Eigenmode Asymptotics in Thin Elastic Plates

M. Dauge, I. Djurdjevic, E. Faou, A. Rössle

Abstract. *In this paper, we investigate the behavior of the vibration modes (eigenvalues) of an isotropic homogeneous plate as its thickness tends to zero. As lateral boundary conditions, we consider clamped or free edge. We prove distinct asymptotics for bending and membrane modes: the smallest bending eigenvalues behave as the square of the thickness whereas the membrane eigenvalues tend to non-zero limits. Moreover, we prove that all these eigenvalues have an expansion in power series with respect to the thickness regardless of their multiplicities or of the multiplicities of the limit in-plane problems.*

Introduction

Our aim is the investigation of modal analysis in thin plates as the thickness parameter ε goes to zero: We consider a family of plates Ω^ε with fixed mean surface ω indexed by their (half-)thickness ε

$$\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon),$$

and study the *eigenmodes* of the plate Ω^ε, that is the eigenvalues Λ^ε and the corresponding eigenvectors $\boldsymbol{u}^\varepsilon$ of the linearized elasticity operator associated with the constitutive material of the plates.

As usual in such a framework, we suppose that the plates are free on their lower and upper faces $\omega \times \{\pm\varepsilon\}$. As conditions on the lateral edge $\partial\omega \times (-\varepsilon, \varepsilon)$, we take into consideration as representative cases of the possible boundary conditions, compare [6], the hard clamped case and the free edge case. These boundary conditions determine admissible spaces of displacements $\mathbf{V}(\Omega^\varepsilon)$. We thus obtain the eigenvalue problems associated with the stress-strain bilinear form $a^\varepsilon(\boldsymbol{u}, \boldsymbol{v}) = \langle \sigma(\boldsymbol{u}) : e(\boldsymbol{v}) \rangle_{\Omega^\varepsilon}$ in the spaces $\mathbf{V}(\Omega^\varepsilon)$:

$$\textit{Find } \Lambda^\varepsilon \textit{ and non-zero } \boldsymbol{u}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon), \quad \forall \boldsymbol{v} \in \mathbf{V}(\Omega^\varepsilon), \quad a^\varepsilon(\boldsymbol{u}, \boldsymbol{v}) = \Lambda^\varepsilon \langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\Omega^\varepsilon},$$

where $\langle \cdot, \cdot \rangle_{\Omega^\varepsilon}$ denotes the usual L^2 scalar product in Ω^ε.

Thanks to the Korn inequality *cf* [8], the form a^ε is positive symmetric with compact resolvent. Thus its spectrum is discrete with only accumulation point at infinity and can be ordered (with the usual repetition convention according to the multiplicity)

$$0 \le \Lambda_1^\varepsilon \le \Lambda_2^\varepsilon \cdots \le \Lambda_\ell^\varepsilon \le \ldots, \quad \lim_{\ell\to\infty} \Lambda_\ell^\varepsilon = +\infty.$$

In [3], CIARLET & KESAVAN study the case of hard clamped *isotropic* plates. Their result shows up the bending dominated behavior of plates at the lowest frequencies. If λ and μ are the Lamé coefficients of the plate material, the associated two-dimensional *bending* operator L^{b} is the biharmonic operator in ω

$$L^{\mathrm{b}} = (\tilde\lambda + 2\mu)\Delta^2, \tag{0.1}$$

with the homogenized Lamé constant $\tilde\lambda$ defined as

$$\tilde\lambda = \frac{2\lambda\mu}{\lambda+2\mu}. \tag{0.2}$$

The result in [3] is that each Λ_ℓ^ε tends to $\frac{\varepsilon^2}{3}\varrho_{\mathrm{b},\ell}$, with $\varrho_{\mathrm{b},\ell}$ the eigenvalue of corresponding rank of the Dirichlet problem for the bending operator L^{b} and that the eigenvectors tend to the Kirchhoff-Love displacement generated by an associated eigenvector of L^{b} (after possible extraction of a subsequence in the case of a multiple eigenvalue).

In [14], NAZAROV extends this result to plates with much more general material law and moreover shows the influence on the three-dimensional spectrum of the associated in-plane membrane operator L^{m} which generates $O(1)$ families of eigenvalues, in contrast to the $O(\varepsilon^2)$ bending family : In the case of an isotropic material with Lamé coefficients λ and μ, L^{m} is the bi-dimensional Lamé operator associated with the Lamé coefficients $\tilde\lambda$ and μ, that is

$$L^{\mathrm{m}} = \mu \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} + (\tilde\lambda + \mu) \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix} \operatorname{div}. \tag{0.3}$$

The modal analysis in [14], and also in [17, 16] where a two-terms asymptotics is constructed, requires an asymptotic analysis of the eigendisplacements, which has to take into account the boundary layer in the neighborhood of the lateral boundary. The assumption (also made in [17, 16, 14]) that the boundary $\partial\omega$ of the mean surface is smooth makes such an analysis easier: If the mean surface is polygonal, special corner layers appear, see [15]. Thus, in order to simplify our analysis (which is also based on asymptotic expansions), we assume that ω *is a smooth domain.*

Moreover, we choose to work with the assumption that the plates are made of a *homogeneous and isotropic material*. This assumption has an important consequence: It allows the splitting of the three-dimensional spectrum in a bending spectrum and a membrane spectrum, in correspondence with the two-dimensional bending and membrane operators. We note that such a splitting is still possible for any *monoclinic* material with rigidity matrix constant along the transverse fibers (this is the framework of [4, 5] where the asymptotic expansion of displacements is proved for clamped plates). Therefore, our present analysis, at least for laterally clamped plates, extends in a natural way to such materials.

To summarize, in this paper we propose a further investigation of eigenmodes in two directions:

(i) Take advantage of the transverse symmetry of plate problems which enable us to split eigenmodes in *bending* and *membrane* eigenmodes $(\Lambda^\varepsilon_{\rm b}, \boldsymbol{u}^\varepsilon_{\rm b})$ and $(\Lambda^\varepsilon_{\rm m}, \boldsymbol{u}^\varepsilon_{\rm m})$.

(ii) Adapt the idea of combined outer and inner expansions to construct asymptotic expansions at any order for bending and membrane eigenmodes.

The main outcome of our study is that the ℓ-th bending eigenvalue of a^ε has a *power series expansion* starting with $\frac{\varepsilon^2}{3}\varrho_{{\rm b},\ell}$ and that the ℓ-th membrane eigenvalue of a^ε has a *power series expansion* starting with the ℓ-th eigenvalue $\varrho_{{\rm m},\ell}$ of the associated in-plane membrane operator $-L^{\rm m}$. These power series expansions do not converge in general.

We emphasize that we prove this result even in the case when the limit eigenvalues are *multiple*: Then it may happen that the corresponding three-dimensional eigenvalues are multiple too, or that they have the same asymptotic expansion but nevertheless differ with each other, or that they have distinct expansions with the same first term.

Our result inspires the following comments:

(i) The limits of the eigenvalues of a^ε are the eigenvalues of the operator

$$K(\varepsilon) := \begin{pmatrix} -L^{\rm m} & 0 \\ 0 & \frac{\varepsilon^2}{3}L^{\rm b} \end{pmatrix} \quad \text{on } \omega\,.$$

This operator is the exact counterpart for plates of the Koiter operator for shells.

(ii) If one considers the eigenvalues Λ^ε_ℓ arranged in non-decreasing order, as is noticed in [2] one sees in the limit only the bending eigenvalues.

(iii) The eigenvalues of the "Koiter" operator $K(\varepsilon)$ do not give a full description of the spectrum of the three-dimensional operator on Ω^ε: In the limit as $\varepsilon \to 0$, most of the three-dimensional eigenvalues go to infinity. The question of organizing them in coherent families behaving for example in $O(\varepsilon^{-2})$ is still open. The authors are glad to acknowledge discussions with Sergei NAZAROV who indicated earlier formal attempts by BERDICHEVSKII [1], see the comments in [14].

Our paper is organized as follows: We introduce in section 1 the different eigenvalue problems in the thin plates Ω^ε, in the scaled plate $\Omega = \omega \times (-1, 1)$ and in their mean surface ω. In section 2, before stating our results concerning the *limits* of the three-dimensional eigenmodes with optimal estimates on their convergence, we recall the notion of quasimode and the classical related results about the spectrum approximation: Thus, as $\varepsilon \to 0$, when a limit two-dimensional eigenvalue is *multiple*, the space of Kirchhoff-Love displacements generated by the corresponding eigenspace is the limit of a *cluster* of three-dimensional eigenspaces.

Sections 3 to 6 are devoted to the construction of three-dimensional quasimodes at any order $\mathcal{O}(\varepsilon^k)$, based on two-dimensional problems, whereas section 7 yields weak convergence results about the three-dimensional problems, in the spirit of [3]. Combining the results of the previous sections, we obtain in section 8 the complete eigenmode asymptotics.

The most original aspects of our approach are the following:

(i) The use of multiple formal series operations in order to transform the initial three-dimensional eigenvalue problem into a two-dimensional one for a new formal operator series, *cf* sections 3 and 4.

(ii) The solution of this eigenvalue problem for the two-dimensional formal operator series by a sequence of nested spectral problems for finite-dimensional self-adjoint operators, *cf* section 5.

1 Eigenmodes for plate models in two or three dimensions

We state the three-dimensional eigenmode problem, split it into membrane and bending problems, and scale it to the fixed reference domain $\Omega = \omega \times (-1, 1)$. Next we introduce the two-dimensional membrane and bending operators as they appear in the limit as $\varepsilon \to 0$ of displacements in thin plates, see [2], [6]: These operators are the models determining the two-dimensional generators of the limit Kirchhoff-Love displacements.

1.a Linearized elasticity

Let us recall that λ and μ are the Lamé constants of the constitutive material of our plates Ω^ε. To each displacement field $\boldsymbol{u} = (u_1, u_2, u_3)$ is associated the linearized strain tensor $e_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. Hooke's law yields the stress tensor

$$\sigma(\boldsymbol{u}) = A\, e(\boldsymbol{u}),$$

where the rigidity matrix $A = (A_{ijkl})$ of the material is given by:

$$A_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

As usual, Latin indices are always taken from $\{1,\, 2,\, 3\}$ while the Greek ones α, β vary in $\{1,\, 2\}$ and we use summation convention. As lateral boundary conditions, we will consider the hard clamped condition and the free boundary condition, although other combinations are possible, *cf* [6].

To hard clamped plates is associated the space of variations $\mathbf{V}_{\mathrm{D}}(\Omega^\varepsilon)$ given by

$$\mathbf{V}_{\mathrm{D}}(\Omega^\varepsilon) := \big\{\boldsymbol{v} \in H^1(\Omega^\varepsilon)^3 \,|\, \boldsymbol{v} = 0 \text{ on } \Gamma_0^\varepsilon := \partial\omega \times (-\varepsilon,\, \varepsilon)\big\}$$

whereas for free plates the space of variations is $\mathbf{V}_{\mathrm{N}}(\Omega^\varepsilon) = H^1(\Omega^\varepsilon)^3$. We agree to denote by $\mathbf{V}(\Omega^\varepsilon)$, either $\mathbf{V}_{\mathrm{D}}(\Omega^\varepsilon)$ or $\mathbf{V}_{\mathrm{N}}(\Omega^\varepsilon)$.

When necessary, we particularize the objects attached to Ω^ε by a $\tilde{}$, e.g. the variables in Ω^ε are denoted by $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \tilde{x}$ with $\tilde{x}_3$ the transverse variable in $(-\varepsilon, \varepsilon)$, and the eigenvectors are denoted by $\tilde{\boldsymbol{u}}^\varepsilon$.

Then the variational formulations of the eigenvalue problems read: Find $\Lambda^\varepsilon \in \mathbb{R}$ and non-zero $\tilde{\boldsymbol{u}}^\varepsilon$ in $\mathbf{V}(\Omega^\varepsilon)$ such that

$$a^\varepsilon(\tilde{\boldsymbol{u}}^\varepsilon, \tilde{\boldsymbol{v}}) = \Lambda^\varepsilon(\tilde{\boldsymbol{u}}^\varepsilon,\, \tilde{\boldsymbol{v}}), \qquad \forall\, \tilde{\boldsymbol{v}} \in \mathbf{V}(\Omega^\varepsilon), \tag{1.1}$$

where a^ε and $(\cdot\,,\,\cdot)$ are the bilinear forms

$$\begin{aligned} a^\varepsilon(\boldsymbol{u},\,\boldsymbol{v}) &:= \int_{\Omega^\varepsilon} \big\{\lambda e_{pp}(\boldsymbol{u}) e_{qq}(\boldsymbol{v}) + 2\mu e_{ij}(\boldsymbol{u}) e_{ij}(\boldsymbol{v})\big\}\,\mathrm{d}\tilde{x} \\ (\boldsymbol{u},\,\boldsymbol{v}) &:= \int_{\Omega^\varepsilon} u_i\, v_i\,\mathrm{d}\tilde{x}. \end{aligned}$$

Korn inequality and the compactness of the embedding $\mathbf{V}(\Omega^\varepsilon) \hookrightarrow L^2(\Omega^\varepsilon)^3$ yield that the eigenvalues Λ^ε are nonnegative and form a discrete set in $\mathbb{R}$ with only accumulation point at infinity, see [12, 7]. Moreover there exists an associated sequence of eigenfunctions which forms an orthogonal basis in both Hilbert spaces $\big(\mathbf{V}(\Omega^\varepsilon),\,(a^\varepsilon+1)^{1/2}\big)$ and $L^2(\Omega^\varepsilon)^3$. In particular for the hard clamped situation the first eigenvalue is positive. For the case of a free plate $\Lambda_1^\varepsilon = 0$ is a six-fold eigenvalue with eigenspace spanned by the rigid motions

$$\begin{pmatrix}1\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\1\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\1\end{pmatrix},\ \begin{pmatrix}-\tilde{x}_3\\0\\\tilde{x}_1\end{pmatrix},\ \begin{pmatrix}0\\-\tilde{x}_3\\\tilde{x}_2\end{pmatrix},\ \begin{pmatrix}\tilde{x}_2\\-\tilde{x}_1\\0\end{pmatrix}. \tag{1.2}$$

1.b Three-dimensional membrane and bending modes

Let S be the transverse symmetry operator defined for $\boldsymbol{u} = (u_1, u_2, u_3)$ in $L^2(\Omega^\varepsilon)$ by:

$$S : \Big(\tilde{x}_3 \mapsto \big(\boldsymbol{u}_*(\tilde{x}_3),\, u_3(\tilde{x}_3)\big)\Big) \quad\longmapsto\quad \Big(\tilde{x}_3 \mapsto \big(\boldsymbol{u}_*(-\tilde{x}_3),\, -u_3(-\tilde{x}_3)\big)\Big), \tag{1.3}$$

where $\boldsymbol{u}_* = (u_1, u_2)$. The bilinear form a^ε is invariant by S, that is

$$\forall \boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}(\Omega^\varepsilon), \qquad \begin{cases} a^\varepsilon(S\boldsymbol{u}, \boldsymbol{v}) = a^\varepsilon(\boldsymbol{u}, S\boldsymbol{v}) \\ a^\varepsilon(S\boldsymbol{u}, S\boldsymbol{v}) = a^\varepsilon(\boldsymbol{u}, \boldsymbol{v}), \end{cases} \tag{1.4}$$

and the same holds for the L^2 scalar product in Ω^ε. From the fact that $S^2 = I$ we conclude that $\mathbf{V}(\Omega^\varepsilon)$ splits into two invariant subspaces, see [12]. So, we have

$$\mathbf{V}(\Omega^\varepsilon) = \mathbf{V}^{\mathrm{m}}(\Omega^\varepsilon) \oplus \mathbf{V}^{\mathrm{b}}(\Omega^\varepsilon)\,, \tag{1.5}$$

where $S\boldsymbol{u} = \boldsymbol{u}$ for all $\boldsymbol{u} \in \mathbf{V}^{\mathrm{m}}(\Omega^\varepsilon)$ and $S\boldsymbol{u} = -\boldsymbol{u}$ for all $\boldsymbol{u} \in \mathbf{V}^{\mathrm{b}}(\Omega^\varepsilon)$. We refer to $\mathbf{V}^{\mathrm{m}}(\Omega^\varepsilon)$ (resp. $\mathbf{V}^{\mathrm{b}}(\Omega^\varepsilon)$) as the *membrane* (resp. *bending*) space, compare [9].

As a consequence of (1.4), the decomposition

$$\boldsymbol{u}_{\mathrm{m}} = \tfrac{1}{2}(I+S)\boldsymbol{u} \quad \text{and} \quad \boldsymbol{u}_{\mathrm{b}} = \tfrac{1}{2}(I-S)\boldsymbol{u} \tag{1.6}$$

is orthogonal with respect to both scalar products a^ε and $(\cdot\,,\,\cdot)$. Thus solutions of (1.1) split in membrane and bending eigenmodes, respectively solutions of

$$\begin{aligned} &a^\varepsilon(\tilde{\boldsymbol{u}}^\varepsilon_{\mathrm{m}},\,\tilde{\boldsymbol{v}}_{\mathrm{m}}) = \Lambda^\varepsilon_{\mathrm{m}}(\tilde{\boldsymbol{u}}^\varepsilon_{\mathrm{m}},\,\tilde{\boldsymbol{v}}_{\mathrm{m}}), \qquad &\forall\, \tilde{\boldsymbol{v}}_{\mathrm{m}} \in \mathbf{V}^{\mathrm{m}}(\Omega^\varepsilon), \\ &a^\varepsilon(\tilde{\boldsymbol{u}}^\varepsilon_{\mathrm{b}},\,\tilde{\boldsymbol{v}}_{\mathrm{b}}) = \Lambda^\varepsilon_{\mathrm{b}}(\tilde{\boldsymbol{u}}^\varepsilon_{\mathrm{b}},\,\tilde{\boldsymbol{v}}_{\mathrm{b}}), \qquad &\forall\, \tilde{\boldsymbol{v}}_{\mathrm{b}} \in \mathbf{V}^{\mathrm{b}}(\Omega^\varepsilon). \end{aligned}$$

Let
$$0 \leq \Lambda^{\varepsilon}_{\mathrm{m},1} \leq \Lambda^{\varepsilon}_{\mathrm{m},2} \cdots \leq \Lambda^{\varepsilon}_{\mathrm{m},\ell} \leq \ldots \tag{1.7a}$$
be the membrane eigenvalues and
$$0 \leq \Lambda^{\varepsilon}_{\mathrm{b},1} \leq \Lambda^{\varepsilon}_{\mathrm{b},2} \cdots \leq \Lambda^{\varepsilon}_{\mathrm{b},\ell} \leq \ldots \tag{1.7b}$$
be the bending eigenvalues and let
$$\widetilde{E}^{\varepsilon}_{\mathrm{m},\ell} \qquad \text{and} \qquad \widetilde{E}^{\varepsilon}_{\mathrm{b},\ell} \tag{1.8}$$
be the corresponding eigenspaces with the convention that if $\Lambda_\ell = \cdots = \Lambda_{\ell+\nu-1}$ has the multiplicity ν, we have $\widetilde{E}_\ell = \cdots = \widetilde{E}_{\ell+\nu-1}$, defining a space of dimension ν. We denote by $\mathcal{L}^{\varepsilon}_{\mathrm{m}}$ and $\mathcal{L}^{\varepsilon}_{\mathrm{b}}$ the set of such indices ℓ, so that we have
$$\bigoplus_{\ell \,\in\, \mathcal{L}^{\varepsilon}_{\mathrm{m}}} \widetilde{E}^{\varepsilon}_{\mathrm{m},\ell} = \mathbf{V}^{\mathrm{m}}(\Omega^{\varepsilon}) \qquad \text{and} \qquad \bigoplus_{\ell \,\in\, \mathcal{L}^{\varepsilon}_{\mathrm{b}}} \widetilde{E}^{\varepsilon}_{\mathrm{b},\ell} = \mathbf{V}^{\mathrm{b}}(\Omega^{\varepsilon}). \tag{1.9}$$

1.c Scalings

In order to study the behavior of eigenmodes as $\varepsilon \to 0$ we introduce a fixed reference configuration Ω as the image of a dilatation along the vertical axis $x_3 = \varepsilon^{-1}\,\tilde{x}_3$, whereas the in-plane variables $(\tilde{x}_1, \tilde{x}_2) = (x_1, x_2)$ are unchanged. Thus we have $\Omega = \omega \times (-1,\, 1)$. For the displacement fields we use the scaling preserving the elastic structure, see [2]
$$u^{\varepsilon}_{\alpha}(x) = \tilde{u}^{\varepsilon}_{\alpha}(\tilde{x}),\ \alpha = 1,2, \qquad u^{\varepsilon}_{3}(x) = \varepsilon\, \tilde{u}^{\varepsilon}_{3}(\tilde{x}). \tag{1.10}$$

The eigenvalue problems then take the form: Find $\Lambda^{\varepsilon} \in \mathbb{R}$ and non-zero $\boldsymbol{u}^{\varepsilon}$ in $\mathbf{V}(\Omega)$ such that
$$a(\varepsilon)(\boldsymbol{u}^{\varepsilon}, \boldsymbol{v}) = \Lambda^{\varepsilon} \big\langle \boldsymbol{u}^{\varepsilon}, \boldsymbol{v} \big\rangle_{\varepsilon}, \qquad \forall\, \boldsymbol{v} \in \mathbf{V}(\Omega), \tag{1.11}$$
where $\mathbf{V}(\Omega)$ is the space correponding to $\mathbf{V}(\Omega^{\varepsilon})$, and $a(\varepsilon)$ and $\langle \cdot\,,\, \cdot \rangle_{\varepsilon}$ are the bilinear forms
$$\begin{aligned} a(\varepsilon)(\boldsymbol{u},\, \boldsymbol{v}) \ &:= \ \int_{\Omega} \big\{ \lambda\, \kappa_{pp}(\varepsilon)(\boldsymbol{u})\, \kappa_{qq}(\varepsilon)(\boldsymbol{v}) + 2\mu\, \kappa_{ij}(\varepsilon)(\boldsymbol{u})\, \kappa_{ij}(\varepsilon)(\boldsymbol{v}) \big\}\, \mathrm{d}x \\ \big\langle \boldsymbol{u},\, \boldsymbol{v} \big\rangle_{\varepsilon} \ &:= \ \int_{\Omega} u_\alpha v_\alpha + \varepsilon^{-2}\, u_3 v_3\, \mathrm{d}x \end{aligned}$$
with the scaled strain tensor $\kappa(\varepsilon)$
$$\kappa_{\alpha\beta}(\varepsilon)(\boldsymbol{u}) = e_{\alpha\beta}(\boldsymbol{u}), \quad \kappa_{\alpha 3}(\varepsilon)(\boldsymbol{u}) = \varepsilon^{-1}\, e_{\alpha 3}(\boldsymbol{u}), \quad \kappa_{33}(\varepsilon)(\boldsymbol{u}) = \varepsilon^{-2}\, e_{33}(\boldsymbol{u}).$$

It is then straightforward that $(\Lambda, \tilde{\boldsymbol{u}}^{\varepsilon})$ solves (1.1) if and only if $(\Lambda, \boldsymbol{u}^{\varepsilon})$ solves (1.11). Of course the splitting membrane–bending is still valid. We do not need to change the notations (1.7) for the eigenvalues and we only introduce the spaces of scaled eigenmodes corresponding to (1.8)
$$E^{\varepsilon}_{\mathrm{m},\ell} \qquad \text{and} \qquad E^{\varepsilon}_{\mathrm{b},\ell}. \tag{1.12}$$

1.d Two-dimensional membrane and bending operators

The bilinear form associated with the two-dimensional membrane operator $-L^{\mathrm{m}}$ (0.3) is defined for $\boldsymbol{\zeta} = (\zeta_1, \zeta_2)$ and $\boldsymbol{\zeta}' = (\zeta_1', \zeta_2')$ in $H^1(\omega)^2$ by:

$$a^{\mathrm{m}}(\boldsymbol{\zeta}, \boldsymbol{\zeta}') = \int_\omega \big\{ \tilde{\lambda}\, e_{\alpha\alpha}(\boldsymbol{\zeta})\; e_{\beta\beta}(\boldsymbol{\zeta}') + 2\mu\, e_{\alpha\beta}(\boldsymbol{\zeta})\; e_{\alpha\beta}(\boldsymbol{\zeta}') \big\}\, \mathrm{d}x_1 \mathrm{d}x_2, \tag{1.13}$$

where we recall that the "homogenized" Lamé coefficient $\tilde{\lambda}$ is equal to $2\lambda\mu(\lambda+2\mu)^{-1}$. The variational spaces $\mathbf{V}^{\mathrm{m}}_{\mathrm{D}}(\omega)$ and $\mathbf{V}^{\mathrm{m}}_{\mathrm{N}}(\omega)$ respectively associated with the clamped and free boundary conditions are $H^1_0(\omega)^2$ and $H^1(\omega)^2$. The eigenvalue problem for $-L^{\mathrm{m}}$ reads: Find $\varrho_{\mathrm{m}} \in \mathbb{R}$ and non-zero $\boldsymbol{\zeta} \in \mathbf{V}^{\mathrm{m}}(\omega)$ such that

$$a^{\mathrm{m}}(\boldsymbol{\zeta}, \boldsymbol{\zeta}') = \varrho_{\mathrm{m}} \int_\omega \zeta_\alpha\, \zeta_\alpha'\, \mathrm{d}x_1 \mathrm{d}x_2, \qquad \forall\, \boldsymbol{\zeta}' \in \mathbf{V}^{\mathrm{m}}(\omega). \tag{1.14}$$

The bilinear form associated with the two-dimensional bending operator L^{b} is defined for η and η' in $H^2(\omega)$ by:

$$a^{\mathrm{b}}(\eta, \eta') = \int_\omega \big\{ \tilde{\lambda}\, \partial_{\alpha\alpha}(\eta)\; \partial_{\beta\beta}(\eta') + 2\mu\, \partial_{\alpha\beta}(\eta)\; \partial_{\alpha\beta}(\eta') \big\}\, \mathrm{d}x_1 \mathrm{d}x_2. \tag{1.15}$$

The variational spaces $\mathbf{V}^{\mathrm{b}}_{\mathrm{D}}(\omega)$ and $\mathbf{V}^{\mathrm{b}}_{\mathrm{N}}(\omega)$ respectively associated with the clamped and free boundary conditions are $H^2_0(\omega)$ and $H^2(\omega)$. The eigenvalue problem for L^{b} reads: Find $\varrho_{\mathrm{b}} \in \mathbb{R}$ and non-zero $\eta \in \mathbf{V}^{\mathrm{b}}(\omega)$ such that

$$a^{\mathrm{b}}(\eta, \eta') = \varrho_{\mathrm{b}} \int_\omega \eta\, \eta'\, \mathrm{d}x_1 \mathrm{d}x_2, \qquad \forall\, \eta' \in \mathbf{V}^{\mathrm{b}}(\omega). \tag{1.16}$$

Both forms a^{m} and a^{b} are nonnegative and symmetric on their variational spaces, compactly embedded in $L^2(\omega)^2$ in the membrane and in $L^2(\omega)$ in the bending case, respectively. Thus the eigenvalues in problems (1.14) and (1.16) are nonnegative real numbers:

Let

$$0 \le \varrho_{\mathrm{m},1} \le \varrho_{\mathrm{m},2} \cdots \le \varrho_{\mathrm{m},\ell} \le \ldots \tag{1.17a}$$

be the two-dimensional membrane eigenvalues and

$$0 \le \varrho_{\mathrm{b},1} \le \varrho_{\mathrm{b},2} \cdots \le \varrho_{\mathrm{b},\ell} \le \ldots \tag{1.17b}$$

be the two-dimensional bending eigenvalues. In order to take into account the multiple eigenvalues, we define the sets of indices $\mathcal{L}_{\mathrm{m}}$ and $\mathcal{L}_{\mathrm{b}}$ similarly as above and we let

$$\forall \ell \in \mathcal{L}_{\mathrm{m}}, \quad F_{\mathrm{m},\ell} \qquad \text{and} \qquad \forall \ell \in \mathcal{L}_{\mathrm{b}}, \quad F_{\mathrm{b},\ell} \tag{1.18}$$

be the corresponding eigenspaces. Thus there holds

$$\bigoplus_{\ell \in \mathcal{L}_{\mathrm{m}}} F_{\mathrm{m},\ell} = \mathbf{V}^{\mathrm{m}}(\omega) \qquad \text{and} \qquad \bigoplus_{\ell \in \mathcal{L}_{\mathrm{b}}} F_{\mathrm{b},\ell} = \mathbf{V}^{\mathrm{b}}(\omega). \tag{1.19}$$

To these eigenspaces we associate the spaces of corresponding Kirchhoff-Love displacements:

$$K_{\mathrm{m},\ell} = \big\{ \boldsymbol{u} = (\boldsymbol{\zeta}, 0)\;; \quad \boldsymbol{\zeta} \in F_{\mathrm{m},\ell} \big\}, \tag{1.20a}$$

$$K_{\mathrm{b},\ell} = \big\{ \boldsymbol{u} = (-x_3\partial_1\eta, -x_3\partial_2\eta, \eta)\;; \quad \eta \in F_{\mathrm{b},\ell} \big\}. \tag{1.20b}$$

2 Main results

2.a Quasimodes

Our result relies on an asymptotic expansion of displacements (including power series and boundary layer Ansätze as in [13, 17] and [6]) which will provide the construction of h-*quasimodes* with $h = \mathcal{O}(\varepsilon^k)$ for any integer k, where we define:

Definition 2.1 Let A be an unbounded self-adjoint operator on a Hilbert space H with domain $D(A)$. For a fixed $h > 0$, a pair $(\Lambda, \boldsymbol{u}) \in \mathbb{R} \times D(A) \setminus \{0\}$ is called a h-*quasimode* of A if there holds

$$\|(A - \Lambda)\boldsymbol{u}\|_H \le h\|\boldsymbol{u}\|_H . \qquad \blacksquare$$

The interest of such a definition relies on the following fact: if $(\Lambda, \boldsymbol{u})$ is a h-quasimode of A, then the distance from Λ to the spectrum of A is less than h, and the distance between $\boldsymbol{u}$ and certain eigenspaces of A can be estimated, *cf* Lemma 2.2. Thus, we will construct $\mathcal{O}(\varepsilon^k)$-quasimodes for the problem on Ω^ε from the eigenmodes of L^{b} and $-L^{\mathrm{m}}$ on the midsurface ω.

In order to state our convergence results and to explain our strategy of proof, we need to recall the notion of "distance" between two subspaces E and F of the same Hilbert space H, *cf* [11] p. 264:

$$\delta(E, F) = \max_{\boldsymbol{u}\in E} \min_{\boldsymbol{v}\in F} \frac{\|\boldsymbol{u}-\boldsymbol{v}\|_H}{\|\boldsymbol{u}\|_H} . \tag{2.1}$$

In general this distance is not symmetric. However, if E and F are finite dimensional subspaces satisfying $\delta(E, F) < 1$ and $\delta(F, E) < 1$, then $\dim E = \dim F$ and $\delta(E, F) = \delta(F, E)$, see KATO [11, Lemma 2.2.1]. In this situation $\delta(E, F)$ is called the *gap* between E and F. Using Definition 2.1 of quasimodes, we can state, see VISHIK & LYUSTERNIK [18, Lemmas 12 & 13, §9]:

Lemma 2.2 *Let A be an unbounded self-adjoint operator on a Hilbert space H with domain $D(A)$ compactly embedded in H. Let $(\Lambda_k)_{k\in\mathbb{N}}$ be the distinct eigenvalues of A and let E_k be the corresponding eigenspaces. Let $h > 0$. If $(\Lambda, \boldsymbol{u})$ is a h*-quasimode *of A, then there holds*

$$\operatorname{dist}\big(\Lambda,\ (\Lambda_k)_{k\in\mathbb{N}}\big) \le h. \tag{2.2}$$

Let K be the (non-empty) set $\{k \in \mathbb{N}\ ;\ |\Lambda - \Lambda_k| \le h\}$. Let M be defined as the minimum $\min_{k\notin K} |\Lambda - \Lambda_k|$. Then there holds

$$\delta\Big(\operatorname{span}\{\boldsymbol{u}\}, \bigoplus_{k\,\in\, K} E_k\Big) \ \le\ \frac{h}{M} . \tag{2.3}$$

2.b Convergence of eigenmodes

We prove in this paper that the three-dimensional eigenvalues $\Lambda^\varepsilon_{\mathrm{m},\ell}$ and $\Lambda^\varepsilon_{\mathrm{b},\ell}$ have an infinite power series expansion, and in particular in the limit:

Theorem 2.3 *As $\varepsilon \to 0$, for any $\ell \geq 1$, $\Lambda^\varepsilon_{\mathrm{m},\ell}$ tends to $\varrho_{\mathrm{m},\ell}$ and there holds the estimate*

$$|\Lambda^\varepsilon_{\mathrm{m},\ell} - \varrho_{\mathrm{m},\ell}| \leq C_\ell\, \varepsilon. \tag{2.4a}$$

As $\varepsilon \to 0$, for any $\ell \geq 1$, $\Lambda^\varepsilon_{\mathrm{b},\ell}$ tends to $\frac{\varepsilon^2}{3}\varrho_{\mathrm{b},\ell}$ and there holds the estimate

$$|\Lambda^\varepsilon_{\mathrm{b},\ell} - \frac{\varepsilon^2}{3}\,\varrho_{\mathrm{b},\ell}| \leq C_\ell\, \varepsilon^3. \tag{2.4b}$$

If $\varrho_{\mathrm{m},\ell}$ is a simple eigenvalue, then the distance $\delta(E^\varepsilon_{\mathrm{m},\ell}, K_{\mathrm{m},\ell})$ is $\mathcal{O}(\varepsilon)$ and similarly for bending modes. The situation is much more complicated in the case of multiple eigenvalues and it is then convenient to introduce *clusters*, as follows.

Definition 2.4 Let $\ell \in \mathcal{L}^{\mathrm{m}}$ correspond to a multiple membrane eigenvalue $\varrho_{\mathrm{m},\ell}$ and let ν be its multiplicity. Then for any ε we introduce the corresponding cluster

$$C^\varepsilon_{\mathrm{m},\ell} = \bigoplus \big\{ E^\varepsilon_{\mathrm{m},k}\,; \quad k \in \mathcal{L}^\varepsilon_{\mathrm{m}}, \ \ \ell \leq k < \ell + \nu \big\}. \tag{2.5}$$

If $\varrho_{\mathrm{m},\ell}$ is a simple eigenvalue, we agree that $C^\varepsilon_{\mathrm{m},\ell} = E^\varepsilon_{\mathrm{m},\ell}$. Similar definitions hold for bending. ∎

Our result reads

Theorem 2.5 *As $\varepsilon \to 0$, for any $\ell \in \mathcal{L}^{\mathrm{m}}$ there holds*

$$\delta_\varepsilon(K_{\mathrm{m},\ell}, C^\varepsilon_{\mathrm{m},\ell}) \leq C_\ell\, \varepsilon. \tag{2.6a}$$

As $\varepsilon \to 0$, for any $\ell \in \mathcal{L}^{\mathrm{b}}$ there holds

$$\delta_\varepsilon(K_{\mathrm{b},\ell}, C^\varepsilon_{\mathrm{b},\ell}) \leq C_\ell\, \varepsilon\,, \tag{2.6b}$$

where δ_ε denotes the gap (2.1) *with respect to the norm $\|\boldsymbol{u}\|_\varepsilon := \sqrt{\langle \boldsymbol{u}, \boldsymbol{u}\rangle_\varepsilon}$.*

2.c Outline of the strategy of proof

The main part of our paper (sections 3-6) is devoted to the construction of $\mathcal{O}(\varepsilon^k)$-quasimodes (for any integer $k > 0$) for problem (1.11) starting with Kirchhoff-Love displacements associated with two-dimensional eigenmodes, and including outer and inner expansion terms (power series and boundary layer series).

As a consequence the two-dimensional spectrum is close to three-dimensional eigenvalues. To have a complete picture, we need the converse information, i.e. that all the smallest three-dimensional eigenvalues are close to the two-dimensional spectrum. Concerning the bending modes in clamped plates, the answer is brought by CIARLET & KESAVAN's result [3], and for the remaining cases (free edge lateral boundary condition or membrane modes) we prove in section 7 that CIARLET & KESAVAN's result can be extended by similar techniques of proofs based on weak convergence arguments.

Thus we can conclude in section 8 with precise statements about eigenmode asymptotics at any order.

3 Outer expansion formal series for quasimodes

The aim of the four following sections is to construct formal series expansions for membrane and bending eigenmodes. In this section, we describe membrane and bending formal series $\boldsymbol{u}(\varepsilon)$ with coefficients $\boldsymbol{u}^k$ in $\mathcal{C}^\infty(\overline{\Omega})^3$ associated with formal eigenvalue series $\Lambda(\varepsilon) = \sum_k \varepsilon^k \Lambda_k$ which solve (in the sense of formal series) the interior equations and the horizontal boundary conditions included in (1.11).

In the next section, we combine the above formal series with a third one, $\boldsymbol{\varphi}(\varepsilon)$, which is a formal series of boundary layer profiles, and find the conditions so that all equations of (1.11) are solved in the sense of formal series. These conditions amount to new two-dimensional formal series eigenproblems.

In section 5, we show that these two-dimensional formal series eigenproblems can be solved and in section 6, we prove that for any k we can keep a finite number of terms in the formal series so that to obtain $\mathcal{O}(\varepsilon^k)$-quasimodes.

3.a Formal series

We fix here a few definitions. As said above, we are going to formulate our problems and exhibit solution operators in formal series algebra. Let us recall that if $A(\varepsilon)$ is a formal series with operator coefficients

$$A(\varepsilon) = \sum_k \varepsilon^k A_k \quad \text{with} \quad A_k \in \mathcal{L}(E, F),$$

with E, F functional spaces, and if $b(\varepsilon)$ and $c(\varepsilon)$ are formal series in E and F

$$b(\varepsilon) = \sum_k \varepsilon^k b^k, \quad b^k \in E, \quad \text{and} \quad c(\varepsilon) = \sum_k \varepsilon^k c^k, \quad c^k \in F,$$

the equation $A(\varepsilon)b(\varepsilon) = c(\varepsilon)$ means that

$$\forall k \in \mathbb{N}, \quad \sum_{\ell=0}^{k} A_{k-\ell} b^\ell = c^k.$$

Our solution operators will depend polynomially on the coefficients Λ_k of the eigenvalue formal series $\Lambda(\varepsilon)$ and we need the following notion of *degree*.

Definition 3.1 Let $d \in \mathbb{Z}$ be an integer. The linear operator A from E to F is said to be *polynomial in* Λ *of degree* d if it is equal to a (finite) sum of terms of the form

$$\Big(\prod_{k=0}^{K} \Lambda_k^{\alpha_k}\Big) A_{[\alpha]} \quad \text{where} \quad A_{[\alpha]} \in \mathcal{L}(E, F) \text{ does not depend on } \Lambda(\varepsilon)$$

with

$$\alpha = (\alpha_0, \ldots, \alpha_K) \in \mathbb{N}^{K+1} \quad \text{and} \quad \sum_{k,\ \alpha_k \neq 0} (k + \alpha_k) \leq d.$$

For $d \leq 0$, it is understood that if A is polynomial in Λ of degree d, it does not depend on Λ. ∎

3.b The problem without lateral boundary conditions

Integrating by parts (1.11), we find a boundary value problem of second order with one boundary condition on each of the horizontal sides on $\Gamma_\pm = \omega \times \{\pm 1\}$ and on the lateral boundary $\Gamma_0 = \partial\omega \times (-1,1)$: There holds for any pair of smooth enough functions $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbf{V}(\Omega)$

$$a(\varepsilon)(\boldsymbol{u},\boldsymbol{v}) = -\big\langle B(\varepsilon)\boldsymbol{u},\,\boldsymbol{v}\big\rangle_{\Omega,\varepsilon} + \big\langle T(\varepsilon)\boldsymbol{u},\,\boldsymbol{v}\big\rangle_{\Gamma,\varepsilon}, \tag{3.1}$$

where Γ is $\Gamma_+ \cup \Gamma_-$ in the clamped case and $\Gamma_+ \cup \Gamma_- \cup \Gamma_0$ in the free case. The scalar products $\langle\cdot,\cdot\rangle_{\Omega,\varepsilon}$ and $\langle\cdot,\cdot\rangle_{\Gamma,\varepsilon}$ specify on Ω and Γ the product $\langle\cdot,\cdot\rangle_\varepsilon$ used in (1.11).

We know that in general, we cannot expect to solve the whole problem by a simple power series Ansatz, but only the part $A(\varepsilon)$ of it obtained by dropping the lateral boundary conditions: let us set

$$A(\varepsilon) = \Big(\varepsilon^2 B(\varepsilon)\;;\;\varepsilon\, T(\varepsilon)\big|_{\Gamma_\pm}\Big). \tag{3.2}$$

Then

$$A(\varepsilon) = A_0 + \varepsilon^2 A_2,$$

where the two operators A_0 and A_2 associate to a displacement $\boldsymbol{u}$ in Ω a volume force in Ω and tractions on the horizontal sides on $\Gamma_\pm$ according to: (Δ_* denotes the horizontal Laplacian $\partial_{11}+\partial_{22}$ and div_* is the horizontal divergence)

$$A_0\boldsymbol{u} = \Big(2\mu\,\partial_3 e_{\alpha 3}(\boldsymbol{u}) + \lambda\,\partial_{\alpha 3}u_3\;,\;(\lambda+2\mu)\partial_{33}u_3\;;\;2\mu\, e_{\alpha 3}(\boldsymbol{u})\big|_{\Gamma_\pm}\;,\;(\lambda+2\mu)\partial_3 u_3\big|_{\Gamma_\pm}\Big)$$

$$A_2\boldsymbol{u} = \Big((\lambda+\mu)\partial_\alpha \operatorname{div}_*\boldsymbol{u}_* + \mu\,\Delta_* u_\alpha\;,\;\lambda\,\partial_3 \operatorname{div}_*\boldsymbol{u}_* + 2\mu\,\partial_\gamma e_{\gamma 3}(\boldsymbol{u})\;;\;0\big|_{\Gamma_\pm}\;,\;\lambda \operatorname{div}_*\boldsymbol{u}_*\big|_{\Gamma_\pm}\Big)$$

the first group of arguments being the in-plane volume forces, the second, the transverse volume force, and similarly for the tractions. In order to have compact formulas, we introduce the embedding operator Π which associates to a vector field $\boldsymbol{f}$ the pair $(\boldsymbol{f};\boldsymbol{g})$ with $\boldsymbol{g}$ the zero traction on $\Gamma_\pm$:

$$\Pi\boldsymbol{f} = \Big(\boldsymbol{f}\;;\;0\big|_{\Gamma_\pm}\Big).$$

Then the formal series formulation of problem (1.11) without lateral boundary conditions reads

Find formal series $\Lambda(\varepsilon)$ *with coefficients* $\Lambda_k \in \mathbb{R}$
$\boldsymbol{u}(\varepsilon)$ *with coefficients* $\boldsymbol{u}^k \in \mathcal{C}^\infty(\overline{\Omega})^3$ *such that:*

$$A(\varepsilon)\boldsymbol{u}(\varepsilon) + \varepsilon^2\Lambda(\varepsilon)\Pi\boldsymbol{u}(\varepsilon) = 0. \tag{3.3}$$

Thus equation (3.3) means that $\big(B(\varepsilon)+\Lambda(\varepsilon)\big)\boldsymbol{u}(\varepsilon) = 0$ in Ω and $T(\varepsilon)\boldsymbol{u}(\varepsilon) = 0$ on $\Gamma_\pm$.

Before giving the whole result, let us comment on the equation of order zero of (3.3):

$$A_0\boldsymbol{u}^0 = 0.$$

It is well known that the solutions of this problem are the Kirchhoff-Love displacements. Thus $\boldsymbol{u}^0$ is generated by a displacement $\boldsymbol{z}^0 = (\boldsymbol{z}^0_*, z^0_3)$ depending only on $x_* \in \omega$, according to the formula:

$$\boldsymbol{u}^0 = U_{\mathrm{KL}}(\boldsymbol{z}^0) = \big(\boldsymbol{z}^0_*(x_*) - x_3 \nabla_* z^0_3(x_*), z^0_3(x_*)\big)$$

with the two-dimensional gradient ∇_*. If $\boldsymbol{\zeta}$ is a in-plane displacement and η a function, the operators

$$U^{\mathrm{m}}_{\mathrm{KL}}(\boldsymbol{\zeta}) := U_{\mathrm{KL}}(\boldsymbol{\zeta}, 0) \quad \text{and} \quad U^{\mathrm{b}}_{\mathrm{KL}}(\eta) := U_{\mathrm{KL}}(0, \eta)$$

take values respectively in the membrane and bending subspaces. There holds

$$U^{\mathrm{m}}_{\mathrm{KL}}(\boldsymbol{\zeta}) = \big(\boldsymbol{\zeta}(x_*),\, 0\big) \quad \text{and} \quad U^{\mathrm{b}}_{\mathrm{KL}}(\eta) = \big(-x_3 \nabla_* \eta(x_*), \eta(x_*)\big)\,. \tag{3.4}$$

In the following, we first prove a general theorem for the formal series solution, then give particular descriptions for the membrane and bending cases.

3.c Formal series solution algorithms

The solvability of three-dimensional equations (3.3) reveals to reduce to the solvability of two-dimensional equations: By integrating the equations of (3.3) with respect to the transverse variable x_3 the determination of the three-dimensional unknown formal series $\boldsymbol{u}(\varepsilon)$ is reduced to the determination of a new two-dimensional unknown formal series of Kirchhoff-Love generators $\boldsymbol{z}(\varepsilon)$.

Theorem 3.2 *Let U_0 be defined as U_{KL} and let U_1 be the operator zero $\mathcal{C}^\infty(\overline{\omega})^3 \to \mathcal{C}^\infty(\overline{\Omega})^3$. Let us denote by $\mathcal{C}^\infty_{(0)}(\overline{\Omega})$ the space of $\mathcal{C}^\infty(\overline{\Omega})^3$-displacements with mean values zero across each fiber, i.e.*

$$\boldsymbol{u} \in \mathcal{C}^\infty_{(0)}(\overline{\Omega}) \quad \Longleftrightarrow \quad \forall x_* \in \omega, \ \int_{-1}^{1} \boldsymbol{u}(x_*, x_3)\,\mathrm{d}x_3 = 0.$$

There exist for each integer $k \geq 0$:

- *a bounded operator L_k: $\mathcal{C}^\infty(\overline{\omega})^3 \to \mathcal{C}^\infty(\overline{\omega})^3$,*
- *a bounded operator U_{k+2}: $\mathcal{C}^\infty(\overline{\omega})^3 \to \mathcal{C}^\infty_{(0)}(\overline{\Omega})$ polynomial in x_3,*

defining the formal operator series $L(\varepsilon)$ and $U(\varepsilon)$ which realize a link between three- and two-dimensional problems in the following way:

If the formal series $\Lambda(\varepsilon) = \sum_k \varepsilon^k \Lambda_k$ and $\boldsymbol{z}(\varepsilon) = \sum_k \varepsilon^k \boldsymbol{z}^k$ solve

$$\big(L(\varepsilon) + \Lambda(\varepsilon)\big)\boldsymbol{z}(\varepsilon) = 0 \quad \textit{in } \omega\,, \tag{3.5}$$

then the three-dimensional formal series

$$\boldsymbol{u}(\varepsilon) := U(\varepsilon)\boldsymbol{z}(\varepsilon) \tag{3.6}$$

is a solution of the three-dimensional eigenvalue formal problem (3.3) *in Ω.*

Proof.

a) Clearly for any $\boldsymbol{z}^0$ and $\boldsymbol{z}^1$ in $\mathcal{C}^\infty(\overline{\omega})^3$ the displacements $\boldsymbol{u}^0$ and $\boldsymbol{u}^1$ defined as $\boldsymbol{u}^0 = U_0(\boldsymbol{z}^0)$ and $\boldsymbol{u}^1 = U_0(\boldsymbol{z}^1)$ satisfy the equations of (3.3) for $k = 0$ and $k = 1$, which are simply $A_0\,\boldsymbol{u}^0 = 0$ and $A_0\,\boldsymbol{u}^1 = 0$.

b) We are going to prove that there exist operators L_ℓ and U_ℓ satisfying the conditions of the Theorem so that there holds the following identity in operator valued formal series with coefficients in $\mathcal{L}\big(\mathcal{C}^\infty(\overline{\omega})^3, \mathcal{C}^\infty(\overline{\Omega})^3 \times \mathcal{C}^\infty(\overline{\Gamma}_\pm)^3\big)$

$$A(\varepsilon) \circ U(\varepsilon) + \varepsilon^2 \Lambda(\varepsilon)\,\Pi \circ U(\varepsilon)0\big|_{\Gamma_\pm}\Big) = \varepsilon^2\,\Pi \circ \big(L(\varepsilon) + \Lambda(\varepsilon)\big).\Lambda(\varepsilon)\big)\,; \qquad (3.7)$$

In the right hand side the composition with the canonical embedding $\mathcal{I} : \mathcal{C}^\infty(\overline{\omega})^3 \hookrightarrow \mathcal{C}^\infty(\overline{\Omega})^3$ is implied: The operator $\Pi \circ L(\varepsilon)$ is *stricto sensu* the operator $\Pi \circ \mathcal{I} \circ L(\varepsilon)$ and takes its values in spaces of functions independent of x_3.

The statement of the Theorem follows immediately from identity (3.7).

To prove (3.7), we are going to show by recurrence that for any $k \geq 0$, we have:

> *The operators L_ℓ are constructed for $\ell = -2, \ldots, k-2$, the operators U_ℓ are constructed for $\ell = 0, \ldots, k$ so that for any $\boldsymbol{z} \in \mathcal{C}^\infty(\overline{\omega})^3$ and any $\ell = 0, \ldots, k$ there holds*

$$A_0 U_\ell \boldsymbol{z} + A_2 U_{\ell-2}\boldsymbol{z} + \sum_{j=0}^{\ell-2} \Lambda_{\ell-2-j}\Pi U_j \boldsymbol{z} = \Pi\big(L_{\ell-2}\boldsymbol{z} + \Lambda_{\ell-2}\boldsymbol{z}).\big). \qquad (3.8)$$

With $L_{-2} = L_{-1} = 0$, (3.8) is true for $k = 0, 1$. Let us assume that (3.8) holds for $\ell = k \geq 1$ and let us construct U_{k+1} and L_{k+1} so that (3.8) holds for $\ell = k+1$.

c) We first consider the transverse component of equation (3.8). For $\ell = k+1$, the problem reads, for any $\boldsymbol{z} \in \mathcal{C}^\infty(\overline{\omega})^3$

$$\Big(A_0(\boldsymbol{v}) + A_2(U_{k-1}\boldsymbol{z}) + \sum_{j=0}^{k-1} \Lambda_{k-1-j}\Pi U_j \boldsymbol{z} - \Lambda_{k-1}\Pi\boldsymbol{z}\Big)_3 = \Big(\Pi L_{k-1}\boldsymbol{z}\Big)_3\Big) \qquad (3.9)$$

with still unknown $\boldsymbol{v}$ and $\left(L_{k-1}\boldsymbol{z}\right)_3$. The equation has the type

$$\begin{cases} (\lambda + 2\mu)\partial_{33}v_3 + F_3(\boldsymbol{z}) &= \left(L_{k-1}\boldsymbol{z}\right)_3 \quad \text{in } \Omega, \\ (\lambda + 2\mu)\partial_3 v_3 + G_3(\boldsymbol{z}) &= 0 \qquad\qquad\quad \text{on } \Gamma_\pm\,. \end{cases}$$

This boundary value problem is solvable if

$$\int_{-1}^{+1} F_3(\boldsymbol{z})(x_*, x_3)\,\mathrm{d}x_3 - G_3(\boldsymbol{z})(x_*, 1) + G_3(\boldsymbol{z})(x_*, -1) = 2\left(L_{k-1}\boldsymbol{z}\right)_3(x_*) \qquad (3.10)$$

for all $x_* \in \omega$, which yields the definition of $\left(L_{k-1}\boldsymbol{z}\right)_3$. The unique solution of this problem with zero mean values across each fiber defines the operators $\left(U_{k+1}\boldsymbol{z}\right)_3$.

d) Next, the in-plane component of equation (3.8) for $\ell = k+1$ reads

$$\Big(A_0(\boldsymbol{v}) + A_2(U_{k-1}\boldsymbol{z}) + \sum_{j=0}^{k-1} \Lambda_{k-1-j}\Pi U_j \boldsymbol{z} - \Lambda_{k-1}\Pi\boldsymbol{z}\Big)_\alpha = \Big(\Pi L_{k-1}\boldsymbol{z}\Big)_\alpha\Big) \qquad (3.11)$$

with still unknown v_α and $\big(L_{k-1}\boldsymbol{z}\big)_\alpha$. This equation has the type

$$\begin{cases} \mu\partial_{33}v_\alpha + (\lambda+\mu)\partial_{\alpha 3}v_3 + F'_\alpha(\boldsymbol{z}) &= (L_{k-1}\boldsymbol{z})_\alpha \quad \text{in } \Omega, \\ \mu(\partial_3 v_\alpha + \partial_\alpha v_3) &= 0 \qquad\qquad\quad \text{on } \Gamma_\pm . \end{cases}$$

Inserting the equality $v_3 = (U_{k+1}\boldsymbol{z})_3$, we obtain an equation of the type $\mu\partial_{33}v_\alpha + F_\alpha(\boldsymbol{z}) = (L_{k-1}\boldsymbol{z})_\alpha$ with boundary conditions $\mu\partial_3 v_\alpha + G_\alpha(\boldsymbol{z}) = 0$. This boundary value problem is solvable if

$$\int_{-1}^{+1} F_\alpha(\boldsymbol{z})(x_*, x_3)\,\mathrm{d}x_3 - G_\alpha(\boldsymbol{z})(x_*, 1) + G_\alpha(\boldsymbol{z})(x_*, -1) = 2(L_{k-1}\boldsymbol{z})_\alpha(x_*) \tag{3.12}$$

for all $x_* \in \omega$, which yields the definition of $(L_{k-1}\boldsymbol{z})_\alpha$. The unique solution of this problem with zero mean values across each fiber defines the operators $(U_{k+1}\boldsymbol{z})_\alpha$. Thus the recurrence step is proved. Whence identity (3.7). ■

The operators U_ℓ and L_ℓ are polynomial in Λ. We explain this in detail in the next subsection.

Remark 3.3 If $\boldsymbol{u}(\varepsilon)$ is a formal series solution of (3.3), then defining $\boldsymbol{z}(\varepsilon)$ by

$$\boldsymbol{z}(\varepsilon) = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}(\varepsilon)\,\mathrm{d}x_3,$$

the formal series $\boldsymbol{z}(\varepsilon)$ satisfies (3.5) and moreover, $\boldsymbol{u}(\varepsilon) = U(\varepsilon)\boldsymbol{z}(\varepsilon)$. ■

3.d Membrane and bending formal series

Let us now recall that our three-dimensional eigenvalue problem commutes with the symmetry operator S defined in (1.3), that the displacements $\boldsymbol{u}$ such that $S\boldsymbol{u} = \boldsymbol{u}$ are the membrane displacements, and those such that $S\boldsymbol{u} = -\boldsymbol{u}$ are the bending ones. When restricted to the displacements $\boldsymbol{z}$ independent of x_3, the membrane displacements are those of the form $(\boldsymbol{\zeta}, 0)$ with $\boldsymbol{\zeta}$ any in-plane displacement, whereas the bending displacements are those of the form $(0, \eta)$ with η any function. There holds

$$U(\varepsilon)\circ S = S\circ U(\varepsilon), \tag{3.13}$$

which means that S commutes with all operators U_k. This can be easily proved by recurrence according to steps **c)** and **d)** of the proof of Theorem 3.2: The main argument is the choice of solutions of the Neumann problem in $[-1, 1]$ by the condition of mean value zero, which preserves the parity.

Thus for any in-plane displacement $\boldsymbol{\zeta}$, the displacement $U(\varepsilon)(\boldsymbol{\zeta}, 0)$ has membrane type, and for any function η, the displacement $U(\varepsilon)(0, \eta)$ has bending type.

Similarly there holds

$$L(\varepsilon)\circ S = S\circ L(\varepsilon). \tag{3.14}$$

For $\boldsymbol{u}$ independent on x_3, S is only the symmetry with respect to the plane generated by the two first components. Hence a consequence of identity (3.14) is that the series $L(\varepsilon)$

is block diagonal with respect to the splitting into in-plane and transverse components:

$$L(\varepsilon) = \begin{pmatrix} L^{\mathrm{m,m}}(\varepsilon) & 0 \\ 0 & L^{\mathrm{b,b}}(\varepsilon) \end{pmatrix}.$$

The solvability of three-dimensional equations (3.3) reveals to reduce to the solvability of two-dimensional equations based on the membrane operator L^{m} (plane stress model) introduced in (0.3) and the bending operator L^{b}, defined in (0.1).

The next theorem collects the results for the membrane and bending dimension reducing process:

Theorem 3.4

(i) *For any in-plane displacement $\boldsymbol{\zeta} \in \mathcal{C}^\infty(\overline{\omega})^2$ let $U^{\mathrm{m}}(\varepsilon)$ be defined as $U^{\mathrm{m}}(\varepsilon)\,\boldsymbol{\zeta} = U(\varepsilon)\,(\boldsymbol{\zeta}, 0)$ and $L^{\mathrm{m}}(\varepsilon)$ be defined as $L^{\mathrm{m}}(\varepsilon)\,\boldsymbol{\zeta} = \big(L(\varepsilon)\,(\boldsymbol{\zeta}, 0)\big)_*$, where $U(\varepsilon)$ and $L(\varepsilon)$ are the formal series appearing in Theorem* 3.2. *Then L^{m}_0 coincides with the membrane operator L^{m} of* (0.3) *and for each integer k, the operators L^{m}_k and U^{m}_{k+2} are polynomial of degree $k-1$ in Λ. Moreover, for any formal series $\boldsymbol{\zeta}(\varepsilon)$ with coefficients in $\mathcal{C}^\infty(\overline{\omega})^2$ solving*

$$\big(L^{\mathrm{m}}(\varepsilon) + \Lambda(\varepsilon)\big)\boldsymbol{\zeta}(\varepsilon) = 0 \quad \textit{in } \omega\,, \tag{3.15}$$

then the three-dimensional formal series

$$\boldsymbol{u}(\varepsilon) := U^{\mathrm{m}}(\varepsilon)\boldsymbol{\zeta}(\varepsilon) \tag{3.16}$$

is a membrane solution of the three-dimensional eigenvalue formal problem (3.3) *in* Ω.

(ii) *For any function $\eta \in \mathcal{C}^\infty(\overline{\omega})$ let $U^{\mathrm{b}}(\varepsilon)$ be defined as $U^{\mathrm{b}}(\varepsilon)\,\eta = U(\varepsilon)\,(0, \eta)$. Let us assume that*

$$\Lambda_0 = \Lambda_1 = 0, \quad \textit{and set} \quad \Lambda(\varepsilon) =: \frac{\varepsilon^2}{3}\Lambda^{\mathrm{b}}(\varepsilon).$$

Then $L_0(0,\eta) = L_1(0,\eta) = 0$ and $\big(L_2(0,\eta)\big)_3$ coincides with $-\frac{1}{3}L^{\mathrm{b}}\eta$ where L^{b} is the bending operator (0.1). *We define the operator valued formal series $L^{\mathrm{b}}(\varepsilon)$ by*

$$\big(L(\varepsilon)\,(0,\eta)\big)_3 =: -\frac{\varepsilon^2}{3}L^{\mathrm{b}}(\varepsilon)\,\eta.$$

Then the operators L^{b}_k and U^{b}_{k+2} are polynomial in Λ^{b} of degree $k-1$. Moreover, if the formal series $\Lambda(\varepsilon) = \sum_{k\geq 2}\varepsilon^k\Lambda_k$ and $\eta(\varepsilon) = \sum_{k\geq 0}\varepsilon^k\eta^k$ solve

$$\big(L^{\mathrm{b}}(\varepsilon) - \Lambda^{\mathrm{b}}(\varepsilon)\big)\eta(\varepsilon) = 0 \quad \textit{in } \omega\,, \tag{3.17}$$

then the three-dimensional formal series

$$\boldsymbol{u}(\varepsilon) := U^{\mathrm{b}}(\varepsilon)\eta(\varepsilon) \tag{3.18}$$

is a bending solution of the three-dimensional eigenvalue formal problem (3.3) *in* Ω.

Proof.

The equation (3.8) for $k=2$ applied successively to $(\boldsymbol{\zeta},0)$ and $(0,\eta)$ reduces to

$$\big(A_0U_2^{\mathrm{m}}\boldsymbol{\zeta}+A_2U_0^{\mathrm{m}}\boldsymbol{\zeta}\big)_\alpha=\big(\Pi L_0(\boldsymbol{\zeta},0)\big)_\alpha\Big)$$

and if $\Lambda_0=0$ to

$$\big(A_0U_2^{\mathrm{b}}\eta+A_2U_0^{\mathrm{b}}\eta\big)_3=\big(\Pi L_0(0,\eta)\big)_3\,,$$

respectively. But computations like those in [6, Lemma 3.2] show that $\big(L_0(0,\eta)\big)_3=0$ and $\big(L_0(\boldsymbol{\zeta},0)\big)_\alpha=\big(L^{\mathrm{m}}\boldsymbol{\zeta}\big)_\alpha$. Obviously, since $U_1=0$ there holds $L_1(\boldsymbol{\zeta},0)=0$, and if $\Lambda_0=\Lambda_1=0$ there holds $L_1(0,\eta)=0$ too, and by a computation we obtain that $\big(L_2(0,\eta)\big)_3=-\frac{1}{3}L^{\mathrm{b}}\eta$.

The assertions concerning the degree are proved by induction.

In the membrane case *(i)* for $k=0$ we have clearly that L_0^{m} and U_0^{m}, U_1^{m}, U_2^{m} are of degree zero in Λ. Suppose that for $\ell=0,\ldots,k$, the operators U_ℓ^{m} and $L_{\ell-2}^{\mathrm{m}}$ are of degree $\ell-3$ in Λ and fix $\boldsymbol{\zeta}\in\mathcal{C}^\infty(\overline{\omega})^2$. The equation (3.8) yields, with the fact that $U_0^{\mathrm{m}}\boldsymbol{\zeta}=(\boldsymbol{\zeta},0)$ and $U_1=0$ that

$$A_0U_{k+1}^{\mathrm{m}}\boldsymbol{\zeta}+A_2U_{k-1}^{\mathrm{m}}\boldsymbol{\zeta}+\sum_{j=2}^{k-1}\Lambda_{k-1-j}\Pi U_j^{\mathrm{m}}\boldsymbol{\zeta}=\Pi\big(L_{k-2}^{\mathrm{m}}\boldsymbol{\zeta},0\big). \tag{3.19}$$

This equation is of the type

$$\partial_{33}U_{k+1}^{\mathrm{m}}\boldsymbol{\zeta}+\tilde{F}(\boldsymbol{\zeta})=\big(L_{k-1}^{m}\boldsymbol{\zeta},0\big)\quad\text{in}\quad\Omega,\qquad\partial_3U_{k+1}^{\mathrm{m}}\boldsymbol{\zeta}+\tilde{G}(\boldsymbol{\zeta})=0\quad\text{on}\quad\Gamma_\pm\,,$$

where the operators $\tilde{G}$ and $\tilde{F}$ are similar to the operators F and G used in the proof of Theorem 3.2. Now, by the recurrence hypothesis, the operators $\tilde{F}$ and $\tilde{G}$ are polynomial in Λ of degree

$$\max_{j=2}^{k-2}\big(k-1-j+1+(j-3)_+\big)=k-2.$$

The equation (3.12) shows that L_{k-1}^{m} is polynomial of degree $k-2$, and we have immediately that U_{k+1}^{m} is also of degree $k-2$. Hence the result for the membrane operators.

For the bending case *(ii)*, we can prove the result in the same manner. Note however that $U_0^{\mathrm{b}}\eta-\eta\neq0$ and the assumption $\Lambda_0=\Lambda_1=0$ is the right one in order that the first non-zero bending operator in $L(\varepsilon)$ does not depend on Λ. ■

k	$(U_k^{\mathrm{m}}\boldsymbol{\zeta})_3$	$(U_k^{\mathrm{m}}\boldsymbol{\zeta})_\alpha$	$(L_k^{\mathrm{m}}\boldsymbol{\zeta})_\alpha$
0	0	ζ_α	$\mu\Delta\zeta_\alpha+(\tilde{\lambda}+\mu)\,\partial_\alpha\operatorname{div}\boldsymbol{\zeta}$
1	0	0	0
2	$\bar{p}_1\operatorname{div}\boldsymbol{\zeta}$	$\bar{p}_2\,\partial_\alpha\operatorname{div}\boldsymbol{\zeta}$	$(c_2\Lambda_0+c_2'\Delta)\,\partial_\alpha\operatorname{div}\boldsymbol{\zeta}$
3	0	0	0
4	$(q_3\Lambda_0+q_3'\Delta)\operatorname{div}\boldsymbol{\zeta}$	$(r_4\Lambda_0+r_4'\Delta)\partial_\alpha\operatorname{div}\boldsymbol{\zeta}$	$(c_4\Lambda_0^2+c_4'\Lambda_2+c_4''\Lambda_0\Delta+c_4''\Delta^2)\,\partial_\alpha\operatorname{div}\boldsymbol{\zeta}$

Table 1. *First rank membrane outer expansion operators.*

Here c_2, c'_2, c_4, c'_4,... denote real numbers, q_3, q'_3 odd polynomials of degree 3 and r_4, r'_4 even polynomials of degree 4, compare [6, §3.3], and, *cf* [6, Lemma 3.2]:

$$\bar{p}_1(x_3) = -\,\tfrac{\tilde{\lambda}}{2\mu}\,x_3, \qquad \bar{p}_2(x_3) = \tfrac{\tilde{\lambda}}{4\mu}\,\big(x_3^2 - \tfrac{1}{3}\big). \tag{3.20}$$

For the bending operators we have, with the assumption $\Lambda_0 = \Lambda_1 = 0$:

k	$(U^{\rm b}_k\eta)_3$	$(U^{\rm b}_k\eta)_\alpha$	$L^{\rm b}_k\eta$
0	η	$-x_3\partial_\alpha\eta$	$(\tilde{\lambda}+2\mu)\Delta^2\eta$
1	0	0	0
2	$\bar{p}_2\,\Delta\eta$	$\bar{p}_3\,\Delta\partial_\alpha\eta$	$(c_3\Lambda_2 + c'_3\Delta)\,\Delta^2\eta$
3	0	0	0
4	$q_4\,\Delta^2\eta$	$(r_5\Lambda_2 + r'_5\Delta^2)\,\partial_\alpha\eta$	$(c_5\Lambda_2^2 + c'_5\Lambda_4 + c''_5\Lambda_2\Delta + c''_5\Delta^2)\,\Delta^2\eta$

Table 2. *First rank bending outer expansion operators.*

Here c_3, c'_3, c_5, c'_5,... are real numbers, q_4 is an even polynomial of degree 4 and r_5, r'_5 are odd polynomials of degree 5 and:

$$\bar{p}_3(x_3) = -\,\tfrac{1}{12\mu}\,\big((\tilde{\lambda}+4\mu)x_3^3 - (5\tilde{\lambda}+12\mu)x_3\big). \tag{3.21}$$

4 Combined inner and outer expansion formal series

In order to fulfill the lateral boundary conditions on $\Gamma_0 = \partial\omega \times (-1,1)$, we have now to combine the formal series $\boldsymbol{u}(\varepsilon)$ satisfying the conditions of Theorem 3.4 with formal series $\boldsymbol{\varphi}(\varepsilon)$ with coefficients in spaces of exponentially decreasing *profiles*, which yield the *boundary layer terms* naturally involved in the solution asymptotics, see [13, 17] and [6].

4.a Inner expansion formal series

We need local coordinates $(r,\, s)$ in a plane neighborhood $\mathcal{U}$ of the lateral boundary $\partial\omega$. Here r denotes the distance to $\partial\omega$ and s the positively oriented arclength on it. The local basis at each point in $\partial\omega$ is given by the unit inner normal $\boldsymbol{n}$ and the tangent unit vector $\boldsymbol{\tau}$. Extending $\boldsymbol{n}$ and $\boldsymbol{\tau}$ into $\mathcal{U}$ we arrive at following relations for the normal and (horizontal) tangential components u_r and u_s of any vector field $\boldsymbol{u}_* = (u_1,\, u_2)$:

$$u_r = n_1u_1 + n_2u_2 \quad \text{and} \quad u_s = (1-\kappa r)(n_2u_1 - n_1u_2), \tag{4.1}$$

where $\kappa = \kappa(s)$ is the curvature of $\partial\omega$ at s from inside ω. We also use the partial derivatives (which, of course, commute with each other)

$$\partial_r = n_1\partial_1 + n_2\partial_2 \quad \text{and} \quad \partial_s = (1-\kappa r)(n_2\partial_1 - n_1\partial_2). \tag{4.2}$$

When restricted to the lateral boundary $\partial\omega$, we will also write u_n and ∂_n instead of u_r and ∂_r. Let the stretched distance to $\partial\omega$ be defined by $t = r/\varepsilon$.

The boundary layer Ansatz has then the form

$$\sum_{k\geq 0} \varepsilon^k \boldsymbol{w}^k(r\varepsilon^{-1}, s, x_3)$$

where $t \mapsto \boldsymbol{w}^k(t, s, x_3)$ is an exponentially decreasing profile as $t \to +\infty$. Here s belongs to the in-plane boundary $\partial\omega$ and (t, x_3) to the half-strip $\Sigma^+ = \mathbb{R}^+ \times (-1, 1)$. In order to preserve the homogeneity of the elasticity system, we scale the profiles $\boldsymbol{w}^k$ back, that is we set $\boldsymbol{\varphi}_*^k = \boldsymbol{w}_*^k$ and $\varphi_3^k = w_3^{k+1}$. In formal series writing this means

$$\boldsymbol{w}(\varepsilon) = W(\varepsilon)\boldsymbol{\varphi}(\varepsilon), \quad \text{with}$$

$$W_0(\boldsymbol{\varphi}_*, \varphi_3) = (\boldsymbol{\varphi}_*, 0), \quad W_1(\boldsymbol{\varphi}_*, \varphi_3) = (0, \varphi_3) \quad \text{and} \quad W_k = 0,\ \forall k \geq 2.$$

In this section we are going to exhibit operator valued formal series determining the Dirichlet or Neumann traces of the in-plane generator formal series $\boldsymbol{\zeta}(\varepsilon)$ so that the Dirichlet or Neumann traces on the lateral boundary Γ_0 of $\boldsymbol{u}(\varepsilon) = U(\varepsilon)\boldsymbol{\zeta}(\varepsilon)$ (with U standing for U^{m} or U^{b}) can be compensated by the corresponding traces of a boundary layer formal series $\boldsymbol{\varphi}(\varepsilon)$ with values in exponentially decreasing function spaces.

To this aim, let $\mathfrak{H}(\Sigma^+)$ be the space of $\mathcal{C}^\infty(\Sigma^+)$ functions φ, which are smooth up to any regular point of the boundary of Σ^+ and are exponentially decreasing as $t \to \infty$ in the following sense

$$\forall i, j, k \in \mathbb{N}, \qquad e^{\delta t}\, t^k\, \partial_t^i \partial_3^j \varphi \in L^2(\Sigma^+)$$

with $\delta > 0$ a fixed number smaller than the smallest exponent arising from the Papkovich-Fadle eigenfunctions, *cf* [10]. With ρ the distance to the two corners of Σ^+, we moreover prescribe the following behavior at the corners for the elements of $\mathfrak{H}(\Sigma^+)$

$$\varphi \in L^2(\Sigma^+) \quad \text{and} \quad \forall i, j \in \mathbb{N},\ i + j \neq 0, \qquad \rho^{i+j-1}\, \partial_t^i \partial_3^j \varphi \in L^2(\Sigma^+).$$

Then we define the corresponding displacement space $\boldsymbol{\mathfrak{H}}(\Sigma^+) := \mathfrak{H}(\Sigma^+)^3$. Our profile formal series $\boldsymbol{\varphi}(\varepsilon)$ will have its coefficients in $\mathcal{C}^\infty\big(\partial\omega, \boldsymbol{\mathfrak{H}}(\Sigma^+)\big)$.

4.b Inner expansion problems and matching of lateral boundary conditions

In variables (t, s, x_3) and unknowns $\boldsymbol{\varphi} = (\varphi_t, \varphi_s, \varphi_3)$ the pair $A(\varepsilon)$ of the interior operator and the horizontal boundary operator become

$$\tilde{A}(\varepsilon)(t\varepsilon, s\,;\, \varepsilon^{-1}\partial_t, \partial_s, \partial_3),$$

where $W(\varepsilon) \circ \tilde{A}(\varepsilon)(r, s; \partial_r, \partial_s, \partial_3) = \big(A_0(\partial_x) + \varepsilon^2 A_2(\partial_x)\big) \circ W(\varepsilon)$ in the neighborhood $\mathcal{U} \times (-1, 1)$ of Γ_0. The Taylor expansion at $t = 0$ of the coefficients of $\tilde{A}(\varepsilon)$ provides the operator valued formal series

$$\mathcal{A}(\varepsilon) = \sum\nolimits_k \varepsilon^k \mathcal{A}_k = \Big(\sum\nolimits_k \varepsilon^k \mathcal{B}_k\ ;\ \sum\nolimits_k \varepsilon^k \mathcal{G}_k\Big)$$

where the $\mathcal{B}_k(t, s\,;\, \partial_t, \partial_s, \partial_3)$ are partial differential systems of order 2 in the stretched domain $\partial\omega \times \Sigma^+$ whereas the $\mathcal{G}_k(t, s\,;\, \partial_t, \partial_s, \partial_3)$ are partial differential systems of order

1 on its horizontal boundaries $\partial\omega \times \gamma_\pm$, where $\gamma_\pm = \mathbb{R}^+ \times \{x_3 = \pm 1\}$ denotes the horizontal boundaries of Σ^+; all operators have polynomial coefficients in t.

The counterpart of the outer eigenproblem (3.3) is

$$\mathcal{A}(\varepsilon)\boldsymbol{\varphi}(\varepsilon) + \varepsilon^2\Lambda(\varepsilon)\Pi\boldsymbol{\varphi}(\varepsilon) = 0 \tag{4.3}$$

for the inner expansion formal series, where we use the embedding operator Π which associates now to a vector field $\boldsymbol{f}$ the pair $\big(\boldsymbol{f}\,; 0\,\big|_{\partial\omega\times\gamma_\pm}\big)$.

It is worthwhile to note that the principal parts $\mathcal{B}_0$ and $\mathcal{G}_0$ split into 2D-Lamé and 2D-Laplace operators in variables (t, x_3) with Neumann boundary conditions, respectively:

$$(\mathcal{B}_0\boldsymbol{\varphi})_t = \mu\,\Delta_{t,3}\varphi_t + (\lambda+\mu)\,\partial_t\big(\mathrm{div}_{t,3}(\varphi_t,\varphi_3)\big), \qquad (\mathcal{G}_0\boldsymbol{\varphi})_t = \mu(\partial_3\varphi_t + \partial_t\varphi_3),$$
$$(\mathcal{B}_0\boldsymbol{\varphi})_3 = \mu\,\Delta_{t,3}\varphi_3 + (\lambda+\mu)\,\partial_3\big(\mathrm{div}_{t,3}(\varphi_t,\varphi_3)\big), \qquad (\mathcal{G}_0\boldsymbol{\varphi})_3 = (\lambda+2\mu)\partial_3\varphi_3 + \lambda\,\partial_t\varphi_t$$

and

$$(\mathcal{B}_0\boldsymbol{\varphi})_s = \mu\,\Delta_{t,3}\varphi_s, \qquad (\mathcal{G}_0\boldsymbol{\varphi})_s = \mu\partial_3\varphi_s\,.$$

Now, it remains to give the equations that should hold in order that $\sum_k \varepsilon^k\boldsymbol{u}^k(x) + \sum_k \varepsilon^k\boldsymbol{w}^k(r\varepsilon^{-1}, s, x_3)$ fulfills the lateral boundary conditions on Γ_0.

Concerning the Dirichlet case, we set

$$D(\varepsilon) = \varepsilon^{-1}D_{-1} + D_0, \quad \text{with}$$
$$D_{-1}(u_n, u_s, u_3) = (0,0,u_3) \quad \text{and} \quad D_0(u_n,u_s,u_3) = (u_n,u_s,0).$$

The Dirichlet boundary condition takes then the form

$$\boldsymbol{\varphi}(\varepsilon)\big|_{t=0} + D(\varepsilon)\boldsymbol{u}(\varepsilon)\big|_{\Gamma_0} = 0. \tag{4.4}$$

In the case of a free plate, the traction operator $\mathcal{T}(\varepsilon)$ on $\boldsymbol{\varphi}(\varepsilon)$ is obtained like $\mathcal{B}(\varepsilon)$ from the change of variables $x \mapsto (t,s,x_3)$ and has only two terms

$$\mathcal{T}(\varepsilon) = \mathcal{T}_0 + \varepsilon\mathcal{T}_1.$$

The main term $\mathcal{T}_0$ is the traction operator associated with $\mathcal{B}_0$ and reads

$$\big(\mathcal{T}_0\boldsymbol{\varphi}\big)_{t,3,s} = \big(\lambda\,\partial_3\varphi_3 + (\lambda+2\mu)\partial_t\varphi_t\,,\ \mu(\partial_t\varphi_3 + \partial_3\varphi_t)\,,\ \mu\,\partial_t\varphi_s\big).$$

The counterpart traction $N(\varepsilon)$ acting on $\boldsymbol{u}(\varepsilon)$ has three terms and reads

$$N(\varepsilon) = \varepsilon^{-1}N_{-1} + N_0 + \varepsilon N_1 \quad \text{with}$$
$$N_{-1}(u_n,u_s,u_3) = \big(\lambda\,\partial_3 u_3\,,\,0\,,\,0\big), \quad N_0(u_n,u_s,u_3) = \big(0\,,\,0\,,\,\mu(\partial_n u_3 + \partial_3 u_n)\big)$$
$$N_1(u_n,u_s,u_3) = \big(\lambda\,\mathrm{div}_*\,\boldsymbol{u}_* + 2\mu\,\partial_n u_n\,,\ \mu(\partial_s u_n + \partial_n u_s + 2\kappa\,u_s)\,,\ 0\big)\,,$$

where $\kappa(s)$ denotes the curvature of $\partial\omega$. The free boundary condition takes the form

$$\mathcal{T}(\varepsilon)\boldsymbol{\varphi}(\varepsilon)\big|_{t=0} + N(\varepsilon)\boldsymbol{u}(\varepsilon)\big|_{\Gamma_0} = 0. \tag{4.5}$$

We can write (4.4) and (4.5) in a unified form as

$$\mathcal{H}(\varepsilon)\boldsymbol{\varphi}(\varepsilon)\big|_{t=0} + H(\varepsilon)\boldsymbol{u}(\varepsilon)\big|_{\Gamma_0} = 0 \tag{4.6}$$

with $\mathcal{H}(\varepsilon)$ defined as Id and $\mathcal{T}(\varepsilon)$ in the clamped and free case respectively, and $H(\varepsilon)$ defined as $D(\varepsilon)$ and $N(\varepsilon)$ in the clamped and free case respectively.

4.c General inner expansion formal series

We need the definition of the image counterpart of the space $\mathfrak{H}$.

Let $\mathfrak{K}(\Sigma^+)$ be the space of triples $(\psi, \psi^\pm) \in \mathcal{C}^\infty(\Sigma^+) \times \mathcal{C}^\infty(\gamma_\pm)$ which satisfy

$$\forall i,j,k \in \mathbb{N}, \qquad e^{\delta t}\, t^k\, \partial_t^i \partial_3^j \psi \in L^2(\Sigma^+) \quad \text{and} \quad e^{\delta t}\, t^k\, \partial_t^i \psi^\pm \in L^2(\gamma_\pm)$$

and

$$\forall i,j \in \mathbb{N}, \qquad \rho^{i+j+1}\, \partial_t^i \partial_3^j \psi \in L^2(\Sigma^+) \quad \text{and} \quad \rho^{i+j+1/2}\, \partial_t^i \psi^\pm \in L^2(\gamma_\pm).$$

Then we define the corresponding displacement space:

$$\boldsymbol{\mathfrak{K}}(\Sigma^+) := \big\{ \boldsymbol{\Psi} = (\boldsymbol{\psi}, \boldsymbol{\psi}^\pm) \in \mathfrak{K}(\Sigma^+)^3 \big\}.$$

According to [6] the operator $\mathcal{A}_0$ has similar properties in both clamped and free cases. We recall here what we need and fix some notations, compare [6, section 5].

Proposition 4.1 *There exists a four-dimensional space $\mathcal{Z}$ of polynomial motions, such that if $\boldsymbol{\Psi}$ belongs to $\mathcal{C}^\infty\big(\partial\omega, \boldsymbol{\mathfrak{K}}(\Sigma^+)\big)$ and $\boldsymbol{v}$ belongs to $\mathcal{C}^\infty(\overline{\Gamma}_0)^3$, then there exist a unique $\boldsymbol{\varphi} \in \mathcal{C}^\infty\big(\partial\omega, \boldsymbol{\mathfrak{H}}(\Sigma^+)\big)$ and a unique $\boldsymbol{Z} \in \mathcal{C}^\infty(\partial\omega, \mathcal{Z})$ such that*

$$\begin{cases} \mathcal{A}_0(\boldsymbol{\varphi}) + \boldsymbol{\Psi} &= 0 \qquad \text{in} \quad \partial\omega \times \big(\Sigma^+ \times \gamma_+ \times \gamma_-\big), \\ \mathcal{H}_0(\boldsymbol{\varphi} - \boldsymbol{Z})\big|_{t=0} + \boldsymbol{v}\big|_{\Gamma_0} &= 0, \end{cases}$$

where $\mathcal{H}_0 = \mathrm{Id}$ in the clamped case, and $\mathcal{H}_0 = \mathcal{T}_0$ in the free case.

Note that, once a basis of $\mathcal{Z}$ is fixed, any $\boldsymbol{Z} \in \mathcal{C}^\infty(\partial\omega, \mathcal{Z})$ determines four coefficients in $\mathcal{C}^\infty(\partial\omega)$ depending on $\boldsymbol{\Psi}$ and $\boldsymbol{v}$ which are the coordinates of $\boldsymbol{Z}$ in this basis. In the free case, these coefficients can be computed explicitly from $\boldsymbol{\Psi}$ and $\boldsymbol{v}$, see §4.e. In the clamped case, the space $\mathcal{Z}$ is generated by the four rigid motions (two membrane and two bending) given in coordinates (t, s, x_3) by:

$$\boldsymbol{Z}_{\mathrm{D}}^{\mathrm{m},1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{Z}_{\mathrm{D}}^{\mathrm{m},2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \boldsymbol{Z}_{\mathrm{D}}^{\mathrm{b},1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \boldsymbol{Z}_{\mathrm{D}}^{\mathrm{b},2} = \begin{pmatrix} -x_3 \\ 0 \\ t \end{pmatrix}.$$

In this subsection, we will consider equation (4.3) under the conditions (4.6). The following result reduces the solution of that problem to a new boundary condition on $\boldsymbol{z}(\varepsilon)$.

Theorem 4.2 *Let $\boldsymbol{\delta}_{-1}$ be the operator zero $\mathcal{C}^\infty(\overline{\omega})^3 \to \mathcal{C}^\infty(\partial\omega, \mathcal{Z})$ in the free case, and $\boldsymbol{\delta}_{-1}$ be the operator $\mathcal{C}^\infty(\overline{\omega})^3 \to \mathcal{C}^\infty(\partial\omega, \mathcal{Z})$ defined by $\boldsymbol{\delta}_{-1}\boldsymbol{z} = \big(z_3\big|_{\partial\omega}\big)\boldsymbol{Z}_{\mathrm{D}}^{\mathrm{b},1}$ in the clamped plate. There exist for each $k \geq 0$*

- *a unique operator $\Phi_k : \mathcal{C}^\infty(\overline{\omega})^3 \to \mathcal{C}^\infty\big(\partial\omega, \boldsymbol{\mathfrak{H}}(\Sigma^+)\big)$,*
- *a unique operator $\boldsymbol{\delta}_k : \mathcal{C}^\infty(\overline{\omega})^3 \to \mathcal{C}^\infty(\partial\omega, \mathcal{Z})$,*

defining operator formal series $\Phi(\varepsilon) = \sum_{k\geq 0} \varepsilon^k \Phi_k$ and $\boldsymbol{\delta}(\varepsilon) = \sum_{k\geq -1} \varepsilon^k \boldsymbol{\delta}_k$ such that:

If the formal series $\boldsymbol{z}(\varepsilon)$ satisfies $\boldsymbol{\delta}(\varepsilon)\boldsymbol{z}(\varepsilon) = 0$, then $\boldsymbol{\varphi}(\varepsilon) = \Phi(\varepsilon)\boldsymbol{z}(\varepsilon)$ is a formal series solution of the problem (4.3) *under the boundary conditions*

$$\mathcal{H}(\varepsilon)\boldsymbol{\varphi}(\varepsilon)\big|_{t=0} + H(\varepsilon)U(\varepsilon)\boldsymbol{z}(\varepsilon)\big|_{\Gamma_0} = 0. \tag{4.7}$$

Proof. We are going to prove the existence of formal operator series $\Phi(\varepsilon) = \sum_{\ell\geq-1} \varepsilon^\ell \Phi_\ell$ and $\boldsymbol{\delta}(\varepsilon) = \sum_{\ell\geq-1} \varepsilon^\ell \boldsymbol{\delta}_\ell$ satisfying the relation

$$\begin{cases} \mathcal{A}(\varepsilon)\Phi(\varepsilon) + \big(\varepsilon^2\Lambda(\varepsilon)\Phi(\varepsilon); 0\,\big|_{\partial\omega\times\gamma_\pm}\big) = 0, \\ \mathcal{H}(\varepsilon)\big(\Phi(\varepsilon) - \boldsymbol{\delta}(\varepsilon)\big)\,\big|_{t=0} + H(\varepsilon)U(\varepsilon)\,\big|_{\Gamma_0} = 0. \end{cases} \tag{4.8}$$

The first relation corresponds to the power ε^{-1} and reads

$$\begin{cases} \mathcal{A}_0\Phi_{-1} + \big(0; 0\,\big|_{\partial\omega\times\gamma_\pm}\big) = 0, \\ \mathcal{H}_0\big(\Phi_{-1} - \boldsymbol{\delta}_{-1}\big)\,\big|_{t=0} + H_{-1}U_0\,\big|_{\Gamma_0} = 0. \end{cases}$$

With Φ_{-1} the operator zero and the definitions of $\boldsymbol{\delta}_{-1}$ given in the theorem, the above identity holds.

Let $k \geq 0$, and suppose that the operators Φ_ℓ and $\boldsymbol{\delta}_\ell$ are constructed for $\ell = -1, \ldots, k$ so that the relations corresponding to the powers ε^ℓ in (4.8) hold for $\ell = -1, \ldots, k$.

Note that in both free and clamped cases, we have $H_{-1} \neq 0$ and $H_k = 0$ for $k \geq 2$, and $\mathcal{H}_k = 0$ for $k \geq 2$. Consider for any fixed $\boldsymbol{z} \in \mathcal{C}^\infty(\overline{\omega})^3$ the problem of finding $\boldsymbol{\psi}$ solution of:

$$\begin{cases} \mathcal{B}_0\boldsymbol{\psi} + \displaystyle\sum_{\ell=1}^{k+1} \mathcal{B}_\ell\Phi_{k+1-\ell}\boldsymbol{z} + \sum_{\ell=0}^{k-1} \Lambda_\ell\Phi_{k-1-\ell}\boldsymbol{z} = 0 & \text{in } \Sigma^+, \\ \mathcal{G}_0\boldsymbol{\psi} + \displaystyle\sum_{\ell=1}^{k+1} \mathcal{G}_\ell\Phi_{k+1-\ell}\boldsymbol{z} = 0 & \text{on } \gamma_\pm, \\ \big(\mathcal{H}_0\boldsymbol{\psi} + \mathcal{H}_1(\Phi_k - \boldsymbol{\delta}_k)\boldsymbol{z}\big)\,\big|_{t=0} + \big(H_{-1}U_{k+2}\boldsymbol{z} + H_0U_{k+1}\boldsymbol{z} + H_1U_k\boldsymbol{z}\big)\,\big|_{\Gamma_0} = 0. \end{cases}$$

Proposition 4.1 shows that there exists a unique solution $\boldsymbol{\psi} = \boldsymbol{\varphi} - \boldsymbol{Z}$ to the above problem. Setting $\Phi_{k+1}\boldsymbol{z} := \boldsymbol{\varphi}$ and $\boldsymbol{\delta}_{k+1}\boldsymbol{z} := \boldsymbol{Z}$ we obtain the relation at the rank $k+1$ in (4.8).

Therefore, if $\boldsymbol{z}(\varepsilon)$ is a formal series satisfying $\boldsymbol{\delta}(\varepsilon)\boldsymbol{z}(\varepsilon) = 0$, the identity (4.8) shows that the equations (4.3) and (4.7) hold. ■

In the next subsection, we study the first terms of the formal series $\Phi(\varepsilon)$ and $\boldsymbol{\delta}(\varepsilon)$ according to the boundary condition imposed, and also to the type of the displacement (membrane or bending).

4.d Clamped case

Let us investigate the first non-zero terms in $\boldsymbol{\delta}(\varepsilon)$ and $\Phi(\varepsilon)$. Let us recall that $\boldsymbol{\delta}_{-1}\boldsymbol{z}$ is equal to $\big(z_3\,\big|_{\partial\omega}\big)\boldsymbol{Z}^{\mathrm{b},1}_{\mathrm{D}}$. In order to determine the next operators Φ_0 and $\boldsymbol{\delta}_0$, we have to consider the problem for all $\boldsymbol{z}$ (remind that $U_1 = 0$):

$$\begin{cases} \mathcal{B}_0\Phi_0\boldsymbol{z} = 0 & \text{in } \Sigma^+, \\ \mathcal{G}_0\Phi_0\boldsymbol{z} = 0 & \text{on } \gamma_\pm, \\ (\Phi_0 - \boldsymbol{\delta}_0)\boldsymbol{z}\,\big|_{t=0} + D_0U_0\boldsymbol{z}\,\big|_{\Gamma_0} = 0 \end{cases}$$

with $D_0 U_0 \boldsymbol{z} = (\boldsymbol{z}_* - x_3 \nabla_* z_3, 0)$. Hence, we conclude that

$$\boldsymbol{\delta}_0 \boldsymbol{z} = \big(z_n\big|_{\partial\omega}\big)\boldsymbol{Z}_{\mathrm{D}}^{\mathrm{m},1} + \big(z_s\big|_{\partial\omega}\big)\boldsymbol{Z}_{\mathrm{D}}^{\mathrm{m},2} + \big(\partial_n z_3\big|_{\partial\omega}\big)\boldsymbol{Z}_{\mathrm{D}}^{\mathrm{b},2}$$

and that $\Phi_0 = 0$. So the boundary layer term only starts with the operator Φ_1 .

More generally, for any $k \geq 1$, we can write that

$$\boldsymbol{\delta}_k \boldsymbol{z} = (\delta_k^{\mathrm{m},1}\boldsymbol{z})\,\boldsymbol{Z}_{\mathrm{D}}^{\mathrm{m},1} + (\delta_k^{\mathrm{m},2}\boldsymbol{z})\,\boldsymbol{Z}_{\mathrm{D}}^{\mathrm{m},2} + (\delta_k^{\mathrm{b},1}\boldsymbol{z})\,\boldsymbol{Z}_{\mathrm{D}}^{\mathrm{b},1} + (\delta_k^{\mathrm{b},2}\boldsymbol{z})\,\boldsymbol{Z}_{\mathrm{D}}^{\mathrm{b},2}.$$

Now, the condition $\boldsymbol{\delta}(\varepsilon)\boldsymbol{z}(\varepsilon) = 0$ is equivalent to $\delta^{\mathrm{m},j}(\varepsilon)\boldsymbol{z}(\varepsilon) = 0$ and $\delta^{\mathrm{b},j}(\varepsilon)\boldsymbol{z}(\varepsilon) = 0$, $j = 1, 2$, in the membrane and bending cases respectively. Thus the solvability of problem (4.3) under boundary conditions (4.4) is guaranteed by a condition of the form

$$\boldsymbol{\gamma}(\varepsilon)\boldsymbol{z}(\varepsilon) = 0,$$

where $\boldsymbol{\gamma}(\varepsilon)$ is a formal operator valued series with continuous coefficients from $\mathcal{C}^\infty(\overline{\omega})^3$ into $\mathcal{C}^\infty(\partial\omega)^4$. Then as a consequence of Theorem 4.2, we obtain:

Theorem 4.3

(i) *Let $\boldsymbol{\gamma}^{\mathrm{m}}(\varepsilon) = \sum_{k\geq 0} \varepsilon^k \boldsymbol{\gamma}_k^{\mathrm{m}}$ be defined for any in-plane displacement $\boldsymbol{\zeta} \in \mathcal{C}^\infty(\overline{\omega})^2$ by $\boldsymbol{\gamma}_k^{\mathrm{m}}\boldsymbol{\zeta} = \big(\delta_k^{\mathrm{m},1}, \delta_k^{\mathrm{m},2}\big)(\boldsymbol{\zeta}, 0) \in \mathcal{C}^\infty(\partial\omega)^2$. We also set $\Phi^{\mathrm{m}}(\varepsilon)\,\boldsymbol{\zeta} = \Phi(\varepsilon)\,(\boldsymbol{\zeta}, 0)$. Here $\Phi(\varepsilon)$ and $\boldsymbol{\delta}(\varepsilon)$ are the formal series appearing in Theorem* 4.2*. Then $\Phi_0^{\mathrm{m}} = 0$ and $\boldsymbol{\gamma}_0^{\mathrm{m}}$ coincides with the Dirichlet traces of the membrane operator L^{m} :*

$$\boldsymbol{\gamma}_0^{\mathrm{m}}\boldsymbol{\zeta} = \big(\zeta_n, \zeta_s\big)\,\big|_{\partial\omega}$$

and for each integer k , the operators Φ_k^{m} and $\boldsymbol{\gamma}_k^{\mathrm{m}}$ are polynomial of degree $k-2$ in Λ . Moreover, for any formal series $\boldsymbol{\zeta}(\varepsilon)$ with coefficients in $\mathcal{C}^\infty(\overline{\omega})^2$ solving

$$\boldsymbol{\gamma}^{\mathrm{m}}(\varepsilon)\boldsymbol{\zeta}(\varepsilon) = 0 \quad \textit{in } \partial\omega\,, \tag{4.9}$$

then the three-dimensional formal series $\boldsymbol{\varphi}(\varepsilon) := \Phi^{\mathrm{m}}(\varepsilon)\boldsymbol{\zeta}(\varepsilon)$ is a membrane solution of the three-dimensional eigenvalue formal problem (4.3) *in $\partial\omega \times \Sigma^+$ with the Dirichlet condition*

$$\boldsymbol{\varphi}(\varepsilon)\big|_{t=0} + D(\varepsilon)U^{\mathrm{m}}(\varepsilon)\boldsymbol{\zeta}(\varepsilon)\big|_{\Gamma_0} = 0.$$

(ii) *Let $\boldsymbol{\gamma}^{\mathrm{b}}(\varepsilon) = \sum_{k\geq 0} \varepsilon^k \boldsymbol{\gamma}_k^{\mathrm{b}}$ be defined for any function $\eta \in \mathcal{C}^\infty(\overline{\omega})$ by $\boldsymbol{\gamma}_k^{\mathrm{b}}\eta = \big(\delta_{k-1}^{\mathrm{b},1}, \delta_k^{\mathrm{b},2}\big)(0, \eta) \in \mathcal{C}^\infty(\partial\omega)^2$ and set $\Phi^{\mathrm{b}}(\varepsilon)\,\eta = \Phi(\varepsilon)\,(0, \eta)$. Then $\Phi_0^{\mathrm{b}} = 0$ and $\boldsymbol{\gamma}_0^{\mathrm{b}}$ coincides with the Dirichlet traces of the bending operator L^{b} :*

$$\boldsymbol{\gamma}_0^{\mathrm{m}}\eta = \big(\eta, \partial_n\eta\big)\,\big|_{\partial\omega}$$

and if $\Lambda_0 = \Lambda_1 = 0$, for each integer k the operators Φ_k^{b} and $\boldsymbol{\gamma}_k^{\mathrm{b}}$ are polynomial of degree $k-2$ in Λ^{b} . Moreover, for any formal series $\eta(\varepsilon)$ with coefficients in $\mathcal{C}^\infty(\overline{\omega})$ solving

$$\boldsymbol{\gamma}^{\mathrm{b}}(\varepsilon)\eta(\varepsilon) = 0 \quad \textit{in } \partial\omega\,, \tag{4.10}$$

then the three-dimensional formal series $\boldsymbol{\varphi}(\varepsilon) := \Phi^{\mathrm{b}}(\varepsilon)\eta(\varepsilon)$ is a bending solution of the three-dimensional eigenvalue formal problem (4.3) *in $\partial\omega \times \Sigma^+$ with the Dirichlet condition*

$$\boldsymbol{\varphi}(\varepsilon)\big|_{t=0} + D(\varepsilon)U^{\mathrm{b}}(\varepsilon)\eta(\varepsilon)\big|_{\Gamma_0} = 0.$$

Remark 4.4 By the same computations as in [6, §6], we obtain that $\boldsymbol{\gamma}_1^{\mathrm{m}}\boldsymbol{\zeta}$ is defined as $(c^{\mathrm{m}}\operatorname{div}\boldsymbol{\zeta},0)\big|_{\partial\omega}$ and that $\boldsymbol{\gamma}_1^{\mathrm{b}}\eta=(0,c^{\mathrm{b}}\Delta\eta)\big|_{\partial\omega}$ with c^{m} and c^{b} non-zero constants only depending on the Lamé coefficients λ and μ. ■

4.e Free case

In this case, a basis of $\mathcal{Z}$ is given by the four displacements

$$\boldsymbol{Z}_{\mathrm{N}}^{\mathrm{m},1}=\begin{pmatrix} t\\ 0\\ \bar p_1\end{pmatrix}\quad \boldsymbol{Z}_{\mathrm{N}}^{\mathrm{m},2}=\begin{pmatrix} 0\\ t\\ 0\end{pmatrix}\quad \boldsymbol{Z}_{\mathrm{N}}^{\mathrm{b},1}=\begin{pmatrix} -2tx_3\\ 0\\ t^2+6t\bar p_2\end{pmatrix}$$

$$\text{and}\quad \boldsymbol{Z}_{\mathrm{N}}^{\mathrm{b},2}=\begin{pmatrix} -3t^2x_3+6\bar p_3\\ 0\\ t^3+6t\bar p_2\end{pmatrix},$$

where the polynomials $\bar p_1$, $\bar p_2$ and $\bar p_3$ are defined in (3.20) and (3.21). Moreover, if $\boldsymbol{\Psi}=(\boldsymbol{\psi},\boldsymbol{\psi}^{\pm})\in\mathcal{C}^\infty\big(\partial\omega,\boldsymbol{\mathfrak{K}}(\Sigma^+)\big)$ and $\boldsymbol{v}\in\mathcal{C}^\infty(\overline{\Gamma}_0)^3$, then the element $\boldsymbol{Z}\in\mathcal{Z}$ given by Proposition 4.1 writes

$$\boldsymbol{Z}=\delta^{\mathrm{m},1}\,\boldsymbol{Z}_{\mathrm{N}}^{\mathrm{m},1}+\delta^{\mathrm{m},2}\,\boldsymbol{Z}_{\mathrm{N}}^{\mathrm{m},2}+\delta^{\mathrm{b},1}\,\boldsymbol{Z}_{\mathrm{N}}^{\mathrm{b},1}+\delta^{\mathrm{b},2}\,\boldsymbol{Z}_{\mathrm{N}}^{\mathrm{b},2}$$

with, see [6, Propositions 5.6 & 5.12]:

$$\begin{aligned}\delta^{\mathrm{m},1}&=\int_{\Sigma^+}\Psi_t\,\mathrm{d}t\mathrm{d}x_3-\int_{\mathbb{R}^+}\big(\psi_t^+-\psi_t^-\big)\,\mathrm{d}t+\int_{-1}^1 v_t\,\mathrm{d}x_3,\\ \delta^{\mathrm{m},2}&=\frac{1}{2\mu}\left(\int_{\Sigma^+}\Psi_s\,\mathrm{d}t\mathrm{d}x_3-\int_{\mathbb{R}^+}\big(\psi_s^+-\psi_s^-\big)\,\mathrm{d}t+\int_{-1}^1 v_s\,\mathrm{d}x_3\right)\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}\delta^{\mathrm{b},1}&=\int_{\Sigma^+}\big(-x_3\Psi_t+t\Psi_3\big)\,\mathrm{d}t\mathrm{d}x_3+\int_{\mathbb{R}^+}\big(\psi_t^++\psi_t^--t(\psi_3^+-\psi_3^-)\big)\,\mathrm{d}t-\int_{-1}^1 x_3v_t\,\mathrm{d}x_3,\\ \delta^{\mathrm{b},2}&=\int_{\Sigma^+}\Psi_3\,\mathrm{d}t\mathrm{d}x_3-\int_{\mathbb{R}^+}\big(\psi_3^+-\psi_3^-\big)\,\mathrm{d}t+\int_{-1}^1 v_3\,\mathrm{d}x_3.\end{aligned}\tag{4.12}$$

In the following, we investigate the first non-zero terms in $\boldsymbol{\delta}(\varepsilon)$ and $\Phi(\varepsilon)$. Remind that in the free case $\boldsymbol{\delta}_{-1}=0$. We now study separately the membrane and bending cases.

(*i*) MEMBRANE CASE. Recall that the Neumann traces of the membrane operator L^{m} are

$$\begin{aligned}T_n^{\mathrm{m}}\boldsymbol{\zeta}&=\tilde\lambda\operatorname{div}\boldsymbol{\zeta}+2\mu\partial_n\zeta_n,\\ T_s^{\mathrm{m}}\boldsymbol{\zeta}&=\mu(\partial_s\zeta_n+\partial_n\zeta_s+2\kappa\zeta_s).\end{aligned}\tag{4.13}$$

As in the clamped case, we define for all k the operators Φ_k^{m} and $\boldsymbol{\delta}_k^{\mathrm{m}}$ by the equations

$$\Phi_k^{\mathrm{m}}\boldsymbol{\zeta}=\Phi_k(\boldsymbol{\zeta},0)\quad\text{and}\quad\boldsymbol{\delta}_k^{\mathrm{m}}\boldsymbol{\zeta}=\boldsymbol{\delta}_k(\boldsymbol{\zeta},0)$$

for $\boldsymbol{\zeta} \in \mathcal{C}^\infty(\overline{\omega})^2$. We have for all $k \geq 0$

$$\boldsymbol{\delta}^{\rm m}_k \boldsymbol{\zeta} = (\delta^{\rm m,1}_k \boldsymbol{\zeta}) \boldsymbol{Z}^{\rm m,1}_{\rm N} + (\delta^{\rm m,2}_k \boldsymbol{\zeta}) \boldsymbol{Z}^{\rm m,2}_{\rm N}.$$

For $k = 0$, the operators satisfy for all $\boldsymbol{\zeta}$ the equation (remind $U_1 = 0$)

$$\begin{cases} \mathcal{B}_0 \Phi^{\rm m}_0 \boldsymbol{\zeta} &= 0 \quad \text{in} \quad \Sigma^+, \\ \mathcal{G}_0 \Phi^{\rm m}_0 \boldsymbol{\zeta} &= 0 \quad \text{on} \quad \gamma_\pm, \\ \mathcal{T}_0 (\Phi^{\rm m}_0 - \boldsymbol{\delta}^{\rm m}_0) \boldsymbol{\zeta}\big|_{t=0} + N_0 U^{\rm m}_0 \boldsymbol{\zeta}\big|_{\Gamma_0} &= 0. \end{cases}$$

As there holds $N_0 U^{\rm m}_0 \boldsymbol{\zeta} = 0$, we obtain $\Phi^{\rm m}_0 = 0$ and $\boldsymbol{\delta}^{\rm m}_0 = 0$. For $k = 1$, we then have

$$\begin{cases} \mathcal{B}_0 \Phi^{\rm m}_1 \boldsymbol{\zeta} &= 0 \quad \text{in} \quad \Sigma^+, \\ \mathcal{G}_0 \Phi^{\rm m}_1 \boldsymbol{\zeta} &= 0 \quad \text{on} \quad \gamma_\pm, \\ \mathcal{T}_0 (\Phi^{\rm m}_1 - \boldsymbol{\delta}^{\rm m}_1) \boldsymbol{\zeta}\big|_{t=0} + (N_{-1} U^{\rm m}_2 \boldsymbol{\zeta} + N_1 U^{\rm m}_0 \boldsymbol{\zeta})\big|_{\Gamma_0} &= 0. \end{cases}$$

A computation shows that in coordinates (t, s, x_3),

$$N_{-1} U^{\rm m}_2 \boldsymbol{\zeta} + N_1 U^{\rm m}_0 \boldsymbol{\zeta} = \big(T^{\rm m}_n \boldsymbol{\zeta} \,, T^{\rm m}_s \boldsymbol{\zeta} \,, 0 \big).$$

From (4.11), we deduce immediately that

$$\delta^{\rm m,1}_1 = 2 T^{\rm m}_n \quad \text{and} \quad \delta^{\rm m,2}_1 = \tfrac{1}{\mu} T^{\rm m}_s.$$

Moreover, we can check that $\Phi^{\rm m}_1 = 0$. Consequently, Theorem 4.2 yields the following result:

Theorem 4.5 *Let $\boldsymbol{\gamma}^{\rm m}(\varepsilon) = \sum_{k \geq 0} \varepsilon^k \boldsymbol{\gamma}^{\rm m}_k$ be defined for any in-plane displacement $\boldsymbol{\zeta} \in \mathcal{C}^\infty(\overline{\omega})^2$ by $\boldsymbol{\gamma}^{\rm m}_k \boldsymbol{\zeta} = \big(\frac{1}{2} \delta^{\rm m,1}_{k+1} \,, \mu \delta^{\rm m,2}_{k+1} \big) \boldsymbol{\zeta}$. Then $\Phi^{\rm m}_0 = \Phi^{\rm m}_1 = 0$ and $\boldsymbol{\gamma}^{\rm m}_0$ coincides with the Neumann traces of the membrane operator $L^{\rm m}$:*

$$\boldsymbol{\gamma}^{\rm m}_0 \boldsymbol{\zeta} = \big(T^{\rm m}_n \boldsymbol{\zeta}, T^{\rm m}_s \boldsymbol{\zeta} \big)\big|_{\partial\omega}$$

and for each integer k, the operators $\Phi^{\rm m}_k$ and $\boldsymbol{\gamma}^{\rm m}_k$ are polynomial of degree $k-1$ in Λ. Moreover, for any formal series $\boldsymbol{\zeta}(\varepsilon)$ with coefficients in $\mathcal{C}^\infty(\overline{\omega})^2$ solving

$$\boldsymbol{\gamma}^{\rm m}(\varepsilon) \boldsymbol{\zeta}(\varepsilon) = 0 \quad \textit{in } \partial\omega \,, \tag{4.14}$$

then the three-dimensional formal series $\boldsymbol{\varphi}(\varepsilon) := \Phi^{\rm m}(\varepsilon) \boldsymbol{\zeta}(\varepsilon)$ is a membrane solution of the three-dimensional eigenvalue formal problem (4.3) *in $\partial\omega \times \Sigma^+$ with the Neumann condition*

$$\mathcal{T}(\varepsilon) \boldsymbol{\varphi}(\varepsilon)\big|_{t=0} + N(\varepsilon) U^{\rm m}(\varepsilon) \boldsymbol{\zeta}(\varepsilon)\big|_{\Gamma_0} = 0.$$

(*ii*) BENDING CASE. As usual, we suppose $\Lambda_0 = \Lambda_1 = 0$. Recall that the Neumann traces of the bending operator L^{b} are

$$\begin{array}{rcl} M_n\eta & = & \tilde\lambda\Delta\eta + 2\mu\partial_{nn}\eta, \\ N_n\eta & = & (\tilde\lambda + 2\mu)\partial_n\Delta\eta + 2\mu\partial_s(\partial_n + \kappa)\partial_s\eta. \end{array} \tag{4.15}$$

We define for all k the operators Φ_k^{b} and $\boldsymbol{\delta}_k^{\mathrm{b}}$ by the equations

$$\Phi_k^{\mathrm{b}}\eta = \Phi_k(0,\eta) \quad \text{and} \quad \boldsymbol{\delta}_k^{\mathrm{b}}\eta = \boldsymbol{\delta}_k(0,\eta)$$

for $\eta \in \mathcal{C}^\infty(\overline{\varpi})$. We have for all $k \geq 0$

$$\boldsymbol{\delta}_k^{\mathrm{b}}\boldsymbol{\zeta} = (\delta_k^{\mathrm{b},1}\boldsymbol{\zeta})\boldsymbol{Z}_{\mathrm{N}}^{\mathrm{b},1} + (\delta_k^{\mathrm{b},2}\boldsymbol{\zeta})\boldsymbol{Z}_{\mathrm{N}}^{\mathrm{b},2}.$$

As in the membrane case, using the fact that $N_0 U_0^{\mathrm{b}}\eta = 0$, we have that $\Phi_0^{\mathrm{b}} = 0$ and $\boldsymbol{\delta}_0^{\mathrm{b}} = 0$. For $k = 1$, we have to consider the problem for all η

$$\left\{ \begin{array}{rcll} \mathcal{B}_0\Phi_1^{\mathrm{b}}\eta & = & 0 & \text{in} \quad \Sigma^+, \\ \mathcal{G}_0\Phi_1^{\mathrm{b}}\eta & = & 0 & \text{on} \quad \gamma_\pm, \\ \mathcal{T}_0(\Phi_1^{\mathrm{b}} - \boldsymbol{\delta}_1^{\mathrm{b}})\eta\big|_{t=0} + (N_{-1}U_2^{\mathrm{b}}\eta + N_1U_0^{\mathrm{b}}\eta)\big|_{\Gamma_0} & = & 0. & \end{array} \right.$$

A computation shows that

$$N_{-1}U_2^{\mathrm{b}}\eta + N_1U_0^{\mathrm{b}}\eta = \big(-x_3M_n\eta\,,\, -x_3 2\mu(\partial_n + \kappa)\partial_s\eta\,,\, 0\big),$$

whence, with (4.12) we deduce

$$\delta_1^{\mathrm{b},1} = \tfrac{2}{3}M_n \quad \text{and} \quad \delta_1^{\mathrm{b},2} = 0.$$

Moreover we have $\Phi_1^{\mathrm{b}} = (0, \Phi_{1,s}^{\mathrm{b}}, 0)$ with, see [6, Section 10], $\Phi_{1,s}^{\mathrm{b}}\eta = \varphi_{\mathrm{Neu}}(\partial_n + \kappa)\partial_s\eta$ where φ_{Neu} is the solution of the Neumann problem in the half-strip Σ^+ with data 0 on $\gamma_\pm$ and $2x_3$ on $\partial\omega$. The function φ_{Neu} is exponentially decreasing and there holds, see [6, Lemma 5.7],

$$\int_0^\infty \varphi_{\mathrm{Neu}}(t, x_3)\,\mathrm{d}t\mathrm{d}x_3 = -\frac{2}{3}. \tag{4.16}$$

At this stage, we still do not have the first non-zero term of the formal series $\delta^{\mathrm{b},2}(\varepsilon)$. It remains now to compute the term $\delta_2^{\mathrm{b},2}$. For $k = 2$, we have the equations for all η

$$\left\{ \begin{array}{rcll} \mathcal{B}_0\Phi_2^{\mathrm{b}}\eta & = & -\mathcal{B}_1\Phi_1^{\mathrm{b}}\eta & \text{in} \quad \Sigma^+, \\ \mathcal{G}_0\Phi_2^{\mathrm{b}}\eta & = & -\mathcal{G}_1\Phi_1^{\mathrm{b}}\eta & \text{on} \quad \gamma_\pm, \\ \mathcal{T}_0(\Phi_2^{\mathrm{b}} - \boldsymbol{\delta}_2^{\mathrm{b}})\eta\big|_{t=0} + \mathcal{T}_1(\Phi_1^{\mathrm{b}} - \boldsymbol{\delta}_1^{\mathrm{b}})\eta\big|_{t=0} + N_0U_2^{\mathrm{b}}\eta\big|_{\Gamma_0} & = & 0. & \end{array} \right.$$

According to (4.12), only the third components are necessary to compute $\delta_2^{\mathrm{b},2}\eta$. We have, see [6, Equation (4.2)], that $(\mathcal{T}_1)_3 = 0$, and with the expression of $\mathcal{B}_1$, [6, Equation (4.6)], and Φ_1^{b}, we have $(\mathcal{B}_1\Phi_1^{\mathrm{b}}\eta)_3 = (\lambda + \mu)\partial_3\partial_s\Phi_{1,s}^{\mathrm{b}}\eta$. Similarly, with the expression of $\mathcal{G}_1$, [6, Equation (4.10)], we have $(\mathcal{G}_1\Phi_1^{\mathrm{b}}\eta)_3 = \lambda\partial_s\Phi_{1,s}^{\mathrm{b}}\eta$. Finally, we have the formula $(N_0U_2^{\mathrm{b}}\eta)_3 = \mu(\bar{p}_2 + \bar{p}_3')\partial_n\Delta\eta$, and using (4.12) and (4.16) , we find that

$$\delta_2^{\mathrm{b},2}\eta = -\frac{2}{3}N_n\eta.$$

As consequence of Theorem 4.2, we have

Theorem 4.6 *Let $\boldsymbol{\gamma}^{\mathrm{b}}(\varepsilon) = \sum_{k\geq 0}\varepsilon^k\boldsymbol{\gamma}_k^{\mathrm{b}}$ be defined for any function $\eta\in\mathcal{C}^\infty(\overline{\omega})$ by $\boldsymbol{\gamma}_k^{\mathrm{b}}\eta = \frac{3}{2}\big(\delta_{k+1}^{\mathrm{b},1}, -\delta_{k+2}^{\mathrm{b},2}\big)\eta$. Then $\Phi_0^{\mathrm{b}} = 0$ and $\boldsymbol{\gamma}_0^{\mathrm{b}}$ coincides with the Neumann traces of the bending operator L^{b} :*

$$\boldsymbol{\gamma}_0^{\mathrm{b}}\eta = \big(M_n\eta, N_n\eta\big)\big|_{\partial\omega}$$

and if $\Lambda_0 = \Lambda_1 = 0$, for each integer k the operators Φ_k^{b} and $\boldsymbol{\gamma}_k^{\mathrm{b}}$ are polynomial of degree $k-1$ in Λ^{b}. Moreover, for any formal series $\eta(\varepsilon)$ with coefficients in $\mathcal{C}^\infty(\overline{\omega})$ solving

$$\boldsymbol{\gamma}^{\mathrm{b}}(\varepsilon)\eta(\varepsilon) = 0 \quad \textit{in } \partial\omega\,, \tag{4.17}$$

then the three-dimensional formal series $\boldsymbol{\varphi}(\varepsilon) := \Phi^{\mathrm{b}}(\varepsilon)\eta(\varepsilon)$ is a bending solution of the three-dimensional eigenvalue formal problem (4.3) *in $\partial\omega\times\Sigma^+$ with the Neumann condition*

$$\mathcal{T}(\varepsilon)\boldsymbol{\varphi}(\varepsilon)\big|_{t=0} + N(\varepsilon)U^{\mathrm{b}}(\varepsilon)\eta(\varepsilon)\big|_{\Gamma_0} = 0.$$

5 Solution of the in-plane eigenvalue problems

Collecting the results in Theorems 3.4 and 4.3 - 4.5, we obtain that in the membrane case, if the formal series $\boldsymbol{\zeta}(\varepsilon)$ and $\Lambda(\varepsilon)$ solve

$$\begin{cases} \big(L^{\mathrm{m}}(\varepsilon) + \Lambda(\varepsilon)\big)\boldsymbol{\zeta}(\varepsilon) = 0 & \text{in} \quad \omega, \\ \boldsymbol{\gamma}^{\mathrm{m}}(\varepsilon)\boldsymbol{\zeta}(\varepsilon) = 0 & \text{on} \quad \partial\omega, \end{cases} \tag{5.1}$$

then the formal series $\boldsymbol{u}(\varepsilon) := U^{\mathrm{m}}(\varepsilon)\boldsymbol{\zeta}(\varepsilon)$ and $\boldsymbol{\varphi}(\varepsilon) := \Phi^{\mathrm{m}}(\varepsilon)\boldsymbol{\zeta}(\varepsilon)$ solve the three-dimensional eigen-problems (3.3) and (4.3) respectively, moreover the sum of their traces on the lateral boundary solve (4.6). We will prove in §6 that, provided (5.1) holds, the partial sums of series $U^{\mathrm{m}}(\varepsilon)\boldsymbol{\zeta}(\varepsilon)$ and $\Phi^{\mathrm{m}}(\varepsilon)\boldsymbol{\zeta}(\varepsilon)$ yield three-dimensional $\mathcal{O}(\varepsilon^k)$ -quasi-modes. We have similar statements for the bending case if the formal series $\eta(\varepsilon)$ and $\Lambda^{\mathrm{b}}(\varepsilon)$ solve

$$\begin{cases} \big(L^{\mathrm{b}}(\varepsilon) - \Lambda^{\mathrm{b}}(\varepsilon)\big)\eta(\varepsilon) = 0 & \text{in} \quad \omega, \\ \boldsymbol{\gamma}^{\mathrm{b}}(\varepsilon)\eta(\varepsilon) = 0 & \text{on} \quad \partial\omega. \end{cases} \tag{5.2}$$

Thus the three-dimensional formal series eigenproblems, including inner and outer parts, reduce to two-dimensional formal series eigenproblems.

5.a Change of unknowns

By a simple change of unknowns we are going to replace the formal series $\boldsymbol{\zeta}(\varepsilon)$ or $\eta(\varepsilon)$ in problems (5.1) or (5.2) by new formal series $\breve{\boldsymbol{\zeta}}(\varepsilon)$ or $\breve{\eta}(\varepsilon)$ which will have to satisfy the homogeneous boundary conditions associated with operators L^{m} or L^{b}.

Theorem 5.1 *Let $\boldsymbol{\gamma}(\varepsilon)$ be any formal series defined by Theorems* 4.3, 4.5 *and* 4.6*. Then there exists an invertible formal series $C(\varepsilon) = \sum_{k\geq 0} \varepsilon^k C_k$ where $C_0 = \mathrm{Id}$ and C_k is continuous : $\mathcal{C}^\infty(\overline{\omega})^p \to \mathcal{C}^\infty(\overline{\omega})^p$ with $p = 1$ and 2 in the bending and membrane case respectively, such that*

$$\boldsymbol{\gamma}_0 \circ C(\varepsilon) = \boldsymbol{\gamma}(\varepsilon).$$

Proof. We only have to prove the existence of the operators C_k satisfying $\boldsymbol{\gamma}_0 \circ C_k = \boldsymbol{\gamma}_k$. We can obviously take $C_0 = \mathrm{Id}$. For $k \geq 1$, it suffices to set $C_k = R_0 \circ \boldsymbol{\gamma}_k$, where R_0 is a lifting operator corresponding to the trace operator $\boldsymbol{\gamma}_0$, that is a continuous operator $\mathcal{C}^\infty(\partial\omega)^2 \to \mathcal{C}^\infty(\overline{\omega})^p$ such that $\boldsymbol{\gamma}_0 \circ R_0 = \mathrm{Id}$. ∎

By composition with $C^{-1}(\varepsilon)$ the inverse formal series to $C(\varepsilon)$ we obtain:

Corollary 5.2 *In both clamped and free plate cases, we have:*

(i) *With the change of unknowns $\breve{\boldsymbol{\zeta}}(\varepsilon) := C(\varepsilon)\boldsymbol{\zeta}(\varepsilon)$, problem* (5.1) *is equivalent to the system*

$$\begin{cases} \big(\breve{L}^{\mathrm{m}}(\varepsilon) + \Lambda(\varepsilon)\big)\breve{\boldsymbol{\zeta}}(\varepsilon) &=& 0 & \quad \textit{in} \quad \omega, \\ \boldsymbol{\gamma}_0^{\mathrm{m}}\breve{\boldsymbol{\zeta}}(\varepsilon) &=& 0 & \quad \textit{on} \quad \partial\omega, \end{cases} \tag{5.3}$$

where $\breve{L}^{\mathrm{m}}(\varepsilon)$ is a formal series with coefficients $\breve{L}^{\mathrm{m}}_\ell : \mathcal{C}^\infty(\overline{\omega})^2 \to \mathcal{C}^\infty(\overline{\omega})^2$ of degree $\ell - 1$ in Λ, and such that $\breve{L}^{\mathrm{m}}_0 = L^{\mathrm{m}}$.

(ii) *With the change of unknowns $\breve{\eta}(\varepsilon) := C(\varepsilon)\eta(\varepsilon)$, problem* (5.2) *is equivalent to the problem*

$$\begin{cases} \big(\breve{L}^{\mathrm{b}}(\varepsilon) - \Lambda^{\mathrm{b}}(\varepsilon)\big)\breve{\eta}(\varepsilon) &=& 0 & \quad \textit{in} \quad \omega, \\ \boldsymbol{\gamma}_0^{\mathrm{b}}\breve{\eta}(\varepsilon) &=& 0 & \quad \textit{on} \quad \partial\omega, \end{cases} \tag{5.4}$$

where $\breve{L}^{\mathrm{b}}(\varepsilon)$ is a formal series with coefficients $\breve{L}^{\mathrm{b}}_\ell : \mathcal{C}^\infty(\overline{\omega}) \to \mathcal{C}^\infty(\overline{\omega})$ of degree $\ell - 1$ in Λ^{b}, and such that $\breve{L}^{\mathrm{b}}_0 = L^{\mathrm{b}}$.

5.b Solution of the plane formal series eigenproblems

In this subsection, we solve the equations (5.3) and (5.4). We first investigate the membrane case. The bending case will be very similar.

Theorem 5.3 *Let Λ_0 be an eigenvalue of the operator $-L^{\mathrm{m}}$ with boundary condition $\boldsymbol{\gamma}_0^{\mathrm{m}}$ and let F_0 be the corresponding eigenspace. There exist an integer $d \leq \dim F_0$ and a splitting of F_0 into d subspaces F^ℓ_∞ :*

$$F_0 = \bigoplus_{\ell=1}^{d} F^\ell_\infty$$

such that for all $\ell = 1, \ldots, d$, there exist

- *a formal series* $\Lambda^\ell(\varepsilon)$ *with real coefficients such that* $\Lambda^\ell_0 = \Lambda_0$,
- *a formal series* $\vartheta^\ell(\varepsilon)$ *with coefficients* $\vartheta^\ell_k : F^\ell_\infty \to \mathcal{C}^\infty(\overline{\omega})^2$

with the following property: For all $\boldsymbol{\zeta}_\infty \in F^\ell_\infty$ *, the formal series* $\breve{\boldsymbol{\zeta}}^\ell(\varepsilon) = \vartheta^\ell(\varepsilon)\boldsymbol{\zeta}_\infty$ *solves problem* (5.3) *with* $\Lambda(\varepsilon) = \Lambda^\ell(\varepsilon)$.

Proof. Let us first note that the formal series $\breve{\boldsymbol{\zeta}}(\varepsilon) = \sum_{k\ge 0} \varepsilon^k \breve{\boldsymbol{\zeta}}^k$ and $\Lambda(\varepsilon) = \sum_{k\ge 0} \varepsilon^k \Lambda_k$ satisfy the equation (5.3) if and only if for each $k \ge 0$

$$\boldsymbol{\gamma}^{\mathrm{m}}_0 \breve{\boldsymbol{\zeta}}^k = 0 \quad \text{and} \quad \big(L^{\mathrm{m}} + \Lambda_0\big)\breve{\boldsymbol{\zeta}}^k = -\sum_{\ell=1}^{k} \big(\breve{L}_\ell + \Lambda_\ell\big)\breve{\boldsymbol{\zeta}}^{k-\ell}. \tag{5.5}$$

For $k = 0$, this equation reads $\big(L^{\mathrm{m}} + \Lambda_0\big)\breve{\boldsymbol{\zeta}}^0 = 0$. Since Λ_0 is an eigenvalue of $-L^{\mathrm{m}}$, the equation is solvable and $\breve{\boldsymbol{\zeta}}^0$ can be chosen at this stage as any element $\boldsymbol{\zeta}^1_0$ in F_0 .

For $k = 1$ we thus take any $\boldsymbol{\zeta}^1_0 \in F_0$ and consider the problem of finding $\boldsymbol{\zeta}$ such that

$$\left\{\begin{array}{rcll} (\breve{L}^{\mathrm{m}} + \Lambda_0)\boldsymbol{\zeta} & = & -(\breve{L}^{\mathrm{m}}_1 + \Lambda_1)\boldsymbol{\zeta}^1_0 & \quad \text{in} \quad \omega, \\ \boldsymbol{\gamma}^{\mathrm{m}}_0\boldsymbol{\zeta} & = & 0 & \quad \text{on} \quad \partial\omega. \end{array}\right. \tag{5.6}$$

This problem is solvable if the right-hand side is orthogonal to F_0 . Defining $\mathfrak{M}_1$ by

$$\mathfrak{M}_1 : F_0 \to F_0 \qquad \forall\, \boldsymbol{\zeta},\ \boldsymbol{\psi} \in F_0, \quad \langle\, \mathfrak{M}_1\, \boldsymbol{\zeta}\,,\, \boldsymbol{\psi}\,\rangle = \langle\, \breve{L}^{\mathrm{m}}_1\, \boldsymbol{\zeta}\,,\, \boldsymbol{\psi}\,\rangle, \tag{5.7}$$

this orthogonality condition reads

$$(\mathfrak{M}_1 + \Lambda_1)\boldsymbol{\zeta}^1_0 = 0. \tag{5.8}$$

But $\mathfrak{M}_1$ is symmetric, as will be proved later on. Hence for any eigenvalue Λ_1 of $\mathfrak{M}_1$ we can take $\boldsymbol{\zeta}^1_0$ as any element $\boldsymbol{\zeta}^2_1$ of the corresponding eigenspace F_1 . Then (5.6) is solvable and admits as solution any element $\boldsymbol{\zeta}^1_{-1}$ of the form $\boldsymbol{\zeta}^2_0 + \theta^1_0\boldsymbol{\zeta}^2_1$, where $\boldsymbol{\zeta}^2_0$ is any element in F_0 and $\theta^1_0 : F_1 \to J_0$ (with J_0 the orthogonal complement of F_0 in $\mathcal{C}^\infty(\overline{\omega})^2$) the solution operator in J_0 of problem (5.6).

Thus we have (partly) proved the two first steps of the following lemma:

Lemma 5.4 *Set* $F_{-1} = \mathcal{C}^\infty(\overline{\omega})^2$, F_0 *as in Theorem* 5.3 *and* J_0 *the orthogonal complement of* F_0 *in* F_{-1} *for the scalar product of* $L^2(\omega)^2$ *. We take* Λ_0 *as in Theorem* 5.3. *There exist for all* $i \ge 1$

- *a real* Λ_i *and orthogonal subspaces* F_i *and* J_i *, such that* $F_i \oplus J_i = F_{i-1}$,
- *for any* $j > i$ *, operators* $\theta^j_i : F_j \to J_i$,

allowing for any fixed $n \ge 0$ *, the construction of solutions of the equations* (5.5) *for* $k \in \{0,\dots,n\}$ *in the following way: Choosing any functions* $\boldsymbol{\zeta}^{n+1}_\ell \in F_\ell$ *for* $\ell = 0,\dots,n$ *, we construct successively the functions* $\boldsymbol{\zeta}^k_\ell$ *for* $k = n,\dots,0$ *by*

$$\forall\, \ell = -1,\dots,k-1\,, \qquad \boldsymbol{\zeta}^k_\ell = \boldsymbol{\zeta}^{k+1}_{\ell+1} + \sum_{j=\ell+2}^{k} \theta^j_{\ell+1}\, \boldsymbol{\zeta}^{k+1}_j. \tag{5.9}$$

Thus the functions $\boldsymbol{\zeta}^0_{-1},\dots,\boldsymbol{\zeta}^n_{-1}$ *depend linearly on the generating functions* $\boldsymbol{\zeta}^{n+1}_0$, ..., $\boldsymbol{\zeta}^{n+1}_n$ *via a lower triangular* $(n+1)\times(n+1)$ *matrix operator* Θ^n *:*

$$\big(\boldsymbol{\zeta}^0_{-1},\dots,\boldsymbol{\zeta}^n_{-1}\big)^\top = \Theta^n\big(\boldsymbol{\zeta}^{n+1}_n,\dots,\boldsymbol{\zeta}^{n+1}_0\big)^\top.$$

Then the functions $\breve{\boldsymbol\zeta}^k := \boldsymbol\zeta^k_{-1}$ *for* $k = 0,\ldots,n$ *satisfy the first* $n+1$ *equations of* (5.5).

Moreover for any $n \geq 0$, Λ_n *is an eigenvalue and* F_n *the associated eigenspace of the symmetric operator* $\mathfrak{M}_n : F_{n-1} \to F_{n-1}$ *defined by*

$$\forall\, \boldsymbol\zeta, \boldsymbol\psi \in F_{n-1}, \quad \langle\, \mathfrak{M}_n\, \boldsymbol\zeta\,,\, \boldsymbol\psi\,\rangle = \langle\, M_n\, \boldsymbol\zeta\,,\, \boldsymbol\psi\,\rangle\,, \tag{5.10}$$

where $\langle\cdot,\cdot\rangle$ *is the* L^2 *scalar product on* $\mathcal{C}^\infty(\overline{\omega})^2$ *and the operator* M_n *is such that the following identity holds for any* $\boldsymbol\zeta^n_\ell \in F_\ell$ *(* $\ell = 0,\ldots,n-1$ *)*

$$\sum_{\ell=1}^{n} \big(\breve{L}^{\mathrm{m}}_\ell + \Lambda_\ell\big)\boldsymbol\zeta^{n-\ell}_{-1} = \sum_{\ell=1}^{n}\big(M_\ell + \Lambda_\ell\big)\boldsymbol\zeta^n_{\ell-1}, \tag{5.11}$$

where the functions $\big(\boldsymbol\zeta^0_{-1},\ldots,\boldsymbol\zeta^{n-1}_{-1}\big)^\top$ *are given by* $\Theta^{n-1}\big(\boldsymbol\zeta^n_{n-1},\ldots,\boldsymbol\zeta^n_0\big)^\top$.

Proof of Lemma 5.4.

a) We will prove the existence of Λ_i, F_i, M_i and θ^j_i by induction.

For $n \in \mathbb{N}$, the induction hypothesis is that Λ_i, F_i, M_i are constructed for $i = 0,\ldots,n$ and operators θ^j_i for $i = 0,\ldots,n-1$ and $j = i+1,\ldots,n$ (thus operators Θ^ℓ for $\ell = 0,\ldots,n$) such that the assertions of the lemma are satisfied.

For $n = 0$, the hypothesis is satisfied clearly by setting $M_0 = L^{\mathrm{m}}$.

Suppose now it holds for $n \geq 0$.

Let us take $n+1$ generating functions $\boldsymbol\zeta^{n+1}_\ell \in F_\ell$, $\ell = 0,\ldots,n$, and let $\big(\boldsymbol\zeta^0_{-1},\ldots,\boldsymbol\zeta^n_{-1}\big)^\top$ be defined as $\Theta^n\big(\boldsymbol\zeta^{n+1}_n,\ldots,\boldsymbol\zeta^{n+1}_0\big)^\top$. The assumption yields that the first $n+1$ equations of (5.5) are satisfied with $\breve{\boldsymbol\zeta}^k := \boldsymbol\zeta^k_{-1}$. In order to construct solutions for the equation of (5.5) corresponding to $k = n+1$, we have to solve the problem of finding Λ_{n+1} in $\mathbb{R}$ and $\boldsymbol\zeta$ satisfying

$$\big(L^{\mathrm{m}} + \Lambda_0\big)\boldsymbol\zeta = -\sum_{\ell=1}^{n+1}\big(\breve{L}^{\mathrm{m}}_\ell + \Lambda_\ell\big)\boldsymbol\zeta^{n+1-\ell}_{-1},$$

with the boundary condition $\boldsymbol\gamma_0\boldsymbol\zeta = 0$ on $\partial\omega$. Note that for $\ell = 1,\ldots,n+1$ the operators $\breve{L}^{\mathrm{m}}_\ell$ only depend on $\Lambda_0,\ldots,\Lambda_{n-1}$ and are well defined at this recursion step. We are going to solve this problem via new conditions on the generating functions $\boldsymbol\zeta^{n+1}_\ell$.

Thanks to formula (5.9) for $k = n$:

$$\forall\, \ell = -1,\ldots,n-1\,, \qquad \boldsymbol\zeta^n_\ell = \boldsymbol\zeta^{n+1}_{\ell+1} + \sum_{j=\ell+2}^{n-1}\theta^j_{\ell+1}\,\boldsymbol\zeta^{n+1}_j + \theta^n_{\ell+1}\,\boldsymbol\zeta^{n+1}_n,$$

we can see that

$$\Theta^n\big(\boldsymbol\zeta^{n+1}_n,\ldots,\boldsymbol\zeta^{n+1}_0\big)^\top = \Theta^{n-1}\big(\boldsymbol\zeta^{n+1}_{n-1},\ldots,\boldsymbol\zeta^{n+1}_0\big)^\top + \mathrm{Op}(\boldsymbol\zeta^{n+1}_n)\,,$$

where Op is a linear operator. Thus formula (5.11) yields that

$$\sum_{\ell=1}^{n}\big(\breve{L}^{\mathrm{m}}_\ell + \Lambda_\ell\big)\boldsymbol\zeta^{n+1-\ell}_{-1} = \sum_{\ell=1}^{n}\big(M_\ell + \Lambda_\ell\big)\boldsymbol\zeta^{n+1}_{\ell-1} + \mathrm{Op}(\boldsymbol\zeta^{n+1}_n),$$

where Op is another linear operator. Therefore there exists a new operator M_{n+1} such that the previous problem takes the form

$$\left\{\begin{array}{rcll} (L^{\mathrm{m}}+\Lambda_0)\boldsymbol{\zeta} & = & -\displaystyle\sum_{\ell=1}^{n}\big(M_\ell+\Lambda_\ell\big)\boldsymbol{\zeta}_{\ell-1}^{n+1}-\big(M_{n+1}+\Lambda_{n+1}\big)\boldsymbol{\zeta}_{n}^{n+1} & \text{in} \quad \omega, \\ \boldsymbol{\gamma}_0\boldsymbol{\zeta} & = & 0 & \text{on} \quad \partial\omega. \end{array}\right. \tag{5.12}$$

We then define the operator $\mathfrak{M}_{n+1}: F_n \to F_n$ by the formula (5.10). Now, problem (5.12) is solvable if the right-hand side is orthogonal to F_0.

By taking the scalar product of the right-hand side of (5.12) with any element $\boldsymbol{\psi}$ in F_n, we have to obtain zero since $F_n \subset F_0$. As $\boldsymbol{\zeta}_{\ell-1}^{n+1}$ belongs to $F_{\ell-1}$, it is in the domain of the operator $\mathfrak{M}_\ell + \Lambda_\ell$. Thus for any $l = 1, \dots, n$

$$\Big\langle \big(M_\ell+\Lambda_\ell\big)\boldsymbol{\zeta}_{\ell-1}^{n+1}\,,\ \boldsymbol{\psi}\Big\rangle = \Big\langle \big(\mathfrak{M}_\ell+\Lambda_\ell\big)\boldsymbol{\zeta}_{\ell-1}^{n+1}\,,\ \boldsymbol{\psi}\Big\rangle = \Big\langle \boldsymbol{\zeta}_{\ell-1}^{n+1}\,,\ \big(\mathfrak{M}_\ell+\Lambda_\ell\big)\boldsymbol{\psi}\Big\rangle = 0,$$

since $\boldsymbol{\psi}$ belongs to $F_n \subset F_\ell$ with F_ℓ the kernel of $\big(\mathfrak{M}_\ell + \Lambda_\ell\big)$. Therefore a necessary condition for the solvability of problem (5.12) is the condition $\big(\mathfrak{M}_{n+1}+\Lambda_{n+1}\big)\boldsymbol{\zeta}_n^0 = 0$.

We postpone the proof of the symmetry of the operator $\mathfrak{M}_{n+1}$ to part **b)** of this demonstration. We then have the existence of an eigenvalue Λ_{n+1} and of its associated eigenspace $F_{n+1} \subset F_n$.

We hence can solve step by step the equations obtained by projections on $F_{n-1}, \dots, F_0$ of the equation of problem (5.12). We first take $\boldsymbol{\zeta}_{n+1}^{n+2} \in F_{n+1}$ and set $\boldsymbol{\zeta}_{n}^{n+1} = \boldsymbol{\zeta}_{n+1}^{n+2}$. Then for any $\boldsymbol{\zeta}_{n}^{n+2} \in F_n$, we take

$$\boldsymbol{\zeta}_{n-1}^{n+1} = \boldsymbol{\zeta}_{n}^{n+2} + \theta_n^{n+1}\boldsymbol{\zeta}_{n+1}^{n+2},$$

where $\theta_n^{n+1}: F_{n+1} \to J_n$ is the solution operator $\boldsymbol{\eta} \mapsto \boldsymbol{\zeta}$ of the problem

$$\boldsymbol{\zeta} \in J_n \quad \text{such that} \quad \forall \boldsymbol{\psi} \in F_n: \quad \Big\langle \big(\mathfrak{M}_n+\Lambda_n\big)\boldsymbol{\zeta}\,,\ \boldsymbol{\psi}\Big\rangle = -\Big\langle \big(M_{n+1}+\Lambda_{n+1}\big)\boldsymbol{\eta}\,,\ \boldsymbol{\psi}\Big\rangle.$$

Note that this problem is solvable since, as $\boldsymbol{\eta} \in F_{n+1}$, for any $\boldsymbol{\psi} \in F_n$ there holds

$$\Big\langle \big(M_{n+1}+\Lambda_{n+1}\big)\boldsymbol{\eta}\,,\ \boldsymbol{\psi}\Big\rangle = \Big\langle \big(\mathfrak{M}_{n+1}+\Lambda_{n+1}\big)\boldsymbol{\eta}\,,\ \boldsymbol{\psi}\Big\rangle = 0.$$

Similarly for any $\boldsymbol{\zeta}_\ell^{n+2} \in F_\ell$, with $l = n-1, \dots, 0$, we take

$$\boldsymbol{\zeta}_{\ell-1}^{n+1} = \boldsymbol{\zeta}_{\ell}^{n+2} + \sum_{j=\ell+1}^{n+1} \theta_\ell^j\,\boldsymbol{\zeta}_j^{n+2},$$

where the operators θ_ℓ^{n+1} are successively constructed according to:

The operator $\theta_i^j: F_j \to J_i$ for $i = j-1, \dots, 0$ is the solution operator $\boldsymbol{\eta} \mapsto \boldsymbol{\zeta}$ of the problem of finding $\boldsymbol{\zeta} \in J_i$ such that for all $\boldsymbol{\psi} \in F_{i-1}$

$$\Big\langle \big(\mathfrak{M}_i+\Lambda_i\big)\boldsymbol{\zeta}\,,\ \boldsymbol{\psi}\Big\rangle = -\Big\langle \big(M_j+\Lambda_j\big)\boldsymbol{\eta}\,,\ \boldsymbol{\psi}\Big\rangle - \sum_{\ell=i+1}^{j-1} \Big\langle \big(M_\ell+\Lambda_\ell\big)\theta_\ell^j\boldsymbol{\eta}\,,\ \boldsymbol{\psi}\Big\rangle.$$

Then the new $\breve{\boldsymbol{\zeta}}^k := \boldsymbol{\zeta}_{-1}^k$ for $k = 0, \dots, n+1$ where

$$\big(\boldsymbol{\zeta}_{-1}^0, \dots, \boldsymbol{\zeta}_{-1}^{n+1}\big)^\top = \Theta^{n+1}\big(\boldsymbol{\zeta}_{n+1}^{n+2}, \dots, \boldsymbol{\zeta}_0^{n+2}\big)^\top$$

satisfy the first $n+2$ equations of (5.5).

b) Let us prove now the symmetry of the operator $\mathfrak{M}_{n+1}$ which is the only thing which remains to be shown in the induction step of part **a)**. We have to show that

$$\forall\,\boldsymbol{\zeta} \in F_n, \quad \forall\,\boldsymbol{\psi} \in F_n, \quad \langle\, M_{n+1}\,\boldsymbol{\zeta}\,,\,\boldsymbol{\psi}\,\rangle = \langle\,\boldsymbol{\zeta}\,,\,M_{n+1}\,\boldsymbol{\psi}\,\rangle\,. \tag{5.13}$$

We are going to prove that (5.13) is indeed a consequence of the symmetry of the three-dimensional bilinear form $a(\varepsilon)$ appearing in (1.11).

Let us introduce the finite formal series

$$\Lambda_{[n]}(\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \Lambda_k$$

which is well-defined at the induction step n. We fix an arbitrary $\boldsymbol{\zeta} \in F_n$ and by the induction hypothesis the operator Θ^n is known. Let us define

$$\big(\boldsymbol{\zeta}^0_{-1},\ldots,\boldsymbol{\zeta}^n_{-1}\big)^\top = \Theta^n\big(\boldsymbol{\zeta},0\ldots,0\big)^\top \quad \text{and} \quad \breve{\boldsymbol{\zeta}}^{[n]}(\varepsilon) = \sum_{\ell=0}^{n} \varepsilon^\ell \boldsymbol{\zeta}^\ell_{-1}\,.$$

$\breve{\boldsymbol{\zeta}}^{[n]}(\varepsilon)$ is a finite formal series depending only on $\boldsymbol{\zeta}$.

Let us now consider the formal series $U^{\mathrm{m}}(\varepsilon)$ defined as the formal series constructed in Theorem 3.4 corresponding to the formal eigenvalue series $\Lambda = \Lambda_{[n]}$. Let us denote by $\breve{U}^{\mathrm{m}}(\varepsilon)$ the compound formal series $U^{\mathrm{m}}(\varepsilon) \circ C^{-1}(\varepsilon)$ where $C(\varepsilon)$ is the lifting formal series of Theorem 5.1 and define

$$\breve{U}^{\mathrm{m}}_{[n]}(\varepsilon) = \sum_{\ell=0}^{n+3} \varepsilon^\ell \breve{U}^{\mathrm{m}}_\ell\,.$$

Coming back to problem (3.3), we can see that there holds

$$\begin{aligned} A(\varepsilon) \circ \breve{U}^{\mathrm{m}}_{[n]}(\varepsilon)\,\big(\breve{\boldsymbol{\zeta}}^{[n]}(\varepsilon)\big) + \varepsilon^2 \Lambda_{[n]}(\varepsilon)\,\Pi \circ \breve{U}^{\mathrm{m}}_{[n]}(\varepsilon)\,\Big(\breve{\boldsymbol{\zeta}}^{[n]}(\varepsilon)\big)\breve{U}^{\mathrm{m}}_{[n]}(\varepsilon)\,\Big|_{\Gamma_\pm}\Big) \\ = \varepsilon^{n+3}\Big(M_{n+1}\boldsymbol{\zeta}, 0\,;\,0\Big|_{\Gamma_\pm}\Big) + \mathcal{O}(\varepsilon^{n+4})\,. \end{aligned}$$

Next, we introduce the formal series $\Phi^{\mathrm{m}}(\varepsilon)$ which is given in Theorem 4.2 corresponding to the formal series $\Lambda = \Lambda_{[n]}$ and $U^{\mathrm{m}} = U^{\mathrm{m}}_{[n]}$. Setting $\breve{\Phi}^{\mathrm{m}}(\varepsilon) = \Phi^{\mathrm{m}}(\varepsilon) \circ C^{-1}(\varepsilon)$ and

$$\breve{\Phi}^{\mathrm{m}}_{[n]}(\varepsilon) = \sum_{\ell=1}^{n+4} \varepsilon^\ell \breve{\Phi}^{\mathrm{m}}_\ell$$

we define the three-dimensional displacement

$$\boldsymbol{u}(\boldsymbol{\zeta}) = \Big(\breve{U}^{\mathrm{m}}_{[n]}(\varepsilon) + \chi(r)\,W(\varepsilon)\breve{\Phi}^{\mathrm{m}}_{[n]}(\varepsilon)\Big)\,\breve{\boldsymbol{\zeta}}^{[n]}(\varepsilon),$$

where χ is a cut-off function with $\chi \equiv 1$ in the vicinity of $\partial\omega$ leading to a well defined sum. For any $\boldsymbol{\psi} \in F_n$ we define $\boldsymbol{u}(\boldsymbol{\psi})$ in the same way.

We come back to the framework of the initial problem (1.11) by an integration by parts, *cf* formulas (3.1) and (3.2). We see that by construction we have $\boldsymbol{u}(\boldsymbol{\zeta}) = 0$ on Γ_0 in the clamped case and $T(\varepsilon)\boldsymbol{u}(\boldsymbol{\zeta}) = \mathcal{O}(\varepsilon^{n+2})$ on Γ_0 in the free case and the same holds for $\boldsymbol{u}(\boldsymbol{\psi})$. Now, for $\boldsymbol{\zeta}$ and $\boldsymbol{\psi}$ in F_n we obtain that

$$\begin{aligned} -a(\varepsilon)\big(\boldsymbol{u}(\boldsymbol{\zeta}),\boldsymbol{u}(\boldsymbol{\psi})\big) + \Lambda_{[n]}(\varepsilon)\big\langle\boldsymbol{u}(\boldsymbol{\zeta}),\boldsymbol{u}(\boldsymbol{\psi})\big\rangle_\varepsilon &= \varepsilon^{n+1}\,\langle\, M_{n+1}\,\boldsymbol{\zeta}\,,\,\boldsymbol{\psi}\,\rangle + \mathcal{O}(\varepsilon^{n+2}) \\ &= \varepsilon^{n+1}\,\langle\,\boldsymbol{\zeta}\,,\,M_{n+1}\,\boldsymbol{\psi}\,\rangle + \mathcal{O}(\varepsilon^{n+2}) \end{aligned}$$

which yields the symmetry of the operator $\mathfrak{M}_{n+1}$. ∎

End of the proof of Theorem 5.3.

To each sequence of nested spaces $F_0 \supset \ldots \supset F_n \supset \ldots$ constructed in Lemma 5.4 is associated a minimal integer $n_0 \geq 0$ and a limit space F_∞ such that for all $n \geq n_0$, $F_n = F_\infty$, because the dimensions of the spaces F_i form a decreasing sequence of non-negative integers. This implies that the operators θ_i^j are zero for $i \geq n_0 + 1$.

The relation (5.9) shows that for fixed n, if $(\Theta^n_{i,1})_{i=1}^{n+1}$ denotes the first column of the matrix Θ^n, we have

$$(\Theta^{n+1}_{1,1}, \ldots, \Theta^{n+1}_{1,n+1})^\top = \Theta^n \circ \big(\mathrm{Id}, \theta^{n+1}_n, \ldots, \theta^{n+1}_1\big)^\top.$$

As the operator Θ^n is lower triangular, the above equality can be written as

$$\Theta^{n+1}_{k,1} = \Theta^n_{k,1} + \sum_{\ell=2}^{k} \Theta^n_{k,\ell} \circ \theta^{n+1}_{n+2-\ell} \quad \text{for} \quad k \leq n+1 .$$

Hence if $n + 2 - k \geq n_0 + 1$, we have $\Theta^{n+1}_{k,1} = \Theta^n_{k,1}$, thus for any $k \in \mathbb{N}$ these operators do not depend on n if $n \geq n_0 + k$.

Let us then define

$$\forall\, k \geq 0, \quad \vartheta_k := \Theta^n_{k-1,1} \quad \text{with } n = n_0 + k .$$

In particular, we have $\vartheta_0 = \mathrm{Id}$. Now if $\boldsymbol{\zeta}_\infty$ belongs to F_∞, we can define the sequence

$$\mathbb{N} \ni k \longmapsto \breve{\boldsymbol{\zeta}}^k := \vartheta_k\, \boldsymbol{\zeta}_\infty.$$

Let us fix $k \in \mathbb{N}$. Choosing $n = n_0 + k$ and setting

$$\boldsymbol{\zeta}^{n+1}_n = \boldsymbol{\zeta}_\infty \quad \text{and} \quad \boldsymbol{\zeta}^{n+1}_\ell = 0 \quad \text{for } \ell = 0, \ldots, n-1 ,$$

we have $\boldsymbol{\zeta}^\ell_{-1} = \Theta^n_{\ell-1,1} \boldsymbol{\zeta}^{n+1}_n$ by definition and thus $\boldsymbol{\zeta}^\ell_{-1} = \breve{\boldsymbol{\zeta}}^\ell$ for $\ell = 0, \ldots, k$. Therefore

$$\big(\breve{\boldsymbol{\zeta}}^0, \ldots, \breve{\boldsymbol{\zeta}}^k, \boldsymbol{\zeta}^{k+1}_{-1}, \ldots, \boldsymbol{\zeta}^n_{-1}\big)^\top = \Theta^n\big(\boldsymbol{\zeta}^{n+1}_n, 0, \ldots, 0\big)^\top.$$

This yields that the sequence $\big(\breve{\boldsymbol{\zeta}}^\ell\big)_\ell$ satisfies the relation (5.5) up to order k, with Λ the formal series defined in Lemma 5.4.

To complete the proof of the theorem, it remains to note that the limit spaces F_∞ of all possible chains of spaces $\big(F_n\big)_n$ generate the whole space F_0: This is a consequence of the fact that at each step of the construction, we choose one of the eigenspaces of a symmetric operator and that eigenspaces generate the whole domain of the operator. ■

For the bending case, we can prove the following theorem, by a standard adaptation of Theorem 5.3:

Theorem 5.5 *Let Λ^{b}_0 be an eigenvalue of the operator L^{b} with boundary condition $\boldsymbol{\gamma}^{\mathrm{b}}_0$ and let F_0 be the corresponding eigenspace. There exists a splitting of F_0 into a finite number d of subspaces F^ℓ_∞, the direct sum of which generates F_0, such that for all $\ell = 1, \ldots, d$, there exist*

- *a formal series $\Lambda^{\mathrm{b},\ell}(\varepsilon)$ with real coefficients such that $\Lambda^{\mathrm{b},\ell}_0 = \Lambda^{\mathrm{b}}_0$,*
- *a formal series $\vartheta^\ell(\varepsilon)$ with coefficients $\vartheta^\ell_k : F^\ell_\infty \to \mathcal{C}^\infty(\overline{\varpi})$,*

with the following property: For all $\eta_\infty \in F^\ell_\infty$, the formal series $\breve\eta^\ell(\varepsilon) = \vartheta^\ell(\varepsilon)\eta_\infty$ solves problem (5.4) *with $\Lambda^{\rm b}(\varepsilon) = \Lambda^{{\rm b},\ell}(\varepsilon)$.*

6 Construction of quasimodes

Now we are going to show by means of error estimates that the formal operator series constructed in sections 3-5 yield $\mathcal{O}(\varepsilon^k)$ -quasimodes for problem (1.11) at any order $k \geq 0$.

Thus let us denote by $\mathfrak{A}(\varepsilon)$ the underlying operator corresponding to the variational formulation (1.11) of the eigenvalue problem. In formulas (3.1) and (3.2), $B(\varepsilon)$ is the interior partial differential operator associated with $\mathfrak{A}(\varepsilon)$ and $T(\varepsilon)$ is the traction on Γ associated with $\mathfrak{A}(\varepsilon)$. We note that the domain of $\mathfrak{A}(\varepsilon)$ is

$$D\big(\mathfrak{A}(\varepsilon)\big) = \big\{\boldsymbol{u} \in \mathbf{V}(\Omega) \mid B(\varepsilon)\boldsymbol{u} \in L^2(\Omega) \quad \text{and} \quad T(\varepsilon)\boldsymbol{u}\big|_\Gamma = 0\big\}.$$

We recall that $\|\cdot\|_\varepsilon$ denotes the norm associated with the scalar product $\langle\cdot,\cdot\rangle_\varepsilon$ in (1.11):

$$\|\boldsymbol{u}\|^2_\varepsilon = \|\boldsymbol{u}_*\|^2_{L^2} + \varepsilon^{-2}\|u_3\|^2_{L^2}\,.$$

Theorem 6.1

(i) *Let $\varrho_{\rm m}$ be an eigenvalue of the membrane operator $-L^{\rm m}$ and let $\boldsymbol{\zeta}$ be an associated eigenvector which belongs to one of the limit subspaces F_∞ constructed in Theorem* 5.3. *Thus let $\big(\Lambda(\varepsilon), \breve{\boldsymbol{\zeta}}(\varepsilon)\big)$ be the eigenpair formal series for problem* (5.3) *such that $\Lambda_0 = \varrho_{\rm m}$ and $\breve{\boldsymbol{\zeta}}^0 = \boldsymbol{\zeta}$. Then for all $N \geq 0$, with*

$$\Lambda_{[N]}(\varepsilon) = \sum_{k=0}^{N} \varepsilon^k \Lambda_k\,, \tag{6.1}$$

there exists a function $\boldsymbol{u}^{[N]}_{\rm m}(\varepsilon)$ such that

$$\|\boldsymbol{u}^{[N]}_{\rm m}(\varepsilon) - U^{\rm m}_{\rm KL}(\boldsymbol{\zeta})\|_\varepsilon \leq C\,\varepsilon\,\|U^{\rm m}_{\rm KL}(\boldsymbol{\zeta})\|_\varepsilon \tag{6.2}$$

and the pair $\big(\Lambda_{[N]}(\varepsilon)\,,\,\boldsymbol{u}^{[N]}_{\rm m}(\varepsilon)\big)$ is a membrane $\mathcal{O}(\varepsilon^{N+1})$ -quasimode of $\mathfrak{A}(\varepsilon)$, that is it satisfies the following estimate with a constant C independent of ε

$$\|\big(\mathfrak{A}(\varepsilon) - \Lambda_{[N]}(\varepsilon)\big)\boldsymbol{u}^{[N]}_{\rm m}(\varepsilon)\|_\varepsilon \leq C\,\varepsilon^{N+1}\|\boldsymbol{u}^{[N]}_{\rm m}\|_\varepsilon\,. \tag{6.3}$$

(ii) *Let $\varrho_{\rm b}$ be an eigenvalue of the bending operator $L^{\rm b}$ and let η be an associated eigenvector which belongs to one of the limit subspaces F_∞ appearing in Theorem* 5.5. *Thus let $\big(\Lambda(\varepsilon), \breve\eta(\varepsilon)\big)$ be the eigenpair formal series associated with problem* (5.4) *such that $\Lambda_0 = \Lambda_1 = 0$, $\Lambda_2 = \frac{1}{3}\varrho_{\rm b}$ and $\breve\eta^0 = \eta$. Then for all $N \geq 0$, with $\Lambda_{[N]}$ defined in* (6.1)*, there exists a function $\boldsymbol{u}^{[N]}_{\rm b}(\varepsilon)$ such that*

$$\|\boldsymbol{u}^{[N]}_{\rm b}(\varepsilon) - U^{\rm b}_{\rm KL}(\eta)\|_\varepsilon \leq C\,\varepsilon\,\|U^{\rm b}_{\rm KL}(\eta)\|_\varepsilon \tag{6.4}$$

and the pair $\big(\Lambda_{[N]}(\varepsilon)\,,\,\boldsymbol{u}^{[N]}_{\rm b}(\varepsilon)\big)$ is a bending $\mathcal{O}(\varepsilon^{N+1})$ -quasimode of $\mathfrak{A}(\varepsilon)$.

Proof. Let $\boldsymbol{\zeta}(\varepsilon)$ be the formal series $C^{-1}(\varepsilon)\breve{\boldsymbol{\zeta}}(\varepsilon)$. Let us now consider the formal outer series $U^{\mathrm{m}}(\varepsilon)\boldsymbol{\zeta}(\varepsilon)$ constructed in Theorem 3.4 and the formal inner series $\Phi^{\mathrm{m}}(\varepsilon)\boldsymbol{\zeta}(\varepsilon)$ constructed in Theorems 4.3 & 4.5. For any integer $k \geq 0$ let us denote by $\big[U^{\mathrm{m}}(\varepsilon)\boldsymbol{\zeta}(\varepsilon)\big]_k$ the term of order k in $U^{\mathrm{m}}(\varepsilon)\boldsymbol{\zeta}(\varepsilon)$ and similarly for Φ^{m}. Then we define the three-dimensional displacement $S_k^{\mathrm{m}}(\boldsymbol{\zeta})$ as

$$S_k^{\mathrm{m}}(\boldsymbol{\zeta})(x) := \big[U^{\mathrm{m}}(\varepsilon)\boldsymbol{\zeta}(\varepsilon)\big]_k(x) + \chi(r)\big[W(\varepsilon)\Phi^{\mathrm{m}}(\varepsilon)\boldsymbol{\zeta}(\varepsilon)\big]_k(r\varepsilon^{-1}, s, x_3). \tag{6.5}$$

This displacement depends linearly on $\boldsymbol{\zeta} \in F_\infty$ and we note that $S_0^{\mathrm{m}}(\boldsymbol{\zeta})$ is simply the membrane Kirchhoff-Love displacement $U_{\mathrm{KL}}^{\mathrm{m}}(\boldsymbol{\zeta}) = (\boldsymbol{\zeta}, 0)$. Then we introduce the main part of our quasimode as

$$\boldsymbol{u}_0^{[N]} = \sum_{k=0}^{N+5} \varepsilon^k\, S_k^{\mathrm{m}}(\boldsymbol{\zeta}). \tag{6.6}$$

We note that for $\varepsilon \leq \varepsilon_N$ with ε_N small enough, there holds

$$c_N \|\boldsymbol{\zeta}\|_{L^2(\omega)} \leq \|\boldsymbol{u}_0^{[N]}\|_{\varepsilon} \leq C_N \|\boldsymbol{\zeta}\|_{L^2(\omega)}$$

with positive constants c_N and C_N independent from ε. Here the L^2-norm of $\boldsymbol{\zeta}$ can be used as well as any other Sobolev norm because $\boldsymbol{\zeta}$ is an eigenvector. By construction we have

$$\big\langle \big(B(\varepsilon) + \Lambda_{[N]}\big)\boldsymbol{u}_0^{[N]} + \varepsilon^{N+1}\Lambda_{N+1}\boldsymbol{u}_0^{[N]},\ \boldsymbol{v}\big\rangle_{\varepsilon} \leq C\varepsilon^{N+1}\|\boldsymbol{u}_0^{[N]}\|_{\varepsilon}\,\|\boldsymbol{v}\|_{\varepsilon}\ .$$

This implies

$$\big\langle \big(B(\varepsilon) + \Lambda_{[N]}\big)\boldsymbol{u}_0^{[N]},\ \boldsymbol{v}\big\rangle_{\varepsilon} \leq C\varepsilon^{N+1}\|\boldsymbol{u}_0^{[N]}\|_{\varepsilon}\,\|\boldsymbol{v}\|_{\varepsilon}\ .$$

Moreover we have $\boldsymbol{u}_0^{[N]} \in \mathbf{V}(\Omega)$ because in the hard clamped case $\boldsymbol{u}_0^{[N]} = 0$ on Γ_0. But in general, $\boldsymbol{u}_0^{[N]}$ does not belong to the domain of $\mathfrak{A}(\varepsilon)$ because $T(\varepsilon)\boldsymbol{u}_0^{[N]}$ is not zero. Let $\boldsymbol{g}(\varepsilon)$ be the trace such that

$$T(\varepsilon)\boldsymbol{u}_0^{[N]}\big|_{\Gamma} = \varepsilon^{N+1}\boldsymbol{g}(\varepsilon).$$

By construction, $\|\boldsymbol{g}_*(\varepsilon)\|_{L^2(\Gamma)}$ and $\varepsilon^{-2}\|\boldsymbol{g}_3(\varepsilon)\|_{L^2(\Gamma)}$ are bounded independently of ε. Let $\boldsymbol{w} \in \mathbf{V}(\Omega)$ be the solution of the problem

$$\forall \boldsymbol{v} \in \mathbf{V}(\Omega), \quad a(\varepsilon)(\boldsymbol{w},\ \boldsymbol{v}) + \langle \boldsymbol{w},\ \boldsymbol{v}\rangle_{\Omega,\varepsilon} = -\langle \boldsymbol{g}(\varepsilon),\ \boldsymbol{v}\rangle_{\Gamma,\varepsilon}.$$

Thanks to Korn inequality on Ω, there holds the estimate

$$\|\boldsymbol{w}\|^2_{H^1(\Omega)} + \|\boldsymbol{w}\|^2_{\Omega,\varepsilon} \leq C\Big(\|\boldsymbol{g}_*(\varepsilon)\|_{L^2(\Gamma)} + \varepsilon^{-2}\|\boldsymbol{g}_3(\varepsilon)\|_{L^2(\Gamma)}\Big)\|\boldsymbol{w}\|_{H^1(\Omega)}\,.$$

Therefore $\|\boldsymbol{w}\|_{\Omega,\varepsilon}$ is bounded independently of ε. Let us define

$$\boldsymbol{u}^{[N]} = \boldsymbol{u}_0^{[N]} + \varepsilon^{N+1}\boldsymbol{w}.$$

Now $T(\varepsilon)\boldsymbol{u}^{[N]} = 0$ on Γ and $\boldsymbol{u}^{[N]}$ belongs to the domain of $\mathfrak{A}(\varepsilon)$. Moreover

$$\big\langle \big(B(\varepsilon) + \Lambda_{[N]}\big)\varepsilon^{N+1}\boldsymbol{w},\ \boldsymbol{v}\big\rangle_{\varepsilon} \leq C\varepsilon^{N+1}\|\boldsymbol{u}_0^{[N]}\|_{\varepsilon}\,\|\boldsymbol{v}\|_{\varepsilon}\ ,$$

and with the fact that $\|\boldsymbol{u}_0^{[N]}\|_{\varepsilon} \leq C\|\boldsymbol{u}^{[N]}\|_{\varepsilon}$ for ε small enough, we conclude that

$$\big\langle \big(B(\varepsilon) + \Lambda_{[N]}\big)\boldsymbol{u}^{[N]},\ \boldsymbol{v}\big\rangle_{\varepsilon} \leq C\varepsilon^{N+1}\|\boldsymbol{u}^{[N]}\|_{\varepsilon}\,\|\boldsymbol{v}\|_{\varepsilon}\ .$$

Whence estimate (6.3).

The estimate (6.2) is then a simple consequence of the structure of the first terms in $\boldsymbol{u}^{[N]}$: As already mentioned, $S_0^{\mathrm{m}}(\boldsymbol{\zeta}) = U_{\mathrm{KL}}^{\mathrm{m}}(\boldsymbol{\zeta})$. Moreover, $S_1^{\mathrm{m}}(\boldsymbol{\zeta})$ is the sum of a membrane Kirchhoff-Love displacement $U_{\mathrm{KL}}^{\mathrm{m}}(\boldsymbol{\zeta}^1)$ and of a boundary layer term which has no transverse component.

The proof for the bending case is similar. ∎

7 The limits of three-dimensional eigenpairs

combined with the result of [3] according to which the ℓ-th scaled three-dimensional eigenvalue $\varepsilon^{-2}\Lambda^{\varepsilon}_{\mathrm{b},\ell}$ tends to the two-dimensional eigenvalue $\frac{1}{3}\varrho_{\mathrm{b},\ell}$ as $\varepsilon \to 0$, the result of Theorem 6.1 will yield the optimal estimates stated in Theorems 2.3 and 2.5. In the following, we recall and adapt the proof in [3] to our situation, particularly for the bending case with free boundary condition and for the membrane case with clamped or free boundary conditions.

7.a Bending eigenvalues

Theorem 7.1 *Let $C_0 > 0$ be a fixed bound. With the arrangement* (1.7b) *of the three-dimensional bending eigenvalues $\Lambda^{\varepsilon}_{\mathrm{b},\ell}$, for any $\ell \geq 1$ let $\mathcal{E}_\ell$ be the set of $\varepsilon > 0$ such that $\varepsilon^{-2}\Lambda^{\varepsilon}_{\mathrm{b},\ell} \leq C_0$. If zero belongs to the closure of $\mathcal{E}_\ell$, there exists a function $C_\ell(\varepsilon) > 0$ tending to zero as $\varepsilon \to 0$ such that the following estimate holds*

$$\forall \varepsilon \in \mathcal{E}_\ell, \quad \min_{\ell' \geq \ell} |\varepsilon^{-2}\Lambda^{\varepsilon}_{\mathrm{b},\ell} - \tfrac{1}{3}\varrho_{\mathrm{b},\ell'}| \leq C_\ell(\varepsilon), \tag{7.1}$$

where the $\varrho_{\mathrm{b},\ell'}$ are the arrangement (1.17b) *of the eigenvalues of L^{b}.*

Proof. We just give the main arguments of the proof. Let $\ell \geq 1$ be fixed such that zero belongs to the closure of $\mathcal{E}_\ell$ and set $\Lambda^\varepsilon = \varepsilon^{-2}\Lambda^{\varepsilon}_{\mathrm{b},\ell}$. Since for $\varepsilon \in \mathcal{E}_\ell$, Λ^ε is bounded by C_0 there exists a sequence $\{\varepsilon_n\} \subset \mathcal{E}_\ell$ which tends to zero such that Λ^{ε_n} tends to a limit Λ as $n \to \infty$. In the following, we will denote by ε such a subsequence of $\mathcal{E}_\ell$.

Let $\boldsymbol{u}(\varepsilon) \in \mathbf{V}(\Omega)$ be an eigenvector of $\Lambda^{\varepsilon}_{\mathrm{b},\ell}$ such that

$$\|u_3(\varepsilon)\|_{L^2(\Omega)} = 1.$$

The displacement $\boldsymbol{u}(\varepsilon)$ has the bending parities and, in the free case, is orthogonal to the rigid motions. Moreover, $\boldsymbol{u}(\varepsilon)$ verifies by definition:

$$\forall \boldsymbol{v} \in \mathbf{V}(\Omega), \qquad a(\varepsilon)\big(\boldsymbol{u}(\varepsilon), \boldsymbol{v}\big) = \Lambda^\varepsilon \, \varepsilon^2 \left\langle \boldsymbol{u}(\varepsilon)\,,\, \boldsymbol{v} \right\rangle_\varepsilon. \tag{7.2}$$

As $\boldsymbol{u}(\varepsilon)$ satisfies homogeneous boundary conditions on the lateral boundary of Ω in the clamped case or is orthogonal to the rigid motions in the free case, Korn inequality yields that

$$\|\boldsymbol{u}(\varepsilon)\|_{H^1(\Omega)^3} \leq C\big(\varepsilon\|\boldsymbol{u}_*(\varepsilon)\|_{L^2(\Omega)^2} + \|u_3(\varepsilon)\|_{L^2(\Omega)}\big),$$

where C is independent of ε. Therefore we deduce that

$$\|\boldsymbol{u}(\varepsilon)\|_{H^1(\Omega)^3} \leq C\|u_3(\varepsilon)\|_{L^2(\Omega)} = C,$$

Hence, there exists $\boldsymbol{u}^0 \in \mathbf{V}(\Omega)$ such that after taking subsequences, $\boldsymbol{u}(\varepsilon)$ tends to $\boldsymbol{u}^0$ weakly in $H^1(\Omega)^3$ and strongly in $L^2(\Omega)^3$. Thus we deduce that

$$\|u_3^0\|_{L^2(\Omega)^3} = 1 \tag{7.3}$$

and that in the free case, $\boldsymbol{u}^0$ is orthogonal to the rigid motions. Moreover $\boldsymbol{u}^0$ has the bending parities.

Now using the method of [3] we take different test functions in (7.2) to deduce information about $\boldsymbol{u}^0$ by passing to the limit. Letting $\boldsymbol{v} = (0, v_3)$ in (7.2) gives $\partial_3 u_3^0 = 0$ in Ω, hence u_3 can be identified with a function $\eta \in H^1(\omega)$. In the clamped case we have $\eta = 0$ on $\partial\omega$, and in the free case η is orthogonal to the bending rigid motions. Taking $\boldsymbol{v} = (v_\alpha, 0)$ in (7.2) yields that $e_{\alpha 3}(\boldsymbol{u}^0) = 0$ in Ω and thus $\boldsymbol{u}^0$ is a bending Kirchhoff-Love displacement: we have $u_\alpha^0 = -x_3\partial_\alpha\eta$ and $u_3^0 = \eta \in H^2(\omega)$ (and $H_0^2(\omega)$ in the clamped case).

Moreover, we have for $\varepsilon \in \mathcal{E}_\ell$ that

$$\varepsilon^{-4}\|\partial_3 u_3(\varepsilon)\|^2_{L^2(\Omega)} \leq C_1\, a(\varepsilon)\big(\boldsymbol{u}(\varepsilon), \boldsymbol{u}(\varepsilon)\big) \leq C_2,$$

where C_1 and C_2 are constants independent of ε. Hence, after extracting a new subsequence, there exists $\chi_{33} \in L^2(\Omega)$ such that $\varepsilon^{-2}\partial_3 u_3(\varepsilon) \to \chi_{33}$ weakly in L^2. Taking $\boldsymbol{v} = (0, \varepsilon^2 v_3)$ as test function in (7.2) gives that

$$\chi_{33} = -\frac{\lambda}{\lambda+2\mu} e_{\alpha\alpha}(\boldsymbol{u}^0) = x_3\frac{\lambda}{\lambda+2\mu}\Delta\eta.$$

Now if $\eta' \in H^2(\omega)$ (and $H_0^2(\omega)$ in the clamped case) and if we take $\boldsymbol{v} = (-x_3\partial_\alpha\eta', \eta')$ as test function in (7.2), we find that

$$\frac{2}{3}\int_\omega \Big(\tilde\lambda\Delta\eta\,\Delta\eta' + 2\mu\partial_{\alpha\beta}\eta\,\partial_{\alpha\beta}\eta'\Big)\mathrm{d}x = 2\Lambda\int_\omega \eta\,\eta'\mathrm{d}x.$$

Hence, as η is orthogonal to the bending rigid motions and $\eta \neq 0$ because of (7.3), we have that $\Lambda = \frac{1}{3}\varrho_{\mathrm{b},\ell'}$ for $\ell' \geq 1$.

As $\varepsilon^{-2}\Lambda^\varepsilon_{\mathrm{b},\ell} \leq \varepsilon^{-2}\Lambda^\varepsilon_{\mathrm{b},\ell'}$ if $\ell \leq \ell'$ we have $\mathcal{E}_{\ell'} \subset \mathcal{E}_\ell$ if $\ell \leq \ell'$. Moreover, the weak limits η and η' issued from orthogonal three-dimensional eigenvector sequences $\boldsymbol{u}(\varepsilon)$ and $\boldsymbol{u}'(\varepsilon)$ are orthogonal to each other, too. Thus taking by diagonal process the same subsequence for all ℓ, we can conclude that $\varepsilon^{-2}\Lambda^\varepsilon_{\mathrm{b},\ell} \to \frac{1}{3}\varrho_{\mathrm{b},\ell'}$ with $\ell' \geq \ell$. Hence, we proved the theorem for a subset $\mathcal{E}'_\ell \subset \mathcal{E}_\ell$ whose closure contains zero. But reproducing the same arguments for $\mathcal{E}_\ell \setminus \mathcal{E}'_\ell$ and using the uniqueness of the limit of $\min_{\ell'\geq\ell}|\varepsilon^{-2}\Lambda^\varepsilon_{\mathrm{b},\ell} - \frac{1}{3}\varrho_{\mathrm{b},\ell'}|$, we show that the theorem holds for the whole set $\mathcal{E}_\ell$. ■

7.b Membrane eigenvalues

Theorem 7.2 *Let $C_0 > 0$ be a fixed bound. With the arrangement* (1.7a) *of the three-dimensional membrane eigenvalues $\Lambda^\varepsilon_{\mathrm{m},\ell}$, for any $\ell \geq 1$ let $\mathcal{E}_\ell$ be the set of $\varepsilon > 0$ such that $\Lambda^\varepsilon_{\mathrm{m},\ell} \leq C_0$. If zero belongs to the closure of $\mathcal{E}_\ell$, there exists a function $C_\ell(\varepsilon) > 0$ tending to zero as $\varepsilon \to 0$ such that the following estimate holds*

$$\forall\varepsilon \in \mathcal{E}_\ell, \quad \min_{\ell'\geq\ell}|\Lambda^\varepsilon_{\mathrm{m},\ell} - \varrho_{\mathrm{m},\ell'}| \leq C_\ell(\varepsilon), \tag{7.4}$$

where the $\varrho_{\mathrm{m},\ell'}$ *are the arrangement* (1.17a) *of the eigenvalues of* $-L^{\mathrm{m}}$.

Proof. The proof is similar as in the bending case: Let $\ell \geq 1$ be fixed and $\boldsymbol{u}(\varepsilon)$ a membrane eigenvector associated with $\Lambda^{\varepsilon}_{\mathrm{m},\ell}$, such that $\|\boldsymbol{u}(\varepsilon)\|_{\varepsilon} = 1$. We prove in the same way as before that $\boldsymbol{u}(\varepsilon)$ converges to a limit $\boldsymbol{u}^0$ weakly in $H^1(\Omega)^3$ and strongly in $L^2(\Omega)^3$, where $\boldsymbol{u}^0 \in \mathbf{V}(\Omega)$ is of the type $\boldsymbol{u}^0 = (\boldsymbol{\zeta}, 0)$ with $\boldsymbol{\zeta} \in H^1(\omega)^2$. In the clamped case, $\boldsymbol{\zeta} \in H^1_0(\omega)$ and in the free case $\boldsymbol{\zeta}$ is orthogonal to the membrane rigid motions. Moreover, as $u_3(\varepsilon)$ is odd in x_3 we have

$$a(\varepsilon)\big(\boldsymbol{u}(\varepsilon), \boldsymbol{u}(\varepsilon)\big) \geq C\varepsilon^{-4}\|\partial_3 u_3(\varepsilon)\|^2_{L^2(\Omega)} \geq C\varepsilon^{-4}\|u_3(\varepsilon)\|^2_{L^2(\Omega)},$$

hence $\varepsilon^{-2}\|u_3(\varepsilon)\|^2_{L^2(\Omega)} \to 0$ as $\varepsilon \to 0$ in $\mathcal{E}_\ell$. Thus, we deduce that $2\|\boldsymbol{\zeta}\|^2_{L^2(\omega)} = 1$, therefore $\boldsymbol{\zeta} \neq 0$.

Now, analogous computations as in the bending case show that

$$\varepsilon^{-2}\partial_3 u_3(\varepsilon) \to -\frac{\lambda}{\lambda + 2\mu} e_{\alpha\alpha}(\boldsymbol{\zeta}) \quad \text{weakly in} \quad L^2(\Omega).$$

We then deduce similarly that $\Lambda^{\varepsilon}_{\mathrm{m},\ell}$ tends to $\varrho_{\mathrm{m},\ell'}$ with $\ell' \geq 1$, and conclude in the same way as for Theorem 7.1. ∎

8 Conclusions

8.a Eigenvalues

Applying Theorem 6.1 at the level $N = 0$, we obtain that for any fixed integer ℓ and for any $\varepsilon > 0$, there exists an integer $\ell(\varepsilon)$ such that the following estimate holds for membrane eigenvalues:

$$\left|\varrho_{\mathrm{m},\ell} - \Lambda^{\varepsilon}_{\mathrm{m},\ell(\varepsilon)}\right| \leq C\varepsilon.$$

Moreover, if the multiplicity of $\varrho_{\mathrm{m},\ell}$ is equal to ν, then there exist ν independent $\mathcal{O}(\varepsilon)$-quasimodes for the three-dimensional problem (1.11). Therefore in the above estimate $\ell(\varepsilon) \geq \ell$ holds.

Conversely, Theorem 7.2 yields that for any fixed integer ℓ and for any $\varepsilon > 0$ in the set $\mathcal{E}_\ell$, there exists an integer $\ell'(\varepsilon) \geq \ell$ such that the following estimate holds for membrane eigenvalues:

$$\left|\Lambda^{\varepsilon}_{\mathrm{m},\ell} - \varrho_{\mathrm{m},\ell'(\varepsilon)}\right| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Therefore $\ell(\varepsilon) = \ell$ and we have proved Theorem 2.3 for membrane eigenvalues. Concerning bending eigenvalues, we apply Theorem 6.1 at the level $N = 2$ and conclude similarly. Thus Theorem 2.3 is proved.

As a consequence of Theorem 6.1 at any level $N \geq 0$ and of Theorem 2.3, we obtain the following asymptotic expansions for the three-dimensional membrane and bending eigenvalues:

Theorem 8.1 *For any integer* $\ell \geq 1$, *the three-dimensional eigenvalues* $\Lambda^{\varepsilon}_{\mathrm{m},\ell}$ *and* $\Lambda^{\varepsilon}_{\mathrm{b},\ell}$ *have infinite power series asymptotic expansions as* $\varepsilon \to 0$

$$\Lambda^{\varepsilon}_{\mathrm{m},\ell} \sim \sum_{k\geq 0} \varepsilon^k \Lambda_{\mathrm{m},\ell\,;\,k} \quad \textit{and} \quad \Lambda^{\varepsilon}_{\mathrm{b},\ell} \sim \sum_{k\geq 0} \varepsilon^k \Lambda_{\mathrm{b},\ell\,;\,k} \tag{8.1}$$

with $\Lambda_{\mathrm{m},\ell\,;\,0} = \varrho_{\mathrm{m},\ell}$ *and* $\Lambda_{\mathrm{b},\ell\,;\,0} = \Lambda_{\mathrm{b},\ell\,;\,1} = 0$, $\Lambda_{\mathrm{b},\ell\,;\,2} = \frac{1}{3}\varrho_{\mathrm{b},\ell}$, *in the following sense: For any* $N \in \mathbb{N}$ *there holds*

$$\big|\Lambda^{\varepsilon}_{\mathrm{m},\ell} - \sum_{k=0}^{N} \varepsilon^k \Lambda_{\mathrm{m},\ell\,;\,k}\big| \ + \ \big|\Lambda^{\varepsilon}_{\mathrm{b},\ell} - \sum_{k=0}^{N} \varepsilon^k \Lambda_{\mathrm{b},\ell\,;\,k}\big| \le C\,\varepsilon^{N+1} \ .$$

8.b Eigendisplacements

Let us investigate membrane eigendisplacements. For this let us fix $\ell \in \mathcal{L}_{\mathrm{m}}$ and let ν be the multiplicity of $\varrho_{\mathrm{m},\ell}$. According to Theorem 5.3 the two-dimensional membrane eigenspace $F_{\mathrm{m},\ell}$ splits into the direct sum of subspaces F^d_∞ for $d = 1,\ldots,\bar d$. With Theorem 6.1 at the level $N = 0$, we can associate with each element $\boldsymbol{\zeta}$ in a F^d_∞ a three-dimensional displacement $\boldsymbol{u}^{[0]}[\boldsymbol{\zeta}](\varepsilon)$ defined as the field $\boldsymbol{u}^{[0]}_{\mathrm{m}}(\varepsilon)$ satisfying estimate (6.2) in Theorem 6.1 and such that $\big(\varrho_{\mathrm{m},\ell}, \boldsymbol{u}^{[0]}[\boldsymbol{\zeta}](\varepsilon)\big)$ is a $\mathcal{O}(\varepsilon)$-quasimode of $\mathfrak{A}(\varepsilon)$.

The other three-dimensional eigenvalues $\Lambda^{\varepsilon}_{\mathrm{m},j}$ with $j \notin \{\ell,\ell{+}1,\ldots,\ell{+}\nu{-}1\}$, being separated from the cluster corresponding to ℓ (i.e. with $j \in \{\ell,\ell+1,\ldots,\ell+\nu-1\}$) by a distance *independent from* ε, we deduce from Lemma 2.2 that

$$\delta_\varepsilon\Big(\mathrm{span}\big\{\boldsymbol{u}^{[0]}[\boldsymbol{\zeta}](\varepsilon) \mid \boldsymbol{\zeta} \in F^d_\infty,\ d = 1,\ldots,\bar d\big\}\ , C^{\varepsilon}_{\mathrm{m},\ell}\Big) \le c\,\varepsilon,$$

where $C^{\varepsilon}_{\mathrm{m},\ell}$ is the cluster space (2.5). With estimate (6.2), we obtain Theorem 2.5 for membrane eigenvectors. The proof for bending eigenvectors is similar, taking account of the fact that the clusters of eigenvalues are $\mathcal{O}(\varepsilon^2)$ separated.

Let us keep $\ell \in \mathcal{L}_{\mathrm{m}}$ fixed. To each subspace F^d_∞ of $F_{\mathrm{m},\ell}$ corresponds a different power series expansion $\sum \varepsilon^k \Lambda^d_k$ of a three-dimensional membrane eigenvalue. There exists N large enough so that all polynomials $\sum_{0\le k\le N} \varepsilon^k \Lambda^d_k$ for $d = 1,\ldots,\bar d$ are distinct from each other and there exists ε_0 small enough so that the functions $\varepsilon \mapsto \sum_{0\le k\le N} \varepsilon^k \Lambda^d_k$ do not cross each other on $(0,\varepsilon_0)$. Thus it is possible to renumber them so that on $(0,\varepsilon_0)$

$$\sum_{0\le k\le N} \varepsilon^k \Lambda^1_k < \ldots < \sum_{0\le k\le N} \varepsilon^k \Lambda^{\bar d}_k.$$

According to this renumbering we introduce a new unified notation for the subspaces F^d_∞ of $F_{\mathrm{m},\ell}$: with ν_d the dimension of F^d_∞

$$F^\infty_{\mathrm{m},\ell} := F^1_\infty, \quad F^\infty_{\mathrm{m},\ell+\nu_1} := F^2_\infty, \quad \ldots \quad F^\infty_{\mathrm{m},\ell+\nu_1+\ldots+\nu_{\bar d-1}} := F^{\bar d}_\infty,$$

and

$$\mathcal{L}^\infty_{\mathrm{m}} := \bigcup_{\ell\in\mathcal{L}_{\mathrm{m}}} \{\ell, \ell+\nu_1, \ldots, \ell+\nu_1+\ldots+\nu_{\bar d-1}\}.$$

Thus to each $j \in \mathcal{L}^\infty_{\mathrm{m}}$ corresponds a power series expansion $\sum_{k\ge0} \varepsilon^k \Lambda_{\mathrm{m},j\,;\,k}$ which is in fact the same as the series appearing in the expansion (8.1) of the three-dimensional eigenvalue $\Lambda^{\varepsilon}_{\mathrm{m},j}$. Moreover with each $\boldsymbol{\zeta}$ in the space $F^\infty_{\mathrm{m},j}$ is associated the three-dimensional eigendisplacement expansion

$$\sum_{k\ge0} \varepsilon^k\, S^{\mathrm{m}}_k(\boldsymbol{\zeta}) \tag{8.2}$$

with $S^{\mathrm{m}}_k(\boldsymbol{\zeta})$ defined in (6.5). Our final result follows.

Theorem 8.2 *Let* $j \in \mathcal{L}^\infty_{\rm m}$ *and let* ν^∞_j *be the asymptotic multiplicity* $\dim F^\infty_{{\rm m},j}$. *For* $\varepsilon_0 > 0$ *small enough, there are only two possibilities:*

(i) *For any* $\varepsilon < \varepsilon_0$ *the multiplicity of* $\Lambda^\varepsilon_{{\rm m},j}$ *is equal to* ν^∞_j. *Then for any two-dimensional eigenvectors* $\boldsymbol{\zeta} \in F^\infty_{{\rm m},j}$ *the series* (8.2) *is the asymptotic expansion of a three-dimensional eigenvector* $\mathbf{u}[\boldsymbol{\zeta}](\varepsilon)$, *which means that for any* $\varepsilon > 0$ *there exists an eigenvector* $\mathbf{u}[\boldsymbol{\zeta}](\varepsilon)$ *such that for all* $N \in \mathbb{N}$ *there holds*

$$\|\mathbf{u}[\boldsymbol{\zeta}](\varepsilon) - \sum_{0\le k\le N} \varepsilon^k\, S^{\rm m}_k(\boldsymbol{\zeta})\|_\varepsilon \le C\,\varepsilon^N\,\|U^{\rm m}_{\rm KL}(\boldsymbol{\zeta})\|_\varepsilon . \tag{8.3}$$

(ii) *For any* $\varepsilon < \varepsilon_0$ *the multiplicity of* $\Lambda^\varepsilon_{{\rm m},j}$ *is* $< \nu^\infty_j$. *Then there exist* ν^∞_j *independent in-plane eigenvectors* $\boldsymbol{\zeta} \in F^\infty_{{\rm m},j}$ *such that the series* (8.2) *are the asymptotic expansion of a three-dimensional eigenvector* $\mathbf{u}[\boldsymbol{\zeta}](\varepsilon)$ *in the sense* (8.3).

Similar statements hold for bending eigenmodes.

Proof. Possibilities *(i)* and *(ii)* cover any situation since the eigenvalues of the three-dimensional problem depend analytically on ε.

By construction of the spaces $F^\infty_{{\rm m},i}$ there exists $J_0 \in \mathbb{N}$ such that for any $J > J_0$ and any $i \neq j$ we have the lower bound, with a $c > 0$:

$$\Big|\sum_{k=0}^{J} \varepsilon^k \Lambda_{{\rm m},i\,;\,k} - \sum_{k=0}^{J} \varepsilon^k \Lambda_{{\rm m},j\,;\,k}\Big| \;\ge\; c\,\varepsilon^{J_0+1}.$$

Then Lemma 2.2 combined with Theorem 6.1 yields that for any $\boldsymbol{\zeta} \in F^\infty_{{\rm m},j}$ and any $J > J_0$ the field $\boldsymbol{u}^{[J]}[\boldsymbol{\zeta}](\varepsilon)$ belonging to a three-dimensional $\mathcal{O}(\varepsilon^{J+1})$-quasimode as defined in Theorem 6.1, satisfies

$$\delta_\varepsilon\Big(\mathrm{span}\big\{\boldsymbol{u}^{[J]}[\boldsymbol{\zeta}](\varepsilon)\big\}\,, E^\varepsilon_{{\rm m},j}\Big) \le c\,\varepsilon^{J-J_0},$$

in situation *(i)*, and

$$\delta_\varepsilon\Big(\mathrm{span}\big\{\boldsymbol{u}^{[J]}[\boldsymbol{\zeta}](\varepsilon)\big\}\,, \bigoplus_{j\le i<j+\nu^\infty_j} E^\varepsilon_{{\rm m},i}\Big) \le c\,\varepsilon^{J-J_0},$$

in situation *(ii)*. Choosing $J = N + J_0$ for any fixed N, we can evaluate explicitly the norm $\|\cdot\|_\varepsilon$ of the difference $\boldsymbol{u}^{[J]}[\boldsymbol{\zeta}](\varepsilon) - \sum_{0\le k\le N} \varepsilon^k\, S^{\rm m}_k(\boldsymbol{\zeta})$, whence the result. ∎

References

[1] V. L. BERDICHEVSKIĬ. Вариационные принципы механики сплошной среды *(Variational principles of continuum mechanics)*. “Nauka”, Moscow 1983.

[2] P. G. CIARLET. *Mathematical Elasticity. Vol. II, Theory of Plates*. North-Holland, Amsterdam 1997.

[3] P. G. CIARLET, S. KESAVAN. Two-dimensional approximation of three-dimensional eigenvalue problems in plate theory. *Comp. Methods Appl. Mech. Engrg.* **26** (1981) 149–172.

[4] M. DAUGE, I. GRUAIS. Asymptotics of arbitrary order for a thin elastic clamped plate. I: Optimal error estimates. *Asymptotic Analysis* **13** (1996) 167–197.

[5] M. DAUGE, I. GRUAIS. Asymptotics of arbitrary order for a thin elastic clamped plate. II: Analysis of the boundary layer terms. *Asymptotic Analysis* **16** (1998) 99–124.

[6] M. DAUGE, I. GRUAIS, A. RÖSSLE. The influence of lateral boundary conditions on the asymptotics in thin elastic plates. To appear in *SIAM Jour. of Math. Anal.* (1999).

[7] N. DUNFORD, J. SCHWARTZ. *Linear Operators – Part II*. Interscience Publishers, New York 1963.

[8] G. DUVAUT, J.-L. LIONS. *Les Inéquations en Mécanique et en Physique*. Dunod, Paris 1972.

[9] K. O. FRIEDRICHS, R. F. DRESSLER. A boundary-layer theory for elastic plates. *Comm. Pure Appl. Math.* **14** (1961) 1–33.

[10] R. D. GREGORY, F. Y. WAN. Decaying states of plane strain in a semi-infinite strip and boundary conditions for plate theory. *J. Elasticity* **14** (1984) 27–64.

[11] T. KATO. Perturbation theory for nullity, deficiency and other quantities of linear operators. *J. Analyse Math.* **6** (1958) 261–322.

[12] T. KATO. *Perturbation Theory for Linear Operators*. Springer-Verlag, Berlin - Heidelberg - New York 1976.

[13] V. G. MAZ'YA, S. A. NAZAROV, B. A. PLAMENEVSKII. *Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten II*. Mathematische Monographien, Band 83. Akademie Verlag, Berlin 1991.

[14] S. A. NAZAROV. Об асимптотике спектра задачи теории упругости для тонкой пластины (On the spectrum asymptotics in the elasticity theory for thin plates). 1998.

[15] S. A. NAZAROV. The spatial structure of the stress field in the neighbourhood of the corner point of a thin plate. *J. Appl. Maths Mechs* **55 (4)** (1991) 523–530.

[16] S. A. NAZAROV. Two-term asymptotics of solutions of spectral problems with singular perturbation. *Math. USSR Sbornik* **69 (2)** (1991) 307–340.

[17] S. A. NAZAROV, I. S. ZORIN. Edge effect in the bending of a thin three-dimensional plate. *Prikl. Matem. Mekhan.* **53 (4)** (1989) 642–650. English translation *J. Appl. Maths. Mechs.* (1989) 500–507.

[18] M. I. VISHIK, L. A. LYUSTERNIK. Regular degeneration and boundary layers for linear differential equtions with small parameter. *Amer. Math. Soc. Transl. (2)* **20** (1962) 239–364.

Authors' addresses

Monique DAUGE, Erwan FAOU :
Institut Mathématique, UMR 6625 du CNRS, Université de Rennes 1,
Campus de Beaulieu, 35042 Rennes, FRANCE.

Ivica DJURDJEVIC, Andreas RÖSSLE :
Mathematical Institute A/6, University of Stuttgart,
Pfaffenwaldring 57, 70550 Stuttgart, GERMANY.