On Friedrichs constant and Horgan-Payne angle for LBB condition

Monique Dauge, Christine Bernardi, Martin Costabel,
and Vivette Girault

Abstract. In dimension 2, the Horgan-Payne angle serves to construct a lower bound for the inf-sup constant of the divergence arising in the so-called LBB condition. This lower bound is equivalent to an upper bound for the Friedrichs constant. Explicit upper bounds for the latter constant can be found using a polar parametrization of the boundary. Revisiting carefully the original paper which establishes this strategy, we found out that some proofs need clarification, and some statements, replacement.

Keywords: LBB condition, inf-sup constant, Friedrichs constant, Horgan-Payne angle.
AMS classification: 30A10, 35Q35.

§1. The inf-sup constant and some general properties

Here we only consider *bounded connected* open sets Ω in $\mathbb{R}^2$, the generic point in $\mathbb{R}^2$ being denoted by $\boldsymbol{x} = (x_1, x_2)$. For such a domain Ω, the inf-sup constant of the divergence associated with Dirichlet boundary conditions, also called LBB constant after Ladyzhenskaya, Babuška [3, 2] and Brezzi [4], is defined as

$$\beta(\Omega) = \inf_{q\in L^2_\circ(\Omega)} \ \sup_{\boldsymbol{v}\in H^1_0(\Omega)^2} \frac{\langle \operatorname{div} \boldsymbol{v}, q\rangle_\Omega}{|\boldsymbol{v}|_{1,\Omega}\ \|q\|_{0,\Omega}}\,. \tag{1}$$

Here

- $L^2_\circ(\Omega)$ stands for the space of square integrable scalar functions q with zero mean value in Ω endowed with its natural norm $\|\cdot\|_{0,\Omega}$ and natural scalar product $\langle\cdot,\cdot\rangle_\Omega$,
- $H^1_0(\Omega)^2$ is the standard Sobolev space of vector functions $\boldsymbol{v} = (v_1, v_2)$ with square integrable gradients and zero traces on the boundary, endowed with its natural semi-norm

$$|\boldsymbol{v}|_{1,\Omega} = \Big(\sum_{k=1}^{2}\sum_{j=1}^{2} \|\partial_{x_j} v_k\|^2_{0,\Omega}\Big)^{1/2}.$$

Since Ω is bounded, by virtue of the Poincaré inequality, the above semi-norm on $H^1_0(\Omega)^2$ is equivalent to the usual norm in $H^1(\Omega)^2$.

We list some elementary properties of $\beta(\Omega)$:

(a) $\beta(\Omega) \geq 0$,

(b) $\beta(\Omega) \le 1$, because of the identity $|\boldsymbol{v}|^2_{1,\Omega} = \|\operatorname{curl}\boldsymbol{v}\|^2_{0,\Omega} + \|\operatorname{div}\boldsymbol{v}\|^2_{0,\Omega}$ for any $\boldsymbol{v} \in H^1_0(\Omega)^2$,

(c) $\beta(\Omega)$ is invariant by translations, dilations, symmetries and rotations by virtue of Piola transform. Thus $\beta(\Omega)$ only depends on the *shape* of Ω.

The constant $\beta(\Omega)$ is positive for Lipschitz domains (see [8, Chap. 1, Section 2.2], which relies on [12, Chap. 3, Lemme 7.1]), and also for domains with less regular boundary like John domains [1]. In contrast, domains with an external cusp (also called thin peak) satisfy $\beta(\Omega) = 0$, see [15, Chap. 15].

Finding calculable lower bounds for $\beta(\Omega)$ is of great interest, since it is involved in any analysis of the Stokes and Navier-Stokes equations with no-slip boundary conditions. Moreover, discrete inf-sup constants between finite dimensional subspaces of $H^1_0(\Omega)^2$ and $L^2_\circ(\Omega)$ are influenced by both the continuous inf-sup constant $\beta(\Omega)$ and the type of chosen (mixed) discrete spaces, see [13] and also [6].

In reference [10], Horgan & Payne design an efficient strategy for calculating lower bounds of $\beta(\Omega)$ in domains Ω whose boundary can be described in polar coordinates (r, θ) by a relation $r = f(\theta)$ with a Lipschitz-continuous function f:

- First, state a relation between $\beta(\Omega)$ and the Friedrichs constant $\Gamma(\Omega)$,
- Second, find bounds for $\Gamma(\Omega)$ using f and its first derivative f'.

In the present paper, we revisit these two steps, with more emphasis on the second one.

§2. The Friedrichs constant

In dimension 2, the coordinates (x_1, x_2) are identified with the complex number $x_1 + ix_2$. Two real valued functions h and g are said to be harmonic conjugate if they are the real and imaginary parts of a holomorphic function $h + ig$. The functions h and g are harmonic conjugate if and only if they satisfy the relations

$$\Delta h = 0, \quad \Delta g = 0, \quad \text{and} \quad \operatorname{grad} h = \operatorname{curl} g \quad \text{in} \quad \Omega.$$

Let $\mathfrak{F}(\Omega)$ denote the space of complex valued $L^2(\Omega)$ holomorphic functions and let $\mathfrak{F}_\circ(\Omega)$ be its subspace of functions with mean value 0.

Definition 1. The *Friedrichs constant* (named after [7]) denoted by $\Gamma(\Omega)$, is the smallest constant $\Gamma \in \mathbb{R} \cup \{\infty\}$ such that for all $h + ig \in \mathfrak{F}_\circ(\Omega)$

$$\|h\|^2_{L^2(\Omega)} \le \Gamma \|g\|^2_{L^2(\Omega)}.$$

Theorem 1 ([10], [5])**.** *Let Ω be any bounded connected domain in $\mathbb{R}^2$. The LBB constant $\beta(\Omega)$ is positive if and only if $\Gamma(\Omega)$ is finite and*

$$\Gamma(\Omega) + 1 = \frac{1}{\beta(\Omega)^2}.$$

This relation between $\beta(\Omega)$ and $\Gamma(\Omega)$ was proved in [10] under additional regularity properties on the domain. A new proof is provided in [5], in which no regularity assumption is needed.

§3. An upper bound for the Friedrichs constant

Let Ω be *strictly star-shaped*, which means that there is an open ball $B \subset \Omega$ such that any segment with one end in B and the other in Ω, is contained in Ω. Let O be the center of B and (r, θ) be polar coordinates centered at O. Let $\theta \mapsto r = f(\theta)$ be the polar parametrization of the boundary $\partial\Omega$, defined on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

Lemma 2 ([11, Lemma 1.1.8]). *Let Ω be a bounded strictly star-shaped domain, and f be a polar parametrization of its boundary as described above. Then f belongs to $W^{1,\infty}(\mathbb{T})$.*

Since $\Gamma(\Omega)$ is invariant by dilation, we may assume without restriction that

$$\boxed{\max_{\theta\in\mathbb{T}} f(\theta) = 1} \tag{2}$$

Following the approach in [10], we are prompted to introduce the following notation.

Notation 3. Under condition (2), let $P = P(\alpha, \theta)$ be the function defined on $\mathbb{R}_+ \times \mathbb{T}$ as

$$P(\alpha, \theta) = \frac{1}{\alpha f(\theta)^2}\Big(1 + \frac{f'(\theta)^2}{f(\theta)^2 - \alpha f(\theta)^4}\Big). \tag{3}$$

Let $M(\Omega)$ and $m(\Omega)$ be the following two positive numbers

$$M(\Omega) = \inf_{\alpha\in(0,1)} \Big\{ \sup_{\theta\in\mathbb{T}} P(\alpha, \theta)\Big\} \quad \text{and} \quad m(\Omega) = \sup_{\theta\in\mathbb{T}} \Big\{ \inf_{\alpha\in\left(0,\frac{1}{f(\theta)^2}\right)} P(\alpha, \theta)\Big\}. \tag{4}$$

Remark 1. Let us choose $\theta \in \mathbb{T}$. Calculating the second derivative of the function $P_\theta : \alpha \mapsto P(\alpha, \theta)$ defined on the interval $(0, \frac{1}{f(\theta)^2})$, we find that P_θ is strictly convex. The function P_θ tends to $+\infty$ as $\alpha \to 0$, and if $f'(\theta) \neq 0$, as $\alpha \to \frac{1}{f(\theta)^2}$. In any case, there exists a unique $\alpha(\theta)$ in $(0, \frac{1}{f(\theta)^2}]$ such that

$$P(\alpha(\theta), \theta) = \inf_{\alpha\in\left(0,\frac{1}{f(\theta)^2}\right)} P(\alpha, \theta).$$

So,

$$m(\Omega) = \sup_{\theta\in\mathbb{T}} P(\alpha(\theta), \theta)\,. \tag{5}$$

Since, in particular, for all $\alpha \in (0, 1)$ and $\theta \in \mathbb{T}$, $P(\alpha(\theta), \theta) \leq P(\alpha, \theta)$, we find that

$$M(\Omega) \geq m(\Omega). \tag{6}$$

The quantity $m(\Omega)$ is the original bound introduced by Horgan-Payne in [10] and $M(\Omega)$ is our modified Horgan-Payne like bound.

Theorem 4 (Estimate (6.24) in [10]). *Let Ω be a bounded strictly star-shaped domain. Its Friedrichs constant satisfies the bound*

$$\Gamma(\Omega) \leq M(\Omega)\,. \tag{7}$$

Proof. We assume for simplicity that the origin O of polar coordinates coincides with the origin $\mathbf{0}$ of Cartesian coordinates. Let $g \in \mathscr{D}(\overline{\Omega})$ be an harmonic function and let $h \in \mathscr{D}(\overline{\Omega})$ be its harmonic conjugate such that $h(\mathbf{0}) = 0$. If we bound the $L^2(\Omega)$ norm of h, we bound a fortiori the $L^2(\Omega)$ norm of $h - \frac{1}{|\Omega|}\int_\Omega h$ which is the harmonic conjugate of g in $L^2_\circ(\Omega)$, hence with minimal $L^2(\Omega)$ norm. The extension of the estimate to all pairs of harmonic conjugate functions in $L^2(\Omega)$ follows from a density argument.

Since $h + ig$ is holomorphic, its square is holomorphic too and we deduce that the function $H := h^2 - g^2$ is harmonic conjugate of $G := 2gh$. Hence equation $\operatorname{grad} H = \operatorname{curl} G$ leads to the relation in polar coordinates

$$\partial_\rho \widetilde{H} = \frac{1}{\rho}\partial_\theta \widetilde{G}$$

where $\widetilde{H}(r,\theta) = H(\boldsymbol{x})$ and $\widetilde{G}(r,\theta) = G(\boldsymbol{x})$ for $\boldsymbol{x} = (r\cos\theta, r\sin\theta)$. Thus for any $\theta \in \mathbb{T}$ and $r \in (0, f(\theta))$ we have

$$\widetilde{H}(r,\theta) - H(\mathbf{0}) = \int_0^r \partial_\rho \widetilde{H}(\rho,\theta)\,\mathrm{d}\rho = \int_0^r \frac{1}{\rho}\partial_\theta \widetilde{G}(\rho,\theta)\,\mathrm{d}\rho\,.$$

We divide by $f(\theta)^2$ and integrate for $\theta \in \mathbb{T}$ and $r \in (0, f(\theta))$:

$$\begin{aligned}
\int_{\mathbb{T}}\int_0^{f(\theta)} \frac{\widetilde{H}(r,\theta) - H(\mathbf{0})}{f(\theta)^2}\, r\mathrm{d}r\mathrm{d}\theta &= \int_{\mathbb{T}}\int_0^{f(\theta)} \frac{1}{f(\theta)^2}\left\{\int_0^r \frac{1}{\rho}\partial_\theta \widetilde{G}(\rho,\theta)\,\mathrm{d}\rho\right\} r\mathrm{d}r\mathrm{d}\theta \\
&= \int_{\mathbb{T}}\int_0^{f(\theta)} \frac{1}{f(\theta)^2}\frac{1}{\rho}\partial_\theta \widetilde{G}(\rho,\theta)\left\{\int_\rho^{f(\theta)} r\mathrm{d}r\right\}\mathrm{d}\rho\mathrm{d}\theta \\
&= \frac{1}{2}\int_{\mathbb{T}}\int_0^{f(\theta)} \frac{f(\theta)^2 - \rho^2}{\rho^2 f(\theta)^2}\partial_\theta \widetilde{G}(\rho,\theta)\,\rho\mathrm{d}\rho\mathrm{d}\theta\,.
\end{aligned}$$

Since the function $f(\theta)^2 - \rho^2$ is 0 on the boundary, integration by parts yields

$$\int_{\mathbb{T}}\int_0^{f(\theta)} \frac{\widetilde{H}(r,\theta) - H(\mathbf{0})}{f(\theta)^2}\, r\mathrm{d}r\mathrm{d}\theta = -\int_{\mathbb{T}}\int_0^{f(\theta)} \frac{f'(\theta)}{f(\theta)^3}\widetilde{G}(\rho,\theta)\,\rho\mathrm{d}\rho\mathrm{d}\theta\,.$$

We set for any $\theta \in \mathbb{T}$

$$t(\theta) = \frac{f'(\theta)}{f(\theta)}\,.$$

Coming back to h and g we find:

$$\int_\Omega \frac{h(\boldsymbol{x})^2}{f(\theta)^2}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \frac{g(\boldsymbol{x})^2 - g(\mathbf{0})^2}{f(\theta)^2}\,\mathrm{d}\boldsymbol{x} - 2\int_\Omega \frac{t(\theta)h(\boldsymbol{x})g(\boldsymbol{x})}{f(\theta)^2}\,\mathrm{d}\boldsymbol{x}. \tag{8}$$

In order to take the best advantage of the previous identity we introduce a parameter

$$\alpha \in (0,1)$$

and write for any $\theta \in \mathbb{T}$ (here we use condition (2) which ensures that $1 - \alpha f(\theta)^2 > 0$)

$$2\big|t(\theta)h(\boldsymbol{x})g(\boldsymbol{x})\big| \le \big\{1 - \alpha f(\theta)^2\big\}h(\boldsymbol{x})^2 + \frac{t(\theta)^2}{1 - \alpha f(\theta)^2}g(\boldsymbol{x})^2$$

and deduce from (8) that (note that the same α is used for all θ)

$$\alpha \int_\Omega h(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x} \leq \int_\Omega \frac{g(\boldsymbol{x})^2}{f(\theta)^2} + \frac{t(\theta)^2}{1-\alpha f(\theta)^2}\,\frac{g(\boldsymbol{x})^2}{f(\theta)^2}\,\mathrm{d}\boldsymbol{x}\,.$$

Thus, for any $\alpha \in (0,1)$

$$\int_\Omega h(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x} \leq \sup_{\theta\in\mathbb{T}}\Big\{\frac{1}{\alpha f(\theta)^2}\Big(1+\frac{t(\theta)^2}{1-\alpha f(\theta)^2}\Big)\Big\}\int_\Omega g(\boldsymbol{x})^2\,\mathrm{d}\boldsymbol{x}\,.$$

Optimizing on $\alpha \in (0,1)$ and coming back to the definition of t and P, we find

$$\int_\Omega h(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x} \leq \inf_{\alpha\in(0,1)}\Big\{\sup_{\theta\in\mathbb{T}} P(\alpha,\theta)\Big\}\int_\Omega g(\boldsymbol{x})^2\,\mathrm{d}\boldsymbol{x},$$

which is nothing else than $\|h\|^2_{0,\Omega} \leq M(\Omega)\,\|g\|^2_{0,\Omega}$, whence the theorem. □

Remark 2. The proof above is due to Horgan and Payne in [10, § 6]. Unfortunately, instead of simply concluding that $M(\Omega)$ is an upper bound for $\Gamma(\Omega)$, they try to show that $M(\Omega)$ coincides with $m(\Omega)$ and this part of their argument is flawed. In the rest of our paper we discuss cases where equality or non-equality holds between these two quantities.

§4. The Horgan-Payne angle

Stoyan in [14] propose an interesting geometrical interpretation of the lower bound on $\beta(\Omega)$ under the condition that $m(\Omega)$ is an upper bound for $\Gamma(\Omega)$.

Notation 5. For $\theta \in \mathbb{T}$, let $\boldsymbol{x}$ be the point $(f(\theta)\cos\theta, f(\theta)\sin\theta)$ in $\partial\Omega$, let $\gamma(\theta) \in [0, \frac{\pi}{2})$ denote the (non-oriented) angle between the line $[\mathbf{0}, \boldsymbol{x}]$ and the outward normal vector to $\partial\Omega$ at $\boldsymbol{x}$. We set

$$\gamma(\Omega) = \sup_{\theta\in\mathbb{T}} \gamma(\theta) \quad \text{and} \quad \omega(\Omega) = \frac{\pi}{2} - \gamma(\Omega)\,. \tag{9}$$

The angle $\omega(\Omega)$ is referred as the Horgan-Payne angle in [14].

Lemma 6. *We have the identities*

$$m(\Omega) = \frac{1+\sin\gamma(\Omega)}{1-\sin\gamma(\Omega)} \quad \text{and} \quad \frac{1}{\sqrt{m(\Omega)+1}} = \sin\frac{\omega(\Omega)}{2}\,. \tag{10}$$

Proof. Let us recall the formulas

$$\cos\gamma(\theta) = \frac{f(\theta)}{\sqrt{f(\theta)^2+f'(\theta)^2}}\,, \quad \sin\gamma(\theta) = \frac{f'(\theta)}{\sqrt{f(\theta)^2+f'(\theta)^2}}\,, \quad \tan\gamma(\theta) = \frac{f'(\theta)}{f(\theta)}\,.$$

Hence we have

$$P(\alpha,\theta) = \frac{1}{\alpha f(\theta)^2}\Big(1+\frac{\tan^2\gamma(\theta)}{1-\alpha f(\theta)^2}\Big). \tag{11}$$

Let θ be chosen. To determine the value $\alpha(\theta)$ which realizes the minimum of $P(\alpha,\theta)$ for $\alpha \in (0, 1/f(\theta)^2]$, cf. (5), we calculate

$$\partial_\alpha P(\alpha,\theta) = -\frac{1}{\alpha^2 f(\theta)^2}\Big(1 + \frac{\tan^2\gamma(\theta)}{1-\alpha f(\theta)^2}\Big) + \frac{1}{\alpha f(\theta)^2}\,\frac{\tan^2\gamma(\theta) f(\theta)^2}{(1-\alpha f(\theta)^2)^2}\,.$$

Setting $\zeta = \alpha f(\theta)^2$, we see that $\partial_\alpha P(\alpha,\theta) = 0$ if and only if

$$\zeta^2 - 2(1+\tan^2\gamma(\theta))\zeta + 1 + \tan^2\gamma(\theta) = 0. \tag{12}$$

We look for $\zeta \in (0, 1]$. The convenient root of equation (12) is

$$\begin{aligned}\alpha(\theta) f(\theta)^2 = \zeta &= 1 + \tan^2\gamma(\theta) - \tan\gamma(\theta)\sqrt{1+\tan^2\gamma(\theta)}\\ &= \frac{1}{1+\sin\gamma(\theta)}\,.\end{aligned} \tag{13}$$

Hence we find

$$P(\alpha(\theta),\theta) = \frac{1+\sin\gamma(\theta)}{1-\sin\gamma(\theta)}\,, \tag{14}$$

whose supremum is attained for the supremum $\gamma(\Omega)$ of $\gamma(\theta)$, whence the first formula in (10). The second formula is obtained using $\sin\frac{\omega(\Omega)}{2} = \sin(\frac{\pi}{4} - \frac{\gamma(\Omega)}{2}) = \frac{1}{\sqrt{2}}(\cos\frac{\gamma(\Omega)}{2} - \sin\frac{\gamma(\Omega)}{2})$. □

As a straightforward consequence of Theorem 1 and Lemma 6, we obtain the following.

Corollary 7. *For any domain such that* $\Gamma(\Omega) \le m(\Omega)$*, the inf-sup constant* $\beta(\Omega)$ *satisfies*

$$\beta(\Omega) \ge \sin\frac{\omega(\Omega)}{2}\,. \tag{15}$$

Remark 3. The estimate (15) is stated in [14] for any strictly star-shaped domain. The reality is that (15) is true if and only if $\Gamma(\Omega) \le m(\Omega)$. The latter estimate is true for some categories of domains as we will see in the next section. We will also exhibit domains for which $m(\Omega)$ is distinct from $M(\Omega)$. In [5] it is proved that, in fact, there exists strictly star-shaped domains such that $\Gamma(\Omega) > m(\Omega)$ (equivalently, $\beta(\Omega) < \sin\frac{\omega(\Omega)}{2}$).

§5. Examples

In this section, we consider some particular shapes of domains, namely ellipses, polygons, and limaçons.

5.1. Disks and ellipses

The equation of an ellipse can always be written in suitable Cartesian coordinates as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with positive coefficients $a \le b$. The constant $\Gamma(\Omega)$ is analytically known, cf. [7], namely

$$\Gamma(\Omega) = \frac{b^2}{a^2} \quad \text{and} \quad \beta(\Omega) = \frac{a}{\sqrt{a^2+b^2}} . \tag{16}$$

In polar coordinates, the parametrization of the ellipse is

$$f(\theta) = ab\left(b^2\cos^2\theta + a^2\sin^2\theta\right)^{-1/2}. \tag{17}$$

i) Let us calculate $m(\Omega)$. We have

$$\tan\gamma(\theta) = \frac{f'(\theta)}{f(\theta)} = \frac{\sin\theta\,\cos\theta\,(b^2-a^2)}{b^2\cos^2\theta + a^2\sin^2\theta} = \frac{\tan\theta\,(b^2-a^2)}{b^2+a^2\tan^2\theta} . \tag{18}$$

The maximal value $\tan\gamma(\Omega)$ of $\tan\gamma(\theta)$ is obtained for

$$\tan\theta = \frac{b}{a} ,$$

hence

$$\tan\gamma(\Omega) = \frac{b^2-a^2}{2ab}$$

from which we deduce

$$\sin\gamma(\Omega) = \frac{b^2-a^2}{b^2+a^2} .$$

Formula (10) then yields

$$m(\Omega) = \frac{b^2}{a^2} .$$

ii) Let us calculate $M(\Omega)$. In order to comply with the condition $\max_{\theta\in\mathbb{T}} f(\theta) = 1$, we set $\tilde a = b/a$ and $\tilde b = 1$, and consider f given by (17) with a, b replaced by $\tilde a$, $\tilde b$. We use formula (11) for P to write:

$$P(\alpha,\theta) = \frac{1}{\alpha}\,\frac{(1+\tan^2\gamma(\theta))\,f(\theta)^{-2} - \alpha}{1-\alpha f(\theta)^2} .$$

From (17) and (18) we deduce

$$(1+\tan^2\gamma(\theta))\,f(\theta)^{-2} = \tilde a^{-2} .$$

Therefore

$$P(\alpha,\theta) = \frac{1}{\alpha}\,\frac{\tilde a^{-2}-\alpha}{1-\alpha f(\theta)^2} .$$

For each $\alpha \in (0,1)$, the supremum in θ of $P(\alpha,\theta)$ is attained for $f(\theta)$ minimum, i.e. in $\theta = 0$ for which $f(\theta) = \tilde a$. We deduce

$$\sup_{\theta\in\mathbb{T}} P(\alpha,\theta) = \frac{1}{\alpha}\,\frac{\tilde a^{-2}-\alpha}{1-\alpha\tilde a^2} = \frac{1}{\alpha}\,\frac{1}{\tilde a^2} = \frac{1}{\alpha}\,\frac{b^2}{a^2} ,$$

hence, taking the infimum over $\alpha \in (0,1)$:

$$M(\Omega) = \frac{b^2}{a^2}\,.$$

Comparing with (16), we finally obtain

$$m(\Omega) = M(\Omega) = \frac{b^2}{a^2} = \Gamma(\Omega)\,. \tag{19}$$

In particular, if Ω is a disk

$$m(\Omega) = M(\Omega) = \Gamma(\Omega) = 1\,. \tag{20}$$

5.2. Star-shaped polygons

A polygon Ω is characterized by the fact that its boundary is a finite union of segments. Let us first investigate the behavior of the function P along a segment.

For ease of computation, we consider a segment I lying on a vertical line of equation $x_1 = d$ with $d > 0$. Note that d is the distance of this line to the origin. Normals to I are horizontal. We find

$$f(\theta) = \frac{d}{\cos\theta} \quad \text{and} \quad \gamma(\theta) = \theta. \tag{21}$$

Hence, under the global condition $\max_{\theta\in\mathbb{T}} f(\theta) = 1$, the contribution to the function P of such a segment is — here we use formula (11),

$$\begin{aligned} P(\alpha,\theta) &= \frac{\cos^2\theta}{\alpha d^2}\,\frac{1-\alpha d^2}{\cos^2\theta - \alpha d^2} \\ &= \frac{1}{\alpha d^2}\,\frac{1-\alpha d^2}{1-\alpha f(\theta)^2}. \end{aligned} \tag{22}$$

For any $\alpha \in (0,1)$, the maximal value of P is attained for $f(\theta)$ maximal, i.e., at an end of the segment I, and this end is the most distant from the origin. That is why we introduce:

Notation 8. For any side I_j, $j = 1,\ldots,J$, of a polygon Ω, we define its radius r_j as the distance between the origin and its most distant endpoint E_j. Denoting by $\tilde I_j$ the line containing I_j, we define d_j as the distance of $\tilde I_j$ to the origin.

The normalization (2) here takes the form $\max_j r_j = 1$. From the previous computation (22) we find the formula

$$M(\Omega) = \inf_{\alpha\in(0,1)} \max_{j=1}^{J} \frac{1}{\alpha d_j^2}\,\frac{1-\alpha d_j^2}{1-\alpha r_j^2} \tag{23}$$

$$= \inf_{\alpha\in(0,1)} \max_{j=1}^{J} \frac{1}{\alpha r_j^2}\,\frac{r_j^2 d_j^{-2} - \alpha r_j^2}{1-\alpha r_j^2}\,. \tag{24}$$

In order to find a similar formula for the quantity $m(\Omega)$, we are going to use (5) and we go back to expression (14) which can be written in function of $\cos\gamma(\theta)$ instead of $\sin\gamma(\theta)$:

$$P(\alpha(\theta),\theta) = \left(\frac{1}{\cos\gamma(\theta)} + \sqrt{\frac{1}{\cos^2\gamma(\theta)} - 1}\right)^2. \tag{25}$$

For the angles θ corresponding to the segment I_j, the supremum of $P(\alpha(\theta),\theta)$ is attained for $\cos\gamma(\theta)$ minimum, i.e. for $\cos\gamma(\theta) = \frac{d_j}{r_j}$. Therefore formula (25) yields

$$m(\Omega) = \max_{j=1}^{J} \left(\frac{r_j}{d_j} + \sqrt{\frac{r_j^2}{d_j^2} - 1} \right)^2 . \tag{26}$$

The maximum is attained when r_j/d_j is maximal.

Proposition 9. *Let Ω be a polygon, with d_j and r_j the distances in Notation 8. We have*

i) If all r_j are equal, then $M(\Omega) = m(\Omega)$.

ii) If all d_j are equal, then $M(\Omega) = m(\Omega)$.

iii) If the largest value of r_j/d_j is attained for two different indices j and k and if $r_j \neq r_k$, then $M(\Omega) > m(\Omega)$.

Proof. i) If all r_j are equal, the normalization $\max_j r_j = 1$ yields that $r_j = 1$. Formula (24) then gives that

$$M(\Omega) = \inf_{\alpha\in(0,1)} \frac{1}{\alpha} \frac{d_{\min}^{-2} - \alpha}{1-\alpha}$$

where $d_{\min}$ is the minimum value of the d_j. The optimization with respect to α provides the optimal value

$$\alpha_0 = \frac{1}{d_{\min}^2} - \sqrt{\frac{1}{d_{\min}^4} - \frac{1}{d_{\min}^2}} \in (0,1)$$

for α, hence the infimum

$$M(\Omega) = \left(\frac{1}{d_{\min}} + \sqrt{\frac{1}{d_{\min}^2} - 1} \right)^2$$

which coincides with $m(\Omega)$ given by (26) since r_j/d_j is maximal for $1/d_{\min}$.

ii) If all d_j are equal, Formula (23) gives that

$$M(\Omega) = \inf_{\alpha\in(0,1)} \frac{1}{\alpha d^2} \frac{1-\alpha d^2}{1-\alpha r_{\max}^2}$$

where d is the common value of the d_j and $r_{\max}$ the maximum value of the r_j. Due to normalization $\max_j r_j = 1$, this formula becomes

$$M(\Omega) = \inf_{\alpha\in(0,1)} \frac{1}{\alpha d^2} \frac{1-\alpha d^2}{1-\alpha} = \inf_{\alpha\in(0,1)} \frac{1}{\alpha} \frac{d^{-2}-\alpha}{1-\alpha} .$$

As in the previous case we find

$$M(\Omega) = \left(\frac{1}{d} + \frac{\sqrt{1-d^2}}{d} \right)^2 ,$$

which coincides with $m(\Omega)$ given by (26) since r_j/d_j is maximal for $1/d$.

iii) For $\ell \in \{j,k\}$, let θ_ℓ be the angle θ corresponding to the end E_ℓ. We have

$$M(\Omega) \geq \min_{\alpha\in(0,1)} \max_{\theta\in\{\theta_j,\theta_k\}} P(\alpha,\theta)\,.$$

And let α_m be the value of $\alpha \in (0,1]$ minimizing $\max_{\theta\in\{\theta_j,\theta_k\}} P(\alpha,\theta)$. We have

$$M(\Omega) \geq \max\{P(\alpha_m,\theta_j), P(\alpha_m,\theta_k)\}\,.$$

Now, still for $\ell \in \{j,k\}$, let α_ℓ be the value of $\alpha \in (0, r_\ell^{-2})$ minimizing $P(\alpha,\theta_\ell)$. Since $r_j/d_j = r_k/d_k$ maximizes the quotients r_i/d_i, we have by (26)

$$\begin{aligned} m(\Omega) &= \left(\frac{r_j}{d_j} + \sqrt{\frac{r_j^2}{d_j^2} - 1}\right)^2 = \left(\frac{r_k}{d_k} + \sqrt{\frac{r_k^2}{d_k^2} - 1}\right)^2 \\ &= P(\alpha_j,\theta_j) = P(\alpha_k,\theta_k)\,. \end{aligned}$$

By (13), α_j and α_k satisfy

$$\alpha_\ell r_\ell^2 = \frac{1}{1+\sin\gamma(\theta_\ell)}\,, \quad \ell = j,k\,.$$

But $\sin\gamma(\theta_j) = \sin\gamma(\theta_k)$ because $\cos\gamma(\theta_\ell) = r_\ell/d_\ell$. Hence, since $r_j \neq r_k$, we have $\alpha_j \neq \alpha_k$, therefore α_m cannot coincide with α_j and α_k at the same time. So, since the functions $\alpha \mapsto P(\alpha,\theta)$ are strictly convex in the interval $(0, f(\theta)^{-2})$, we deduce

$$M(\Omega) \geq \max\{P(\alpha_m,\theta_j), P(\alpha_m,\theta_k)\} > P(\alpha_j,\theta_j) = P(\alpha_k,\theta_k) = m(\Omega),$$

and conclude that $M(\Omega) > m(\Omega)$ as announced in the proposition. □

Here are examples for the three situations *i) – iii)* investigated in Proposition 9.

Example 1. In each of the examples below, the center **0** of polar and Cartesian coordinates is chosen at the *barycenter* of the domain.

i) If Ω is a regular polygon or a rectangle, then all r_j are equal, thus $M(\Omega) = m(\Omega)$.

ii) If Ω is a triangle or a rhombus, then all d_j are equal, thus $M(\Omega) = m(\Omega)$.

iii) See Figure 1: For this hexagonal domain, the quotients r_j/d_j are all equal to $\sqrt{2}$, but $r_1 = 1$ and $r_2 = 1/\sqrt{2}$ (with $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{4}$). Therefore $m(\Omega) < M(\Omega)$.

5.3. Limaçons

Limaçons of Pascal (named after Etienne Pascal, father of Blaise Pascal) are curves defined in polar coordinates by a formula of the type

$$f_\varepsilon(\theta) = a\,(1+\varepsilon\cos\theta), \quad a>0,\ \varepsilon>0. \tag{27}$$

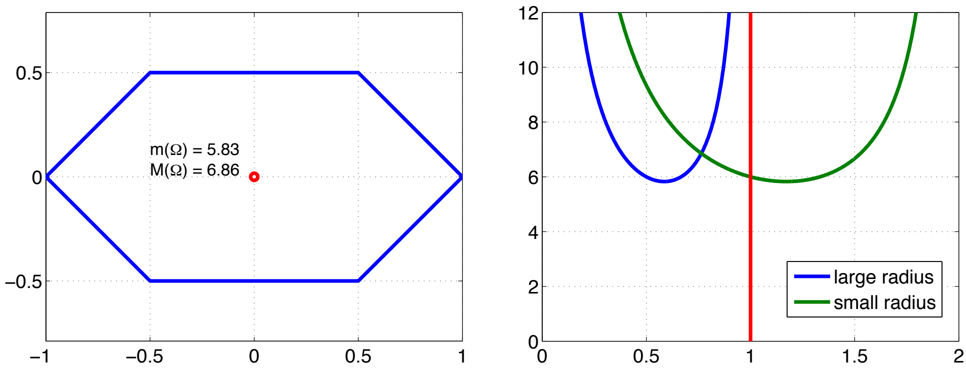

Figure 1: Example where $M(\Omega) > m(\Omega)$. Domain Ω with center of coordinates, left. Plot of $\alpha \mapsto P(\alpha, \theta_j)$ for $\alpha \in (0, \frac{1}{f(\theta_j)^2})$, $j = 1, 2$, right.

Such a curve is simple if ε is less than 1, and so defines the boundary of a domain Ω_ε. In [10] the case of such limaçons is considered. Here the constant $\Gamma(\Omega_\varepsilon)$ is analytically known, [9, 10], which provides an explicit formula for $\beta(\Omega_\varepsilon)$ via Theorem 1

$$\Gamma(\Omega_\varepsilon) = \frac{2+\varepsilon^2}{2-\varepsilon^2} \quad \text{and} \quad \beta(\Omega_\varepsilon) = \frac{\sqrt{2-\varepsilon^2}}{2} \,. \tag{28}$$

This example can serve as a benchmark for bounds m and M. We have computed by a Matlab program the two constants $m(\Omega_\varepsilon)$ and $M(\Omega_\varepsilon)$. It happens that as soon as ε is not zero, i.e., Ω_ε is not a circle, these two constants are distinct, see the top two curves in Figure 2. The other curves are explained below.

Here comes the question of the choice of polar coordinates defining $m(\Omega)$ and $M(\Omega)$. For limaçons, the first choice is to consider the polar coordinates in which the domain is defined by (27). But, considering that Ω_ε intersects the horizontal axis between the points $-a + a\varepsilon$ and $a + a\varepsilon$, choosing new polar coordinates (r', θ') centered at $O' = (a\varepsilon, 0)$ appears more judicious. A new equation

$$r' = f'_\varepsilon(\theta')$$

is associated with Ω_ε, leading to new quantities

$$m'(\Omega_\varepsilon) \quad \text{and} \quad M'(\Omega_\varepsilon)\,.$$

In fact these new quantities are very different from the old ones. We have observed that $m'(\Omega_\varepsilon)$ and $M'(\Omega_\varepsilon)$ do coincide, and are much smaller than $m(\Omega_\varepsilon)$ and $M(\Omega_\varepsilon)$, see Figure 2. Moreover, the asymptotic behavior of $m'(\Omega_\varepsilon) = M'(\Omega_\varepsilon)$ as $\varepsilon \to 0$ is very good. Indeed, using formula (28) allows us to compute the difference $m'(\Omega_\varepsilon) - \Gamma(\Omega_\varepsilon)$. We have found numerical evidence, see Figure 3, for the asymptotic behavior

$$m'(\Omega_\varepsilon) - \Gamma(\Omega_\varepsilon) = M'(\Omega_\varepsilon) - \Gamma(\Omega_\varepsilon) = O(\varepsilon^3).$$

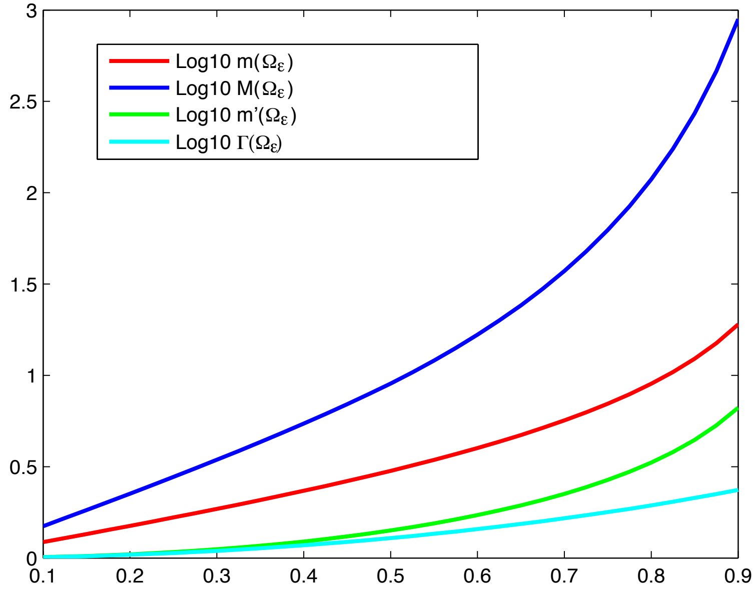

Figure 2: Plot of $\varepsilon \mapsto \log_{10}\{m(\Omega_\varepsilon),\ M(\Omega_\varepsilon),\ m'(\Omega_\varepsilon) = M'(\Omega_\varepsilon),\ \Gamma(\Omega_\varepsilon)\}$ for $\varepsilon = 0.1$ to 0.9.

5.4. Decentered disks

We end this section by a curiosity which sheds some light on the discrepancy between $m(\Omega)$ and $M(\Omega)$ and their dependency on the center of polar coordinates. Let Ω be a disk. We have seen in (20) that we have the optimal values $m(\Omega) = M(\Omega) = \Gamma(\Omega) = 1$.

Now, we consider decentered disks, moving off the center of polar coordinates by a relative amount δ with respect to the radius of the disk. We can assume that the new center lies on the horizontal axis. This defines new versions of the constants, denoted $m[\delta](\Omega)$ and $M[\delta](\Omega)$. It is not very hard to prove the following
(i) The maximal value of the angle γ occurs for $\theta_0 = \frac{\pi}{2}$, so $\sin\gamma = \delta$. Hence, cf (10),

$$m[\delta](\Omega) = \frac{1+\delta}{1-\delta}\,.$$

This value is the same for the limaçon (27) choosing $\varepsilon = \delta$, see [10, (6.34)].
(ii) For any $\alpha \in (0,1)$, the max in θ of $P(\alpha,\theta)$ is attained for $\theta = \pi$, then the inf in α corresponds to $P(1,\pi)$. Hence

$$M[\delta](\Omega) = \frac{1}{f(\pi)^2} = \Big(\frac{1+\delta}{1-\delta}\Big)^2 = m[\delta](\Omega)^2 > m[\delta](\Omega).$$

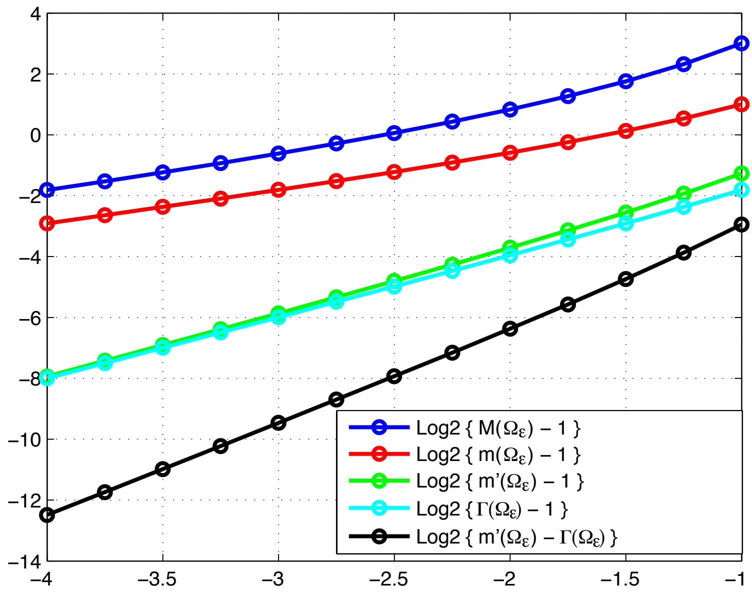

Figure 3: Plot of $\log_2 \varepsilon \mapsto \log_2 \{M(\Omega_\varepsilon)-1,\ m(\Omega_\varepsilon)-1,\ m'(\Omega_\varepsilon)-1,\ \Gamma(\Omega_\varepsilon)-1,\ m'(\Omega_\varepsilon)-\Gamma(\Omega_\varepsilon)\}$ for $\varepsilon = 0.0625$ to 0.5.

References

[1] Acosta, G., Durán, R. G., and Muschietti, M. A. Solutions of the divergence operator on John domains. *Adv. Math. 206*, 2 (2006), 373–401. Available from: http://dx.doi.org/10.1016/j.aim.2005.09.004, doi:10.1016/j.aim.2005.09.004.

[2] Babuška, I. The finite element method with Lagrangian multipliers. *Numer. Math. 20* (1972/73), 179–192.

[3] Babuška, I., and Aziz, A. K. Survey lectures on the mathematical foundations of the finite element method. In *The mathematical foundations of the finite element method with applications to partial differential equations (Proc. Sympos., Univ. Maryland, Baltimore, Md., 1972)*. Academic Press, New York, 1972, pp. 1–359. With the collaboration of G. Fix and R. B. Kellogg.

[4] Brezzi, F. On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge 8*, R-2 (1974), 129–151.

[5] Costabel, M., and Dauge, M. On the inequalities of Babuška–Aziz, Friedrichs and Horgan–Payne. In preparation.

[6] Dobrowolski, M. On the LBB condition in the numerical analysis of the Stokes equations. *Appl. Numer. Math. 54*, 3-4 (2005), 314–323. Available from: http://dx.doi.org/10.1016/j.apnum.2004.09.005, doi:10.1016/j.apnum.2004.09.005.

[7] Friedrichs, K. On certain inequalities and characteristic value problems for analytic functions and for functions of two variables. *Trans. Amer. Math. Soc. 41*, 3 (1937), 321–364. Available from: http://dx.doi.org/10.2307/1989786, doi:10.2307/1989786.

[8] Girault, V., and Raviart, P. *Finite Element Methods for the Navier–Stokes Equations, Theory and Algorithms*. Springer series in Computational Mathematics, 5. Springer-Verlag, Berlin, 1986.

[9] Horgan, C. O. Inequalities of Korn and Friedrichs in elasticity and potential theory. *Z. Angew. Math. Phys. 26* (1975), 155–164.

[10] Horgan, C. O., and Payne, L. E. On inequalities of Korn, Friedrichs and Babuška-Aziz. *Arch. Rational Mech. Anal. 82*, 2 (1983), 165–179. Available from: http://dx.doi.org/10.1007/BF00250935, doi:10.1007/BF00250935.

[11] Maz'ya, V. *Sobolev spaces with applications to elliptic partial differential equations*, augmented ed., vol. 342 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. Available from: http://dx.doi.org/10.1007/978-3-642-15564-2, doi:10.1007/978-3-642-15564-2.

[12] Nečas, J. *Les méthodes directes en théorie des équations elliptiques*. Masson-Academia, Paris-Prague, 1967.

[13] Stoyan, G. Towards discrete Velte decompositions and narrow bounds for inf-sup constants. *Comput. Math. Appl. 38*, 7-8 (1999), 243–261. Available from: http://dx.doi.org/10.1016/S0898-1221(99)00254-0, doi:10.1016/S0898-1221(99)00254-0.

[14] Stoyan, G. Iterative Stokes solvers in the harmonic Velte subspace. *Computing 67*, 1 (2001), 13–33. Available from: http://dx.doi.org/10.1007/s006070170014, doi:10.1007/s006070170014.

[15] Tartar, L. *An introduction to Navier-Stokes equation and oceanography*, vol. 1 of *Lecture Notes of the Unione Matematica Italiana*. Springer-Verlag, Berlin, 2006.

Monique Dauge and Martin Costabel
IRMAR
CNRS et Université de Rennes 1
Campus de Beaulieu
35042 RENNES Cedex, FRANCE
monique.dauge@univ-rennes1.fr
martin.costabel@univ-rennes1.fr

Christine Bernardi and Vivette Girault
Laboratoire Jacques-Louis Lions
CNRS et Université Pierre et Marie Curie
4 place Jussieu
75252 PARIS Cedex 05, FRANCE
bernardi@ann.jussieu.fr
girault@ann.jussieu.fr