# Higher Order Bending and Membrane Responses of Thin Linearly Elastic Plates

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**Abstract.** The limit behaviors of three-dimensional displacements in thin linearly elastic plates, as the half-thickness $\varepsilon$ tends to zero, is now known for various lateral boundary conditions, see [1] [5]. In the generic case one obtains that the leading term of the asymptotic series $u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \dots$ of the scaled displacement is a Kirchhoff-Love field. In this note we investigate the case where this leading term vanishes, giving the structure of the first non-vanishing term $u^k$ and an error estimate for its deviation from the scaled solution $u(\varepsilon)$ multiplied by $\varepsilon^{-k}$. There are essentially only three new cases (uncoupling in membrane and bending). Finally, in these situations a boundary layer term of the *same* order as the actual leading term appears in a generic way.

### Réactions d'ordre élevé des plaques minces linéairement élastiques

**Résumé.** *Le déplacement d'une plaque mince élastique a une limite lorsque sa demi-épaisseur $\varepsilon$ tend vers zéro, voir* [1] [5]. *On obtient en général que le déplacement mis à l'échelle se développe en série $u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \dots$ et que son premier terme $u^0$ est un déplacement de Kirchhoff-Love. Dans cette note, nous étudions les cas où ce premier terme disparaît, laissant la place à un nouveau terme dominant $u^k$ . Il n'existe essentiellement que trois cas génériques, chacun d'entre eux se découplant en membrane et flexion. Enfin, dans chacune de ces situations, apparaît un terme de couche limite du* même *ordre que le terme dominant.*

**Version française abrégée.** Nous considérons une famille de plaques linéairement élastiques $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$ , encastrées le long de leur bord latéral $\partial\omega \times (-\varepsilon, \varepsilon)$ . Sous des hypothèses convenables quant aux données [1, 9] et après changement d'échelle pour se ramener à la configuration de référence $\Omega = \omega \times (-1, 1)$ , on sait [1, 2, 6], [4, 5], que le déplacement mis à l'échelle $\boldsymbol{u}(\varepsilon)$ tend vers une limite dans $H^1(\Omega)$ qui est un déplacement de Kirchhoff-Love $\boldsymbol{u}^0_{\mathrm{KL}}$ issu de la solution d'un problème bi-dimensionnel sur la surface moyenne $\omega$ dont le second membre est constitué de certains *moments* des données du problème initial sur $\Omega$ .

Par la méthode des développements asymptotiques raccordés [8], [4, 5], on montre que $\boldsymbol{u}(\varepsilon)$ admet un développement asymptotique relativement au paramètre d'épaisseur

$$(*) \qquad \boldsymbol{u}(\varepsilon) = \boldsymbol{u}^0_{\mathrm{KL}} + \varepsilon \boldsymbol{u}^1 + \varepsilon^2 \boldsymbol{u}^2 + \dots,$$

où les termes non principaux $\boldsymbol{u}^k$ sont la somme d'une partie Kirchhoff-Love, d'une partie ayant des moyennes nulles sur chaque fibre $x_* \times (-1,1)$ et d'un terme de couche limite qui vit près du bord latéral.

Dans cette note, nous nous intéressons au cas où $\boldsymbol{u}^0_{\mathrm{KL}}$ s'annule sans que les données soient identiquement nulles. Sous les hypothèses 'canoniques' pré-citées sur les données (voir §1), nous montrons qu'alors la série $(*)$ ne peut commencer qu'à $\varepsilon^2\boldsymbol{u}^2$ (Cas I), ou à $\varepsilon^3\boldsymbol{u}^3$ (Cas II), ou à $\varepsilon^4\boldsymbol{u}^4$ (Cas III).

Nous caractérisons précisément les données produisant ces types de réponse de la plaque et nous décrivons la structure du terme dominant $\boldsymbol{u}^k$ dans chacun des cas ( $k = 2,3,4$ resp.), donnant une formule explicite pour la partie à moyenne nulle sur les fibres transverses. Dans chacun des trois cas, nous avons une estimation en norme $H^1(\Omega)$ du type $\|\varepsilon^{-k}\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^k\| \le C\sqrt{\varepsilon}$ .

Le cas I se produit sous de simples conditions de moments nuls sur les données (par exemple, s'il n'y a que des forces volumiques, on se trouve dans le cas I si le moment d'ordre $0$ par rapport à la variable transverse $x_3$ est nul pour chaque composante de la force et si le moment d'ordre $1$ est de même nul pour les composantes planes de la force. Les cas II et III se produisent quand les composantes planes de la force sont nulles et que les moments d'ordre $0,1,2$ de la composante transverse sont nul. Nous donnons des exemples explicites de forces ou de chargements de type membrane ou flexion, produisant les réactions de chacun des trois cas.

Nous classifions ainsi tous les types de réponses, et les 'cas' exhibés ne sont génériquement jamais des déplacements de Kirchhoff-Love seuls: on a génériquement un terme de couche limite du même ordre, et il peut même se produire que pour certaines forces particulières (du Cas II) le terme 'dominant' $\boldsymbol{u}^k$ soit réduit à une couche limite.

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## 1 The plate problem

Our point of departure is the problem of 3D linearized elasticity, applied to a family of thin plates $\Omega^\varepsilon$ . We consider $\Omega^\varepsilon = \omega\times(-\varepsilon,\, \varepsilon)$ with $\omega \subset \mathbb{R}^2$ a regular domain and $\varepsilon_0 > \varepsilon > 0$ . We assume that the plates are constituted of a homogeneous, isotropic material with Lamé constants $\lambda$ and $\mu$ . The rigidity matrix is given by $A = (A_{ijkl}) = \lambda\,\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ . Latin indices are always taken from $\{1,\, 2,\, 3\}$ and Greek ones $\alpha$ , $\beta$ from $\{1,\, 2\}$ . In this note, we restrict our analysis to plates whose lateral boundary $\partial\omega \times (-\varepsilon, +\varepsilon)$ is hard clamped. On lower and upper face $\Gamma^\varepsilon_\pm := \omega \times \{\pm\varepsilon\}$ we take boundary conditions of traction type. As standard, let $\boldsymbol{u} = (u_1, u_2, u_3)$ denote the displacement field, and let $e(\boldsymbol{u})$ denote the associated linearized strain tensor $e_{ij}(\boldsymbol{u}) = \frac{1}{2}\,(\partial_i u_j + \partial_j u_i)$ . We assume the applied volume forces $\boldsymbol{f}^\varepsilon$ to be in $\mathscr{C}^\infty(\overline{\Omega^\varepsilon})^3$ and applied surface forces $\boldsymbol{g}^\varepsilon$ belonging to $\mathscr{C}^\infty(\overline{\omega})^3$ . Then, the variational formulation of the 3D problem consists in finding the unique solution of

$$(1)\quad \left\{\begin{array}{l} \boldsymbol{u}^\varepsilon \in V(\Omega^\varepsilon) := \{\boldsymbol{v} \in H^1(\Omega^\varepsilon)^3 \mid \quad v = 0 \text{ on } \partial\omega \times (-\varepsilon,\, \varepsilon)\} \text{ such that} \\ \forall\, \boldsymbol{v} \in V(\Omega^\varepsilon), \quad \displaystyle\int_{\Omega^\varepsilon} A\, e(\boldsymbol{u}^\varepsilon) : e(\boldsymbol{v}) = \int_{\Omega^\varepsilon} \boldsymbol{f}^\varepsilon \cdot \boldsymbol{v} \;+ \int_{\Gamma^\varepsilon_+} \boldsymbol{g}^{\varepsilon,+} \cdot \boldsymbol{v} \;- \int_{\Gamma^\varepsilon_-} \boldsymbol{g}^{\varepsilon,-} \cdot \boldsymbol{v}. \end{array}\right.$$

Scaling and assumptions on the data. We stretch the plates along the vertical axis to obtain a fixed reference domain:

$x^\varepsilon = (x_*, x_3^\varepsilon) \in \Omega^\varepsilon \mapsto x = (x_*, x_3) \in \Omega := \omega \times (-1, +1)$, where the in-plane coordinates $x_* := (x_1, x_2)$ remain unchanged and $x_3 = \varepsilon^{-1} x_3^\varepsilon$. Likewise we scale the displacement field $\boldsymbol{u}^\varepsilon$ into a new field $\boldsymbol{u}(\varepsilon) : u_\alpha(\varepsilon)(x) = u_\alpha^\varepsilon(x^\varepsilon),\ u_3(\varepsilon)(x) = \varepsilon\, u_3^\varepsilon(x^\varepsilon)$. Finally, suitable assumptions on the data [1, 9] are given by $f_\alpha^\varepsilon(x^\varepsilon) = f_\alpha(x),\ \varepsilon^{-1} f_3^\varepsilon(x^\varepsilon) = f_3(x);\ \varepsilon^{-1} g_\alpha^{\varepsilon,\pm}(x^\varepsilon) = \overset{\pm}{g_\alpha}(x_*),\ \varepsilon^{-2} g_3^{\varepsilon,\pm}(x^\varepsilon) = \overset{\pm}{g_3}(x_*)$. With these scalings and assumptions, $\boldsymbol{u}(\varepsilon)$ tends to a limit as $\varepsilon \to 0$, which is a Kirchhoff-Love displacement $\boldsymbol{u}^0_{\mathrm{KL}}$, see [1, 2, 6].

## 2 The asymptotics of the plate problem solution

We recall here results from [4, 5], based on the method of compound asymptotics, see [7, 8], on regularity results and on error estimates. In order to describe the boundary-layer phenomenon, we introduce local coordinates $(r, s, x_3)$ in a tubular neighborhood of $\partial\omega$. Here $s$ denotes a curvilinear abscissa along $\partial\omega$ and $r$ is the distance to $\partial\omega$. The normal and tangential components of $\boldsymbol{u}$ are $u_n = n_\alpha u_\alpha$ and $u_s = (n_2 u_1 - n_1 u_2)$, resp. where $n_\alpha$ are the Euclidean components of the inner unit normal $\boldsymbol{n}$.

The asymptotic expansion of $\boldsymbol{u}(\varepsilon)$ combines a power series like $\sum_k \varepsilon^k \underline{\boldsymbol{u}}^k(x)$ *(outer expansion)* and a boundary layer series like $\sum_k \varepsilon^k \boldsymbol{w}^k(r\varepsilon^{-1}, s, x_3)$ *(inner expansion)*. In the sequel, $t$ denotes $r\,\varepsilon^{-1}$. We use also the splitting of coordinates, components and gradients in plane and vertical variables $x = (x_*, x_3)$, $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_*, \zeta_3)$ and $\nabla = (\nabla_*, \partial_3)^\top$. More precisely, the expansion contains three types of terms:

- $\boldsymbol{u}^k_{\mathrm{KL}}$ : Kirchhoff-Love displacements with 'generating functions' $\boldsymbol{\zeta}^k = (\boldsymbol{\zeta}^k_*, \zeta^k_3)$, i.e. $\boldsymbol{u}^k_{\mathrm{KL}}(x) = \Big(\boldsymbol{\zeta}^k_*(x_*) - x_3 \nabla_* \zeta^k_3(x_*), \zeta^k_3(x_*)\Big)$,
- $\boldsymbol{v}^k$ : displacements with zero mean value: $\forall x_* \in \overline{\omega},\ \int_{-1}^{+1} \boldsymbol{v}^k(x_*, x_3)\, dx_3 = 0$,
- $\boldsymbol{w}^k$ : exponentially decreasing profiles as $t \to +\infty$,

according to: $\boldsymbol{u}(\varepsilon)(x) \simeq \boldsymbol{u}^0 + \varepsilon \boldsymbol{u}^1(x, \frac{r}{\varepsilon}) + \cdots + \varepsilon^k \boldsymbol{u}^k(x, \frac{r}{\varepsilon}) + \cdots$ where $\boldsymbol{u}^0 = \boldsymbol{u}^0_{\mathrm{KL}}$ and

$$\boldsymbol{u}^k(x, t) = \boldsymbol{u}^k_{\mathrm{KL}} + \boldsymbol{v}^k + \chi(r)\, \boldsymbol{w}^k(t, s, x_3) \quad \text{for} \quad k \geq 1, \tag{2}$$

with $\chi$ a cut-off function equal to $1$ in a neighborhood of $\partial\omega$.

## 3 The generic leading terms in the asymptotics

The presence of the leading term or the structure of the following ones are closely dependent of different *moments*. This is the reason for the introduction of the notation $[f]_m$ for the $m$-th moment with respect to $x_3$ of $f$.

The generating functions $\boldsymbol{\zeta}^k$ are solutions of membrane and bending problems:

$$L^{\mathrm{m}} \boldsymbol{\zeta}^k_* = \boldsymbol{R}^k_{\mathrm{m}} \qquad \text{and} \qquad L^{\mathrm{b}} \zeta^k_3 = R^k_{\mathrm{b}}, \tag{3}$$

where $L^{\mathrm{m}} \boldsymbol{\zeta}_* := \mu\, \boldsymbol{\Delta}_* \boldsymbol{\zeta}_* + (\tilde{\lambda} + \mu) \nabla_* \operatorname{div}_* \boldsymbol{\zeta}_*$ and $L^{\mathrm{b}} \zeta_3 := (\tilde{\lambda} + 2\mu) \Delta^2_* \zeta_3$ with the 'homogenized' Lamé coefficient $\tilde{\lambda} := 2\lambda\mu(\lambda + 2\mu)^{-1}$. These problems, when completed with the 'correct' Dirichlet traces of $\boldsymbol{\zeta}^k$, provide exactly the membrane and bending models of a plate. In particular we have for $k = 0$

$$\boldsymbol{R}^0_{\mathrm{m}} := -\frac{1}{2} \Big( [\boldsymbol{f}_*]_0 + \boldsymbol{g}^+_* - \boldsymbol{g}^-_* \Big), \tag{4}$$

(5) $$R_{\rm b}^0 := \frac{3}{2}\Big([f_3]_0 + g_3^+ - g_3^- + \mathrm{div}_*([\boldsymbol{f}_*]_1 + \boldsymbol{g}_*^+ + \boldsymbol{g}_*^-)\Big)\,,$$

and the Dirichlet traces $\zeta_n^0$, $\zeta_s^0$, $\zeta_3^0$ and $\partial_n\zeta_3^0$ are zero.

Concerning the term of order $1$, we have $\boldsymbol{u}^1 = \boldsymbol{u}_{\rm KL}^1 + \boldsymbol{w}^1$ with $w_3^1 = 0$. The generating function $\boldsymbol{\zeta}^1$ solves (3) with $\boldsymbol{R}_{\rm m}^1 = 0$, $R_{\rm b}^1 = 0$ and with Dirichlet traces $\zeta_n^1 = c_1\,\mathrm{div}_*\,\boldsymbol{\zeta}_*^0$, $\zeta_s^1 = 0$, $\zeta_3^1 = 0$, $\partial_n\zeta_3^1 = c_4\,\Delta_*\zeta_3^0$ on $\partial\omega$, where $c_1 = c_1(\lambda,\mu)$ and $c_4 = c_4(\lambda,\mu)$ are universal coefficients. Likewise, the boundary-layer term $\boldsymbol{w}^1$ is present only if the traces of $\mathrm{div}_*\,\boldsymbol{\zeta}_*^0$ and $\Delta_*\zeta_3^0$ are non-zero.

## 4 When the generic leading term is zero

The 'leading term' is zero if and only if $\boldsymbol{R}_{\rm m}^0$ and $R_{\rm b}^0$ are zero. Then it follows that $\boldsymbol{u}_{\rm KL}^1$ and $\boldsymbol{w}^1$ are zero, too. The asymptotics then 'starts' with $\boldsymbol{u}^2$ according to (2). For its description, we need several functions of the data $(\boldsymbol{f},\boldsymbol{g}^\pm)$. First, let us introduce the primitives

$$\oint^{x_3} u\;dy_3 \;:=\; \int_{-1}^{x_3} u(y_3)\;dy_3 - \frac{1}{2}\int_{-1}^{+1}\int_{-1}^{z_3} u(y_3)\,dy_3\,dz_3,$$

$$⨍^{y_3} u\;dz_3 \;:=\; \frac{1}{2}\Big(\int_{-1}^{y_3} u(z_3)\;dz_3 - \int_{y_3}^{+1} u(z_3)\;dz_3\Big)\,.$$

The function $G = G(\boldsymbol{f},\boldsymbol{g}^\pm)$ is then defined in $\mathscr{C}^\infty(\overline{\Omega})^3$ by

(6) $$G_3 = 0 \quad\text{and}\quad G_\alpha(x_*,x_3) = -\frac{1}{\mu}\Big(\oint^{x_3}(⨍^{y_3} f_\alpha)\,dy_3 - x_3\,\frac{g_\alpha^+ + g_\alpha^-}{2}\Big),$$

the operator $F\,:\,\boldsymbol{v}\mapsto F(\boldsymbol{v})$ is defined from $\mathscr{C}^\infty(\overline{\Omega})^3$ into $\mathscr{C}^\infty(\overline{\omega})^3$ by

(7) $$F_\alpha(\boldsymbol{v}) = \frac{\tilde\lambda}{2}\int_{-1}^{+1}⨍^{y_3}\partial_{\alpha\beta}e_{\beta3}(\boldsymbol{v})\;dz_3\,dy_3 \quad\text{and}\quad F_3(\boldsymbol{v}) = \mu\int_{-1}^{+1}\partial_\beta e_{\beta3}(\boldsymbol{v})\;dy_3,$$

and the function $H = H(\boldsymbol{f},\boldsymbol{g}^\pm)$, defined in $\mathscr{C}^\infty(\overline{\Omega})^3$, by

(8) $$H_3 = -\frac{1}{\lambda+2\mu}\Big(\oint^{x_3}(⨍^{y_3} f_3)\,dy_3 - x_3\,\frac{g_3^+ + g_3^-}{2}\Big) \quad\text{and}$$
$$H_\alpha = -\oint^{x_3}\Big(\partial_\alpha H_3 + \frac{1}{\mu}\,y_3\,F_*(G) + \frac{\lambda}{\mu}⨍^{y_3}\Big\{\partial_{\alpha3}H_3 - \frac{1}{2}\int_{-1}^{+1}\partial_{\alpha3}H_3\,dz_3\Big\}\Big)dy_3\,.$$

And finally let

(9) $$\boldsymbol{R}_{\rm m}^2 = F_*(G) - \frac{\tilde\lambda}{4\mu}\nabla_*\Big([f_3]_1 + g_3^+ + g_3^-\Big) \quad\text{and}$$
$$R_{\rm b}^2 = 3F_3(H) + \frac{3\mu(3\lambda+4\mu)}{2(\lambda+2\mu)}\Delta_*\,\mathrm{div}_*[G_*]_1.$$

## 5 Case I: $\boldsymbol{u}(\varepsilon) = \varepsilon^2\boldsymbol{u}^2 + \varepsilon^3\boldsymbol{u}^3 + \dots$

Let $\|\cdot\|$ denote the $H^1$-norm.

THEOREM 5.1. – *Let* $\boldsymbol{f}$ , $\boldsymbol{g}^{\pm}$ *be such that* $\boldsymbol{R}^0_{\mathrm{m}} = R^0_{\mathrm{b}} = 0$ *and that either* $\boldsymbol{R}^2_{\mathrm{m}}$ , *or* $R^2_{\mathrm{b}}$ *or* $G$ *is not identically zero. Then*

(i) $\boldsymbol{u}(\varepsilon)$ *starts with* $\varepsilon^2\boldsymbol{u}^2$ *and there holds the estimate* $\|\varepsilon^{-2}\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^2\| \le C\sqrt{\varepsilon}$ .

(ii) $\boldsymbol{u}^2 = \boldsymbol{u}^2_{\mathrm{KL}} + G + \chi\boldsymbol{w}^2$ *with the generating function* $\boldsymbol{\zeta}^2$ *solving* (3) *for* $k = 2$ *with boundary conditions* $\zeta^2_s = 0$ , $\zeta^2_n = d_1(s)$ , $\zeta^2_3 = 0$ , $\partial_n\zeta^2_3 = d_2(s)$ *on* $\partial\omega$ , *with functions* $d_\alpha \in \mathscr{C}^\infty(\partial\omega)$ *only depending on* $G_n$ , $\lambda$ *and* $\mu$ . *Generically,* $\boldsymbol{w}^2$ *is non-trivial.*

Notice that the leading term of $\boldsymbol{u}(\varepsilon)$ is the superposition of a Kirchhoff-Love term $\boldsymbol{u}^2_{\mathrm{KL}}$ and a non-Kirchhoff-Love term $G$ and last but not least close to the lateral boundary there is additionally a boundary-layer profile $\boldsymbol{w}^2$ active.

Let us recall that any displacement field can be split into a membrane field and a bending field, having the parities (even, even, odd) and (odd, odd, even) with respect to the transverse variable $x_3$ respectively.

MEMBRANE EXAMPLE. Let $(f_3,\, g^{\pm}_3) = 0$ , $\boldsymbol{f}_* = -(1,\,1)^\top$ , $\boldsymbol{g}^{\pm}_* = \pm(1,\,1)^\top$ . Then $G_\alpha$ is a non-zero multiple of $(x_3^2 - \frac{1}{3})$ and independent of $x_*$ . Moreover, we have $\boldsymbol{R}^0_{\mathrm{m}} = R^0_{\mathrm{b}} = 0$ and also $\boldsymbol{R}^2_{\mathrm{m}} = R^2_{\mathrm{b}} = 0$ . Since $\boldsymbol{u}(\varepsilon)$ has the membrane parities, $\zeta^2_3$ is zero. But, in general $d_1$ is a non-zero constant function and $\boldsymbol{\zeta}^2_*$ and $\boldsymbol{w}^2$ are non-zero, too. Within the hierarchic plate models, the interior part of the leading term is of type $(2,2,0)$ .

BENDING EXAMPLE. Let $(f_3,\, g^{\pm}_3) = 0$ , $\boldsymbol{f}_* = -3x_3(1,\,1)^\top$ , $\boldsymbol{g}^{+}_* = \boldsymbol{g}^{-}_* = (1,\,1)^\top$ . Then $G_\alpha$ is an odd non-zero cubic polynomial in $x_3$ and independent of $x_*$ . Moreover, we have $\boldsymbol{R}^0_{\mathrm{m}} = R^0_{\mathrm{b}} = 0$ and also $\boldsymbol{R}^2_{\mathrm{m}} = R^2_{\mathrm{b}} = 0$ . Conversely to the previous example, $\zeta^2_*$ is zero and $\zeta^2_3$ is non-zero. Far away from the lateral boundary, the leading term of $\boldsymbol{u}(\varepsilon)$ is of type $(3,3,0)$ .

## 6 Case II: $\boldsymbol{u}(\varepsilon) = \varepsilon^3\boldsymbol{u}^3 + \varepsilon^4\boldsymbol{u}^4 + \dots$ and Case III: $\boldsymbol{u}(\varepsilon) = \varepsilon^4\boldsymbol{u}^4 + \varepsilon^5\boldsymbol{u}^5 + \dots$

LEMMA 6.1. – *There holds* $\boldsymbol{R}^0_{\mathrm{m}} = \boldsymbol{R}^2_{\mathrm{m}} = R^0_{\mathrm{b}} = R^2_{\mathrm{b}} = 0$ *and* $G = 0$ *if and only if*

$$(10)\qquad (\boldsymbol{f}_*,\, \boldsymbol{g}^{\pm}_*) = 0 \ \textit{and}\ [f_3]_0 + g^+_3 - g^-_3 = \nabla_*([f_3]_1 + g^+_3 + g^-_3) = \Delta_*([f_3]_2 - [f_3]_0) = 0.$$

If (10) holds, $\boldsymbol{u}^k = 0$ for $k = 0,1,2$ and $\boldsymbol{u}(\varepsilon)$ 'starts' by $\varepsilon^3\boldsymbol{u}^3 + \varepsilon^4\boldsymbol{u}^4$ with

$$(11)\qquad \boldsymbol{u}^3 = \boldsymbol{u}^3_{\mathrm{KL}} + \chi\boldsymbol{w}^3 \quad\text{and}\quad \boldsymbol{u}^4 = \boldsymbol{u}^4_{\mathrm{KL}} + H + \chi\boldsymbol{w}^4.$$

Remarkably, the series starts at $\varepsilon^3$ as soon as the trace of $H_3$ on $\partial\omega\times(-1,1)$ is non-zero.

THEOREM 6.2. – *Let* $\boldsymbol{f}$ , $\boldsymbol{g}^{\pm}$ *be non-zero and such that* (10) *holds. Let* $h_3$ *be the trace of* $H_3$ *on the lateral boundary* $\partial\omega\times(-1,1)$ .

*a) If* $h_3 \not\equiv 0$ , *then*

(i) $\boldsymbol{u}(\varepsilon)$ *starts with* $\varepsilon^3\boldsymbol{u}^3$ *and there holds the estimate* $\|\varepsilon^{-3}\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^3\| \le C\sqrt{\varepsilon}$ .

(ii) $\boldsymbol{u}^3 = \boldsymbol{u}^3_{\mathrm{KL}} + \chi\boldsymbol{w}^3$ *with generating function* $\boldsymbol{\zeta}^3$ *solving* (3) *for* $k = 3$ *with* $\boldsymbol{R}^3_{\mathrm{m}} = 0$ *and* $R^3_{\mathrm{b}} = 0$ *and with boundary conditions* $\zeta^3_s = 0$ , $\zeta^3_n = d_1(s)$ , $\zeta^3_3 = 0$ , $\partial_n\zeta^3_3 = d_2(s)$ *on* $\partial\omega$ , *with functions* $d_\alpha \in \mathscr{C}^\infty(\partial\omega)$ *only depending on* $h_3$ , $\lambda$ *and* $\mu$ . *Generically,* $\boldsymbol{w}^3$ *is non-trivial. Moreover there exist data such that* $\boldsymbol{w}^3$ *is non-zero while* $\boldsymbol{\zeta}^3$ *is zero, or vice versa.*

*b) If* $h_3 \equiv 0$ , *then*

(i) $\boldsymbol{u}(\varepsilon)$ *starts with* $\varepsilon^4\boldsymbol{u}^4$ *and there holds the estimate* $\|\varepsilon^{-4}\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^4\| \le C\sqrt{\varepsilon}$ .

(ii) $\boldsymbol{u}^4 = \boldsymbol{u}^4_{\mathrm{KL}} + H + \chi \boldsymbol{w}^4$ *with the generating function* $\boldsymbol{\zeta}^4$ *solving* (3) *for* $k = 4$ *with* $\boldsymbol{R}^4_{\mathrm{m}} = F_*(H)$ ,

$$R^4_{\mathrm{b}} = \frac{3\mu}{4(\lambda+2\mu)}\Big\{(6\lambda+8\mu)\,\Delta_*\,\mathrm{div}_*[H_*]_1 + \lambda\,\Delta^2_*[H_3]_2\Big\}$$

*and with boundary conditions* $\zeta^4_s = 0$ , $\zeta^4_n = d_1(s)$ , $\zeta^4_3 = 0$ , $\partial_n \zeta^4_3 = d_2(s)$ *on* $\partial\omega$ , *with functions* $d_\alpha \in \mathscr{C}^\infty(\partial\omega)$ *only depending on* $\partial_n H_3$ , $\lambda$ *and* $\mu$ *. Generically,* $\boldsymbol{w}^4$ *is non-trivial.*

Membrane examples. *a)* $f_3 = 0$ and $g^{\pm}_3 = 1$ is an example for case II, for which $H_3 = (\lambda+2\mu)^{-1}x_3$ .

*b)* Taking any non-zero $\psi \in C^\infty_0(\omega)$ and letting $g^{\pm}_3 = \psi(x_*)$ and $f_3 = -3\psi(x_*)\,x_3$ yields an example for case III, for which $H_3$ is an odd cubic polynomial in $x_3$ and $H_\alpha$ is an even polynomial of degree 4 . Here $\boldsymbol{w}^4 = 0$ since $\partial_n H_3$ is zero on $\partial\omega \times (-1, 1)$ .

Bending examples. *a)* $f_3 = (x^2_3 - 1/3)$ and $g^{\pm}_3 = 0$ is an example for case II, for which $H_3$ is a non-zero polynomial of degree 4 in $x_3$ .

*b)* Taking any non-zero $\psi \in C^\infty_0(\omega)$ and letting $g^{\pm}_3 = 0$ and $f_3 = \psi(x_*)(35x^4_3 - 30x^2_3 + 3)$ yields an example for case III, for which $\boldsymbol{w}^4 = 0$ .

Finally, under considered applied forces, $\boldsymbol{u}(\varepsilon)$ asymptotics cannot begin with $\varepsilon^5\boldsymbol{u}^5$ :

Lemma 6.3. – *There holds* $\boldsymbol{R}^0_{\mathrm{m}} = R^0_{\mathrm{b}} = 0$ *and* $G = H = 0$ (*and then* $\boldsymbol{R}^2_{\mathrm{m}} = R^2_{\mathrm{b}} = 0$) *if and only if* $\boldsymbol{f} = 0$ *and* $\boldsymbol{g}^{\pm} = 0$ *.*

Thus, among the data like those in §1, we have found all non-trivial cases, such that the asymptotic expansion of $\boldsymbol{u}(\varepsilon)$ does not start with $\varepsilon^0\boldsymbol{u}^0$ .

Smooth data and the use of boundary layer profiles allow to give sharp estimates on the asymptotics of $\boldsymbol{u}(\varepsilon)$ up to an arbitrary order, and also $L^2$-estimates for the error in strains or stresses, see [3, 4, 5].

A similar analysis works for various other lateral boundary conditions as well, see [5].

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