \text{(3.1a)}$$

$$\forall \boldsymbol{z} \in \mathbf{H}^1 \times \mathrm{H}^2(S), \qquad |z_3|_{2\,;\,S} \;\leq\; A\big(|\boldsymbol{\rho}|_{0\,;\,S} + \kappa_1 |\boldsymbol{z}|_{1\,;\,S} + \kappa_2^2 |\boldsymbol{z}|_{0\,;\,S}\big). \quad \text{(3.1b)}$$

Proof. Let us recall (see e.g. [25, 13]) that we have the following relation for the commutation of two covariant derivatives: For all z_α,

$$\mathrm{D}_\alpha \mathrm{D}_\beta z_\sigma - \mathrm{D}_\beta \mathrm{D}_\alpha z_\sigma = R_{\beta\alpha\sigma\nu} z^\nu \qquad (3.2)$$ ERiemann

where the Riemann tensor $R_{\beta\alpha\sigma\nu}$ of S is given by

$$R_{\beta\alpha\sigma\nu} = b_{\beta\nu} b_{\alpha\sigma} - b_{\alpha\nu} b_{\beta\sigma}.$$

Using this relation, and setting $\overline{\gamma}_{\alpha\sigma} = \frac{1}{2}(\mathrm{D}_\alpha z_\sigma + \mathrm{D}_\sigma z_\alpha)$, we have

$$\mathrm{D}_\alpha \mathrm{D}_\beta z_\sigma = \mathrm{D}_\alpha \overline{\gamma}_{\beta\sigma} - \mathrm{D}_\sigma \overline{\gamma}_{\alpha\beta} + \mathrm{D}_\beta \overline{\gamma}_{\sigma\alpha} - \tfrac{1}{2} R_{\beta\alpha\sigma\nu} z^\nu + \tfrac{1}{2} R_{\sigma\alpha\beta\nu} z^\nu + \tfrac{1}{2} R_{\sigma\beta\alpha\nu} z^\nu.$$

This formula clearly implies that there exists an adimensional constant A such that

$$|z_\sigma|_{2\,;\,S} \leq A\big(|\overline{\gamma}|_{1\,;\,S} + \kappa_1^2 |\boldsymbol{z}|_{0\,;\,S}\big).$$

As $\gamma_{\alpha\beta} = \overline{\gamma}_{\alpha\beta} - b_{\alpha\beta} z_3$, this finally gives (3.1a).
The estimate (3.1b) is an easy consequence of the expression (2.7) of $\rho_{\alpha\beta}$. ∎

From the previous estimates, we are going to deduce bounds for $|z_\sigma|^{(\boldsymbol{b})}_{n\,;\,S}$ and $|z_3|^{(\boldsymbol{b})}_{n\,;\,S}$ by induction over n . In the remaining part of this section, we use the following notation: $f \lesssim g$ means that there exist an adimensional constant A such that $f \le Ag$. When $f \lesssim g$ and $g \lesssim f$ we write $f \simeq g$.

Applying the estimates (3.1b) and (3.1a) to $\mathrm{D}_\delta \boldsymbol{z}$ in combination with (3.2) and then using induction, we find the estimates for any $n \ge 2$:

$$|z_\sigma|_{n\,;\,S} \;\lesssim\; |\boldsymbol{\gamma}|_{n-1\,;\,S} + \sum_{1\le j\le n} \kappa_j^j |\boldsymbol{z}|_{n-j\,;\,S} \tag{3.3a}$$

$$|z_3|_{n\,;\,S} \;\lesssim\; |\boldsymbol{\rho}|_{n-2\,;\,S} + \sum_{1\le j\le n} \kappa_j^j |\boldsymbol{z}|_{n-j\,;\,S}\,. \tag{3.3b}$$

Combining (3.3a) and (3.3b) for $n, n-1, \ldots, 0$ we obtain for all $n \ge 2$ and all $\boldsymbol{z} \in \mathbf{H}^n \times \mathrm{H}^n(S)^3$

$$|z_\sigma|^{(\boldsymbol{b})}_{n\,;\,S} \;\lesssim\; |\boldsymbol{\gamma}|^{(\boldsymbol{b})}_{n-1\,;\,S} + \sum_{1\le j\le n-2} \kappa_j^j |\boldsymbol{\rho}|_{n-2-j\,;\,S} + \kappa_{n-1}^{n-1} |\boldsymbol{z}|_{1\,;\,S} + \kappa_n^n |\boldsymbol{z}|_{0\,;\,S} \tag{3.4a}$$

$$|z_3|^{(\boldsymbol{b})}_{n\,;\,S} \;\lesssim\; |\boldsymbol{\rho}|^{(\boldsymbol{b})}_{n-2\,;\,S} + \sum_{1\le j\le n-1} \kappa_j^j |\boldsymbol{\gamma}|_{n-1-j\,;\,S} + \kappa_{n-1}^{n-1} |\boldsymbol{z}|_{1\,;\,S} + \kappa_n^n |\boldsymbol{z}|_{0\,;\,S}\,. \tag{3.4b}$$

We can eliminate the terms $\kappa_{n-1}^{n-1} |\boldsymbol{z}|_{1\,;\,S} + \kappa_n^n |\boldsymbol{z}|_{0\,;\,S}$ with Poincaré type estimates, see [3]. Indeed, we can prove that for a given $n \ge 1$ there exist an adimensional constant A_n such that

$$\forall \boldsymbol{z} \in \mathbf{H}^1_0 \times \mathrm{H}^2_0(S), \quad \kappa_n |\boldsymbol{z}|_{1\,;\,S} + \kappa_n^2 |\boldsymbol{z}|_{0\,;\,S} \le A_n \Big(|\boldsymbol{\rho}|_{0\,;\,S} + \kappa_n |\boldsymbol{\gamma}|_{0\,;\,S} \Big). \tag{3.5}$$

Ecoer

Combining this with (3.4a)-(3.4b) we obtain

$$|z_\sigma|^{(\boldsymbol{b})}_{n\,;\,S} \;\lesssim\; |\boldsymbol{\gamma}|^{(\boldsymbol{b})}_{n-1\,;\,S} + \kappa_n \sum_{0\le j\le n-3} \kappa_n^j |\boldsymbol{\rho}|_{n-3-j\,;\,S}\,, \tag{3.6a}$$

$$|z_3|^{(\boldsymbol{b})}_{n\,;\,S} \;\lesssim\; |\boldsymbol{\rho}|^{(\boldsymbol{b})}_{n-2\,;\,S} + \kappa_n \sum_{0\le j\le n-2} \kappa_n^j |\boldsymbol{\gamma}|_{n-2-j\,;\,S}\,. \tag{3.6b}$$

Then we use the definition (2.15) of the wave lengths together with (2.16), and deduce from the previous inequalities that there exists an adimensional constant A_n such that

$$|\boldsymbol{z}|^{(\boldsymbol{b})}_{n\,;\,S} \le A \Big(|\boldsymbol{\gamma}|_{0\,;\,S} \sum_{j+k=n-1} L_{n-1}^{-k} \kappa_n^j \;+\; |\boldsymbol{\rho}|_{0\,;\,S} \sum_{j+k=n-2} L_{n-1}^{-k} \kappa_n^j \Big). \tag{3.7}$$

3E4

Remark 3.2 The adimensional constants A in (3.7) depend on n and on S via the Poincaré type estimates (3.5). We can prove that these constants remain bounded in a family $\{S_\delta\}$ of shallow shells in the sense of [9, 5], see also [1], as $\delta \to 0$.

Using relations (2.22) we find immediately that $\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}] \simeq E\varepsilon(|\boldsymbol{\gamma}|^2_{0\,;S} + \varepsilon^2|\boldsymbol{\rho}|^2_{0\,;S})$. Hence

$$|\boldsymbol{\gamma}|^2_{0\,;S} \lesssim AE^{-1}\varepsilon^{-1}\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}] \quad\text{and}\quad |\boldsymbol{\rho}|^2_{0\,;S} \lesssim AE^{-1}\varepsilon^{-3}\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}]. \tag{3.8}$$ 3E13

Estimate (3.7) combined with (3.8) yields the following energy estimates for any two-dimensional displacement $\boldsymbol{z}$ satisfying the conditions of the clamped boundary:

Theorem 3.3 *For any $n \geq 2$, there exists an adimensional constant $A > 0$ so that for any $\boldsymbol{z} \in \mathbf{H}^1_0 \times \mathrm{H}^2_0(S)$ satisfying $\boldsymbol{\gamma} \in \mathrm{H}^{n-1}(S)$ and $\boldsymbol{\rho} \in \mathrm{H}^{n-2}(S)$, there holds for any $\varepsilon > 0$:*

$$\left(|\boldsymbol{z}|^{(\boldsymbol{b})}_{n\,;S}\right)^2 \leq \frac{AE^{-1}}{\varepsilon^3}\Big(\sum_{i,j,k\in G_n} \frac{\varepsilon^{2i}\kappa_n^{2j}}{L^{2k}_{n-1}}\Big)\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}], \tag{3.9}$$ 3E14

where E is the Young modulus, κ_n the n-th constant estimating the curvature, L_{n-1} is the global wave length of $\boldsymbol{z}$, cf *Definitions 2.1 and 2.4, and where*

$$G_n = \{(i,j,k) \in \mathbb{N}^3 \mid i \in \{0,1\},\ j+k = i+n-2\}. \tag{3.10}$$ EGn

4 Energy of the reconstructed displacement

In this section, we prove Theorem 2.7. The proof is organized in three steps.

Proof of Theorem 2.7. Step 1. The proof is easier when using the shifted displacement $\mathsf{W}\boldsymbol{z}$, *see* (2.18), corresponding to the reconstructed displacement $\mathsf{U}\boldsymbol{z}$. For any three-dimensional displacement $\boldsymbol{u}$, we recall that $\mathsf{E}^{\varepsilon}_{\mathsf{3D}}[\boldsymbol{u}]$ denotes its quadratic energy, *cf* (2.4). If $\boldsymbol{w}$ is the shifted displacement associated with $\boldsymbol{u}$ we denote the corresponding energy by $\widetilde{\mathsf{E}}^{\varepsilon}_{\mathsf{3D}}[\boldsymbol{w}]$ which is defined so that $\widetilde{\mathsf{E}}^{\varepsilon}_{\mathsf{3D}}[\boldsymbol{w}] = \mathsf{E}^{\varepsilon}_{\mathsf{3D}}[\boldsymbol{u}]$. Hence we have

$$\widetilde{\mathsf{E}}^{\varepsilon}_{\mathsf{3D}}[\boldsymbol{w}] = \int_{\Omega^\varepsilon} A^{ijk\ell}\tilde{e}_{ij}(\boldsymbol{w})\tilde{e}_{k\ell}(\boldsymbol{w})\,\mathrm{d}V, \tag{4.1}$$ EEtilde

where the modified strain tensor $\tilde{e}_{ij}(\boldsymbol{w})$ is defined so that $\tilde{e}_{ij}(\boldsymbol{w}) = e_{ij}(\boldsymbol{u})$. In normal coordinates we have the following expressions for the tensor $\tilde{e}_{ij}(\boldsymbol{w})$, see [14]:

$$\tilde{e}^3_3(\boldsymbol{w}) = \partial_{x_3} w_3, \tag{4.2a}$$

$$\tilde{e}^3_\beta(\boldsymbol{w}) = \tfrac{1}{2}\big(\partial_{x_3} w_\beta - x_3 b^\alpha_\beta \partial_{x_3} w_\alpha + \theta_\beta(\boldsymbol{w})\big), \tag{4.2b}$$

$$\tilde{e}^\alpha_\beta(\boldsymbol{w}) = \gamma^\alpha_\beta(\boldsymbol{w}) + \sum_{n=1}^{\infty} x_3^n (b^n)^\alpha_\delta \gamma^\delta_\beta(\boldsymbol{w}) + \sum_{n=1}^{\infty} n x_3^n (b^{n-1})^\alpha_\delta \Lambda^\delta{}_{\cdot\beta}(\boldsymbol{w}), \tag{4.2c}$$

where $\theta_\beta(\boldsymbol{z}) = \mathrm{D}_\beta z_3 + b^\alpha_\beta z_\alpha$ and $\Lambda_{\alpha\beta}(\boldsymbol{z}) = \frac{1}{2}(b^\sigma_\alpha \mathrm{D}_\sigma z_\beta - b^\sigma_\beta \mathrm{D}_\alpha z_\sigma)$.

Using the definition of the rigidity tensor, we obtain

$$\widetilde{\mathsf{E}}^{\varepsilon}_{\mathsf{3D}}[\boldsymbol{w}] = \int_{\Omega^\varepsilon} \Big[(\lambda+2\mu)\tilde{e}^3_3(\boldsymbol{w})\tilde{e}^3_3(\boldsymbol{w}) + 2\lambda\tilde{e}^3_3(\boldsymbol{w})\tilde{e}^\alpha_\alpha(\boldsymbol{w}) + \lambda\tilde{e}^\alpha_\alpha(\boldsymbol{w})\tilde{e}^\beta_\beta(\boldsymbol{w}) \\ + 4\mu a^{\alpha\beta}(x_3)\tilde{e}^3_\alpha(\boldsymbol{w})\tilde{e}^3_\beta(\boldsymbol{w}) + 2\mu\tilde{e}^\beta_\alpha(\boldsymbol{w})\tilde{e}^\alpha_\beta(\boldsymbol{w})\Big]\,\mathrm{d}V, \quad (4.3)$$

E3D

where $a^{\alpha\beta}(x_3) = a^{\sigma\nu}(\mu^{-1}(x_3))^\alpha_\sigma(\mu^{-1}(x_3))^\beta_\nu$ is the inverse of the metric tensor of the surface at the level x_3 in the shell, see [25, 14]. The inverse $(\mu^{-1}(x_3))^\alpha_\sigma$ of the shifter can be expanded as

$$(\mu^{-1}(x_3))^\alpha_\sigma = \sum_{k=0}^{\infty} x_3^k (b^k)^\alpha_\sigma.$$

Moreover, in a given normal coordinate system (x_σ, x_3) on $S\times(-\varepsilon,\varepsilon)$, the Riemannian volume $\mathrm{d}V$ can be written $\mathrm{d}V = |\det a_{\alpha\beta}(x_3)|^{1/2}\mathrm{d}x_\sigma\,\mathrm{d}x_3$, where $a_{\alpha\beta}(x_3)$ is the inverse tensor of $a^{\alpha\beta}(x_3)$ defined above. With the definition of the shifter, we hence can write

$$\mathrm{d}V = \frac{|\det a_{\alpha\beta}(x_3)|^{1/2}}{|\det a_{\alpha\beta}(0)|^{1/2}}\,\mathrm{d}S\,\mathrm{d}x_3 = (1+h(x_3))\,\mathrm{d}S\,\mathrm{d}x_3 \quad (4.4)$$

ERvol

where $h(x_3)$ is a convergent power series in x_3, provided that $|x_3| < R$, such that $h(0) = 0$, and with adimensional function coefficients globally defined on S. This implies in particular that

$$|h(x_3)| \le A\frac{\varepsilon}{R} \quad (4.5)$$

Eh

uniformly in Ω^ε, and for an adimensional constant A.

Thus, we reduce the proof to showing that $\big|\tilde{\mathsf{E}}^{\varepsilon}_{\mathsf{3D}}[\mathsf{W}\boldsymbol{z}] - \mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}]\big|$ is bounded by the right hand side (2.23). We note that $\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}]$ is associated with the material law of Lamé coefficients $2\mu p$ and μ (we recall that $p = \lambda(\lambda+2\mu)^{-1}$) and writes

$$\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}] = 2\varepsilon\int_S \Big[2\mu p\gamma^\alpha_\alpha(\boldsymbol{z})\gamma^\beta_\beta(\boldsymbol{z}) + 2\mu\gamma^\beta_\alpha(\boldsymbol{z})\gamma^\alpha_\beta(\boldsymbol{z})\Big]\mathrm{d}S \\ + \frac{2}{3}\varepsilon^3\int_S \Big[2\mu p\rho^\alpha_\alpha(\boldsymbol{z})\rho^\beta_\beta(\boldsymbol{z}) + 2\mu\rho^\beta_\alpha(\boldsymbol{z})\rho^\alpha_\beta(\boldsymbol{z})\Big]\mathrm{d}S. \quad (4.6)$$

E2D

STEP 2. We are going to calculate each term forming $\mathsf{E}^{\varepsilon}_{\mathsf{3D}}[\mathsf{W}\boldsymbol{z}]$ with the help of the splitting of $\mathsf{W}\boldsymbol{z}$ into the sum of a displacement of Kirchhoff-Love type $\mathsf{W}^{\mathsf{KL}}\boldsymbol{z}$ and of a complementary term $\mathsf{W}^{\mathsf{cmp}}\boldsymbol{z}$ which is a transverse quadratic displacement, *cf* (1.4):

$$\mathsf{W}^{\mathsf{KL}}\boldsymbol{z} = \begin{cases} z_\sigma - x_3\theta_\sigma(\boldsymbol{z}), \\ z_3, \end{cases} \quad \text{and} \quad \mathsf{W}^{\mathsf{cmp}}\boldsymbol{z} = \begin{cases} 0, \\ -x_3 p\gamma^\alpha_\alpha(\boldsymbol{z}) + \frac{x_3^2}{2}p\rho^\alpha_\alpha(\boldsymbol{z}). \end{cases}$$

Lemma 4.1 *With the minimal principal radius of curvature* $R = \kappa_1^{-1}$*, we have:*

$$\tilde{e}_i^3(\mathsf{W}^{\mathsf{KL}}\boldsymbol{z}) \;=\; 0 \quad \textit{for} \quad i = \sigma, 3 \tag{4.7a}$$

$$\tilde{e}_\sigma^\alpha(\mathsf{W}^{\mathsf{KL}}\boldsymbol{z}) \;=\; \gamma_\sigma^\alpha(\boldsymbol{z}) - x_3\big(\rho_\sigma^\alpha(\boldsymbol{z}) - 2b_\delta^\alpha\gamma_\sigma^\delta(\boldsymbol{z})\big) + \sum_{n=1}^{\infty} x_3^{1+n}(P_n^{\mathsf{KL}})_\sigma^\alpha(\boldsymbol{z}), \tag{4.7b}$$

and

$$\tilde{e}_3^3(\mathsf{W}^{\mathsf{cmp}}\boldsymbol{z}) \;=\; -p\gamma_\alpha^\alpha(\boldsymbol{z}) + px_3\rho_\alpha^\alpha(\boldsymbol{z}), \tag{4.8a}$$

$$2\tilde{e}_\sigma^3(\mathsf{W}^{\mathsf{cmp}}\boldsymbol{z}) \;=\; -x_3p\mathrm{D}_\sigma\gamma_\alpha^\alpha(\boldsymbol{z}) + \tfrac{x_3^2}{2}p\mathrm{D}_\sigma\rho_\alpha^\alpha(\boldsymbol{z}), \tag{4.8b}$$

$$\tilde{e}_\sigma^\alpha(\mathsf{W}^{\mathsf{cmp}}\boldsymbol{z}) \;=\; x_3pb_\sigma^\alpha\gamma_\delta^\delta(\boldsymbol{z}) + \sum_{n=1}^{\infty} x_3^{1+n}(P_n^{\mathsf{cmp}})_\sigma^\alpha(\boldsymbol{z}), \tag{4.8c}$$

where the tensors $(P_n^{\mathsf{KL}})(\boldsymbol{z})$ *and* $(P_n^{\mathsf{cmp}})(\boldsymbol{z})$ *satisfy the estimates, for all* $n \geq 1$*,*

$$|(P_n^{\mathsf{KL}})(\boldsymbol{z})|_{0\,;S} + |(P_n^{\mathsf{cmp}})(\boldsymbol{z})|_{0\,;S} \leq \frac{An}{R^n}\big(|\boldsymbol{\rho}(\boldsymbol{z})|_{0\,;S} + \frac{1}{R}|\boldsymbol{\gamma}(\boldsymbol{z})|_{0\,;S}\big), \tag{4.9}$$ Epq

for an adimensional constant A*.*

Equation (4.7a) justifies the denomination of $\mathsf{W}^{\mathsf{KL}}\boldsymbol{z}$ after Kirchhoff-Love.

Proof. It is clear that $\tilde{e}_3^3(\mathsf{W}^{\mathsf{KL}}\boldsymbol{z}) = \partial_{x_3}z_3 = 0$. Using equality (4.2b), we calculate

$$2\tilde{e}_\sigma^3(\mathsf{W}^{\mathsf{KL}}\boldsymbol{z}) = -\theta_\sigma(\boldsymbol{z}) + x_3b_\sigma^\alpha\theta_\alpha(\boldsymbol{z}) + \theta_\sigma(\boldsymbol{z}) - x_3b_\sigma^\alpha\theta_\alpha(\boldsymbol{z}) = 0,$$

which yields (4.7a).

The equation (4.8a) is clear. The expression (4.2b) of the operator $\tilde{e}_\sigma^3(\boldsymbol{w})$ yields (4.8b).

To obtain (4.7b) we first note that

$$\Lambda_{\alpha\beta}(\mathsf{W}^{\mathsf{KL}}\boldsymbol{z}) = \Lambda_{\alpha\beta}(\boldsymbol{z}) - \frac{x_3}{2}(b_\alpha^\sigma\mathrm{D}_\sigma\theta_\beta(\boldsymbol{z}) - b_\beta^\sigma\mathrm{D}_\alpha\theta_\sigma(\boldsymbol{z}))$$

and hence as $\mathrm{D}_\sigma\theta_\beta(\boldsymbol{z}) = \rho(\boldsymbol{z}) + b_\sigma^\nu b_{\nu\beta}z_3 - b_\sigma^\nu\mathrm{D}_\beta z_\nu$ we have

$$|\Lambda_{\alpha\beta}(\mathsf{W}^{\mathsf{KL}}\boldsymbol{z})|_{0\,;S} \leq \frac{1}{R}|z_\sigma|_{1\,;S} + \frac{x_3}{R}|\boldsymbol{\rho}(\boldsymbol{z})|_{0\,;S} + \frac{x_3}{R^2}|z_\sigma|_{1\,;S} + \frac{x_3}{R^3}|z_3|_{0\,;S}$$

With expression (4.2c) we compute that

$$\tilde{e}_\sigma^\alpha(\mathsf{W}^{\mathsf{KL}}\boldsymbol{z}) = \gamma_\sigma^\alpha(\boldsymbol{z}) + x_3b_\delta^\alpha\gamma_\sigma^\delta(\boldsymbol{z}) + x_3\Lambda^\alpha_{\cdot\,\sigma}(\boldsymbol{z}) - x_3\overline{\rho}_\sigma^\alpha(\boldsymbol{z}) + \sum_{n=1}^{\infty} x_3^{1+n}(P_n^{\mathsf{KL}})_\sigma^\alpha(\boldsymbol{z}),$$

where $\overline{\rho}_{\alpha\beta} = \frac{1}{2}(\mathrm{D}_\alpha\theta_\beta + \mathrm{D}_\beta\theta_\alpha)$ and where the tensors $(P_n^{\mathsf{KL}})(\boldsymbol{z})$ satisfy the estimate

$$|(P_n^{\mathsf{KL}})(\boldsymbol{z})|_{0\,;S} \leq \frac{An}{R^n}\big(|\boldsymbol{\rho}(\boldsymbol{z})|_{0\,;S} + \frac{1}{R}|\boldsymbol{z}|_{1\,;S} + \frac{1}{R^2}|\boldsymbol{z}|_{0\,;S}\big).$$

But we have

$$\overline{\rho}^{\alpha}_{\beta} - \Lambda^{\alpha\cdot}_{\cdot\beta} = \rho^{\alpha}_{\beta} - b^{\alpha}_{\sigma}\gamma^{\sigma}_{\beta}.$$

Moreover, using (3.5) with $n = 1$, we have that for all $\boldsymbol{z} \in \mathbf{H}^1_0 \times \mathrm{H}^2_0(S)$,

$$\frac{1}{R}|\boldsymbol{z}|_{1\,;S} + \frac{1}{R^2}|\boldsymbol{z}|_{0\,;S} \leq A_1\big(|\boldsymbol{\rho}(\boldsymbol{z})|_{0\,;S} + \frac{1}{R}|\boldsymbol{\gamma}(\boldsymbol{z})|_{0\,;S}\big).$$

Therefore we get (4.7b). The proof of (4.8c) is similar. ∎

STEP 3. Gathering the previous results and setting $(P_n)(\boldsymbol{z}) = (P^{\mathsf{KL}}_n)(\boldsymbol{z}) + (P^{\mathsf{cmp}}_n)(\boldsymbol{z})$, we find that

$$\begin{aligned}
\tilde{e}^3_3(\mathsf{W}\boldsymbol{z}) &= -p\gamma^{\alpha}_{\alpha}(\boldsymbol{z}) + px_3\rho^{\alpha}_{\alpha}(\boldsymbol{z}),\\
\tilde{e}^3_{\sigma}(\mathsf{W}\boldsymbol{z}) &= -\tfrac{x_3}{2}p\mathrm{D}_{\sigma}\gamma^{\alpha}_{\alpha}(\boldsymbol{z}) + \tfrac{x_3^2}{4}p\mathrm{D}_{\sigma}\rho^{\alpha}_{\alpha}(\boldsymbol{z}),\\
\tilde{e}^{\alpha}_{\sigma}(\mathsf{W}\boldsymbol{z}) &= \gamma^{\alpha}_{\sigma}(\boldsymbol{z}) - x_3\big(\rho^{\alpha}_{\sigma}(\boldsymbol{z}) - pb^{\alpha}_{\sigma}\gamma^{\delta}_{\delta}(\boldsymbol{z}) - 2b^{\alpha}_{\delta}\gamma^{\delta}_{\sigma}(\boldsymbol{z})\big) + \textstyle\sum_{n=1}^{\infty} x_3^{1+n}(P_n)^{\alpha}_{\sigma}(\boldsymbol{z})
\end{aligned}$$

where $(P_n)(\boldsymbol{z})$ satisfies the estimate (4.9).

We compute now the different contributions in the integral (4.3). The previous computations yield a convergent series expansion of each term in powers of x_3. Therefore each contribution in the integral (4.3) has also a convergent series expansion in powers of x_3. When integrating with respect to x_3 from $-\varepsilon$ to ε, the odd powers of x_3 have no contribution. Based on this remark we immediately obtain, first:

$$\int_{\Omega^{\varepsilon}} (\lambda + 2\mu)\tilde{e}^3_3(\mathsf{W}\boldsymbol{z})\tilde{e}^3_3(\mathsf{W}\boldsymbol{z})\,\mathrm{d}V = 2\varepsilon(\lambda+2\mu)p^2\int_S \gamma^{\alpha}_{\alpha}(\boldsymbol{z})\gamma^{\beta}_{\beta}(\boldsymbol{z})\,\mathrm{d}S + \frac{2\varepsilon^3}{3}(\lambda+2\mu)p^2\int_S \rho^{\alpha}_{\alpha}(\boldsymbol{z})\rho^{\beta}_{\beta}(\boldsymbol{z})\,\mathrm{d}S + Q_0(\varepsilon,\boldsymbol{z})$$

where the term $Q_0(\varepsilon,\boldsymbol{z})$ is due to the function h is (4.4) and thus satisfies, using (3.8)

$$|Q_0(\varepsilon,\boldsymbol{z})| \leq A\frac{\varepsilon}{R}\,\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}].$$

Then:

$$\int_{\Omega^{\varepsilon}} 2\lambda\tilde{e}^3_3(\mathsf{W}\boldsymbol{z})\tilde{e}^{\alpha}_{\alpha}(\mathsf{W}\boldsymbol{z})\,\mathrm{d}V = -4\varepsilon\lambda p\int_S \Big[\gamma^{\alpha}_{\alpha}(\boldsymbol{z})\gamma^{\beta}_{\beta}(\boldsymbol{z}) + \frac{\varepsilon^2}{3}\rho^{\alpha}_{\alpha}(\boldsymbol{z})\rho^{\beta}_{\beta}(\boldsymbol{z})\Big]\,\mathrm{d}S + Q_1(\varepsilon,\boldsymbol{z})$$

where

$$Q_1(\varepsilon,\boldsymbol{z}) = \int_{\Omega^{\varepsilon}}\Big(2x_3^2\lambda p\rho^{\alpha}_{\alpha}(\boldsymbol{z})\big(pb^{\nu}_{\nu}\gamma^{\delta}_{\delta}(\boldsymbol{z}) + 2b^{\nu}_{\delta}\gamma^{\delta}_{\nu}(\boldsymbol{z})\big) - 2\lambda x_3^2 p\gamma^{\alpha}_{\alpha}(\boldsymbol{z})(P_1)^{\nu}_{\nu}(\boldsymbol{z}) + 2\lambda x_3^4 p\rho^{\alpha}_{\alpha}(\boldsymbol{z})(P_2)^{\nu}_{\nu}(\boldsymbol{z}) + \text{h.o.t.}\Big)\,\mathrm{d}V.$$

Hence using (4.9) and (4.5) we see that $Q_1(\varepsilon, \boldsymbol{z})$ satisfies:

$$|Q_1(\varepsilon, \boldsymbol{z})| \le AE\Big(\frac{\varepsilon^2}{R}|\boldsymbol{\gamma}|^2_{0\,;\,S} + \frac{\varepsilon^4}{R}|\boldsymbol{\rho}|^2_{0\,;\,S} + \frac{\varepsilon^3}{R}|\boldsymbol{\gamma}|_{0\,;\,S}\,|\boldsymbol{\rho}|_{0\,;\,S}\Big),$$

where we used the fact that $\varepsilon R^{-1} < 1$. As we have

$$\frac{\varepsilon^3}{R}|\boldsymbol{\gamma}|_{0\,;\,S}\,|\boldsymbol{\rho}|_{0\,;\,S} \le A\Big(\frac{\varepsilon^2}{R}|\boldsymbol{\gamma}|^2_{0\,;\,S} + \frac{\varepsilon^4}{R}|\boldsymbol{\rho}|^2_{0\,;\,S}\Big)$$

we get using (3.8)

$$|Q_1(\varepsilon, \boldsymbol{z})| \le A\frac{\varepsilon}{R}\,\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}].$$

Similarly we compute that:

$$\int_{\Omega^\varepsilon} \lambda \tilde{e}^{\alpha}_{\alpha}(\mathsf{W}\boldsymbol{z})\tilde{e}^{\beta}_{\beta}(\mathsf{W}\boldsymbol{z})\,\mathrm{d}V = 2\varepsilon\lambda\int_S \Big[\gamma^{\alpha}_{\alpha}(\boldsymbol{z})\gamma^{\beta}_{\beta}(\boldsymbol{z}) + \frac{\varepsilon^2}{3}\rho^{\alpha}_{\alpha}(\boldsymbol{z})\rho^{\beta}_{\beta}(\boldsymbol{z})\Big]\,\mathrm{d}S + Q_2(\varepsilon, \boldsymbol{z})$$

where, again, we have

$$|Q_2(\varepsilon, \boldsymbol{z})| \le A\frac{\varepsilon}{R}\,\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}].$$

We also have

$$\int_{\Omega^\varepsilon} 4\mu a^{\alpha\beta}(x_3)\tilde{e}^{3}_{\alpha}(\mathsf{W}\boldsymbol{z})\tilde{e}^{3}_{\beta}(\mathsf{W}\boldsymbol{z})\,\mathrm{d}V = Q_3(\varepsilon, \boldsymbol{z}),$$

with:

$$|Q_3(\varepsilon, \boldsymbol{z})| \le AE\Big(\varepsilon^3|\boldsymbol{\gamma}|^2_{1\,;\,S} + \varepsilon^5|\boldsymbol{\rho}|^2_{1\,;\,S}\Big)$$

and thus using the definition (2.15) of L_1 and the estimates (3.8),

$$|Q_3(\varepsilon, \boldsymbol{z})| \le AE\frac{\varepsilon^2}{L_1^2}\,\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}].$$

Finally, we have:

$$\int_{\Omega^\varepsilon} 2\mu \tilde{e}^{\alpha}_{\beta}(\mathsf{W}\boldsymbol{z})\tilde{e}^{\beta}_{\alpha}(\mathsf{W}\boldsymbol{z})\,\mathrm{d}V = 4\varepsilon\mu\int_S \Big[\gamma^{\alpha}_{\beta}(\boldsymbol{z})\gamma^{\beta}_{\alpha}(\boldsymbol{z}) + \frac{\varepsilon^2}{3}\rho^{\alpha}_{\beta}(\boldsymbol{z})\rho^{\beta}_{\alpha}(\boldsymbol{z})\Big]\,\mathrm{d}S + Q_4(\varepsilon, \boldsymbol{z})$$

where, again:

$$|Q_4(\varepsilon, \boldsymbol{z})| \le A\frac{\varepsilon}{R}\,\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}].$$

Finally, using the relation: $\lambda - 2\lambda p + p^2(\lambda + 2\mu) = 2\mu p$, we find that

$$\begin{aligned}\tilde{E}^{\varepsilon}_{\mathsf{3D}}[\mathsf{W}\boldsymbol{z}] = 2\varepsilon\int_S \Big[2\mu p\gamma^{\alpha}_{\alpha}(\boldsymbol{z})\gamma^{\beta}_{\beta}(\boldsymbol{z}) + 2\mu\gamma^{\beta}_{\alpha}(\boldsymbol{z})\gamma^{\alpha}_{\beta}(\boldsymbol{z})\Big]\,\mathrm{d}S \\ + \frac{2\varepsilon^3}{3}\int_S \Big[2\mu p\rho^{\alpha}_{\alpha}(\boldsymbol{z})\rho^{\beta}_{\beta}(\boldsymbol{z}) + 2\mu\rho^{\beta}_{\alpha}(\boldsymbol{z})\rho^{\alpha}_{\beta}(\boldsymbol{z})\Big]\,\mathrm{d}S + Q(\varepsilon, \boldsymbol{z}) \quad (4.10)\end{aligned}$$ 4E9

where $Q(\varepsilon, \boldsymbol{z})$ is the sum $\sum_{\ell=1}^{4} Q_\ell(\varepsilon, \boldsymbol{z})$, and thus

$$|Q(\varepsilon, \boldsymbol{z})| \le A\Big(\frac{\varepsilon}{R} + \frac{\varepsilon^2}{L_1^2}\Big)\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}].$$

But, compared with (4.6), the right-hand side of (4.10) writes $\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}] + Q(\varepsilon, \boldsymbol{z})$. Hence we have

$$\tilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\mathsf{W}\boldsymbol{z}] - \mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}] = Q(\varepsilon, \boldsymbol{z}),$$

and this yields the result. ∎

Remark 4.2 The part $\mathsf{U}^{\mathsf{cmp}}\boldsymbol{z}$ has a significant energy. If we evaluate the energy of $\mathsf{U}^{\mathsf{KL}}\boldsymbol{z}$ instead of the full $\mathsf{U}\boldsymbol{z}$, we obtain the plain strain energy $2\varepsilon b^\varepsilon_{\mathsf{2D}}(\boldsymbol{z}, \boldsymbol{z})$ of $\boldsymbol{z}$ defined below instead of the plain stress energy $2\varepsilon a^\varepsilon_{\mathsf{2D}}(\boldsymbol{z}, \boldsymbol{z})$: Recall that, *cf* (2.8) $a^\varepsilon_{\mathsf{2D}}(\boldsymbol{z}, \boldsymbol{z})$ is equal to

$$\int_S \tilde\lambda \gamma^\alpha_\alpha(\boldsymbol{z})\gamma^\beta_\beta(\boldsymbol{z}) + 2\mu\gamma^\beta_\alpha(\boldsymbol{z})\gamma^\alpha_\beta(\boldsymbol{z})\,\mathrm{d}S + \frac{\varepsilon^2}{3}\int_S \tilde\lambda \rho^\alpha_\alpha(\boldsymbol{z})\rho^\beta_\beta(\boldsymbol{z}) + 2\mu\rho^\beta_\alpha(\boldsymbol{z})\rho^\alpha_\beta(\boldsymbol{z})\,\mathrm{d}S,$$

and let us define $b^\varepsilon_{\mathsf{2D}}(\boldsymbol{z}, \boldsymbol{z})$ as

$$\int_S \lambda \gamma^\alpha_\alpha(\boldsymbol{z})\gamma^\beta_\beta(\boldsymbol{z}) + 2\mu\gamma^\beta_\alpha(\boldsymbol{z})\gamma^\alpha_\beta(\boldsymbol{z})\,\mathrm{d}S + \frac{\varepsilon^2}{3}\int_S \lambda \rho^\alpha_\alpha(\boldsymbol{z})\rho^\beta_\beta(\boldsymbol{z}) + 2\mu\rho^\beta_\alpha(\boldsymbol{z})\rho^\alpha_\beta(\boldsymbol{z})\,\mathrm{d}S.$$

Using the previous computations, we can show that

$$\big|\mathsf{E}^\varepsilon_{\mathsf{3D}}[\mathsf{U}^{\mathsf{KL}}\boldsymbol{z}] - 2\varepsilon b^\varepsilon_{\mathsf{2D}}(\boldsymbol{z}, \boldsymbol{z})\big| \le AE\Big(\frac{\varepsilon^2}{R}|\boldsymbol{\gamma}|^2_{0\,;S} + \frac{\varepsilon^4}{R}|\boldsymbol{\rho}|^2_{0\,;S}\Big).$$

5 Outline of the proof of the main estimate

To prove (2.24), we have to take lateral Dirichlet boundary conditions on Γ^ε_0 into account. As $\mathsf{U}\boldsymbol{z}$ does not satisfy these boundary conditions in general, we will add a correction term $\boldsymbol{u}^{\mathsf{cor}}$ to it so that $\mathsf{U}\boldsymbol{z} + \boldsymbol{u}^{\mathsf{cor}}$ is zero on Γ^ε_0.

The plan of the proof of (2.24) originates from the following

Theorem 5.1 *Let $\boldsymbol{u}$ be solution of problem (P_{3D}), $\boldsymbol{z}$ the solution of problem (P_{2D}) and $\boldsymbol{u}^{\mathsf{cor}}$ constructed so that $\mathsf{U}\boldsymbol{z} + \boldsymbol{u}^{\mathsf{cor}} \in V(\Omega^\varepsilon)$. If we have the estimates*

$$\forall \boldsymbol{v} \in V(\Omega^\varepsilon) \qquad a^\varepsilon_{\mathsf{3D}}(\boldsymbol{u} - \mathsf{U}\boldsymbol{z}, \boldsymbol{v}) \le B_1^{1/2}\,\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]^{1/2}, \qquad (5.1) \qquad \boxed{\mathtt{5E4}}$$

and

$$\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{u}^{\mathsf{cor}}] \le B_2, \qquad (5.2) \qquad \boxed{\mathtt{5E5}}$$

then there holds

$$\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{u} - \mathsf{U}\boldsymbol{z}] \le \big(B_1^{1/2} + 2B_2^{1/2}\big)^2. \qquad (5.3) \qquad \boxed{\mathtt{5E6}}$$

Proof. Let $\boldsymbol{u}^{\mathsf{new}} = \mathsf{U}\boldsymbol{z} + \boldsymbol{u}^{\mathsf{cor}} \in V(\Omega^\varepsilon)$. Since $\boldsymbol{u} - \mathsf{U}\boldsymbol{z} = (\boldsymbol{u} - \boldsymbol{u}^{\mathsf{new}}) + \boldsymbol{u}^{\mathsf{cor}}$, we start from the triangle inequality

$$\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{u} - \mathsf{U}\boldsymbol{z}]^{1/2} \le \mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{u} - \boldsymbol{u}^{\mathsf{new}}]^{1/2} + \mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{u}^{\mathsf{cor}}]^{1/2}. \tag{5.4}$$

5E7

The last term of the rhs is bounded by $B_2^{1/2}$. As for the first one we write

$$\begin{aligned}\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{u} - \boldsymbol{u}^{\mathsf{new}}] &= a^\varepsilon_{\mathsf{3D}}(\boldsymbol{u} - \boldsymbol{u}^{\mathsf{new}}, \boldsymbol{u} - \boldsymbol{u}^{\mathsf{new}}) \\ &= a^\varepsilon_{\mathsf{3D}}(\boldsymbol{u} - \mathsf{U}\boldsymbol{z}, \boldsymbol{u} - \boldsymbol{u}^{\mathsf{new}}) + a^\varepsilon_{\mathsf{3D}}(\boldsymbol{u}^{\mathsf{cor}}, \boldsymbol{u} - \boldsymbol{u}^{\mathsf{new}}).\end{aligned}$$

Since $\boldsymbol{u} - \boldsymbol{u}^{\mathsf{new}}$ belongs to $V(\Omega^\varepsilon)$, we may use (5.1) and obtain:

$$\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{u} - \boldsymbol{u}^{\mathsf{new}}] \le \left(B_1^{1/2} + \mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{u}^{\mathsf{cor}}]^{1/2}\right) \mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{u} - \boldsymbol{u}^{\mathsf{new}}]^{1/2},$$

whence, using (5.2) again

$$\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{u} - \boldsymbol{u}^{\mathsf{new}}]^{1/2} \le B_1^{1/2} + B_2^{1/2}.$$

With (5.4) this gives the estimate (5.3). ∎

Thus, to obtain (2.24), it is sufficient to prove estimates (5.1)-(5.2) with B_1, $B_2 \lesssim A_S(\varepsilon, \boldsymbol{z}, \boldsymbol{f}^{\mathsf{rem}})$ with

$$A_S(\varepsilon, \boldsymbol{z}, \boldsymbol{f}^{\mathsf{rem}}) = B_S(\varepsilon; \boldsymbol{z})\, \mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}] + \mathsf{D}^2 E^{-1} \|\boldsymbol{f}^{\mathsf{rem}}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)}, \tag{5.5}$$

Eest2

where $B_S(\varepsilon; \boldsymbol{z})$ is defined in (2.24). In §7, we do this for B_1 and in §8 we construct the correction term $\boldsymbol{u}^{\mathsf{cor}}$ and prove that $B_2 \lesssim A_S(\varepsilon, \boldsymbol{z}, \boldsymbol{f}^{\mathsf{rem}})$.

6 Formal series reduction

We had rather to work with the shifted displacement $\boldsymbol{w}$ associated with $\boldsymbol{u}$, cf. §2.f. The 3D displacement $\boldsymbol{w}$ satisfies for all $v \in V(\Omega^\varepsilon)$

$$\int_{\Omega^\varepsilon} A^{ijk\ell} \tilde{e}_{ij}(\boldsymbol{w})\, e_{k\ell}(\boldsymbol{v})\, \mathrm{d}V = \int_{\Omega^\varepsilon} \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}V, \tag{6.1}$$

Pbw

with the shifted strains $\tilde{e}_{ij}(\boldsymbol{w})$ defined in (4.2a)-(4.2c). Integrating by part in (6.1), we find that the shifted displacement $\boldsymbol{w}$ is solution of the boundary value problem

$$\left\{\begin{array}{rcll} \mathsf{L}\boldsymbol{w} &=& \boldsymbol{f} & \text{in} \quad \Omega^\varepsilon \\ \mathsf{T}\boldsymbol{w} &=& 0 & \text{on} \quad \Gamma^\varepsilon_\pm \\ \boldsymbol{w} &=& 0 & \text{on} \quad \Gamma^\varepsilon_0, \end{array}\right. \tag{6.2}$$

5E0

where the coefficients of the operators L and T express in terms of the normal coordinate x_3, the covariant derivative D_α and the curvature tensor $b_{\alpha\beta}$, see [14]. The operator L is of degree 2, while T is of degree 1.

6.A Scaling in the 3D boundary value problem

The formal series approach of [14, 13] relies on the scaling $X_3 = \varepsilon^{-1}x_3$ which transforms problem (6.2) into a problem posed on a domain independent of ε, with operators which are power series of ε. This allows a formal series reduction of the 3D problem.

The scaling $x_3 \mapsto X_3 = \varepsilon^{-1}x_3$ is one-to-one from the shell Ω^ε onto the manifold $\Omega := S \times (-1,1)$ and we denote by $\Gamma_\pm$ its upper and lower faces $S \times \{\pm 1\}$ and by Γ_0 its lateral boundary $\partial S \times (-1,1)$. Likewise $V(\Omega)$ denotes the space of $\boldsymbol{v} \in H^1(\Omega)^3$ which satisfies the Dirichlet boundary condition $\boldsymbol{v}\big|_{\Gamma_0} = 0$.

In the following, we denote by $\underline{\boldsymbol{u}}$ the displacement $\boldsymbol{u}$ viewed on the manifold Ω. In a local coordinate system (x_α) on S, this means that $\underline{\boldsymbol{u}}(x_\alpha, X_3) = \boldsymbol{u}(x_\alpha, x_3)$ for $X_3 = \varepsilon^{-1}x_3$. Similarly, $\underline{\boldsymbol{w}}$ and $\underline{\boldsymbol{f}}$ correspond to the shifted displacement $\boldsymbol{w}$ and the loading forces $\boldsymbol{f}$. To denote the displacements $\mathsf{U}\boldsymbol{z}$ and $\mathsf{W}\boldsymbol{z}$ on Ω, we use the notations $\mathsf{U}(\varepsilon)\boldsymbol{z}$ and $\mathsf{W}(\varepsilon)\boldsymbol{z}$ so that we have with (2.18)

$$\mathsf{W}(\varepsilon)\boldsymbol{z} = \begin{cases} z_\sigma - \varepsilon X_3 \theta_\sigma(\boldsymbol{z}), \\ z_3 - \varepsilon X_3 p\, \gamma_\alpha^\alpha(\boldsymbol{z}) + \varepsilon^2 \frac{X_3^2}{2} p\, \rho_\alpha^\alpha(\boldsymbol{z}), \end{cases} \tag{6.3}$$

2EWeps

and a similar formula for $\mathsf{U}(\varepsilon)\boldsymbol{z}$.

In the same way, we define the three dimensional energy on Ω by the formula

$$\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\underline{\boldsymbol{u}}] = \mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{u}]$$

involving the scaled strain tensor $e_{ij}(\varepsilon)(\underline{\boldsymbol{u}}) = e_{ij}(\boldsymbol{u})$, and associated with the bilinear form $a_{\mathsf{3D}}(\varepsilon)(\cdot\,,\cdot)$ on $V(\Omega)$. In particular, we will often use the relation, for $\boldsymbol{v} \in \mathrm{H}^1(\Omega)$,

$$\|e_{ij}(\varepsilon)(\boldsymbol{v})\|^2_{\mathrm{L}^2(\Omega)} \simeq \varepsilon^{-1}E^{-1}\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}]. \tag{6.4}$$

EEe

Note that with these notations, Korn inequalities (2.20) read, for $\boldsymbol{v} \in V(\Omega)$,

$$\begin{aligned} \|\boldsymbol{v}\|^2_{\mathrm{L}^2(\Omega)} &\leq \mathsf{D}^4 E^{-1}\varepsilon^{-3}\,\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}] \\ \|\mathrm{D}_\alpha \boldsymbol{v}\|^2_{\mathrm{L}^2(\Omega)} &\leq \mathsf{D}^2 E^{-1}\varepsilon^{-3}\,\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}] \\ \|\partial_{X_3} \boldsymbol{v}\|^2_{\mathrm{L}^2(\Omega)} &\leq \mathsf{D}^2 E^{-1}\varepsilon^{-1}\,\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}] \end{aligned} \tag{6.5}$$

Korn3Deps

The scaled displacement $\underline{\boldsymbol{u}} \in V(\Omega)$ is solution of the variational problem

$$\forall\, \boldsymbol{v} \in V(\Omega), \quad a_{\mathsf{3D}}(\varepsilon)(\underline{\boldsymbol{u}}, \boldsymbol{v}) = \varepsilon \langle \underline{\boldsymbol{f}}, \boldsymbol{v} \rangle_{\mathrm{L}^2(\Omega)}. \tag{6.6}$$

ipart

The 3D interior operator $\mathsf{L}(x_\alpha, x_3; \mathrm{D}_\alpha, \partial_3)$ in problem (6.2) is transformed into the operator $\mathsf{L}(\varepsilon)$

$$\mathsf{L}(\varepsilon)(x_\alpha, X_3; \mathrm{D}_\alpha, \partial_{X_3}) := \mathsf{L}(x_\alpha, \varepsilon X_3; \mathrm{D}_\alpha, \varepsilon^{-1}\partial_{X_3}),$$

and similarly for the boundary operators $\mathsf{T}(x_\alpha, x_3; \mathrm{D}_\alpha, \partial_3)$ and $\mathsf{T}(\varepsilon)$. Note that on the manifold Ω, the variable X_3 and the partial derivative ∂_{X_3} are adimensional. The operators $\mathsf{L}(\varepsilon)$ and $\mathsf{T}(\varepsilon)$ expand in power series of ε:

$$\mathsf{L}(\varepsilon) = \varepsilon^{-2} \sum_{k=0}^{\infty} \varepsilon^k \mathsf{L}^k \quad \text{and} \quad \mathsf{T}(\varepsilon) = \varepsilon^{-1} \sum_{k=0}^{\infty} \varepsilon^k \mathsf{T}^k, \tag{6.7}$$

ELTk

where L^k and T^k are intrinsic operators in Ω which are polynomial in X_3 and in ∂_{X_3} with coefficients $\boldsymbol{b}$-homogeneous operators of degree k, see [14, Thm. 3.3].

So problem (6.2) is equivalent to the problem

$$\begin{cases} \mathsf{L}(\varepsilon)\underline{\boldsymbol{w}} & = & \underline{\boldsymbol{f}} & \text{in} & \Omega \\ \mathsf{T}(\varepsilon)\underline{\boldsymbol{w}} & = & 0 & \text{on} & \Gamma_\pm \\ \underline{\boldsymbol{w}} & = & 0 & \text{on} & \Gamma_0 . \end{cases} \tag{6.8}$$

5E0bis

Moreover there holds for all $\boldsymbol{v} \in V(\Omega)$ and all $\boldsymbol{u}^* \in \mathrm{H}^1(\Omega)^3$

$$a_{3\mathsf{D}}(\varepsilon)(\boldsymbol{u}^*, \boldsymbol{v}) = -\,\varepsilon \left\langle \mathsf{L}(\varepsilon)\boldsymbol{w}^*, \boldsymbol{v} \right\rangle_{\mathrm{L}^2(\Omega)} - \varepsilon \left\langle \mathsf{T}(\varepsilon)\boldsymbol{w}^*, \boldsymbol{v} \right\rangle_{\mathrm{L}^2(\Gamma_\pm)}. \tag{6.9}$$

6E7

Here, $\boldsymbol{w}^*$ is the scaled shifted displacement corresponding to $\boldsymbol{u}^*$ i.e.

$$u_3^* = w_3^* \quad \text{and} \quad u_\sigma^* = (\delta_\sigma^\beta - \varepsilon X_3 b_\sigma^\beta) w_\beta^*. \tag{6.10}$$

Eshif

For instance (6.9) holds with $\boldsymbol{u}^* = \mathsf{U}(\varepsilon)\boldsymbol{z}$ and $\boldsymbol{w}^* = \mathsf{W}(\varepsilon)\boldsymbol{z}$.

6.B Solution of transverse problems

The treatment of the first two equations of (6.8) can be performed by solving Neumann problems in X_3 and introducing suitable compatibility conditions. This can be done in a fully exact way without approximation using the formalismus of formal Laurent series and formal power series, as follows.

With expansions (6.7) we associate the *formal Laurent series*

$$\mathsf{L}(\mathsf{X}) = \sum_{k \geq -2} \mathsf{L}^{k+2} \mathsf{X}^k \quad \text{and} \quad \mathsf{T}(\mathsf{X}) = \sum_{k \geq -1} \mathsf{T}^{k+1} \mathsf{X}^k.$$

The three-dimensional formal series system $\{\mathsf{L}(\mathsf{X}), \mathsf{T}(\mathsf{X})\}$ can be reduced to a two dimensional one, cf. [14, Thm. 4.1]:

Theorem 6.1 *There exist two unique formal power series* $\mathsf{V}(\mathsf{X}) = \sum_{k \geq 0} \mathsf{V}^k \mathsf{X}^k$ *and* $\mathsf{A}(\mathsf{X}) = \sum_{k \geq 0} \mathsf{A}^k \mathsf{X}^k$ *satisfying the following three conditions:*

1. *The coefficients* V^k *are reconstruction operators acting from* $\Sigma(S)$ *with values in* $\mathscr{C}^\infty(I, \Sigma(S))$, *and such that for all* $\boldsymbol{z} \in \Sigma(S)$

$$\mathsf{V}^0 \boldsymbol{z} = \boldsymbol{z} \quad \text{and} \quad \forall k \geq 1, \;\; \mathsf{V}^k \boldsymbol{z} = 0 \;\; \text{on} \;\; S,$$

2. *The coefficients* A^k *are 2D operators acting from* $\Sigma(S)$ *into itself,*
3. *There holds the formal series equation*

$$\begin{cases} \mathsf{L}(\mathsf{X})\,\mathsf{V}(\mathsf{X}) & = & -\mathcal{I}\circ\mathsf{A}(\mathsf{X}) & \text{in} \quad \Omega \\ \mathsf{T}(\mathsf{X})\,\mathsf{V}(\mathsf{X}) & = & 0 & \text{on} \quad \Gamma_{\pm}. \end{cases} \tag{6.11}$$

Efso

Here, the operator $\mathcal{I}$ *is the natural embedding operator from* $\Sigma(S)$ *to the space* $\mathscr{C}^\infty(I,\Sigma(S))$ *and the product between two formal series is the standard Cauchy product.*

Thus the equation (6.11) means that there holds

$$\mathsf{L}^0\,\mathsf{V}^0 = 0, \quad \mathsf{L}^0\,\mathsf{V}^1 + \mathsf{L}^1\,\mathsf{V}^0 = 0, \quad \sum_{j+k=i} \mathsf{L}^j\,\mathsf{V}^k = -\mathcal{I}\circ\mathsf{A}^{i-2}, \quad i \geq 2 \tag{6.12a}$$

$$\sum_{j+k=i} \mathsf{T}^j\,\mathsf{V}^k = 0, \quad i \geq 0. \tag{6.12b}$$

Following the proof of [14, Thm. 4.1], we can see that the term V^k of the formal series $\mathsf{V}(\mathsf{X})$ is polynomial in X_3 and $\boldsymbol{b}$-homogeneous of degree k. We have

$$\mathsf{V}^1_\sigma\boldsymbol{z} = -X_3\theta_\sigma(\boldsymbol{z}), \quad \mathsf{V}^1_3\boldsymbol{z} = -p\,X_3\gamma^\alpha_\alpha(\boldsymbol{z})$$

and

$$\mathsf{V}^2_\sigma\boldsymbol{z} = \frac{X_3^2}{2}p\mathrm{D}_\sigma\gamma^\alpha_\alpha(\boldsymbol{z}), \quad \mathsf{V}^2_3\boldsymbol{z} = p\,\frac{X_3^2}{2}\big(\rho^\alpha_\alpha(\boldsymbol{z}) - pb^\alpha_\alpha\gamma^\beta_\beta(\boldsymbol{z}) - 2b^\beta_\alpha\gamma^\alpha_\beta(\boldsymbol{z})\big)$$

as first terms. Actually, the reconstruction operator $\mathsf{W}(\varepsilon):\ \boldsymbol{z} \to \mathsf{W}(\varepsilon)\boldsymbol{z}$ in (6.3) coincides with $\mathsf{V}^0 + \varepsilon\mathsf{V}^1 + \varepsilon^2(\mathsf{V}^2 - \mathsf{v}^2)$ where v^2 is a residual part of the operator V^2.

The term A^k of the formal series $\mathsf{A}(\mathsf{X})$ is a $\boldsymbol{b}$-homogeneous operator of degree $k+2$. The zero-th order term A^0 coincides with the membrane operator M, A^1 is zero, so that $\mathsf{A}(\varepsilon) = \mathsf{M} + \varepsilon^2\mathsf{A}^2 + \cdots$. Moreover, adapting the proof of [14, Prop. 4.5] we obtain the following estimate for the difference $\mathsf{A}^2 - \mathsf{B}$ where B is the Koiter bending operator: If $\boldsymbol{z}$ and $\boldsymbol{\eta} \in \Sigma(S)$ and $\boldsymbol{\eta}$ satisfies the boundary condition $\boldsymbol{\eta}\big|_{\partial S} = 0$,

$$\Big|\big\langle(\mathsf{A}^2 - \mathsf{B})\boldsymbol{z},\boldsymbol{\eta}\big\rangle_{\mathrm{L}^2(S)}\Big| \lesssim E\,\Big(\big|\boldsymbol{\gamma}(\boldsymbol{z})\big|_{2\,;\,S}\,\big|\boldsymbol{\gamma}(\boldsymbol{\eta})\big|_{0\,;\,S} + \kappa_2^2|\boldsymbol{z}|^{(\boldsymbol{b})}_{1\,;\,S}\,\big|\boldsymbol{\gamma}(\boldsymbol{\eta})\big|_{0\,;\,S}$$
$$+\kappa_1|z_3|_{2\,;\,S}\,\big|\boldsymbol{\gamma}(\boldsymbol{\eta})\big|_{0\,;\,S} + \kappa_1\big|\boldsymbol{\gamma}(\boldsymbol{z})\big|_{1\,;\,S}\,|\boldsymbol{\eta}|^{(\boldsymbol{b})}_{1\,;\,S} + \kappa_2^2\big|\boldsymbol{\gamma}(\boldsymbol{z})\big|_{0\,;\,S}\,|\boldsymbol{\eta}|^{(\boldsymbol{b})}_{1\,;\,S}\Big), \tag{6.13}$$

E1

where the constants κ_j, $j = 1, 2$ are defined in (2.13).

7 INNER ESTIMATE

In this section, we prove the following result:

Proposition 7.1 *With the definitions of Section 2, let* $\kappa = \kappa_5$, $\mathsf{r} = 1/\kappa$ *and* $L = L_4$. *For* $\boldsymbol{v} \in V(\Omega)$, *we have the estimate*

$$a_{\mathsf{3D}}(\varepsilon)(\underline{\boldsymbol{u}} - \mathsf{U}(\varepsilon)\boldsymbol{z}, \boldsymbol{v}) \lesssim B_1^{1/2}\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}]^{1/2}$$

where

$$B_1 = \mathsf{D}^2 E^{-1}\varepsilon\|\underline{\boldsymbol{f}}^{\mathsf{rem}}\|^2_{\mathrm{L}^2(\Omega)} + B_S^1(\varepsilon;\boldsymbol{z})\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}], \tag{7.1}$$

EB1

where $\underline{\boldsymbol{f}}^{\mathsf{rem}} = \underline{\boldsymbol{f}} - \boldsymbol{g}$ *and*

$$B_S^1(\varepsilon;\boldsymbol{z}) = \Big(1 + \frac{\mathsf{D}^4}{\mathsf{r}^4}\Big)\Big(\sum_F \frac{\varepsilon^{2k}}{L^{2i}\mathsf{r}^{2j}} + \frac{\varepsilon^4\mathsf{D}^2}{L^6} + \frac{\varepsilon^6\mathsf{D}^2}{L^8}\Big) \tag{7.2}$$

BS1

where F *is the finite set* $\{(i,j,k) \in \mathbb{N}^3 \,|\, i+j = k,\ k \in \{1,2,3,4\}\}$.

Scaling back to Ω^ε, the previous result implies that for $\boldsymbol{v} \in V(\Omega^\varepsilon)$, we have

$$a^{\varepsilon}_{\mathsf{3D}}(\boldsymbol{u} - \mathsf{U}\boldsymbol{z}, \boldsymbol{v}) \lesssim B_1^{1/2}\mathsf{E}^{\varepsilon}_{\mathsf{3D}}[\boldsymbol{v}]^{1/2}$$

where B_1 is given by (7.1). Note that we have $\varepsilon\|\underline{\boldsymbol{f}}^{\mathsf{rem}}\|^2_{\mathrm{L}^2(\Omega)} = \|\boldsymbol{f}^{\mathsf{rem}}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)}$.

Before starting the proof of the proposition, let us prove that B_1 in (7.1) satisfies $B_1 \lesssim A_S(\varepsilon, \boldsymbol{z}, \boldsymbol{f}^{\mathsf{rem}})$ given by (5.5) and (2.24) under the hypothesis $\varepsilon < \mathsf{r}$ and $\varepsilon^2/L^2 \leq M$ of Theorem 2.9.

First, the ratio D/r is an adimensional constant depending on S only. Hence,

$$B_S^1(\varepsilon;\boldsymbol{z}) \lesssim \sum_F \frac{\varepsilon^{2k}}{L^{2i}\mathsf{r}^{2j}} + \frac{\varepsilon^4\mathsf{D}^2}{L^6} + \frac{\varepsilon^6\mathsf{D}^2}{L^8}.$$

Now it is clear that under the hypothesis $\varepsilon^2/L^2 \leq M$ we have

$$\sum_F \frac{\varepsilon^{2k}}{L^{2i}\mathsf{r}^{2j}} \lesssim \Big(\frac{\varepsilon^2}{\mathsf{r}^2} + \frac{\varepsilon^2}{L^2}\Big)(1 + M^3) \quad \text{and} \quad \frac{\varepsilon^6\mathsf{D}^2}{L^8} \leq M\frac{\varepsilon^4\mathsf{D}^2}{L^6}$$

and this proves that $B_1 \lesssim A_S(\varepsilon, \boldsymbol{z}, \boldsymbol{f}^{\mathsf{rem}})$ with an adimensional constant satisfying (2.25).

The leading idea of the proof of Proposition 7.1 is to replace $\mathsf{U}(\varepsilon)\boldsymbol{z}$ with a more precise reconstructed displacement $\mathsf{U}^{\mathsf{asy}}(\varepsilon)\boldsymbol{z}$: Working in shifted displacement, we define the new reconstruction operator $\mathsf{W}^{\mathsf{asy}}(\varepsilon)$ as the first five terms of the formal series $\mathsf{V}(\mathsf{X})$ introduced in Theorem 6.1:

$$\mathsf{W}^{\mathsf{asy}}(\varepsilon) = \mathsf{V}^0 + \varepsilon\mathsf{V}^1 + \varepsilon^2\mathsf{V}^2 + \varepsilon^3\mathsf{V}^3 + \varepsilon^4\mathsf{V}^4. \tag{7.3}$$

5E1

To this operator corresponds the operator $\mathsf{U}^{\mathsf{asy}}(\varepsilon)$ as in (6.10).

We first prove the following lemma:

Lemma 7.2 *For* $\boldsymbol{v} \in V(\Omega)$, *we have the estimate*

$$a_{3\mathsf{D}}(\varepsilon)(\underline{\boldsymbol{u}} - \mathsf{U}^{\mathsf{asy}}(\varepsilon)\boldsymbol{z}, \boldsymbol{v}) \lesssim B_1^{1/2}\, \mathsf{E}_{3\mathsf{D}}(\varepsilon)[\boldsymbol{v}]^{1/2}$$

where $\mathsf{U}^{\mathsf{asy}}(\varepsilon)\boldsymbol{z}$ *is given in shifted components by the displacement* $\mathsf{W}^{\mathsf{asy}}(\varepsilon)\boldsymbol{z}$ *defined in* (7.3) *and* B_1 *by* (7.1).

Remark 7.3 Owing to the fact that $\varepsilon < \mathsf{r} \leq R$, together with (4.5), we will always assume that ε is sufficiently small, so that $|h(x_3)| \leq 1/2$. Hence, the volume form $\mathrm{d}V$ is equivalent (up to constant) to the form $\mathrm{d}S\,\mathrm{d}x_3$. We will make a constant use of this relation in the sequel.

Proof of Lemma 7.2. Let $\boldsymbol{v} \in V(\Omega)$. We split $a_{3\mathsf{D}}(\varepsilon)(\underline{\boldsymbol{u}} - \mathsf{U}^{\mathsf{asy}}(\varepsilon)\boldsymbol{z}, \boldsymbol{v})$ into two terms. Since $\boldsymbol{u}$ is solution of $(P_{3\mathsf{D}})$, eq. (6.6) yields

$$a_{3\mathsf{D}}(\varepsilon)(\underline{\boldsymbol{u}}, \boldsymbol{v}) = \varepsilon\langle \underline{\boldsymbol{f}}, \boldsymbol{v}\rangle_{\mathrm{L}^2(\Omega)}.$$

For the second term, using (6.9) we obtain

$$a_{3\mathsf{D}}(\varepsilon)(\mathsf{U}^{\mathsf{asy}}(\varepsilon)\boldsymbol{z}, \boldsymbol{v}) = -\varepsilon\left\langle \mathsf{L}(\varepsilon)\mathsf{W}^{\mathsf{asy}}(\varepsilon)\boldsymbol{z}, \boldsymbol{v}\right\rangle_{\mathrm{L}^2(\Omega)} - \varepsilon\left\langle \mathsf{T}(\varepsilon)\mathsf{W}^{\mathsf{asy}}(\varepsilon)\boldsymbol{z}, \boldsymbol{v}\right\rangle_{\mathrm{L}^2(\Gamma_\pm)}.$$

By definition of $\mathsf{W}^{\mathsf{asy}}(\varepsilon)$, and using (6.12a), we find

$$\begin{aligned}-\mathsf{L}(\varepsilon)\mathsf{W}^{\mathsf{asy}}(\varepsilon) = \mathsf{M} + \varepsilon^2\mathsf{A}^2 + &\\ + \varepsilon^3(\mathsf{L}^1\mathsf{V}^4 + \mathsf{L}^2\mathsf{V}^3 + \mathsf{L}^3\mathsf{V}^2 + \mathsf{L}^4\mathsf{V}^1 + \mathsf{L}^5\mathsf{V}^0) + \varepsilon^4 &\sum_{0\leq i\leq 4} \overline{\mathsf{L}}^i(\varepsilon)\mathsf{V}^i,\end{aligned} \tag{7.4}$$

E6

where the operators $\overline{\mathsf{L}}^i(\varepsilon)$ are given by the convergent power series

$$\overline{\mathsf{L}}^i(\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \mathsf{L}^{k+6-i} \tag{7.5}$$

Eps

and define operators of order 2 in D_α. The convergence of these series rely on the uniform estimates for all $n \geq 3$ (see [14, Thm. 3.3])

$$\begin{aligned}\|\mathsf{L}^n\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} \lesssim n\,E\Big(&\kappa_1^{n-2}\|\mathrm{D}_{[2]}\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} + \kappa_2^{n-1}\|\mathrm{D}_{[1]}\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} + \kappa_2^n\|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}\\ &+ \kappa_1^{n-1}\|\partial_{X_3}\mathrm{D}_{[1]}v_\alpha\|_{\mathrm{L}^2(\Omega)} + \kappa_1^n\|\partial_{X_3}\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}\Big).\end{aligned} \tag{7.6}$$

EestL

Similarly, using (6.12b), we find

$$-\mathsf{T}(\varepsilon)\mathsf{W}^{\mathsf{asy}}(\varepsilon) = \varepsilon^4 \sum_{0\leq i\leq 4} \overline{\mathsf{T}}^i(\varepsilon)\mathsf{V}^i$$

where $\overline{\mathsf{T}}^i(\varepsilon)$ are of order 1 in D_σ and 0 in ε and depend of the operators T^k of (6.7). These series are given as convergent operators series, owing to estimates similar to (7.6).

Since $\boldsymbol{z}$ is solution of $(P_{2\mathsf{D}})$, we have

$$\mathsf{M}\boldsymbol{z} + \varepsilon^2\mathsf{A}^2\boldsymbol{z} = \boldsymbol{g} + \varepsilon^2(\mathsf{A}^2 - \mathsf{B})\boldsymbol{z}.$$

Putting all together, we find

$$a_{3\mathsf{D}}(\varepsilon)(\underline{\boldsymbol{u}} - \mathsf{U}^{\mathsf{asy}}(\varepsilon)\boldsymbol{z}, \boldsymbol{v}) = \varepsilon\,\langle \boldsymbol{f} - \boldsymbol{g}, \boldsymbol{v}\rangle_{\mathrm{L}^2(\Omega)} \tag{7.7a}$$
$$+\ \varepsilon^3\big\langle\,(\mathsf{A}^2 - \mathsf{B})\boldsymbol{z}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)} \tag{7.7b}$$
$$+\ \varepsilon^4\big\langle\,\mathsf{L}^1\mathsf{V}^4\boldsymbol{z} + \mathsf{L}^2\mathsf{V}^3\boldsymbol{z}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)} \tag{7.7c}$$
$$+\ \varepsilon^4\big\langle\,\mathsf{L}^3\mathsf{V}^2\boldsymbol{z} + \mathsf{L}^4\mathsf{V}^1\boldsymbol{z} + \mathsf{L}^5\mathsf{V}^0\boldsymbol{z}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)} \tag{7.7d}$$
$$+\ \varepsilon^5 \sum_{1\le i\le 4} \Big[\big\langle\overline{\mathsf{L}}^i(\varepsilon)\mathsf{V}^i\boldsymbol{z}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)} + \big\langle\overline{\mathsf{T}}^i(\varepsilon)\mathsf{V}^i\boldsymbol{z}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Gamma_\pm)}\Big]. \tag{7.7e}$$

The proof of Lemma 7.2 consists in estimating each term in the above right hand side: Term (7.7a) in Sublemma 7.3.1, term (7.7b) in Sublemma 7.3.2, terms (7.7c) in Sublemma 7.3.3, and the remaining terms (7.7d) and (7.7e) in the end of this proof of Lemma 7.2.

Sublemma 7.3.1 *For $\boldsymbol{v} \in V(\Omega)$, we have the estimate:*

$$\varepsilon\langle\underline{\boldsymbol{f}} - \boldsymbol{g}, \boldsymbol{v}\rangle_{\mathrm{L}^2(\Omega)} \lesssim \mathsf{D}E^{-1/2}\varepsilon\|\underline{\boldsymbol{f}}^{\mathsf{rem}}\|_{\mathrm{L}^2(\Omega)}\,\mathsf{E}_{3\mathsf{D}}(\varepsilon)[\boldsymbol{v}]^{1/2}. \tag{7.8}$$

Efg

Proof of Sublemma 7.3.1. Let G be the mean value operator

$$\mathsf{G}\boldsymbol{v} = \frac{1}{2}\int_{-1}^{1}\boldsymbol{v}(X_3)\mathrm{d}X_3$$

With (2.10), we obtain $\boldsymbol{g} = \mathsf{G}\underline{\boldsymbol{f}}$ and we compute

$$\big|\varepsilon\langle\underline{\boldsymbol{f}} - \boldsymbol{g}, \boldsymbol{v}\rangle_{\mathrm{L}^2(\Omega)}\big| = \big|\varepsilon\langle\underline{\boldsymbol{f}} - \mathsf{G}\underline{\boldsymbol{f}}, \boldsymbol{v} - \mathsf{G}\boldsymbol{v}\rangle_{\mathrm{L}^2(\Omega)}\big| \le \varepsilon\|\underline{\boldsymbol{f}} - \boldsymbol{g}\|_{\mathrm{L}^2(\Omega)}\,\|\boldsymbol{v} - \mathsf{G}\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}\,.$$

Using the Bramble-Hilbert Lemma on $(-1,1)$, together with the fact that X_3 is an adimensional variable, we get, taking advantage of Remark 7.3,

$$\|\boldsymbol{v} - \mathsf{G}\boldsymbol{v}\|^2_{\mathrm{L}^2(\Omega)} \lesssim \|\partial_{X_3}\boldsymbol{v}\|^2_{\mathrm{L}^2(\Omega)}\,.$$

Combining this with Korn inequality (6.5) we finally find

$$\|\boldsymbol{v} - \mathsf{G}\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} \lesssim \mathsf{D}E^{-1/2}\varepsilon^{-1/2}\mathsf{E}_{3\mathsf{D}}(\varepsilon)[\boldsymbol{v}]^{1/2}.$$

We conclude using $\underline{\boldsymbol{f}}^{\mathsf{rem}} = \underline{\boldsymbol{f}} - \boldsymbol{g}$. ∎

Sublemma 7.3.2 *For $\boldsymbol{v} \in V(\Omega)$, we have the estimate:*

$$\Big|\varepsilon^3\big\langle(\mathsf{A}^2 - \mathsf{B})\boldsymbol{z}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\Big| \lesssim B^1_S(\varepsilon;\boldsymbol{z})^{1/2}\mathsf{E}^{\varepsilon}_{2\mathsf{D}}[\boldsymbol{z}]^{1/2}\mathsf{E}_{3\mathsf{D}}(\varepsilon)[\boldsymbol{v}]^{1/2}, \tag{7.9}$$

where $B^1_S(\varepsilon;\boldsymbol{z})$ is given by (7.2).

Proof of Sublemma 7.3.2. Using (6.13), we have for a 3D displacement $\boldsymbol{v}$ satisfying the homogeneous lateral boundary condition

$$\Big|\big\langle(\mathsf{A}^2-\mathsf{B})\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\Big| \lesssim E\Big(|\boldsymbol{\gamma}(\boldsymbol{z})|_{2\,;\,S}+\kappa_2^2|\boldsymbol{z}|^{(\boldsymbol{b})}_{1\,;\,S}+\kappa_1|z_3|_{2\,;\,S}\Big)\|\boldsymbol{\gamma}(\boldsymbol{v})\|_{\mathrm{L}^2(\Omega)} + E\Big(\kappa_1|\boldsymbol{\gamma}(\boldsymbol{z})|_{1\,;\,S}+\kappa_2^2|\boldsymbol{\gamma}(\boldsymbol{z})|_{0\,;\,S}\Big)\big(\|\mathrm{D}_\alpha\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}+\kappa_1\|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}\big). \quad (7.10)$$ E78

But for any $\boldsymbol{v}$ we have in non-shifted components (see (4.2c) and [14, Prop. 3.2])

$$\gamma_{\alpha\beta}(\boldsymbol{v}) = e_{\alpha\beta}(\varepsilon)(\boldsymbol{v}) - \varepsilon X_3\big(c_{\alpha\beta}v_3+v_\delta \mathrm{D}_\alpha b^\delta_\beta\big).$$

Thus we have

$$\|\boldsymbol{\gamma}(\boldsymbol{v})\|_{\mathrm{L}^2(\Omega)} \lesssim \|e_{\alpha\beta}(\varepsilon)(\boldsymbol{v})\|_{\mathrm{L}^2(\Omega)} + \varepsilon\kappa_2^2\|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}\,. \quad (7.11)$$ E79

Combining (7.11) with Korn inequalities (6.5) in (7.10) we find

$$\Big|\big\langle(\mathsf{A}^2-\mathsf{B})\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\Big| \lesssim \varepsilon^{-3/2}E^{1/2}\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}]^{1/2}\Big\{ \varepsilon\Big(|\boldsymbol{\gamma}(\boldsymbol{z})|_{2\,;\,S}+\kappa_2^2|\boldsymbol{z}|^{(\boldsymbol{b})}_{1\,;\,S}+\kappa_1|z_3|_{2\,;\,S}\Big)(1+\mathsf{D}^2\kappa_2^2) + \Big(\kappa_1|\boldsymbol{\gamma}(\boldsymbol{z})|_{1\,;\,S}+\kappa_2^2|\boldsymbol{\gamma}(\boldsymbol{z})|_{0\,;\,S}\Big)(\mathsf{D}+\kappa_1\mathsf{D}^2)\Big\}.$$

Using (3.5) and the definition 2.4 of the wavelength L_n we find using (6.4)

$$\Big|\big\langle(\mathsf{A}^2-\mathsf{B})\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\Big| \lesssim \varepsilon^{-3/2}E^{1/2}\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}]^{1/2}\Big\{ \varepsilon\Big((L_2^{-2}+\kappa_2^2)|\boldsymbol{\gamma}(\boldsymbol{z})|_{0\,;\,S}+\kappa_2|\boldsymbol{\rho}(\boldsymbol{z})|_{0\,;\,S}\Big)(1+\mathsf{D}^2\kappa_2^2) + (\kappa_1L_1^{-1}+\kappa_2^2)|\boldsymbol{\gamma}(\boldsymbol{z})|_{0\,;\,S}(\mathsf{D}+\kappa_1\mathsf{D}^2)\Big\}$$

Using (3.8) we find

$$\Big|\varepsilon^3\big\langle(\mathsf{A}^2-\mathsf{B})\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\Big| \lesssim a_S(\varepsilon,\boldsymbol{z})\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]^{1/2}\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}]^{1/2}$$

where

$$a_S(\varepsilon,\boldsymbol{z}) = \varepsilon\big(\varepsilon L_2^{-2}+\varepsilon\kappa_2^2+\kappa_2\big)(1+\mathsf{D}^2\kappa_2^2)+\varepsilon(\kappa_1L_1^{-1}+\kappa_2^2)(\mathsf{D}+\kappa_1\mathsf{D}^2).$$

Using the definition of $\mathsf{r}=1/\kappa_5$ and $L=L_4$ we can take

$$a_S(\varepsilon,\boldsymbol{z}) = \Big(1+\frac{\mathsf{D}^2}{\mathsf{r}^2}\Big)\Big(\frac{\varepsilon}{L}+\frac{\varepsilon^2}{L^2}+\frac{\varepsilon}{\mathsf{r}}+\frac{\varepsilon^2}{\mathsf{r}^2}\Big)$$

and we get the result. ∎

Sublemma 7.3.3 *For $\boldsymbol{v}\in V(\Omega)$, we have the estimates*

$$\Big|\varepsilon^4\big\langle\mathsf{L}^1\mathsf{V}^4\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\Big| + \Big|\varepsilon^4\big\langle\mathsf{L}^2\mathsf{V}^3\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\Big| \lesssim B^1_S(\varepsilon;\boldsymbol{z})^{1/2}\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]^{1/2}\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}]^{1/2},$$

where $B^1_S(\varepsilon;\boldsymbol{z})$ is given by (7.2).

Proof of Sublemma 7.3.3. The operators V^3 and V^4 are polynomials in X_3 with 2D operator coefficients. These operators are $\boldsymbol{b}$-homogeneous operators of degree 3 and 4 respectively. If A is a 2D operator acting form $\Sigma(S)$ into itself, it can be viewed as a block 2×2 matrix

$$A = \begin{pmatrix} A_{\sigma\alpha} & A_{\sigma 3} \\ A_{3\alpha} & A_{33} \end{pmatrix}$$

and the inequality

$$\deg A \leq \begin{pmatrix} a_{**} & a_{*3} \\ a_{3*} & a_{33} \end{pmatrix}$$

means that $\deg A_{\sigma\alpha} \leq a_{**}$, etc..., where $\deg$ means the order as partial differential operator. With these notations, we have (see [14, Prop. 4.2]) that

$$\deg \mathsf{V}^3 \leq \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \quad \text{and} \quad \deg \mathsf{V}^4 \leq \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \tag{7.12}$$

7Edeg

(i) Using the expression of the operator L^1 (see [14, Thm. 3.3]),

$$\begin{aligned} \mathsf{L}^1_\sigma(\boldsymbol{w}) &= -\mu b^\alpha_\alpha \partial_{X_3} w_\sigma + (\lambda+\mu)\mathrm{D}_\sigma \partial_{X_3} w_3 - X_3 \mu b^\alpha_\sigma \partial^2_{X_3} w_\alpha, \\ \mathsf{L}^1_3(\boldsymbol{w}) &= -\mu b^\alpha_\alpha \partial_{X_3} w_3 + (\lambda+\mu)\gamma^\alpha_\alpha(\partial_{X_3}\boldsymbol{w}), \end{aligned}$$

and the identity $\gamma^\alpha_\alpha(\boldsymbol{u}) = \mathrm{D}^\alpha u_\alpha - b^\alpha_\alpha u_3$, we find

$$\begin{aligned} \big\langle \mathsf{L}^1\mathsf{V}^4\boldsymbol{z}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)} = &\int_\Omega \big(-\mu b^\alpha_\alpha \partial_{X_3}\mathsf{V}^4_\sigma \boldsymbol{z} + (\lambda+\mu)\mathrm{D}_\sigma\partial_{X_3}\mathsf{V}^4_3\boldsymbol{z} - X_3\mu b^\alpha_\sigma \partial^2_{X_3}\mathsf{V}^4_\alpha\boldsymbol{z}\big)\, v^\sigma \mathrm{d}V \\ &+ \int_\Omega \big(-(\lambda+2\mu) b^\alpha_\alpha \partial_{X_3}\mathsf{V}^4_3\boldsymbol{z} + (\lambda+\mu)\mathrm{D}^\alpha\partial_{X_3}\mathsf{V}^4_\alpha\boldsymbol{z}\big)\, v_3\,\mathrm{d}V. \end{aligned}$$

Using the fact that $\boldsymbol{v}\big|_{\Gamma_0} = 0$ we can integrate by parts with respect to the surfacic derivative D_σ, and we obtain (we omit $\mathrm{d}V$):

$$\begin{aligned} \big\langle \mathsf{L}^1\mathsf{V}^4\boldsymbol{z}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)} = &-\int_\Omega (\mu b^\alpha_\alpha \partial_{X_3}\mathsf{V}^4_\sigma\boldsymbol{z} + X_3\mu b^\alpha_\sigma \partial^2_{X_3}\mathsf{V}^4_\alpha\boldsymbol{z})\, v^\sigma - \int_\Omega (\lambda+\mu)(\partial_{X_3}\mathsf{V}^4_3\boldsymbol{z})\,\mathrm{D}_\sigma v^\sigma \\ &- \int_\Omega (\lambda+2\mu)(\partial_{X_3}\mathsf{V}^4_3\boldsymbol{z})\, b^\alpha_\alpha v_3 - \int_\Omega (\lambda+\mu)(\partial_{X_3}\mathsf{V}^4_\alpha\boldsymbol{z})\,\mathrm{D}^\alpha v_3 \end{aligned}$$

and hence

$$\begin{aligned} \big\langle \mathsf{L}^1\mathsf{V}^4\boldsymbol{z}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)} = &-\int_\Omega (\mu b^\alpha_\alpha \partial_{X_3}\mathsf{V}^4_\sigma\boldsymbol{z} + X_3\mu b^\alpha_\sigma \partial^2_{X_3}\mathsf{V}^4_\alpha\boldsymbol{z})\, v^\sigma - \int_\Omega \mu(\partial_{X_3}\mathsf{V}^4_3\boldsymbol{z})\, b^\alpha_\alpha v_3 \\ &- \int_\Omega (\lambda+\mu)(\partial_{X_3}\mathsf{V}^4_3\boldsymbol{z})\,\gamma^\sigma_\sigma(\boldsymbol{v}) - \int_\Omega (\lambda+\mu)(\partial_{X_3}\mathsf{V}^4_\alpha\boldsymbol{z})\,\mathrm{D}^\alpha v_3. \end{aligned}$$

The operator V_3^4 is $\boldsymbol{b}$-homogeneous of degree 4, and of orders of derivative 3 in z_σ and 4 in z_3. By integration by parts using the boundary condition on $\boldsymbol{v}$, we obtain

$$\Big|\int_\Omega \mu(\partial_{X_3}\mathsf{V}_3^4\boldsymbol{z})b_\alpha^\alpha v_3\Big| \lesssim E\big(\sum_{j=0}^{3}\kappa_4^{4-j}|\boldsymbol{z}|_{j\,;\,S}\big)\,\kappa_1\|v_3\|_{\mathrm{L}^2(\Omega)} + E\kappa_1|z_3|_{3\,;\,S}\,\|\mathrm{D}_\alpha v_3\|_{\mathrm{L}^2(\Omega)}$$
$$\lesssim E\big(\sum_{j=0}^{3}\kappa_4^{4-j}|\boldsymbol{z}|_{j\,;\,S}\big)\big(\|\mathrm{D}_\alpha v_3\|_{\mathrm{L}^2(\Omega)} + \kappa_4\|v_3\|_{\mathrm{L}^2(\Omega)}\big).$$

As the operator V_α^4 is $\boldsymbol{b}$-homogeneous of degree 4, and of orders of derivative 3 in z_3 and 4 in z_α, we obtain

$$\Big|\big\langle \mathsf{L}^1\mathsf{V}^4\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\Big| \lesssim E\big(|z_\alpha|_{4\,;\,S} + \sum_{j=0}^{3}\kappa_4^{4-j}|\boldsymbol{z}|_{j\,;\,S}\big)\big(\|\mathrm{D}_\alpha\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} + \kappa_4\|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}\big)$$
$$+ E\big(|z_3|_{4\,;\,S} + \sum_{j=0}^{3}\kappa_4^{4-j}|\boldsymbol{z}|_{j\,;\,S}\big)\|\gamma_{\alpha\beta}(\boldsymbol{v})\|_{\mathrm{L}^2(\Omega)}. \quad (7.13) \qquad \boxed{\texttt{E241}}$$

Using (7.11), the Korn inequalities (6.5) and inequality (3.8) we find

$$\Big|\big\langle \mathsf{L}^1\mathsf{V}^4\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\Big| \lesssim \varepsilon^{-3/2}E^{1/2}\big(|z_\alpha|_{4\,;\,S} + \sum_{j=0}^{3}\kappa_4^{4-j}|\boldsymbol{z}|_{j\,;\,S}\big)\big(\mathsf{D}+\mathsf{D}^2\kappa_4\big)\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}]^{1/2}$$
$$+ \varepsilon^{-1/2}E^{1/2}\big(|z_3|_{4\,;\,S} + \sum_{j=0}^{3}\kappa_4^{4-j}|\boldsymbol{z}|_{j\,;\,S}\big)\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}]^{1/2}. \quad (7.14) \qquad \boxed{\texttt{E242}}$$

Recall that $\kappa = \kappa_5$ and $L = L_4$. Here, because of asymmetry between surfacic and transverse components, we do not use estimates (3.9): We obtain sharper estimates using directly (3.6a) and (3.6b),

$$|z_\alpha|_{4\,;\,S} + \sum_{j=0}^{3}\kappa_4^{4-j}|\boldsymbol{z}|_{j\,;\,S} \lesssim \sum_{j=0}^{3}\kappa^{3-j}|\boldsymbol{\gamma}(\boldsymbol{z})|_{j\,;\,S} + \sum_{j=0}^{1}\kappa^{2-j}|\boldsymbol{\rho}(\boldsymbol{z})|_{j\,;\,S}$$
$$\lesssim \varepsilon^{-3/2}E^{-1/2}\Big(\varepsilon\sum_{j=0}^{3}\kappa^{3-j}L^{-j} + \sum_{j=0}^{1}\kappa^{2-j}L^{-j}\Big)\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}]^{1/2}$$

Similarly, we have

$$|z_3|_{4\,;\,S} + \sum_{j=0}^{3}\kappa_4^{4-j}|\boldsymbol{z}|_{j\,;\,S} \lesssim \sum_{j=0}^{2}\kappa^{2-j}|\boldsymbol{\rho}(\boldsymbol{z})|_{j\,;\,S} + \sum_{j=0}^{2}\kappa^{3-j}|\boldsymbol{\gamma}(\boldsymbol{z})|_{j\,;\,S}$$
$$\lesssim \varepsilon^{-3/2}E^{-1/2}\Big(\sum_{j=0}^{2}\kappa^{2-j}L^{-j} + \varepsilon\sum_{j=0}^{2}\kappa^{3-j}L^{-j}\Big)\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}]^{1/2}.$$

Combining these estimates with (7.14) we find

$$\left|\left\langle \varepsilon^4 \mathsf{L}^1 \mathsf{V}^4 \boldsymbol{z}, \boldsymbol{v}\right\rangle_{\mathrm{L}^2(\Omega)}\right| \lesssim b_S(\boldsymbol{z}, \varepsilon) \mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}]^{1/2} \mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}]^{1/2}$$

with

$$\begin{aligned} b_S(\boldsymbol{z}, \varepsilon) = \varepsilon\big(\mathsf{D} + \mathsf{D}^2\kappa\big)\Big(\varepsilon L^{-3} + \varepsilon \sum_{j=0}^{2} \kappa^{3-j} L^{-j} + \sum_{j=0}^{1} \kappa^{2-j} L^{-j}\Big) \\ + \varepsilon^2\Big(\sum_{j=0}^{2} \kappa^{2-j} L^{-j} + \varepsilon \sum_{j=0}^{2} \kappa^{3-j} L^{-j}\Big) \end{aligned}$$

and we check that

$$b_S(\boldsymbol{z}, \varepsilon) \lesssim \frac{\varepsilon^2}{L^3}\Big(\mathsf{D} + \frac{\mathsf{D}^2}{\mathsf{r}}\Big) + \Big(1 + \frac{\mathsf{D}^2}{\mathsf{r}^2}\Big)\Big(\sum_{\substack{i+j=k \\ k\in\{1,2\}}} \frac{\varepsilon^k}{\mathsf{r}^i L^j}\Big)$$

and hence $b_S(\boldsymbol{z}, \varepsilon) \lesssim B^1_S(\boldsymbol{z}; \varepsilon)^{1/2}$. Note that we only need the introduction of κ_4 and L_3 to obtain this estimate.

(ii) Similarly, using the degrees (7.12) of V^3 and the expression of L^2 cf. [14, Prop. 3.3]

$$\begin{aligned} \mathsf{L}^2_\sigma(\boldsymbol{w}) = -\mu X_3 c^\alpha_\alpha \partial_{X_3} w_\sigma + \mu X_3 b^\alpha_\alpha b^\beta_\sigma \partial_{X_3} w_\beta - \mu b^\alpha_\alpha \mathrm{D}_\sigma w_3 - \mu b^\beta_\beta b^\alpha_\sigma w_\alpha + \lambda \mathrm{D}_\sigma \gamma^\alpha_\alpha(\boldsymbol{w}) \\ + 2\mu \mathrm{D}_\alpha \gamma^\alpha_\sigma(\boldsymbol{w}), \end{aligned}$$

$$\mathsf{L}^2_3(\boldsymbol{w}) = -\mu X_3 c^\alpha_\alpha \partial_{X_3} w_3 + (\lambda+\mu) b^\beta_\alpha \gamma^\alpha_\beta\big(\partial_{X_3}(X_3 \boldsymbol{w})\big) + \mu b^\beta_\alpha \gamma^\alpha_\beta(\boldsymbol{w}) + \mu \mathrm{D}^\alpha \theta_\alpha(\boldsymbol{w}),$$

we have after integration by parts that

$$\begin{aligned} \big\langle \mathsf{L}^2 \mathsf{V}^3 \boldsymbol{z}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)} = &- \int_\Omega (\mu X_3 c^\alpha_\alpha \partial_{X_3} \mathsf{V}^3_\sigma \boldsymbol{z} - \mu X_3 b^\alpha_\alpha b^\beta_\sigma \partial_{X_3} \mathsf{V}^3_\beta \boldsymbol{z} + \mu b^\beta_\beta b^\alpha_\sigma \mathsf{V}^3_\alpha \boldsymbol{z}) v^\sigma \,\mathrm{d}V \\ &+ \int_\Omega (\mu b^\alpha_\alpha \mathsf{V}^3_3 \boldsymbol{z} - \lambda \gamma^\alpha_\alpha(\mathsf{V}^3\boldsymbol{z})) \mathrm{D}_\sigma v^\sigma \,\mathrm{d}V - \int_\Omega 2\mu \gamma^\alpha_\sigma(\mathsf{V}^3\boldsymbol{z}) \mathrm{D}_\alpha v^\sigma \,\mathrm{d}V \\ &- \int_\Omega (\lambda+2\mu)(c^\alpha_\alpha \partial_3(X_3 \mathsf{V}^3_3 \boldsymbol{z}) v_3 \,\mathrm{d}V - \int_\Omega \big((\lambda+\mu)\partial_{X_3}(X_3 \mathsf{V}^3_\alpha \boldsymbol{z})\big)(\mathrm{D}^\beta b^\alpha_\beta v_3) \,\mathrm{d}V \\ &- \int_\Omega (\mu \mathsf{V}^3_\alpha \boldsymbol{z})(\mathrm{D}^\beta b^\alpha_\beta v_3) \,\mathrm{d}V - \int_\Omega \mu \theta_\alpha(\mathsf{V}^3\boldsymbol{z})(\mathrm{D}^\alpha v_3) \,\mathrm{d}V. \end{aligned}$$

Using the relation

$$\tau^{\alpha\beta} \mathrm{D}_\alpha w_\beta = \tau^\beta_\alpha \gamma^\alpha_\beta(\boldsymbol{w}) + \tau^\beta_\alpha b^\alpha_\beta w_3$$

valid for any symmetric tensor $\tau_{\alpha\beta}$, and using integration by parts, we find the same estimate as in (7.13) which yields the result. ■

End of proof of Lemma 7.2. We now prove that the remaining terms (7.7d) and (7.7e) in equation (7.7) can be estimated by terms of the form

$$B^1_S(\varepsilon; \boldsymbol{z})^{1/2} \mathsf{E}^{\varepsilon}_{\mathsf{3D}}[\boldsymbol{z}]^{1/2} \mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}]$$

where the expression of the bound B^1_S is given by (7.2).

Using [14, Thm. 3.3], we can prove like for the estimates (7.6) the following uniform bound for all $n \geq 3$, $\boldsymbol{w} \in \mathrm{H}^1(\Omega)^3$ and $\boldsymbol{v} \in V(\Omega)$:

$$\left|\langle \mathsf{L}^n \boldsymbol{w}, \boldsymbol{v}\rangle_{\mathrm{L}^2(\Omega)}\right| \lesssim nE\,\kappa_2^{n-2}\big(\|\mathrm{D}_\alpha \boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} + \kappa_2\|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}\big)\times$$
$$\big(\|\mathrm{D}_\alpha \boldsymbol{w}\|_{\mathrm{L}^2(\Omega)} + \kappa_2\|\boldsymbol{w}\|_{\mathrm{L}^2(\Omega)} + \|\partial_{X_3}\mathrm{D}_\alpha \boldsymbol{w}\|_{\mathrm{L}^2(\Omega)} + \kappa_2\|\partial_{X_3}\boldsymbol{w}\|_{\mathrm{L}^2(\Omega)}\big).$$

Recall that for all $n \geq 0$ the operators V^n are $\boldsymbol{b}$-homogeneous of degree n. Hence the uniform estimates for all $n \geq 3$ and all $i \in \{0,1,2,3,4\}$

$$\begin{aligned}
&\left|\langle \mathsf{L}^n \mathsf{V}^i \boldsymbol{z}, \boldsymbol{v}\rangle_{\mathrm{L}^2(\Omega)}\right| \\
&\quad\lesssim nE\kappa_2^{n-2}\big(\|\mathrm{D}_\alpha \boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} + \kappa_2\|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}\big)\big(|\boldsymbol{z}|^{(\boldsymbol{b})}_{i+1\,;\,S} + \kappa_2|\boldsymbol{z}|^{(\boldsymbol{b})}_{i\,;\,S}\big) \qquad (7.15)\ \boxed{\texttt{ELbn}} \\
&\quad\lesssim nE^{1/2}\varepsilon^{-3/2}\kappa_2^{n-2}(\mathsf{D} + \mathsf{D}^2\kappa_2)\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}]^{1/2}\cdot\big(|\boldsymbol{z}|^{(\boldsymbol{b})}_{i+1\,;\,S} + \kappa_2|\boldsymbol{z}|^{(\boldsymbol{b})}_{i\,;\,S}\big),
\end{aligned}$$

using Korn inequalities (6.5).

This estimate yields immediately that

$$\left|\varepsilon^4\big\langle \mathsf{L}^3\mathsf{V}^2\boldsymbol{z} + \mathsf{L}^4\mathsf{V}^1\boldsymbol{z} + \mathsf{L}^5\mathsf{V}^0\boldsymbol{z}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\right| \lesssim \varepsilon^{5/2}E^{1/2}\kappa_2(\mathsf{D} + \mathsf{D}^2\kappa_2)\Big(\sum_{i+j=3}\kappa_2^i|\boldsymbol{z}|^{(\boldsymbol{b})}_{j\,;\,S}\Big)\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}]^{1/2},$$

and using the a priori estimate (3.9) we get

$$\left|\varepsilon^4\big\langle \mathsf{L}^3\mathsf{V}^2\boldsymbol{z} + \mathsf{L}^4\mathsf{V}^1\boldsymbol{z} + \mathsf{L}^5\mathsf{V}^0\boldsymbol{z}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\right| \lesssim \varepsilon(\mathsf{D}\kappa + \mathsf{D}^2\kappa^2)\Big(\sum_{i,j,k\in G_3}\varepsilon^k\kappa^i L^{-j}\Big)\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}]^{1/2}\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}]^{1/2},$$

where G_3 is given by the formula (3.10).

But the constant in the right-hand side can be written

$$\Big(\frac{\mathsf{D}}{\mathsf{r}} + \frac{\mathsf{D}^2}{\mathsf{r}^2}\Big)\Big(\sum_{\substack{i+j=k+1\\ k\in\{0,1\}}}\frac{\varepsilon^{k+1}}{\mathsf{r}^i L^j}\Big) \lesssim B^1_S(\varepsilon;\boldsymbol{z})^{1/2}$$

after a change of index $k \mapsto k-1$ in the sum. This yields the result. Note we only need κ_3 and L_2 to obtain this bound .

The operators $\overline{\mathsf{L}}^i$ in the term (7.7e) are the power series (7.5) of the operators L^k. Using (7.15) and the fact that $\varepsilon \le \mathsf{r}$, we derive that

$$\Big|\varepsilon^5\Big\langle \sum_{i=0}^{4} \overline{\mathsf{L}}^i(\varepsilon)\mathsf{V}^i\boldsymbol{z}, \boldsymbol{v}\Big\rangle_{\mathrm{L}^2(\Omega)}\Big| \lesssim \varepsilon^{7/2}E^{1/2}\mathsf{D}(1+\mathsf{D}\kappa_2)\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}]^{1/2}\cdot\sum_{i=0}^{4}(|\boldsymbol{z}|^{(\boldsymbol{b})}_{i+1\,;\,S} + \kappa_2|\boldsymbol{z}|^{(\boldsymbol{b})}_{i\,;\,S})$$

Owing to (3.9) for $n = 5$, this shows that

$$\Big|\varepsilon^5\Big\langle \sum_{i=0}^{4} \overline{\mathsf{L}}^i(\varepsilon)\mathsf{V}^i\boldsymbol{z}, \boldsymbol{v}\Big\rangle_{\mathrm{L}^2(\Omega)}\Big| \lesssim \varepsilon^2\mathsf{D}(1+\mathsf{D}\kappa_3)\Big(\sum_{i,j,k\in G_5}\varepsilon^k\kappa^iL^{-j}\Big)\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}]^{1/2}\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}]^{1/2}.$$

But the constant in the right hand side is smaller than

$$\Big(1+\frac{\mathsf{D}}{\mathsf{r}}\Big)\sum_{\substack{i+j=k+1\\ k\in\{2,3\}}}\frac{\varepsilon^k\mathsf{D}}{\mathsf{r}^iL^j} \lesssim \Big(1+\frac{\mathsf{D}}{\mathsf{r}}\Big)\Big(\frac{\varepsilon^2\mathsf{D}}{L^3}+\frac{\varepsilon^3\mathsf{D}}{L^4}\Big) + \Big(\frac{\mathsf{D}}{\mathsf{r}}+\frac{\mathsf{D}^2}{\mathsf{r}^2}\Big)\sum_{\substack{i+j=k\\ k\in\{2,3\}}}\frac{\varepsilon^k}{\mathsf{r}^iL^j}$$

after separating in the sum the terms where $i = 0$ and those for $i \ge 1$. This term is dominated by $B^1_S(\varepsilon;\boldsymbol{z})^{1/2}$. Note that we only need κ_5 and L_4 to obtain this bound. The estimate for the traction terms involving the operators $\overline{\mathsf{T}}^i(\varepsilon)$ can be done similarly, and this proves the lemma. ∎

We can now prove the main result of this section.

Proof of Proposition 7.1. Using Lemma 7.2 we have

$$\begin{aligned} a_{\mathsf{3D}}(\varepsilon)(\underline{\boldsymbol{u}} - \mathsf{U}(\varepsilon)\boldsymbol{z}, \boldsymbol{v}) &= a_{\mathsf{3D}}(\varepsilon)(\underline{\boldsymbol{u}} - \mathsf{U}^{\mathsf{asy}}(\varepsilon\boldsymbol{z}, \boldsymbol{v}) + a_{\mathsf{3D}}(\varepsilon)(\mathsf{U}(\varepsilon)\boldsymbol{z} - \mathsf{U}^{\mathsf{asy}}(\varepsilon)\boldsymbol{z}, \boldsymbol{v}) \\ &\lesssim \big(B_1^{1/2} + \mathsf{E}_{\mathsf{3D}}(\varepsilon)[\mathsf{U}(\varepsilon)\boldsymbol{z} - \mathsf{U}^{\mathsf{asy}}(\varepsilon)\boldsymbol{z}]^{1/2}\big)\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\boldsymbol{v}]^{1/2}. \end{aligned}$$

Thus the proposition is proved provided we show $\mathsf{E}_{\mathsf{3D}}(\varepsilon)[\mathsf{U}(\varepsilon)\boldsymbol{z} - \mathsf{U}^{\mathsf{asy}}(\varepsilon)\boldsymbol{z}] \lesssim B_1$, or equivalently using the shifted energy (4.1) scaled on Ω,

$$\widetilde{\mathsf{E}}_{\mathsf{3D}}(\varepsilon)[\mathsf{W}(\varepsilon)\boldsymbol{z} - \mathsf{W}^{\mathsf{asy}}(\varepsilon)\boldsymbol{z}] \lesssim B_1.$$

By definition, we have

$$\mathsf{W}^{\mathsf{asy}}(\varepsilon)\boldsymbol{z} = \mathsf{W}(\varepsilon)\boldsymbol{z} + \varepsilon^2\mathsf{v}^2\boldsymbol{z} + \varepsilon^3\mathsf{V}^3\boldsymbol{z} + \varepsilon^4\mathsf{V}^4\boldsymbol{z} \tag{7.16}$$

En1

where

$$\mathsf{v}^2\boldsymbol{z} = \begin{cases} \frac{X_3^2}{2}p\mathrm{D}_\sigma\gamma^\alpha_\alpha(\boldsymbol{z}), \\ \frac{X_3^2}{2}\big(-p^2b^\alpha_\alpha\gamma^\beta_\beta(\boldsymbol{z}) - 2pb^\beta_\alpha\gamma^\alpha_\beta(\boldsymbol{z})\big). \end{cases} \tag{7.17}$$

Evv2

We now successively estimate the energy of the three terms $\mathsf{v}^2\boldsymbol{z}$, $\mathsf{V}^3\boldsymbol{z}$ and $\mathsf{V}^4\boldsymbol{z}$.

Using (4.3) scaled on the manifold Ω and owing to Remark 7.3, we have for all $\boldsymbol{v} \in \mathrm{H}^1(\Omega)$

$$\widetilde{\mathsf{E}}_{3\mathsf{D}}(\varepsilon)[\boldsymbol{v}] \lesssim \varepsilon E\Big(\|\tilde{e}^{\alpha}_{\beta}(\varepsilon)(\boldsymbol{v})\|^2_{\mathrm{L}^2(\Omega)} + \|\tilde{e}^{3}_{\beta}(\varepsilon)(\boldsymbol{v})\|^2_{\mathrm{L}^2(\Omega)} + \|\tilde{e}^{3}_{3}(\varepsilon)(\boldsymbol{v})\|^2_{\mathrm{L}^2(\Omega)} \Big)$$

where $\tilde{e}^i_j(\varepsilon)$ is the deformation tensor (4.2a)-(4.2c) scaled on Ω.

(i) Using (4.2a)-(4.2c) and (7.17), we obtain

$$\begin{aligned}
\|\tilde{e}^3_3(\varepsilon)(\mathsf{v}^2\boldsymbol{z})\|^2_{\mathrm{L}^2(\Omega)} &\lesssim \varepsilon^{-2}\kappa_1^2|\boldsymbol{\gamma}|^2_{0\,;\,S}\,,\\
\|\tilde{e}^3_\sigma(\varepsilon)(\mathsf{v}^2\boldsymbol{z})\|^2_{\mathrm{L}^2(\Omega)} &\lesssim (\varepsilon^{-2}+\kappa_1^2)|\boldsymbol{\gamma}|^2_{1\,;\,S} + \kappa_2^4|\boldsymbol{\gamma}|^2_{0\,;\,S}\\
\|\tilde{e}^\alpha_\beta(\varepsilon)(\mathsf{v}^2\boldsymbol{z})\|^2_{\mathrm{L}^2(\Omega)} &\lesssim \big(|\boldsymbol{\gamma}|^{(\boldsymbol{b})}_{2\,;\,S}\big)^2,
\end{aligned}$$

provided that ε is sufficiently small ($\varepsilon < \kappa_1^{-1} = R$) to ensure the convergence of the series in (4.2c). Hence we have

$$\widetilde{\mathsf{E}}_{3\mathsf{D}}(\varepsilon)[\mathsf{v}^2\boldsymbol{z}] \lesssim E\Big(\varepsilon|\boldsymbol{\gamma}|^2_{2\,;\,S} + (\varepsilon^{-1}+\varepsilon\kappa_1^2)|\boldsymbol{\gamma}|^2_{1\,;\,S} + (\varepsilon^{-1}\kappa_1^2+\varepsilon\kappa_2^4)|\boldsymbol{\gamma}|^2_{0\,;\,S}\Big)$$

Multiplying by ε^4 and using (3.8), we find

$$\widetilde{\mathsf{E}}_{3\mathsf{D}}(\varepsilon)[\varepsilon^2\mathsf{v}^2\boldsymbol{z}] \lesssim \Big(\varepsilon^4L^{-4} + (\varepsilon^2+\varepsilon^4\kappa^2)L^{-2} + \varepsilon^2\kappa^2 + \varepsilon^4\kappa^4\Big)\mathsf{E}^\varepsilon_{2\mathsf{D}}[\boldsymbol{z}] \tag{7.18}$$

Ev2

with $L = L_4$ and $\kappa = \kappa_5$, and thus we have

$$\widetilde{\mathsf{E}}_{3\mathsf{D}}(\varepsilon)[\varepsilon^2\mathsf{v}^2\boldsymbol{z}] \lesssim B^1_S(\varepsilon;\boldsymbol{z})\mathsf{E}^\varepsilon_{2\mathsf{D}}[\boldsymbol{z}]$$

where $B^1_S(\varepsilon;\boldsymbol{z})$ is given in (7.2).

(ii) We recall that the operator V^3 is $\boldsymbol{b}$-homogeneous of order 3 and that we have the bound (7.12) for the orders of the derivatives of V^3. We deduce that

$$\begin{aligned}
\|\tilde{e}^3_3(\varepsilon)(\mathsf{V}^3\boldsymbol{z})\|^2_{\mathrm{L}^2(\Omega)} &\lesssim \varepsilon^{-2}\Big(|z_\alpha|_{3\,;\,S} + \sum_{j=0}^{2}\kappa_3^{3-j}|\boldsymbol{z}|_{j\,;\,S}\Big)^2\\
\|\tilde{e}^3_\alpha(\varepsilon)(\mathsf{V}^3\boldsymbol{z})\|^2_{\mathrm{L}^2(\Omega)} &\lesssim \varepsilon^{-2}\Big(|z_3|_{3\,;\,S} + \sum_{j=0}^{2}\kappa_3^{3-j}|\boldsymbol{z}|_{j\,;\,S}\Big)^2 + \Big(|z_\alpha|_{4\,;\,S} + \sum_{j=0}^{3}\kappa_4^{4-j}|\boldsymbol{z}|_{j\,;\,S}\Big)^2\\
\|\tilde{e}^\alpha_\beta(\varepsilon)(\mathsf{V}^3\boldsymbol{z})\|^2_{\mathrm{L}^2(\Omega)} &\lesssim \Big(|z_3|_{4\,;\,S} + \sum_{j=0}^{3}\kappa_4^{4-j}|\boldsymbol{z}|_{j\,;\,S}\Big)^2.
\end{aligned}$$

Hence we have

$$\widetilde{\mathsf{E}}_{3\mathsf{D}}(\varepsilon)[\mathsf{V}^3\boldsymbol{z}] \lesssim E\varepsilon^{-1}\Big(\sum_{j=0}^{3}\kappa_3^{3-j}|\boldsymbol{z}|_{j\,;\,S}\Big)^2 + E\varepsilon\Big(\sum_{j=0}^{4}\kappa_4^{4-j}|\boldsymbol{z}|_{j\,;\,S}\Big)^2.$$

Multiplying by ε^6 and using (3.9) we find

$$\widetilde{\mathsf{E}}_{3\mathsf{D}}(\varepsilon)[\varepsilon^3\mathsf{V}^3\boldsymbol{z}] \lesssim \Big(\sum_{i,j,k\in G_3}\frac{\varepsilon^{2k+2}\kappa_3^{2i}}{L_2^{2j}}+\sum_{i,j,k\in G_4}\frac{\varepsilon^{2k+4}\kappa_4^{2i}}{L_3^{2j}}\Big)\mathsf{E}^{\varepsilon}_{2\mathsf{D}}[\boldsymbol{z}],$$

where for all n, $G_n=\{(i,j,k)\in\mathbb{N}^3\,|\,k\in\{0,1\},\ i+j=k+n-2\}$. We can write the previous equation as

$$\widetilde{\mathsf{E}}_{3\mathsf{D}}(\varepsilon)[\varepsilon^3\mathsf{V}^3\boldsymbol{z}] \lesssim \Big(\sum_{i,j,k\in G}\frac{\varepsilon^{2k}\kappa^{2i}}{L^{2j}}\Big)\mathsf{E}^{\varepsilon}_{2\mathsf{D}}[\boldsymbol{z}],$$

where $G=\{(i,j,k)\in\mathbb{N}^3\,|\,k\in\{1,2,3\},\ i+j=k\}$. This shows that

$$\widetilde{\mathsf{E}}^{\varepsilon}_{3\mathsf{D}}[\varepsilon^3\mathsf{V}^3\boldsymbol{z}] \lesssim B^1_S(\varepsilon;\boldsymbol{z})\mathsf{E}^{\varepsilon}_{2\mathsf{D}}[\boldsymbol{z}]$$

where $B^1_S(\varepsilon;\boldsymbol{z})$ is given in (7.2).

(iii) On the same way, we easily find :

$$\widetilde{\mathsf{E}}_{3\mathsf{D}}(\varepsilon)[\mathsf{V}^4\boldsymbol{z}] \lesssim E\varepsilon^{-1}\Big(\sum_{j=0}^{4}\kappa_4^{4-j}|\boldsymbol{z}|_{j\,;\,S}\Big)^2+E\varepsilon\Big(\sum_{j=0}^{5}\kappa_5^{5-j}|\boldsymbol{z}|_{j\,;\,S}\Big)^2,$$

whence the result after multiplying by ε^8 and using (3.9). Note that we used κ_5 and L_4 to obtain this result.

8 Estimate for the corrector term

The goal of this section is to construct a displacement $\boldsymbol{u}^{\mathsf{cor}}$ satisfying the equation (5.2) with $B_2\lesssim B_S(\varepsilon,\boldsymbol{z},\boldsymbol{f}^{\mathsf{rem}})$, and such that $\mathsf{U}\boldsymbol{z}+\boldsymbol{u}^{\mathsf{cor}}\in V(\Omega^\varepsilon)$. In shifted displacements, this amounts to construct $\boldsymbol{w}^{\mathsf{cor}}$ such that $\mathsf{W}\boldsymbol{z}+\boldsymbol{w}^{\mathsf{cor}}$ satisfies lateral Dirichlet conditions and satisfying the same estimates

We recall from Definition 2.8 that r is the geodesic distance to ∂S in S, s the arc-length along ∂S, and d defines the tubular neighborhood $(r,s)\in[0,\mathsf{d}]\times\partial S$.

We introduce the adimensional variable $T=r/\varepsilon$. Let $\chi(T)$ be an adimensional $\mathscr{C}^\infty$ cut-off function defined on $[0,\infty)$ satisfying $\chi(T)\equiv 1$ for $T\in[0,\frac12]$ and $\chi(T)\equiv 0$ for all $T\geq 1$.

Consider now the displacement $\mathsf{W}\boldsymbol{z}$ as defined in (2.18). As $\boldsymbol{z}$ satisfies the boundary conditions $\boldsymbol{z}\big|_{\partial S}=0$ and $\partial_r z_3\big|_{\partial S}=0$ we have that $\mathsf{W}_\alpha\boldsymbol{z}\big|_{\Gamma_0^\varepsilon}=0$ and

$$\mathsf{W}_3\boldsymbol{z}\big|_{\Gamma_0^\varepsilon}=-px_3\gamma^\alpha_\alpha\big|_{\partial S}+p\frac{x_3^2}{2}\rho^\alpha_\alpha\big|_{\partial S}$$

We thus set

$$w_\alpha^{\mathsf{cor}} = 0 \quad \text{and} \quad w_3^{\mathsf{cor}} = \chi(\varepsilon^{-1}r)\big(-px_3\gamma_\alpha^\alpha\big|_{\partial S} + p\frac{x_3^2}{2}\rho_\alpha^\alpha\big|_{\partial S}\big). \tag{8.1}$$ Ewcor

Note that this term is non zero only in the region where $r \leq \varepsilon$. By definition, we have that $\boldsymbol{w} + \boldsymbol{w}^{\mathsf{cor}} \in V(\Omega^\varepsilon)$. It remains to estimate the energy of $\boldsymbol{w}^{\mathsf{cor}}$.

Proposition 8.1 *Let $\boldsymbol{w}^{\mathsf{cor}}$ defined by the equation (8.1), then we have the estimate*

$$\widetilde{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}^{\mathsf{cor}}] \lesssim E\varepsilon^2\Big(|\boldsymbol{\gamma}|^2_{0\,;\partial S} + \varepsilon^2|\boldsymbol{\rho}|^2_{0\,;\partial S}\Big)\Big(1+\frac{\varepsilon^2}{R^2}\Big) + E\varepsilon^4\Big(|\boldsymbol{\gamma}|^2_{1\,;\partial S} + \varepsilon^2|\boldsymbol{\rho}|^2_{1\,;\partial S}\Big) \tag{8.2}$$

where $R = \kappa_1^{-1}$.

Using the definitions of ℓ and the fact that $\varepsilon < R$, this estimate proves that

$$\widetilde{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}^{\mathsf{cor}}] \lesssim \breve{B}_S(\varepsilon;\boldsymbol{z})\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]$$

where

$$\breve{B}_S(\varepsilon;\boldsymbol{z}) = \frac{\varepsilon}{\ell}\left(1+\frac{\varepsilon^2}{\ell^2}\right)$$

and this yields (2.24) provided that $\varepsilon/\ell \leq M$.

Proof of Proposition 8.1. Using the fact that only the transverse component of $\boldsymbol{w}^{\mathsf{cor}}$ is non zero, we have using (4.2a)-(4.2c) that

$$\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}^{\mathsf{cor}}] \lesssim E\|\partial_3 w_3^{\mathsf{cor}}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} + E\|\mathrm{D}_\sigma w_3^{\mathsf{cor}}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} + E\frac{1}{R^2}\|w_3^{\mathsf{cor}}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)}.$$

Let us recall that in the coordinate system (r,s) in a tubular neighborhood of ∂S, the metric satisfies $a_{rr}(r,s) = 1$, $a_{rs}(r,s) = 0$, and $a_{ss}(0,s) = 1$. This implies that the Riemannian volume (4.4) on the tubular neighbourhood $\{r \in (0,\mathsf{d}), s \in \partial S, x_3 \in (-\varepsilon,\varepsilon)\}$ can be written

$$\mathrm{d}V = \mathrm{d}r\,\mathrm{d}s\,\mathrm{d}x_3(1 + j(r,s,x_3))$$

where $j(r,s,x_3)$ is an adimensional convergent power series in r and x_3 provided $r < \mathsf{d}$ and $|x_3| < R$, satisfying $j(0,s,0) = 0$ and with function coefficients defined on ∂S. For $r \in (0,\varepsilon)$, $s \in \partial S$ and $x_3 \in (-\varepsilon,\varepsilon)$, we can always assume for instance that

$$|1 + j(r,s,x_3)| \leq 3/2\,.$$

Let us decompose $w_3^{\mathsf{cor}} = \Phi\boldsymbol{z} + \Psi\boldsymbol{z}$ where

$$\Phi\boldsymbol{z} = -px_3\gamma_\alpha^\alpha\big|_{\partial S}\,\chi(\varepsilon^{-1}r) \quad \text{and} \quad \Psi\boldsymbol{z} = p\frac{x_3^2}{2}\rho_\alpha^\alpha\big|_{\partial S}\,\chi(\varepsilon^{-1}r).$$

We see that

$$\|\Phi \boldsymbol{z}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} = p^4\varepsilon^4 \int_{-1}^{1}\int_{\partial S}\int_0^1 X_3^2 \chi(\varepsilon T)^2 \big(\gamma^\alpha_\alpha(\boldsymbol{z})\,\big|_{\partial S}\big)^2 |1 + j(\varepsilon T, s, \varepsilon X_3)| \,\mathrm{d}T\,\mathrm{d}s\,\mathrm{d}X_3.$$

This implies immediately

$$\|\Phi \boldsymbol{z}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} \lesssim \varepsilon^4 |\boldsymbol{\gamma}|^2_{0\,;\,\partial S}\,.$$

The same calculation with Ψ yields

$$\|w_3^{\mathsf{cor}}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} \lesssim \varepsilon^4 |\boldsymbol{\gamma}|^2_{0\,;\,\partial S} + \varepsilon^6 |\boldsymbol{\rho}|^2_{0\,;\,\partial S}\,.$$

Similarly, we easily see that

$$\|\partial_3 w_3^{\mathsf{cor}}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} \lesssim \varepsilon^2 |\boldsymbol{\gamma}|^2_{0\,;\,\partial S} + \varepsilon^4 |\boldsymbol{\rho}|^2_{0\,;\,\partial S}\,.$$

Note that with the change of coordinate $(r, s, x_3) \mapsto (T, s, X_3)$, the term $\|\mathrm{D}_\sigma w_3^{\mathsf{cor}}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)}$ has to be understood as (see formula (4.3))

$$\int_{\Omega^\varepsilon} a^{\alpha\beta}(x_3)(\mathrm{D}_\alpha w_3^{\mathsf{cor}})(\mathrm{D}_\beta w_3^{\mathsf{cor}})\,\mathrm{d}V.$$

As $\varepsilon \le \varepsilon_0 < \mathsf{d}$, we can assume that the metric on S in coordinates (r, s) is $\mathcal{O}(\varepsilon/\mathsf{d})$ close to the identity. So, for $r \in (0, \varepsilon)$, $s \in \partial S$ and $x_3 \in (-\varepsilon, \varepsilon)$ with $\varepsilon < \min(R, \mathsf{d})$, this yields

$$\|\mathrm{D}_\sigma w_3^{\mathsf{cor}}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} \lesssim \int_0^\varepsilon \int_{\partial S}\int_{-\varepsilon}^{\varepsilon} \Big((\partial_r w_3^{\mathsf{cor}})^2 + (\partial_s w_3^{\mathsf{cor}})^2\Big)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}x_3.$$

But we have

$$\partial_s \Phi \boldsymbol{z} = -p x_3 (\partial_s \gamma^\alpha_\alpha)\,\big|_{\partial S}\, \chi(\varepsilon^{-1} r)$$

and

$$\partial_r \Phi \boldsymbol{z} = -\varepsilon^{-1} p x_3 \gamma^\alpha_\alpha\,\big|_{\partial S}\, (\partial_T \chi)(\varepsilon^{-1} r)$$

where here $\partial_T \chi$ is an adimensional function with support in $T \in (0, 1)$. This shows that

$$\|\mathrm{D}_\sigma \Phi \boldsymbol{z}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} \lesssim \varepsilon^2 |\boldsymbol{\gamma}|^2_{0\,;\,\partial S} + \varepsilon^4 |\boldsymbol{\gamma}|^2_{1\,;\,\partial S}\,,$$

and similarly

$$\|\mathrm{D}_\sigma \Psi \boldsymbol{z}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} \lesssim \varepsilon^4 |\boldsymbol{\rho}|^2_{0\,;\,\partial S} + \varepsilon^6 |\boldsymbol{\rho}|^2_{1\,;\,\partial S}\,.$$

Collecting together the previous estimates yields the result. ■

9 Conclusion: Optimality of the main estimate

To conclude our paper, we apply estimate (2.24) to families $(\boldsymbol{u}^\varepsilon)$ and $(\boldsymbol{z}^\varepsilon)$ of solutions of problems (P_{3D}) and (P_{2D}) for each $\varepsilon \in (0, \varepsilon_0]$ for a smooth fixed load $\boldsymbol{f}$

independent of the transverse variable x_3. Thus $\boldsymbol{f}$ has the form

$$\boldsymbol{f}(x_\alpha, x_3) = \boldsymbol{g}(x_\alpha), \quad \forall x_\alpha \in S, \quad \forall x_3 \in [-\varepsilon_0, \varepsilon_0],$$

with a smooth surface load $\boldsymbol{g}$ independent of ε. Hence for all $\varepsilon \in (0, \varepsilon_0]$, $\boldsymbol{g}$ is the mean value (2.10) of $\boldsymbol{f}$ across the shell Ω^ε. We investigate separately the cases of plates and elliptic shells, and provide a generalization to shallow shells.

9.A Plates

A family of plates (Ω^ε) is defined by its mean surface S which is a domain of $\mathbb{R}^2$. Thus the normal coordinates (x_α, x_3) are globally defined by a Cartesian coordinate system. Hence, the metric is the flat metric and the curvature vanishes on S. Consequently, we have $\kappa_j = 0$ for all $j \geq 1$, thus $\mathsf{r} = \infty$. Moreover the membrane and change of curvature tensor reduce to

$$\gamma_{\alpha\beta}(\boldsymbol{z}) = \tfrac{1}{2}(\partial_\alpha z_\beta + \partial_\beta z_\alpha) \quad \text{and} \quad \rho_{\alpha\beta}(\boldsymbol{z}) = \partial_{\alpha\beta} z_3.$$

This shows that the Koiter operator decouples into the restrictions M_* and B_3 of the membrane and bending operators acting on the surfacic and transverse components $\boldsymbol{z}_*$ and z_3 respectively:

$$\mathsf{K}(\varepsilon) = \begin{pmatrix} \mathsf{M}_* & 0 \\ 0 & \varepsilon^2 \mathsf{B}_3 \end{pmatrix}.$$

Thus the solution of the problem (P_{2D}) is given by

$$\boldsymbol{z}^\varepsilon = (\boldsymbol{z}_{\mathsf{M}}, 0) + \varepsilon^{-2}(0, z_{\mathsf{B}})$$

where the membrane and bending parts $\boldsymbol{z}_{\mathsf{M}} \in \mathbf{H}^1_0(S)$ and $z_{\mathsf{B}} \in \mathrm{H}^2_0(S)$ solve the equations $\mathsf{M}_* \boldsymbol{z}_{\mathsf{M}} = \boldsymbol{g}_*$ and $\mathsf{B}_3 z_{\mathsf{B}} = g_3$. Hence the wave lengths L and ℓ associated with $\boldsymbol{z}^\varepsilon$ are in fact independent on ε.

Estimate (2.24) with $\boldsymbol{f}^{\mathsf{rem}} = \mathbf{0}$ then yields

$$\mathsf{E}^\varepsilon_{\mathsf{3D}}\big[\boldsymbol{u}^\varepsilon - \mathsf{U}\boldsymbol{z}^\varepsilon\big] \leq b_S(\boldsymbol{g})\, \varepsilon\, \mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}^\varepsilon] \tag{9.1}$$

Eestplates

where $b_S(\boldsymbol{g})$ has the dimension of the inverse of a length.

In [11], it is shown that the displacement $\boldsymbol{u}^\varepsilon$ admits a complete two scale asymptotic expansion in powers of ε. This expansion includes regular terms bounded independently of ε, and boundary layer term exponentially decreasing with respect to r/ε where r is the distance to ∂S. Relying on this result, we can prove that the following optimal estimates holds true, see [11, § 12.2]:

$$b'_S(\boldsymbol{g})\, \varepsilon\, \mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{u}^\varepsilon] \;\leq\; \mathsf{E}^\varepsilon_{\mathsf{3D}}\big[\boldsymbol{u}^\varepsilon - \mathsf{U}\boldsymbol{z}^\varepsilon\big] \;\leq\; b_S(\boldsymbol{g})\, \varepsilon\, \mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{u}^\varepsilon] \tag{9.2a}$$

$$a'_S(\boldsymbol{g})\, \mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}^\varepsilon] \;\leq\; \mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{u}^\varepsilon] \;\leq\; a_S(\boldsymbol{g})\, \mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}^\varepsilon], \tag{9.2b}$$

where $b_S(\boldsymbol{g})$, $b'_S(\boldsymbol{g})$ have the dimension of the inverse of a length and $a_S(\boldsymbol{g})$, $a'_S(\boldsymbol{g})$ are adimensional. In relation with the generic non-cancellation of the traces of $\gamma^\alpha_\alpha = \operatorname{div} \boldsymbol{z}_{\mathsf{M}}$ or $\rho^\alpha_\alpha = \Delta z_{\mathsf{B}}$, the constant $b'_S(\boldsymbol{g})$ is generically non-zero. This shows how (9.1) is optimal in the case of plates.

9.B Elliptic shells

In the case of elliptic shells, the curvature tensor $b_{\alpha\beta}$ satisfies an estimate of the form $b_{\alpha\beta}\xi^\alpha\xi^\beta \geq c\xi^\alpha\xi_\alpha$ for all vector field ξ^α on S and for a uniform constant c independent on ξ. This implies that the constant r is a positive number.

Using the result in [15], it is possible to estimate the behaviour of the constants L and ℓ with ε. In [15], it is shown that $\boldsymbol{z}^\varepsilon$ admit a multi-scale asymptotic expansion

$$\boldsymbol{z}^\varepsilon \simeq \boldsymbol{\zeta}^0 + \varepsilon^{1/2}(\boldsymbol{Z}^{1/2} + \boldsymbol{\zeta}^{1/2}) + \varepsilon(\boldsymbol{Z}^1 + \boldsymbol{\zeta}^1)\cdots$$

in powers of $\varepsilon^{1/2}$, where the regular terms $\zeta^{k/2}$ are uniformly bounded in ε, and where the terms $\boldsymbol{Z}^{k/2}$ are boundary layer terms. These terms are sums of functions that are tensor products of smooth functions of $s \in \partial S$ and exponentially decreasing functions with respect to the variable $r\sqrt{b_{ss}}/\sqrt{\varepsilon}$ where b_{ss} is the (non zero) curvature along the boundary ∂S (see equation (1.12) in [15]). This shows that $L \simeq (\varepsilon R_\partial)^{1/2}$ where R_∂ denote the maximum of curvature radius along the boundary ∂S, and that ℓ is a positive constant independent of ε. Hence, estimate (2.24) with $\boldsymbol{f}^{\mathsf{rem}} = \boldsymbol{0}$ yields as before

$$\mathsf{E}^\varepsilon_{\mathsf{3D}}\big[\boldsymbol{u}^\varepsilon - \mathsf{U}\boldsymbol{z}^\varepsilon\big] \leq b_S(\boldsymbol{g})\,\varepsilon\,\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}^\varepsilon].$$

As in the case of plates, this estimate turns to be optimal, see Theorem 1.4 in [15].

9.C Shallow shells

Shallow shells in the sense of [9] are shells for which the mean surface $S = S^\varepsilon$ depend on ε in such a way that the curvature tensor is of order $B\varepsilon$ where B has now the dimension m^{-2}. The limit surface S^0 is a domain of $\mathbb{R}^2$. The constant r is hence of order $(B\varepsilon)^{-1}$. Denote by $(x^\varepsilon_\alpha, x^\varepsilon_3)$ normal coordinates to S^ε, for $\varepsilon \in [0, \varepsilon_0]$.

We define a regular family of loads $\boldsymbol{f}^\varepsilon$ on Ω^ε in the following way. For a fixed smooth surface load $\boldsymbol{G}$ given in a neighborhood of S^0 in $\mathbb{R}^2$, we define the field $\boldsymbol{F}$ by $\boldsymbol{F}(x^0_\alpha, x^0_3) = \boldsymbol{G}(x^0_\alpha)$. Then we set

$$\boldsymbol{f}^\varepsilon = \boldsymbol{F}\big|_{\Omega^\varepsilon} \quad \text{and} \quad \boldsymbol{g}^\varepsilon = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \boldsymbol{f}^\varepsilon(x^\varepsilon_3)\,\mathrm{d}x^\varepsilon_3,$$

and $\boldsymbol{f}^\varepsilon$ is the load in problem (P_{3D}) on Ω^ε, while $\boldsymbol{g}^\varepsilon$ is the right hand side of problem (P_{2D}) on S^ε.

In this situation, the Koiter model can be seen as an operator which couples the membrane and bending operators for plates through low order terms, and it can be shown that $\boldsymbol{z}^\varepsilon$ admits a complete asymptotic expansion in powers of ε with regular terms only. Hence, the constant ℓ and L are independent of ε, and estimate (2.24) yields an estimate similar to (9.1).

In [2], it has been shown that the three dimensional displacement $\boldsymbol{u}^\varepsilon$ admits a complete asymptotic expansion in powers of ε with regular bounded terms and boundary layer terms exponentially decreasing in r/ε. Using this result, it can again be shown that estimates of the form (9.2) hold true in the case of shallow shells. This shows that (2.24) is optimal in this case.

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