Full Asymptotic Expansions for Thin Elastic Free Plates

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Abstract. The case of a linearly elastic plate with free boundary conditions on the lateral side is investigated as the half-thickness ε tends to zero. As in hard clamped plates, the generic leading term of the asymptotic expansion of the scaled displacement is a Kirchhoff-Love field with in-plane generating functions satisfying classical bending and membrane problems of Neumann type, compare [1]. The first boundary layer profile is of bending type, so that in the case of a membrane load the convergence of the 3D solution to the 2D limit one is of improved accuracy. Conditions under which the asymptotic expansion 'starts later' are given and the structure of the first non-vanishing term is studied.

Développements asymptotiques complets pour des plaques élastiques minces libres

Résumé. *Nous considérons une plaque linéairement élastique avec la condition de traction nulle sur le côté latéral. Comme pour une plaque encastrée, lorsque la demi-épaisseur* ε *tends vers zéro, le terme dominant générique dans le développement asymptotique du déplacement est un terme de Kirchhoff-Love dont les générateurs bidimensionnels sont solutions des problèmes de Neumann pour les opérateurs de membrane et de flexion classiques, voir* [1]. *Le premier profil de couche limite est un terme de flexion pure. Ainsi, dans le cas d'un chargement membranaire, le taux de convergence du déplacement tridimensionnel vers sa limite est amélioré. Nous étudions aussi les cas où le terme dominant générique s'annule.*

Version française abrégée. Nous prenons le cadre d'étude habituel pour les plaques minces, *cf* [1]-[4], et considérons plus spécifiquement ici les conditions latérales de traction nulle (plaque libre). Les équations linéarisées du déplacement sont celles de Lamé auxquelles on ajoute des conditions de compatibilité et d'orthogonalité par rapport aux déplacements rigides.

Sous les hypothèses standard de régularité et de comportement par rapport à ε (demi-épaisseur de la plaque) sur les données, nous établissons un développement asymptotique complet du déplacement mis à l'échelle $\boldsymbol{u}(\varepsilon)$

$$\boldsymbol{u}(\varepsilon) = \boldsymbol{u}^0_{\rm KL} + \varepsilon(\boldsymbol{u}^1_{\rm KL} + \boldsymbol{w}^1) + \varepsilon^2(\boldsymbol{u}^2_{\rm KL} + \boldsymbol{v}^2 + \boldsymbol{w}^2) + \dots \qquad (*)$$

où les $\boldsymbol{u}^k_{\rm KL}$ sont des déplacements de Kirchhoff-Love, les $\boldsymbol{v}^k$ des déplacements à moyenne nulle sur chaque fibre transverse et les $\boldsymbol{w}^k$ des termes de couche limite le long du bord latéral, à profil exponentiellement décroissant.

Comme dans les plaques encastrées, les déplacements bi-dimensionnels $\boldsymbol{\zeta}^k = (\boldsymbol{\zeta}_*^k, \zeta_3)$ générant les $\boldsymbol{u}_{\mathrm{KL}}^k$ sont solutions d'équations de flexion et de membrane standard, l'originalité étant que les conditions aux limites pour l'équation de flexion sont non nulles en général, voir aussi [1]. De plus, et encore contrairement aux plaques encastrées, la structure des termes dans le développement $(*)$ dépend fortement des symétries du chargement: pour un *chargement membranaire* (cas étudié dans [7]) les termes en ε, c'est-à-dire $\boldsymbol{u}_{\mathrm{KL}}^1$ et $\boldsymbol{w}^1$, sont nuls, d'où une convergence améliorée de $\boldsymbol{u}(\varepsilon)$ vers sa limite $\boldsymbol{u}_{\mathrm{KL}}^0$:

$$\|\boldsymbol{u}(\varepsilon) - \boldsymbol{u}_{\mathrm{KL}}^0\|_{H^1} \leq C\varepsilon^{3/2} \quad \text{et} \quad \|\boldsymbol{u}(\varepsilon) - \boldsymbol{u}_{\mathrm{KL}}^0\|_{L^2} \leq C\varepsilon^2.$$

Continuant l'investigation commencée dans [2] des réponses d'ordre élevé, nous obtenons pour les plaques libres des résultats à la fois similaires aux plaques encastrées (essentiellement trois types de réponses quand $\boldsymbol{u}_{\mathrm{KL}}^0$ est nul) et différents: les conditions sur les chargements provoquant l'annulation de $\boldsymbol{u}_{\mathrm{KL}}^0$, ou des termes d'ordre 0, 1 et 2, sont plus contraignantes.

Ainsi, pour assurer l'annulation de $\boldsymbol{u}_{\mathrm{KL}}^0$, il faut ajouter aux conditions suffisantes pour une plaque encastrée, qu'un moment flexionnel normal soit nul sur le bord de la surface moyenne. Quant aux conditions assurant que les termes d'ordre 0, 1 et 2 sont nuls, elles ajoutent aux conditions déjà nécessaires pour les plaques encastrées des conditions d'annulation d'autre moments sur le bord de la surface moyenne.

En conclusion, la nature flexionnelle des plaques se manifeste plus fortement lorsque les conditions aux limites latérales sont libres plutôt qu'encastrées, et plus précisément lorsque la composante verticale de la traction est imposée plutôt que la composante verticale du déplacement, voir [4].

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1 The free plate problem

In this Note, we study the free plate problem in the framework of three-dimensional linearized elastostatics. We investigate a family of thin elastic plates $\Omega^\varepsilon = \omega \times (-\varepsilon, +\varepsilon)$, where $\omega \subset \mathbb{R}^2$ is a regular domain and $\varepsilon_0 > \varepsilon > 0$. We restrict ourselves to the consideration of plates which are constituted of a homogeneous, isotropic material with Lamé constants λ and μ. We assume that the boundary conditions on the upper and lower faces $\Gamma_\pm^\varepsilon := \omega \times \{-\varepsilon, +\varepsilon\}$ are of traction type and on the lateral side $\Gamma_0^\varepsilon := \partial\omega \times (-\varepsilon, +\varepsilon)$ we consider the case of free boundary conditions, that is zero tractions. Let us denote by $\boldsymbol{u} = (u_1, u_2, u_3)$ the displacement field and by $e(\boldsymbol{u})$ the associated linearized strain tensor $e_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. We assume that the applied forces $\boldsymbol{f}^\varepsilon$ belong to $\mathscr{C}^\infty(\overline{\Omega^\varepsilon})^3$ and the applied surface forces $\boldsymbol{g}^{\varepsilon,\pm}$ to be given in $\mathscr{C}^\infty(\overline{\omega})^3$. Then, the variational formulation of the 3D problem consists in finding

$$\begin{cases} \boldsymbol{u}^\varepsilon \in V(\Omega^\varepsilon) := H^1(\Omega^\varepsilon)^3 \text{ such that} \\ \forall\, \boldsymbol{v} \in V(\Omega^\varepsilon), \quad \displaystyle\int_{\Omega^\varepsilon} A\, e(\boldsymbol{u}^\varepsilon) : e(\boldsymbol{v}) = \int_{\Omega^\varepsilon} \boldsymbol{f}^\varepsilon \cdot \boldsymbol{v} \, + \int_{\Gamma_+^\varepsilon} \boldsymbol{g}^{\varepsilon,+} \cdot \boldsymbol{v} \, - \int_{\Gamma_-^\varepsilon} \boldsymbol{g}^{\varepsilon,-} \cdot \boldsymbol{v}\,. \end{cases} \tag{1}$$

If the right hand side satisfies the compatibility condition

$$\forall\, \boldsymbol{v} \in \mathcal{R}(\Omega^\varepsilon), \quad \int_{\Omega^\varepsilon} \boldsymbol{f}^\varepsilon \cdot \boldsymbol{v} \, + \int_{\Gamma_+^\varepsilon} \boldsymbol{g}^{\varepsilon,+} \cdot \boldsymbol{v} \, - \int_{\Gamma_-^\varepsilon} \boldsymbol{g}^{\varepsilon,-} \cdot \boldsymbol{v} = 0, \tag{2}$$

where $\mathcal{R}(\Omega^\varepsilon)$ denotes the six-dimensional space of rigid motions, then there exists a unique solution $\boldsymbol{u}^\varepsilon$ to (1) satisfying the orthogonality condition $\int_{\Omega^\varepsilon} \boldsymbol{u}^\varepsilon \cdot \boldsymbol{v} = 0$ for all $\boldsymbol{v} \in \mathcal{R}(\Omega^\varepsilon)$. Our objective is to state the construction and the justification of the complete asymptotics of $\boldsymbol{u}^\varepsilon$ as $\varepsilon \to 0$.

2 Scalings and Ansatz terms

By means of a dilatation along the vertical axis $x_3 = \varepsilon^{-1} x_3^\varepsilon$ we obtain a fixed reference domain: $x^\varepsilon = (x_*, x_3^\varepsilon) \in \Omega^\varepsilon \mapsto x = (x_*, x_3) \in \Omega := \omega \times (-1, +1)$, where the in-plane coordinates $x_* := (x_1, x_2)$ remain unchanged. As standard, the displacement field $\boldsymbol{u}^\varepsilon$ is scaled into a new field $\boldsymbol{u}(\varepsilon)$: $u_\alpha(\varepsilon)(x) = u_\alpha^\varepsilon(x^\varepsilon)$, $u_3(\varepsilon)(x) = \varepsilon\, u_3^\varepsilon(x^\varepsilon)$, because of homogeneity reasons. Suitable assumptions on the data [1, 6] are made by $f_\alpha^\varepsilon(x^\varepsilon) = f_\alpha(x)$, $\varepsilon^{-1} f_3^\varepsilon(x^\varepsilon) = f_3(x)$; $\varepsilon^{-1} g_\alpha^{\varepsilon,\pm}(x^\varepsilon) = g_\alpha^{\pm}(x_*)$, $\varepsilon^{-2} g_3^{\varepsilon,\pm}(x^\varepsilon) = g_3^{\pm}(x_*)$.

For the asymptotic expansion of $\boldsymbol{u}(\varepsilon)$ we use a mixed Ansatz combining a power series (*outer expansion*) and a boundary layer series $\sum_k \varepsilon^k \boldsymbol{w}^k(r\varepsilon^{-1}, s, x_3)$ (*inner expansion*), see [2, 3, 4]. Here (r, s) are local coordinates in a tubular neighborhood of $\partial\omega$. We use ∂_n (∂_s) to indicate derivation in the direction of the normal (tangent) to $\partial\omega$ and also the index n (s) to denote the normal (tangent) component of a vector. Using the splitting of components and gradients in plane and vertical variables $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_*, \zeta_3)$ and $\nabla = (\nabla_*, \partial_3)^\top$, the expansion contains three types of terms:

- $\boldsymbol{u}_{\mathrm{KL}}^k$: Kirchhoff-Love displacements with 'generating functions' $\boldsymbol{\zeta}^k = (\boldsymbol{\zeta}_*^k, \zeta_3^k)$,
- $\boldsymbol{v}^k$: displacements with zero mean value: $\forall x_* \in \overline{\omega}$, $\int_{-1}^{+1} \boldsymbol{v}^k(x_*, x_3)\, dx_3 = 0$,
- $\boldsymbol{w}^k$: exponentially decreasing profiles as $t \to +\infty$,

according to: $\boldsymbol{u}(\varepsilon)(x) \simeq \boldsymbol{u}_{\mathrm{KL}}^0 + \varepsilon \boldsymbol{u}^1(x, \frac{r}{\varepsilon}) + \cdots + \varepsilon^k \boldsymbol{u}^k(x, \frac{r}{\varepsilon}) + \cdots$ with

$$\boldsymbol{u}^k(x,t) = \boldsymbol{u}_{\mathrm{KL}}^k + \boldsymbol{v}^k + \chi(r)\, \boldsymbol{w}^k(t, s, x_3) \quad \text{for} \quad k \geq 1, \tag{3}$$

where t denotes $r\varepsilon^{-1}$ and χ is a cut-off function equal to 1 in a neighborhood of $\partial\omega$.

3 The asymptotic expansion

Following general principles according to the *method of compound asymptotic expansions*, see e.g. [5], we are able to construct recursively the complete expansion of $\boldsymbol{u}(\varepsilon)$. We emphasize that the derivation of the 'correct' natural boundary conditions in the limit problems for the generators $\boldsymbol{\zeta}^k$ is subordinated to the existence of exponentially decaying boundary layer profiles. Denoting by $\|\cdot\|$ the H^1-norm and by $[f]_m$ the m-th moment of f with respect to x_3, let us now quote the main results. Detailed calculations can be found in [4].

The solution $\boldsymbol{u}(\varepsilon)$ can be uniquely decomposed as a sum of a *bending part* $\boldsymbol{u}_{\mathrm{b}}(\varepsilon)$ and a *membrane part* $\boldsymbol{u}_{\mathrm{m}}(\varepsilon)$ which satisfy the parity properties with respect to x_3:

$$\begin{aligned} u_{\mathrm{b},\alpha}(x_*, x_3) &= -\,u_{\mathrm{b},\alpha}(x_*, -x_3), \quad \alpha = 1,2, & u_{\mathrm{b},3}(x_*, x_3) &= u_{\mathrm{b},3}(x_*, -x_3)\,, \\ u_{\mathrm{m},\alpha}(x_*, x_3) &= u_{\mathrm{m},\alpha}(x_*, -x_3), \quad \alpha = 1,2, & u_{\mathrm{m},3}(x_*, x_3) &= -\,u_{\mathrm{m},3}(x_*, -x_3)\,. \end{aligned}$$

As well known, $\boldsymbol{u}_{\mathrm{b}}$ and $\boldsymbol{u}_{\mathrm{m}}$ can be computed independently of each other provided that the load functional is split correspondingly into bending and membrane parts, *cf* [9, 8].

Bending part. The generating functions ζ_3^k are solutions of bending problems

$$L^{\mathrm{b}}\zeta_3^k = R_{\mathrm{b}}^k \quad \text{in } \omega\,, \tag{4}$$

where $L^{\mathrm{b}}\zeta_3 := (\tilde\lambda{+}2\mu)\Delta_*^2\zeta_3$ with the 'homogenized' Lamé coefficient $\tilde\lambda := 2\lambda\mu(\lambda{+}2\mu)^{-1}$. These problems are completed with the natural boundary conditions associated to L^{b}

$$\begin{array}{rcl} M_n(\zeta_3^k) &=& \tilde\lambda\Delta_*\zeta_3^k + 2\mu\partial_{nn}\zeta_3^k \\ N_n(\zeta_3^k) &=& (\tilde\lambda+2\mu)\partial_n\Delta_*\zeta_3^k + 2\mu\partial_s(\partial_n+\kappa)\partial_s\zeta_3^k \end{array} \quad \text{on } \partial\omega\,, \tag{5}$$

where κ denotes the curvature of $\partial\omega$. Since we are now concerned with limit problems of Neumann type, compatibility conditions have to be satisfied in order to guarantee the existence of the generators $\boldsymbol{\zeta}_3^k$ and hence to ensure that terms of arbitrary order in the asymptotics can be calculated.

Theorem 3.1. – *The process of constructing the terms in the asymptotic expansion is recursive. The* 3D *compatibility conditions* (2) *imply the solvability of the* 2D *limit problems* (4) *for the* ζ_3^k. *Moreover the above Ansatz is justified by the estimate*

$$\forall\, N \geq 0 \qquad \|\boldsymbol{u}_{\mathrm{b}}(\varepsilon)(x) - \boldsymbol{u}^0_{\mathrm{KL,b}}(x) - \sum_{k=1}^{N}\varepsilon^k\boldsymbol{u}^k_{\mathrm{b}}(x,t)\| \leq C(N)\,\varepsilon^{N+1/2} \quad . \tag{6}$$

For the first limit problem we have in particular

$$R^0_{\mathrm{b}} := \frac{3}{2}\Big([f_3]_0 + g_3^+ - g_3^- + \mathrm{div}_*([\boldsymbol{f}_*]_1 + \boldsymbol{g}_*^+ + \boldsymbol{g}_*^-)\Big)$$

and the Neumann boundary conditions are given by $N_n(\zeta_3^0) = \frac{3}{2}\left([f_n]_1 + g_n^+ + g_n^-\right)$ and $M_n(\zeta_3^0) = 0$ on $\partial\omega$. The nonhomogeneity of the boundary condition $N_n(\zeta_3^0)$ is known, compare [1, p. 66, Th. 1.7.2], and can be interpreted as a prescribed moment on the lateral side generated by f_n and $g_n^{\pm}$.

In the next step, we have $\boldsymbol{u}^1_{\mathrm{b}} = \boldsymbol{u}^1_{\mathrm{KL,b}} + \boldsymbol{w}^1_{\mathrm{b}}$. The generating function ζ_3^1 solves (4) with $R^1_{\mathrm{b}} = 0$ and Neumann traces $M_n(\zeta_3^1) = c_1\,\partial_s(\partial_n+\kappa)\partial_s\zeta_3^0$ and $N_n(\zeta_3^1) = d_1(s)$ on $\partial\omega$, where $c_1 = c_1(\lambda,\mu)$ is a universal coefficient and $d_1(s) \in \mathscr{C}^\infty(\partial\omega)$ depends on special traces of ζ_3^0 as well as on f_n and $g_n^{\pm}$. The first boundary layer profile $\boldsymbol{w}^1_{\mathrm{b}}$ has only its s-component being non-zero. It holds $w^1_s = (\partial_n+\kappa)\partial_s\zeta_3^0(s)\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}$, where $\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}(t,x_3)$ is the exponentially decaying solution as $t\to+\infty$ of $(\partial_{tt}+\partial_{33})\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}} = 0$ in $\mathbb{R}^+\times(-1,1)$, $\partial_3\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}(t,\pm1) = 0$ for $t\in\mathbb{R}^+$ and $\partial_t\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}(0,x_3) = 2x_3$ for $x_3\in(-1,+1)$. Due to the corners $(0,\pm1)$ the regularity of $\bar\varphi^{\mathrm{s}}_{\mathrm{Neu}}$ is almost H^3 but not H^3.

Membrane part. The generators $\boldsymbol{\zeta}_*^k$ are solutions of membrane problems

$$L^{\mathrm{m}}\boldsymbol{\zeta}_*^k = \boldsymbol{R}^k_{\mathrm{m}} \quad \text{in } \omega\,, \tag{7}$$

where $L^{\mathrm{m}}\boldsymbol{\zeta}_* := \mu\,\boldsymbol{\Delta}_*\boldsymbol{\zeta}_* + (\tilde\lambda+\mu)\nabla_*\,\mathrm{div}_*\,\boldsymbol{\zeta}_*$ with Neumann boundary conditions. These natural traces, that is the stresses associated to L^{m}, read

$$\begin{array}{rcl} T^{\mathrm{m}}_s(\boldsymbol{\zeta}_*^k) &=& \mu(\partial_s\zeta_n^k + \partial_n\zeta_s^k + 2\kappa\zeta_s^k) \\ T^{\mathrm{m}}_n(\boldsymbol{\zeta}_*^k) &=& \tilde\lambda\,\mathrm{div}_*\,\boldsymbol{\zeta}_*^k + 2\mu\partial_n\zeta_n^k \end{array} \quad \text{on } \partial\omega\,. \tag{8}$$

The analog to Theorem 3.1 holds, guaranteeing the solvability of these limit problems.

The generator $\boldsymbol{\zeta}_*^0$ fulfills problem (7) for $k=0$ with $\boldsymbol{R}_{\rm m}^0 := -\frac{1}{2}([\boldsymbol{f}_*]_0 + \boldsymbol{g}_*^+ - \boldsymbol{g}_*^-)$ and homogeneous Neumann boundary conditions $T_s^{\rm m}(\boldsymbol{\zeta}_*^0) = T_n^{\rm m}(\boldsymbol{\zeta}_*^0) = 0$ on $\partial\omega$. Remarkably, the problem for $\boldsymbol{\zeta}_*^1$ is a completely homogeneous one, such that $\boldsymbol{\zeta}_*^1 \equiv 0$ is valid and the first boundary layer profile is zero, *cf* [7]. This implies in particular that $\boldsymbol{u}_{\rm m}(\varepsilon)$ converges to the usual limit Kirchhoff-Love displacement with improved accuracy:

$$\|\boldsymbol{u}_{\rm m}(\varepsilon) - \boldsymbol{u}_{\rm KL,m}^0\| \le C\,\varepsilon^{3/2} \quad .$$

4 Higher order responses

Since the boundary condition $N_n(\zeta_3^0)$ is not homogeneous and moreover we have to take into account six compatibility conditions to ensure the existence of a solution to the 3D problem at all, we have in contrast to the hard clamped case [2] a priori nine conditions in order to enforce that the 'leading term' in the asymptotic expansion vanishes. By the following Lemma these nine conditions are reduced to three.

LEMMA 4.1. – *The first term* $\boldsymbol{u}_{\rm KL}^0$ *vanishes if and only if*

$$\boldsymbol{R}_{\rm m}^0 = R_{\rm b}^0 = 0 \textit{ in } \omega \quad \textit{and} \quad [f_n]_1 + g_n^+ + g_n^- = 0 \textit{ on } \partial\omega \;. \tag{9}$$

Moreover, the 3D *compatibility conditions* (2) *follow automatically from* (9).

At this stage one could imagine that the expansion 'starts' with $\varepsilon\boldsymbol{u}^1$, since at the first glance it is not obvious at all that the boundary condition $N_n(\zeta_3^1)$ is homogeneous. But it turns out that under the conditions (9), there holds $N_n(\zeta_3^1) = 0$.

LEMMA 4.2. – *If* (9) *holds, then* $\boldsymbol{u}^1$ *vanishes identically as well and the expansion 'starts' with* $\varepsilon^2\boldsymbol{u}^2$.

With the help of the special primitives $\oint^{x_3}$ and $⨍^{y_3}$, see [2, 4], let us define the function $G = G(\boldsymbol{f}, \boldsymbol{g}^\pm)$ in $\mathscr{C}^\infty(\overline{\Omega})^3$ by

$$G_3 = 0 \quad \text{and} \quad G_\alpha(x_*, x_3) = -\frac{1}{\mu}\Big(\oint^{x_3}(⨍^{y_3} f_\alpha)\,dy_3 - x_3\,\frac{g_\alpha^+ + g_\alpha^-}{2}\Big),$$

and the function $H = H(\boldsymbol{f}, \boldsymbol{g}^\pm)$ in $\mathscr{C}^\infty(\overline{\Omega})^3$ by

$$H_3 = -\frac{1}{\lambda+2\mu}\Big(\oint^{x_3}(⨍^{y_3} f_3)\,dy_3 - x_3\,\frac{g_3^+ + g_3^-}{2}\Big) \quad \text{and}$$
$$H_\alpha = -\oint^{x_3}\Big(\partial_\alpha H_3 + \frac{1}{\mu}\,y_3\,F_*(G) + \frac{\lambda}{\mu}⨍^{y_3}\Big\{\partial_{\alpha 3}H_3 - \frac{1}{2}\int_{-1}^{+1}\partial_{\alpha 3}H_3\,dz_3\Big\}\Big)dy_3\,,$$

which will be of use in the following. See [2, 4] for the definition of F_*.

THEOREM 4.3. – *Let* $\boldsymbol{f}$, $\boldsymbol{g}^\pm$ *be such that* (9) *holds and that either* $\boldsymbol{R}_{\rm m}^2$, $R_{\rm b}^2$, G, $T_s^{\rm m}(\boldsymbol{\zeta}_*^2)$, $T_n^{\rm m}(\boldsymbol{\zeta}_*^2)$, $M_n(\zeta_3^2)$ *or* $N_n(\zeta_3^2)$ *is not identically zero. Then*
(i) $\boldsymbol{u}(\varepsilon)$ *starts with* $\varepsilon^2\boldsymbol{u}^2$ *and there holds the estimate* $\|\varepsilon^{-2}\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^2\| \le C\sqrt{\varepsilon}$.
(ii) $\boldsymbol{u}^2 = \boldsymbol{u}_{\rm KL}^2 + G + \chi\boldsymbol{w}^2$ *with the generating function* $\boldsymbol{\zeta}^2$ *solving* (4) *and* (7) *for* $k=2$ *with boundary conditions* $T_s^{\rm m}(\boldsymbol{\zeta}_*^2) = d_2(s)$, $T_n^{\rm m}(\boldsymbol{\zeta}_*^2) = d_3(s)$, $M_n(\zeta_3^2) = d_4(s)$ *and* $N_n(\zeta_3^2) = d_5(s)$ *on* $\partial\omega$, *with functions* $d_j \in \mathscr{C}^\infty(\partial\omega)$, $j = 2, \dots 5$, *depending on* G, H, λ *and* μ. *Generically,* $\boldsymbol{w}^2$ *is non-trivial. The expressions for the higher order* $\boldsymbol{R}_{\rm m}^k$ *and* $R_{\rm b}^k$ *are given in* [2].

In the next Lemma we deal with the question under which conditions the first three terms in the expansion vanish.

Lemma 4.4. – *The first three terms* $\boldsymbol{u}^k$, $k=0,1,2$, *in the expansion vanish if and only if the generators* $\boldsymbol{\zeta}^k$, $k=0,1,2$, *and additionally* G *are zero. This is exactly the case if there holds*

$$(\boldsymbol{f}_*, \boldsymbol{g}_*^\pm) = 0 \quad \textit{and} \quad [f_3]_0 + g_3^+ - g_3^- = [f_3]_1 + g_3^+ + g_3^- = [f_3]_2 - [f_3]_0 = 0. \qquad (10)$$

Compared to the hard clamped situation [2], condition (10) is stronger. Let us recall that for the hard clamped plate we obtained the conditions

$$(\boldsymbol{f}_*,\ \boldsymbol{g}_*^\pm) = 0 \text{ and } [f_3]_0 + g_3^+ - g_3^- = \nabla_*([f_3]_1 + g_3^+ + g_3^-) = \Delta_*([f_3]_2 - [f_3]_0) = 0. \quad (11)$$

We can see that (10) holds if (11) holds and the following traces are zero:

$$[f_3]_1 + g_3^+ + g_3^- = [f_3]_2 - [f_3]_0 = 0 \quad \text{on} \quad \partial\omega.$$

Note that the first one of these conditions is of membrane type and the second one of bending type.

Theorem 4.5. – *Let* $\boldsymbol{f}$, $\boldsymbol{g}^\pm$ *be such that* (10) *holds, but* $H \not\equiv 0$.
a) If $\partial_3 H_3 \neq 0$ *on* $\partial\omega \times (-1,+1)$, *then* $\boldsymbol{u}(\varepsilon)$ *starts with* $\varepsilon^3\boldsymbol{u}^3$ *and the estimate* $\|\varepsilon^{-3}\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^3\| \leq C\sqrt{\varepsilon}$ *holds. We have* $\boldsymbol{u}^3 = \boldsymbol{u}^3_{\mathrm{KL}} + \chi\boldsymbol{w}^3$ *and* $\boldsymbol{w}^3$ *is generically non-trivial.*

b) If $\partial_3 H_3 = 0$ *on* $\partial\omega \times (-1,+1)$, *then* $\boldsymbol{u}(\varepsilon)$ *starts with* $\varepsilon^4\boldsymbol{u}^4$ *and the estimate* $\|\varepsilon^{-4}\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^4\| \leq C\sqrt{\varepsilon}$ *holds.* $\boldsymbol{u}^4 = \boldsymbol{u}^4_{\mathrm{KL}} + H + \chi\boldsymbol{w}^4$ *is valid with generically non-trivial* $\boldsymbol{w}^4$.

With [2, Lemma 6.3] it is impossible that the expansion starts with $\varepsilon^5\boldsymbol{u}^5$. Thus all non-trivial cases of loads (of the class introduced in §2) are found, such that the generic leading term in the asymptotic expansion vanishes.

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