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Analytical and Numerical Treatment of Singularities in PDE

Edge-Corner interaction inside polyhedral singularities

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Domains and Equations

Domain Ω : 3D polyhedral *(or with curved faces, conical points, in higher dimensions)*

Inner Operator L : 2nd order strongly elliptic, scalar or matrix-valued *(or ord. $2m$),* defined via a 1st order form a (usual ingredients: grad, div, curl, strain e).

Coefficients of a : constant, or piecewise-constant in a polyhedral domain decomposition Ω_j of Ω *(or piecewise-smooth).*

Boundary conditions B : Dirichlet or Neumann on each face (of Ω or of Ω_j [a]) defining a subspace V of H^1 (except for Maxwell)

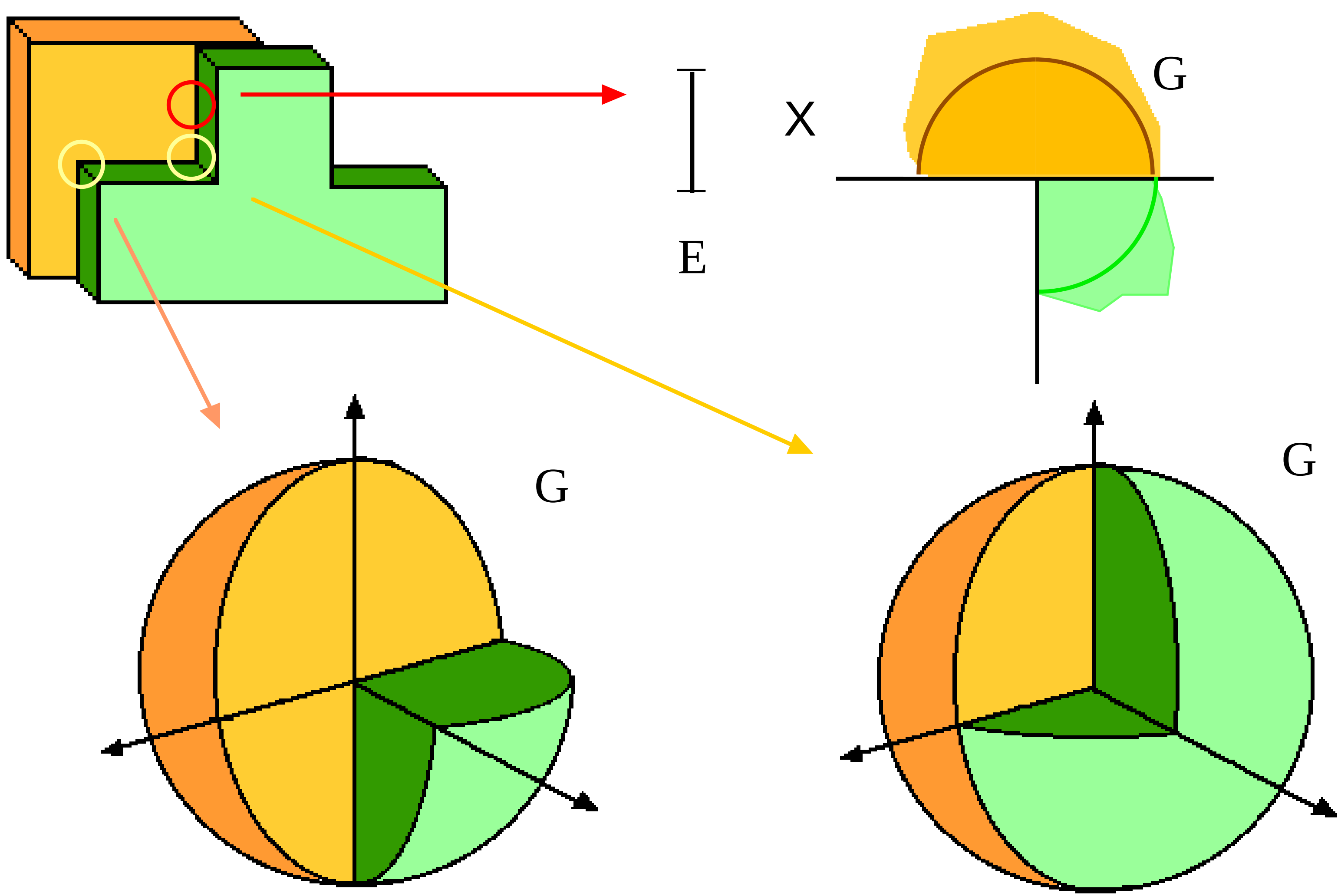
$$(\mathcal{P}) \qquad u \in V, \quad \forall v \in V, \qquad a(u,v) = \int_\Omega f\, v\, dx$$

i.e.

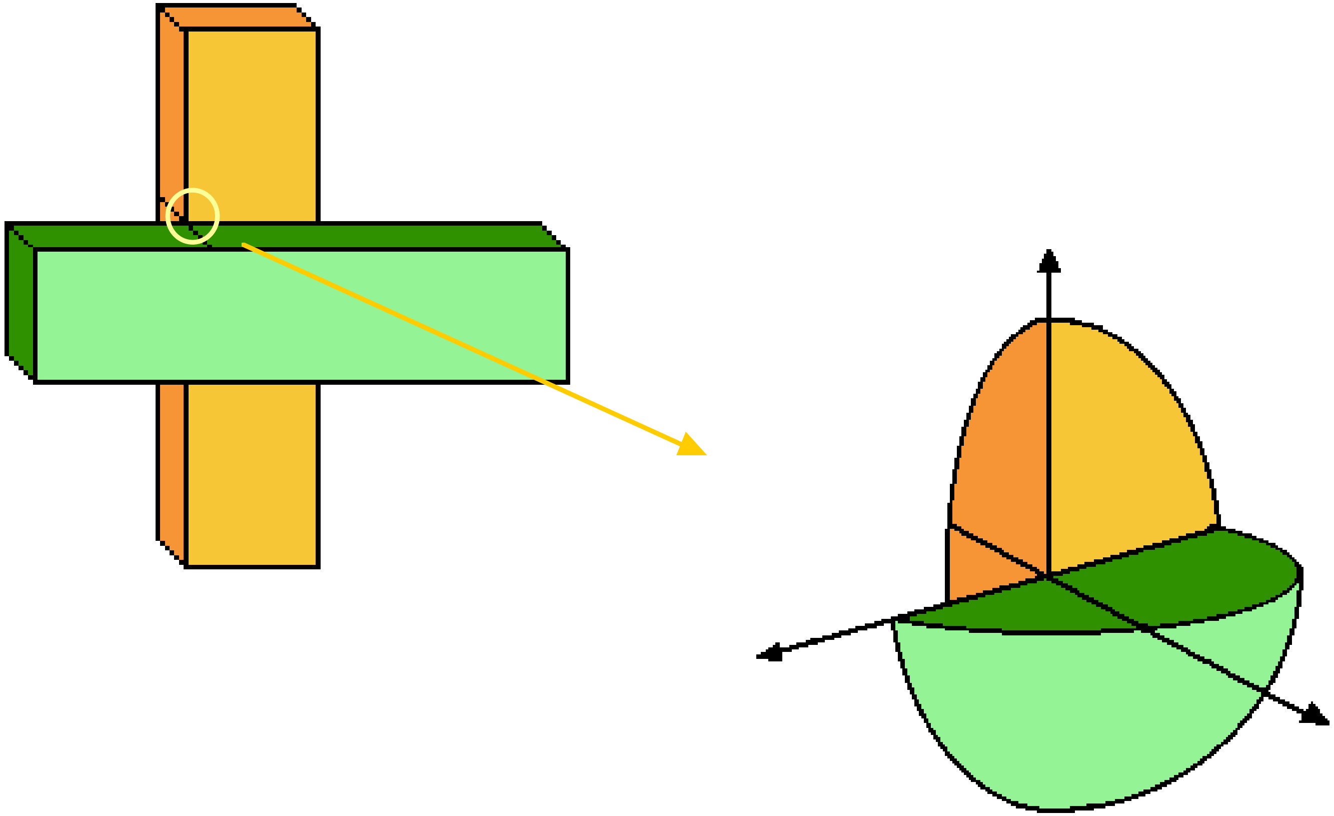
$$(\mathcal{P}) \qquad \begin{cases} Lu = f & \text{in } \Omega_j, \\ Bu = 0 & \text{on } \partial\Omega \cap \partial\Omega_j, \qquad [u] = [Nu] = 0 \ \text{ on } \partial\Omega_j \setminus \partial\Omega \end{cases}$$

[a] *Careful! Mixed boundary conditions on a smooth domain are a trap!*

Vertices and Edges



A non-Lipschitz polyhedron



Example: Electrostatic Potential in (Heterogeneous) Media

Domain Ω, 3D-polyhedron. $\Omega = \cup_j \Omega_j$, Ω_j 3D-polyhedron.

Each Ω_j is formed by an homogeneous medium: Constant electric permittivity ε_j.

Electrostatic potential u solution of the variational problem:

$$(\mathcal{P}) \qquad u \in H_0^1(\Omega), \quad \forall v \in H_0^1(\Omega), \qquad a(u,v) = \int_\Omega f\, v\, dx$$

with the bilinear form a

$$a(u,v) = \sum_j \int_{\Omega_j} \varepsilon_j \operatorname{grad} u \cdot \operatorname{grad} v \; dx.$$

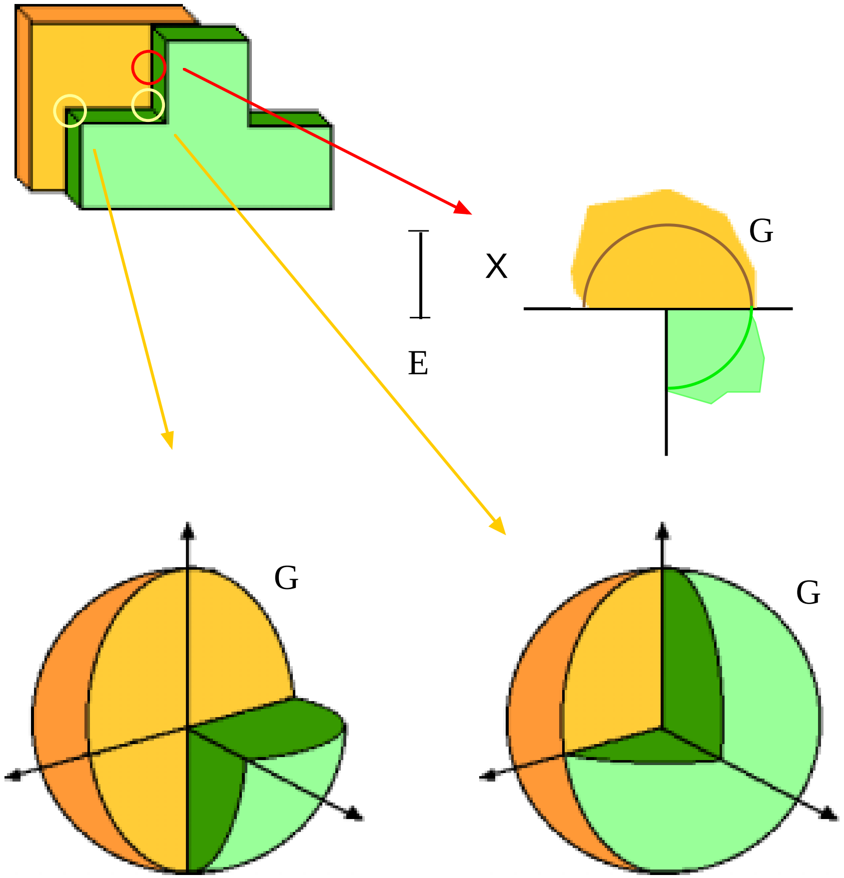
Corresponding electrostatic field (solution of Maxwell equations at zero frequency)

$$\boxed{E = \operatorname{grad} u}$$

Vertex and edge singularities

For a polyhedron Ω and an operator L with constant coefficients.

With curved faces or variable coefficients, the edge singularities vary along their edge.



Corner c $\longleftrightarrow$ Polar coord. $(\rho, \vartheta) \in \mathbb{R}_+ \times G_{\mathrm{c}}$ **centered in** c.

$$\begin{cases} \textit{Singularity Exponents} & \lambda \in \Lambda[\mathrm{c}] \\ \textit{Singular Functions} & U_{\mathrm{c},\lambda} = \rho^{\lambda}\, u_{\mathrm{c},\lambda}(\vartheta) \end{cases}$$

Edge e $\longleftrightarrow$ Cylindrical coord.

$(r, \theta, z) \in \mathbb{R}_+ \times (0, \omega_{\mathrm{e}}) \times \mathrm{e}$.

$$\begin{cases} \textit{Singularity Exponents} & \lambda \in \Lambda[\mathrm{e}] \\ \textit{Singular Functions} & U_{\mathrm{e},\lambda} = r^{\lambda}\, u_{\mathrm{e},\lambda}(\theta) \end{cases}$$

How to find Singular Functions

Corner $\mathbf{c}$: Singular Function $U_{\mathbf{c},\lambda} = \rho^{\lambda}\, u_{\mathbf{c},\lambda}(\vartheta)$ in the cone

$$\Gamma_{\mathbf{c}} = \{\mathbb{R}^3 \ni x \simeq (\rho, \vartheta) \in \mathbb{R}_+ \times G_{\mathbf{c}}\}$$

Edge $\mathbf{e}$: Singular Function $U_{\mathbf{e},\lambda} = r^{\lambda}\, u_{\mathbf{e},\lambda}(\theta)$ in the sector

$$\Gamma_{\mathbf{e}} = \{\mathbb{R}^2 \ni y \simeq (r, \theta) \in \mathbb{R}_+ \times (0, \omega_{\mathbf{e}})\}$$

Singularities are special solutions of problem $(\mathcal{P})$:

- **With zero right hand sides,**
- **In infinite cones $\Gamma_{\mathbf{c}}$ (for the corner $\mathbf{c}$) or sectors $\Gamma_{\mathbf{e}}$ (for the edge $\mathbf{e}$),**
- **Homogeneous in ρ (for corners) or in r (for edges).**

The problem of finding the singularities is a spectral problem

★ **in the spherical polygon $G_{\mathbf{c}}$ for the corner $\mathbf{c}$,**

★ **in the intervals $(0, \omega_{\mathbf{e}})$ for the edge $\mathbf{e}$.**

. . . / . . .

Vertex Mellin Symbols

Vertex $\mathbf{c}$ (of Ω or Ω_j). In polar coordinates $(\rho, \vartheta) \in \mathbb{R}_+ \times G_{\mathbf{c}}$

$$L = \rho^{-2}\mathcal{L}_{\mathbf{c}}(\vartheta; \rho\partial_\rho, \partial_\vartheta), \qquad \text{and} \qquad B = \rho^{-\deg B}\mathcal{B}_{\mathbf{c}}(\vartheta; \rho\partial_\rho, \partial_\vartheta).$$

Mellin Symbol and associated problem:

$$\mathbb{C} \ni \lambda \longmapsto \mathcal{L}_{\mathbf{c}}(\vartheta; \lambda, \partial_\vartheta) =: \mathcal{L}_{\mathbf{c}}(\lambda) \quad \text{and} \quad \mathbb{C} \ni \lambda \longmapsto \mathcal{B}_{\mathbf{c}}(\vartheta; \lambda, \partial_\vartheta) =: \mathcal{B}_{\mathbf{c}}(\lambda).$$

$$\mathcal{P}[\mathbf{c}, \lambda] \quad \begin{cases} \mathcal{L}_{\mathbf{c}}(\lambda)\, u = f & \text{in } G_{\mathbf{c}}, \\ \mathcal{B}_{\mathbf{c}}(\lambda)\, u = 0 & \text{on } \partial G_{\mathbf{c}}, \qquad [u] = [\mathcal{N}_{\mathbf{c}} u] = 0 \quad \text{on } \partial G_{\mathbf{c},j} \setminus \partial G_{\mathbf{c}} \end{cases}$$

The ellipticity of problem $(\mathcal{P})$ implies the solvability of $\mathcal{P}[\mathbf{c}, \lambda]$ in $H^1(G_{\mathbf{c}})$ except for λ in a discrete set $\Lambda[\mathbf{c}]$. We repeat each $\lambda \in \Lambda[\mathbf{c}]$ according to $\dim \ker \mathcal{P}[\mathbf{c}, \lambda]$ and denote by $u_{\mathbf{c},\lambda}$ a basis of $\ker \mathcal{P}[\mathbf{c}, \lambda]$.

The *singularity exponents* λ and the associated base *singular functions* at $\mathbf{c}$ are

$$\lambda \in \Lambda[\mathbf{c}] \quad \text{with} \quad \operatorname{Re}\lambda > -\tfrac{1}{2} \qquad \text{and} \qquad U_{\mathbf{c},\lambda} = \rho^{\lambda} u_{\mathbf{c},\lambda}(\vartheta).$$

Edge Mellin Symbols

Edge e (of Ω or of Ω_j). In cylindrical coordinates $(r,\theta,z)\in\mathbb{R}_+\times(0,\omega)\times\mathrm{e}$,

$$L = r^{-2}\mathcal{L}_{\mathrm{e}}(\theta; r\partial_r,\partial_\theta) + r^{-1}\mathcal{L}_1(\theta; r\partial_r,\partial_\theta,\partial_z) + \mathcal{L}_2(\theta; r\partial_r,\partial_\theta,\partial_z).$$

Mellin Symbol and associated problem:

$$\mathbb{C}\ni\lambda \longmapsto \mathcal{L}_{\mathrm{e}}(\theta;\lambda,\partial_\theta) =: \mathcal{L}_{\mathrm{e}}(\lambda) \quad\text{and}\quad \mathbb{C}\ni\lambda \longmapsto \mathcal{B}_{\mathrm{e}}(\theta;\lambda,\partial_\theta) =: \mathcal{B}_{\mathrm{e}}(\lambda).$$

$$\mathcal{P}[\mathrm{e},\lambda]\quad\begin{cases} \mathcal{L}_{\mathrm{e}}(\lambda)\,u = f & \text{in } G_{\mathrm{e}},\\ \mathcal{B}_{\mathrm{e}}(\lambda)\,u = 0 & \text{on } \partial G_{\mathrm{e}}, \qquad [u]=[\mathcal{N}_{\mathrm{e}}u]=0 \quad \text{on } \partial G_{\mathrm{e},j}\setminus\partial G_{\mathrm{e}} \end{cases}$$

The ellipticity of problem $(\mathcal{P})$ implies the solvability of $\mathcal{P}[\mathrm{e},\lambda]$ in $H^1(G_{\mathrm{e}})$ except for λ in a discrete set $\Lambda[\mathrm{e}]$. We repeat each $\lambda\in\Lambda[\mathrm{c}]$ according to $\dim\ker\mathcal{P}[\mathrm{e},\lambda]$ and denote by $u_{\mathrm{e},\lambda}$ a basis of $\ker\mathcal{P}[\mathrm{e},\lambda]$.

The *singularity exponents* λ and the associated base *singular functions* along e are

$$\lambda\in\Lambda[\mathrm{e}] \quad\text{with}\quad \operatorname{Re}\lambda>0 \qquad\text{and}\qquad U_{\mathrm{e},\lambda} = r^\lambda\, u_{\mathrm{e},\lambda}(\theta).$$

Singular Functions of the Laplace operator

When Ω has a constant coefficient ε.

Edge e with opening ω_{e} $\{G_{\mathrm{e}} = (0, \omega_{\mathrm{e}})\}$

$$\begin{cases} \textit{Singularity Exponents} & \lambda \in \Lambda[\mathrm{e}] = \left\{ \dfrac{\ell\pi}{\omega_{\mathrm{e}}} \;\middle|\; \ell \in \mathbb{N} \right\} \\ \textit{Singular Functions} & U_{\mathrm{e},\lambda} = r^{\lambda} \sin(\lambda\theta). \end{cases}$$

Corner c with solid angle G_{c}.

Eigenpairs of the Laplace-Beltrami Dirichlet problem on G_{c}: $\{\mu_\ell, v_\ell(\vartheta)\}$

$$\begin{cases} \textit{Singularity Exponents} & \lambda \in \Lambda[\mathrm{c}] = \left\{ -\dfrac{1}{2} + \sqrt{\mu_\ell + \dfrac{1}{4}} \;\middle|\; \ell \in \mathbb{N} \right\} \\ \textit{Singular Functions} & U_{\mathrm{c},\lambda} = \rho^{\lambda_\ell}\, v_\ell(\vartheta). \end{cases}$$

Shift Theorem

For smooth domains, there holds $\boxed{f \in H^{\sigma-1}(\Omega) \quad \Longrightarrow \quad u \in H^{\sigma+1}(\Omega)}$.

Let $H^s_\star(\Omega)$ be the closure in $H^s(\Omega)$ of the smooth functions with their support outside the union of edges $\bigcup \bar{e}$. There holds

$$\boxed{f \in H^{\sigma-1}_\star(\Omega) \quad \Longrightarrow \quad u \in H^{\sigma+1}_\star(\Omega)}$$

if and only if $\sigma < \sigma[\Omega, a]$

$$\boxed{\sigma[\Omega, a] = \min\Big\{ \min_{\mathrm{c\ corner}} \xi_{\mathrm{c}} + \tfrac{1}{2} \ , \min_{\mathrm{e\ edge}} \xi_{\mathrm{e}} \Big\}}$$

with

ξ_{c} the least (real part) $> -\frac{1}{2}$ of the exponents $\lambda \in \Lambda[\mathrm{c}]$ and

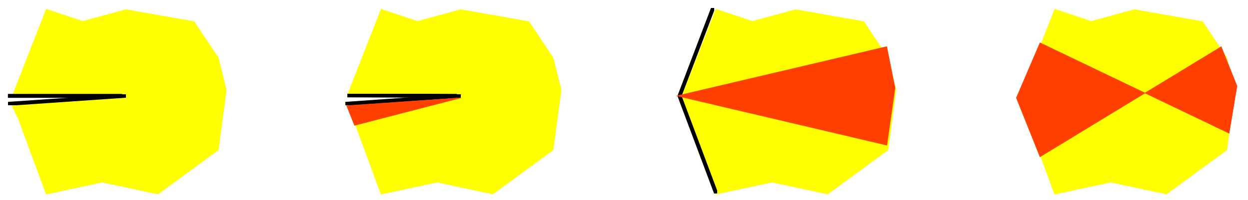
ξ_{e} the least (real part) > 0 of the exponents $\lambda \in \Lambda[\mathrm{e}]$.

The shift Theo. in standard Sobolev spaces has to take polynomial RHS into account.

Optimal Minima for ξ_{e}

(Regularity $H^{1+\xi_{\mathrm{e}}}$ for potential and $H^{\xi_{\mathrm{e}}}$ for electric field) :

Exterior Angle	1 material	2 materials	3 materials	4 materials
$\leq \frac{\pi}{2}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$
Convex	$\mathbf{1}$	$\mathbf{\frac{1}{2}}$	$\mathbf{0}$	$\mathbf{0}$
Any	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{1}{4}}$	$\mathbf{0}$	$\mathbf{0}$
None	∞	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{1}{4}}$	$\mathbf{0}$



Expansion Theorem: first layer of singularities

For $\sigma > \sigma[\Omega, a]$ with $\sigma < \boxed{1 + \sigma[\Omega, a]}$ and a smooth RHS $f \in H^\infty_\star(\Omega)$

$$u - \Big(\sum_{\mathrm{c\ corner}} \chi_\mathrm{c} u_\mathrm{c} + \sum_{\mathrm{e\ edge}} \chi_\mathrm{e} u_\mathrm{e} \Big) \in H^{\sigma+1}_\star(\Omega),$$

with the cut-off $\chi_\mathrm{c} = \chi_\mathrm{c}(\rho)$ and $\chi_\mathrm{e} = \chi_\mathrm{e}(r/\rho)$. The corner and edge contributions are:

$$u_\mathrm{c} = \sum_{\substack{\lambda \in \Lambda[\mathrm{c}] \\ -1/2 < \lambda < \sigma - 1/2}} \gamma_{\mathrm{c},\lambda} U_{\mathrm{c},\lambda} \quad \text{and} \quad u_\mathrm{e} = \sum_{\substack{\lambda \in \Lambda[\mathrm{e}] \\ 0 < \lambda < \sigma}} \gamma_{\mathrm{e},\lambda}(z) U_{\mathrm{e},\lambda}$$

where

- $\gamma_{\mathrm{c},\lambda} \in \mathbb{R}$ are the vertex coefficients
- $\mathrm{e} \ni z \mapsto \gamma_{\mathrm{e},\lambda}$ are the edge coefficients along e. They are $\mathcal{C}^\infty(e)$ (inside e).

Note: This expansion is valid if the inverses of Mellin symbols have poles of degree 1 in the exponents of singularities. If there exist poles with degree ≥ 2, other singular functions with logarithmic terms $\rho^\lambda \log^k \rho$ or $r^\lambda \log^k r$ appear.

Expansion Theorem: more layers of singularities

For $\sigma \geq \boxed{1+\sigma[\Omega, a]}$ and a smooth RHS $f \in H^{\infty}_{\star}(\Omega)$

$$u - \Big(\sum_{\mathrm{c\ corner}} \chi_{\mathrm{c}} u_{\mathrm{c}} + \sum_{\mathrm{e\ edge}} \chi_{\mathrm{e}} u_{\mathrm{e}} \Big) \in H^{\sigma+1}_{\star}(\Omega),$$

with the corner and edge contributions:

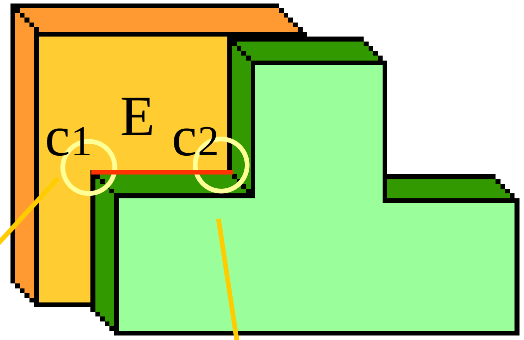
$$u_{\mathrm{c}} = \sum_{\substack{\lambda \in \Lambda[\mathrm{c}] \\ -1/2 < \lambda < \sigma - 1/2}} \gamma_{\mathrm{c},\lambda} U_{\mathrm{c},\lambda} \quad \text{and} \quad u_{\mathrm{e}} = \sum_{\substack{\lambda \in \Lambda[\mathrm{e}] \\ 0 < \lambda + p < \sigma}} \sum_{p \in \mathbb{N}_0} \mathrm{d}_z^p \gamma_{\mathrm{e},\lambda}(z)\, U^p_{\mathrm{e},\lambda}$$

where

- $\gamma_{\mathrm{c},\lambda} \in \mathbb{C}$ are the vertex coefficients (with more terms)
- $\mathrm{e} \ni z \mapsto \gamma_{\mathrm{e},\lambda}$ are the edge coefficients along e (with more terms)
- For $p = 0$, $U^0_{\mathrm{e},\lambda} = U_{\mathrm{e},\lambda}$.
  For $p \geq 1$, the new singularities $U^p_{\mathrm{e},\lambda}$ have the form $U^p_{\mathrm{e},\lambda} = r^{\lambda+p}\, u^p_{\mathrm{e},\lambda}(\theta)$.

Vertex – Edge interaction: the geometry

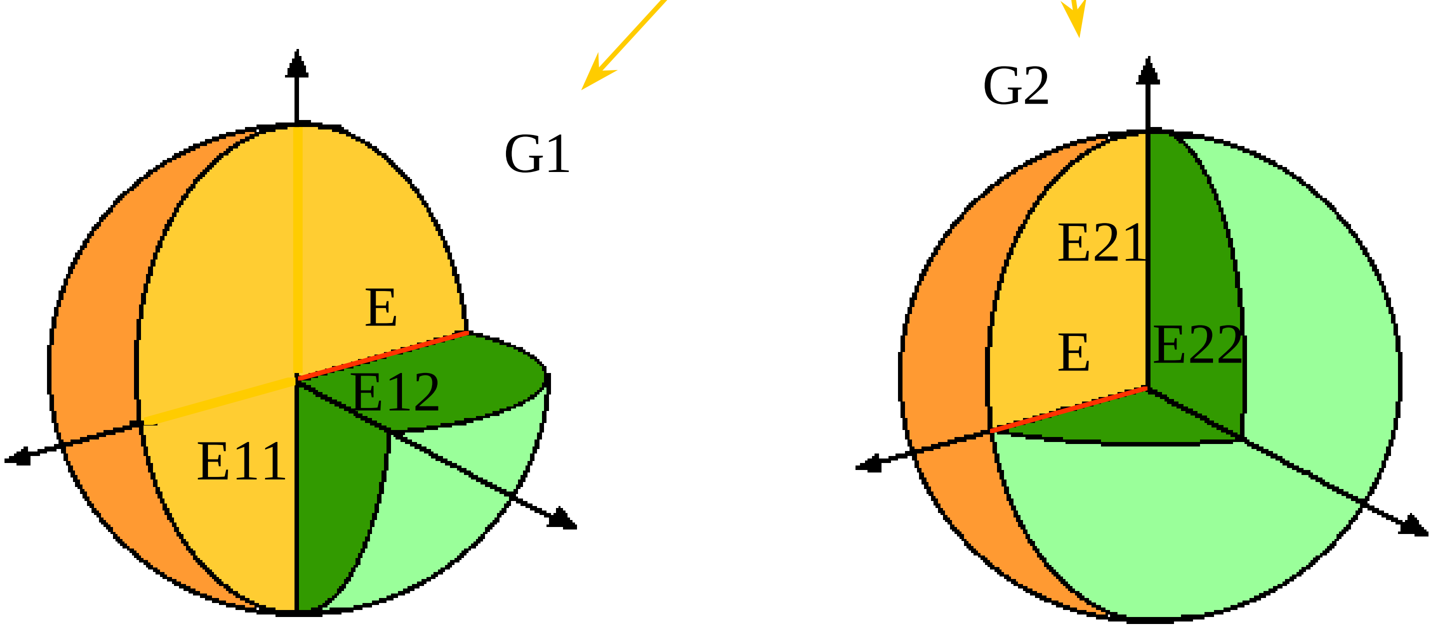
The edge E has the vertices $\mathrm{c}1$ and $\mathrm{c}2$ as end points.



Vertices $\mathrm{c}1$ – $\mathrm{c}2$ $\longleftrightarrow$ spherical polygons $G1$ – $G2$.

Corners of $G1$ $\longleftrightarrow$ edges E, $E11$ and $E12$.

Corners of $G2$ $\longleftrightarrow$ edges E, $E21$ and $E22$.



Vertex – Edge interaction: edge coefficients

We have

$$u - \Big(\sum_{\mathrm{c\ corner}} \chi_{\mathrm{c}} u_{\mathrm{c}} + \sum_{\mathrm{e\ edge}} \chi_{\mathrm{e}} u_{\mathrm{e}} \Big) \in H^{\sigma+1}_{\star}(\Omega),$$

with the corner and edge contributions:

$$u_{\mathrm{c}} = \sum_{\substack{\lambda \in \Lambda[\mathrm{c}] \\ -1/2 < \lambda < \sigma - 1/2}} \gamma_{\mathrm{c},\lambda} U_{\mathrm{c},\lambda} \quad \text{and} \quad u_{\mathrm{e}} = \sum_{\substack{\nu \in \Lambda[\mathrm{e}] \\ 0 < \nu + p < \sigma}} \sum_{p \in \mathbb{N}_0} \mathrm{d}_z^p \gamma_{\mathrm{e},\nu}(z)\, U^p_{\mathrm{e},\nu}$$

For the edge $\mathrm{e} = \{z \in (z_1, z_2)\}$, the edge coefficients satisfy

$$\boxed{\delta^{\alpha-\sigma}(z)\, \mathrm{d}_z^\alpha \gamma_{\mathrm{e},\nu} \in L^2(e), \quad \forall \alpha \in \mathbb{N}_0}$$

where $\delta = (z - z_1)(z_2 - z)$ is $\simeq$ the distance to the endpoints of e.

When σ increases, the corner coefficients are only more numerous.

When σ increases, the edge coefficients *depend on σ*. Should write instead:

Vertex – Edge interaction: edge coefficients

We have

$$u - \Big(\sum_{\mathrm{c\ corner}} \chi_{\mathrm{c}} u_{\mathrm{c}} + \sum_{\mathrm{e\ edge}} \chi_{\mathrm{e}} u_{\mathrm{e}} \Big) \in H^{\sigma+1}_{\star}(\Omega),$$

with the corner and edge contributions:

$$u_{\mathrm{c}} = \sum_{\substack{\lambda \in \Lambda[\mathrm{c}] \\ -1/2 < \lambda < \sigma - 1/2}} \gamma_{\mathrm{c},\lambda} U_{\mathrm{c},\lambda} \quad \text{and} \quad u_{\mathrm{e}} = \sum_{\substack{\nu \in \Lambda[\mathrm{e}] \\ 0 < \nu + p < \sigma}} \sum_{p \in \mathbb{N}_0} \mathrm{d}_z^p \gamma^{(\sigma)}_{\mathrm{e},\nu}(z)\, U^p_{\mathrm{e},\nu}$$

For the edge $\mathrm{e} = \{z \in (z_1, z_2)\}$, the edge coefficients satisfy

$$\boxed{\delta^{\alpha-\sigma}(z)\, \mathrm{d}_z^{\alpha} \gamma^{(\sigma)}_{\mathrm{e},\nu} \in L^2(e), \quad \forall \alpha \in \mathbb{N}_0}$$

where $\delta = (z - z_1)(z_2 - z)$ is $\simeq$ the distance to the endpoints of e.

When σ increases, the corner coefficients are only more numerous.

When σ increases, the edge coefficients $\gamma^{(\sigma)}_{\mathrm{e},\nu}$ are more and more flat.

Vertex – Edge interaction: stress intensity functions

We have

$$u - \Big(\sum_{\mathrm{c\ corner}} \chi_{\mathrm{c}} \widetilde{u}_{\mathrm{c}} + \sum_{\mathrm{e\ edge}} \chi_{\mathrm{e}} \widetilde{u}_{\mathrm{e}} \Big) \in H^{\sigma+1}_{\star}(\Omega),$$

with the corner and edge contributions:

$$\widetilde{u}_{\mathrm{c}} = \sum_{\substack{\lambda \in \Lambda[\mathrm{c}] \\ -1/2 < \lambda < \sigma - 1/2}} \gamma_{\mathrm{c},\lambda} \widetilde{U}^{(\sigma)}_{\mathrm{c},\lambda} \quad \text{and} \quad \widetilde{u}_{\mathrm{e}} = \sum_{\substack{\nu \in \Lambda[\mathrm{e}] \\ 0 < \nu + p < \sigma}} \sum_{p \in \mathbb{N}_0} \mathrm{d}^p_z \widetilde{\gamma}_{\mathrm{e},\nu}(z)\, U^p_{\mathrm{e},\nu}$$

For the edge $\mathrm{e} = \{z \in (z_1, z_2)\}$, the edge coefficients satisfy

$$\boxed{\delta^{\alpha}(z)\, \mathrm{d}^{\alpha}_z \widetilde{\gamma}_{\mathrm{e},\nu} \in L^2(e), \quad \forall \alpha \in \mathbb{N}_0}$$

where $\delta = (z - z_1)(z_2 - z)$ is $\simeq$ the distance to the endpoints of e.

When σ increases, the corner functions are more and more flat.

The edge coefficients (stress intensity functions) $\widetilde{\gamma}_{\mathrm{e},\nu}$ are more numerous.

Vertex – Edge interaction: expansion of corner singularities

Case $\sigma < \sigma[\Omega, a] + 1$. In the corner contribution

$$u_{\mathbf{c}} = \sum_{\substack{\lambda \in \Lambda[\mathbf{c}] \\ -1/2 < \lambda < \sigma - 1/2}} \gamma_{\mathbf{c},\lambda} U_{\mathbf{c},\lambda}(x) = \sum_{\substack{\lambda \in \Lambda[\mathbf{c}] \\ -1/2 < \lambda < \sigma - 1/2}} \boxed{\gamma_{\mathbf{c},\lambda}\, \rho^{\lambda}}\, U_{\mathbf{c},\lambda}\Big(\frac{x}{\rho}\Big)$$

the angular function $U_{\mathbf{c},\lambda}$ has singularities at the corners of $G_{\mathbf{c}}$ ($\leftrightarrow$ edges $\overline{\mathbf{e}} \ni \mathbf{c}$):

$$U_{\mathbf{c},\lambda}\Big(\frac{x}{\rho}\Big) - \sum_{\overline{\mathbf{e}} \ni \mathbf{c}} \sum_{\substack{\nu \in \Lambda[\mathbf{e}] \\ 0 < \nu < \sigma}} \boxed{a^{\mathbf{c},\lambda}_{\mathbf{e},\nu}}\, U_{\mathbf{e},\nu}\Big(\frac{r}{\rho}, \theta\Big) = \widetilde{U}^{(\sigma)}_{\mathbf{c},\lambda} \in H^{\sigma+1}_{\star}(G_{\mathbf{c}}).$$

Note that $U_{\mathbf{e},\nu}(\frac{r}{\rho}, \theta) = \boxed{\rho^{-\nu}}\, U_{\mathbf{e},\nu}(r, \theta)$. Expansion of the corner singularities

$$u_{\mathbf{c}} - \sum_{\substack{\lambda \in \Lambda[\mathbf{c}] \\ -1/2 < \lambda < \sigma - 1/2}} \sum_{\overline{\mathbf{e}} \ni \mathbf{c}} \sum_{\substack{\nu \in \Lambda[\mathbf{e}] \\ 0 < \nu < \sigma}} \boxed{\gamma_{\mathbf{c},\lambda}\, \rho^{\lambda}}\, \boxed{a^{\mathbf{c},\lambda}_{\mathbf{e},\nu}}\, \boxed{\rho^{-\nu}}\, U_{\mathbf{e},\nu}(r, \theta) = \widetilde{u}_{\mathbf{c}}.$$

Vertex – Edge interaction: expansion of Stress Intensity Functions

The expansion of the corner singularities

$$u_{\mathrm{c}} - \sum_{\substack{\lambda \in \Lambda[\mathrm{c}] \\ -1/2 < \lambda < \sigma - 1/2}} \sum_{\overline{\mathrm{e}} \ni \mathrm{c}} \sum_{\substack{\nu \in \Lambda[\mathrm{e}] \\ 0 < \nu < \sigma}} \boxed{\gamma_{\mathrm{c},\lambda}\, \rho^{\lambda}} \; \boxed{a^{\mathrm{c},\lambda}_{\mathrm{e},\nu}} \; \boxed{\rho^{-\nu}} \; U_{\mathrm{e},\nu}(r,\theta) = \widetilde{u}_{\mathrm{c}}$$

gives the stress intensity functions of u_{c} along the edges e such that $\overline{\mathrm{e}} \ni \mathrm{c}$:

They are

$$\boxed{a^{\mathrm{c},\lambda}_{\mathrm{e},\mu}} \; \boxed{\gamma_{\mathrm{c},\lambda}\, \rho^{\lambda}} \; \boxed{\rho^{-\nu}}$$

Note that $\rho = z - z_1$ is a suitable coordinate along the edge(s) e.

Therefore, we find the expansion at the corner c of the stress intensity functions:

$$\widetilde{\gamma}_{\mathrm{e},\nu}(\rho) - \sum_{\substack{\lambda \in \Lambda[\mathrm{c}] \\ -1/2 < \lambda < \sigma - 1/2}} \boxed{a^{\mathrm{c},\lambda}_{\mathrm{e},\mu}\; \gamma_{\mathrm{c},\lambda}\, \rho^{\lambda}\; \rho^{-\nu}} = \gamma^{(\sigma)}_{\mathrm{e},\nu}(\rho)$$

Are Stress Intensity Functions bounded ?

The interesting SIF are those associated with the first edge exponent ν_1 .

The SIF $\widetilde{\gamma}_{\mathrm{e},\nu}$ is bounded iff $\lambda_1 \geq \nu_1$ with λ_1 the first corner exponent.

- **Example of Laplace Dirichlet.**

  Fichera corner: $\lambda_1 \simeq 0.45418$ and $\nu_1 = \pi/(3\pi/2) = 2/3$.

  Screen problem: $\Gamma_{\mathrm{c}} = \mathbb{R}^3 \setminus S_\omega$ (S_ω plane sector of opening ω).

  $\nu_1 = 1/2$ and $\lambda_1 = \lambda_1(\omega)$ where λ_1 is increasing

$$\lambda_1(0) = 0 \quad \text{and} \quad \lambda_1(2\pi) = 1 \quad (\text{and also} \quad \lambda_1(\pi) = 1/2).$$

- **Maxwell (electric).**

Same difference $\lambda_1 - \nu_1$ than for Laplace-Dirichlet, since the first electric singularities are the gradients of the Laplace-Dirichlet singularities:

$$E_{\mathrm{c},\lambda} = \nabla_{x,y,z} U^{\Delta,\mathrm{Dir}}_{\mathrm{c},\lambda+1} \qquad \text{and} \qquad E_{\mathrm{e},\lambda} = \nabla_{x,y} U^{\Delta,\mathrm{Dir}}_{\mathrm{e},\lambda+1}.$$