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Problems of Corner Singularities

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Vertex and edge singularities

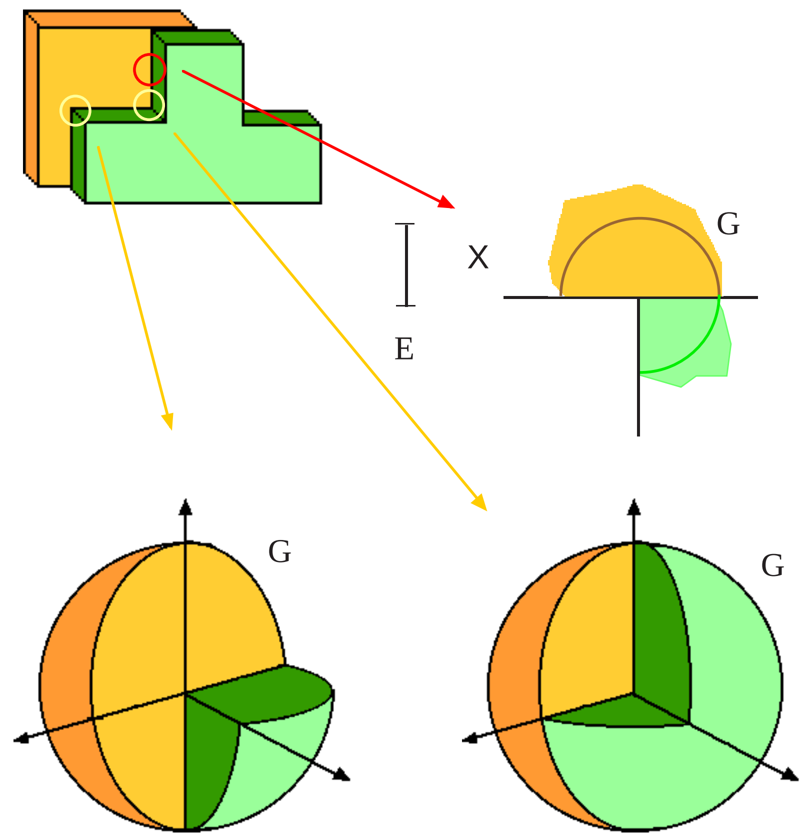
For a polyhedral domain Ω and a second order strongly elliptic operator L with boundary conditions B.

Vertex v $\longleftrightarrow$ Polar coord. $(\rho,\vartheta)\in\mathbb{R}_+\times G$ centered in v.

$$\begin{cases} \textit{Singularity Exponents} & \lambda\in\mathcal{E}[\mathrm{v}] \\ \textit{Singular Functions} & U_{\mathrm{v},\lambda}=\rho^{\lambda}\,u_{\mathrm{v},\lambda}(\vartheta) \end{cases}$$

Edge $E\ni\mathrm{e}$ $\longleftrightarrow$ Cylindrical coord. $(r,\theta,z)\in\mathbb{R}_+\times(0,\omega)\times T_{\mathrm{e}}A$.

$$\begin{cases} \textit{Singularity Exponents} & \lambda\in\mathcal{E}[\mathrm{e}] \\ \textit{Singular Functions} & U_{\mathrm{e},\lambda}=r^{\lambda}\,u_{\mathrm{e},\lambda}(\theta) \end{cases}$$



What are Singularities doing ? They ...

Invalidate usual regularity results.
Prevent regularity consuming proofs.
Stop construction algo. in singular perturbation expansions.

For (pure) Mathematicians

Make numerical schemes slow...
...Or wrong.

For (applied) Mathematicians

Attract lightnings.
Concentrate stresses.
Grow cracks.
Destroy ill-computed structures.

For Engineers ... and Users.

Change locally (?) physical laws (plasticity)

For Physicists.

Regularity Theorem

There holds for all thermal and elasticity pbs (use piecewise – H^s for multi-material)

$$\boxed{f \in H^{\sigma-1}(\Omega) \quad \Longrightarrow \quad u \in H^{\sigma+1}(\Omega)}$$

if and only if $\sigma < \sigma[\Omega, L, B]$

$$\boxed{\sigma[\Omega, L, B] = \min\left\{ \min_{\mathrm{v\ vertex}} \xi_{\mathrm{v}} + \tfrac{1}{2} \;,\; \min_{\mathrm{e\ in\ edges}} \xi_{\mathrm{e}} \right\}}$$

with

ξ_{v} **the least real part** $> -\frac{1}{2}$ **of the exponents** $\lambda \in \mathcal{E}[\mathrm{v}]$.

and

ξ_{e} **the least real part** > 0 **of the exponents** $\lambda \in \mathcal{E}[\mathrm{e}]$.

For regularized Maxwell, the above limits are $-\frac{3}{2}$ and -1.

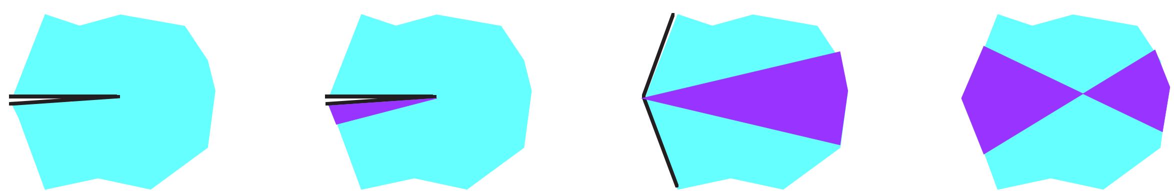
Example of Electrostatic Potential in Heterogeneous Media

Associated with the bilinear form (ε_j is the electric permittivity of material Ω_j)

$$a(u,v) = \sum_j \int_{\Omega_j} \varepsilon_j \,\mathrm{grad}\, u \cdot \mathrm{grad}\, v \; dx, \qquad \text{for} \quad u,v \in \overset{\circ}{H}{}^1(\Omega).$$

Optimal Minima for ξ_{e} (Regularity $H^{1+\xi_{\mathrm{e}}}$ for potential and $H^{\xi_{\mathrm{e}}}$ for electric field) :

Exterior Angle	1 material	2 materials	3 materials	4 materials
$\leq \frac{\pi}{2}$	2	1	0	0
Convex	1	$\frac{1}{2}$	0	0
Any	$\frac{1}{2}$	$\frac{1}{4}$	0	0
None	∞	$\frac{1}{2}$	$\frac{1}{4}$	0



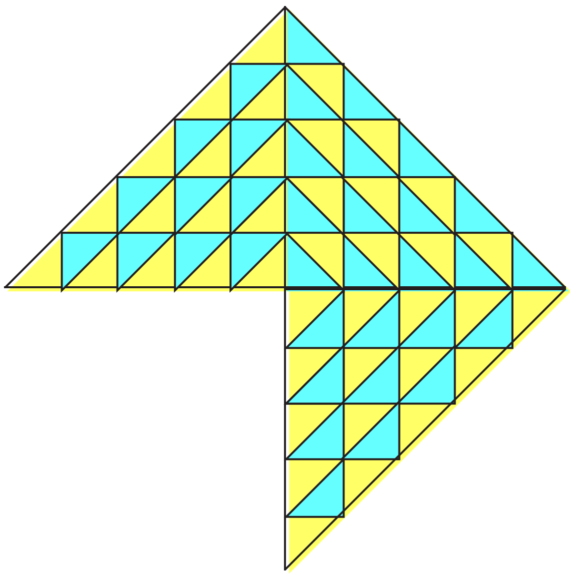
Joint work with Martin Costabel & Serge Nicaise about Maxwell transmission (1998)

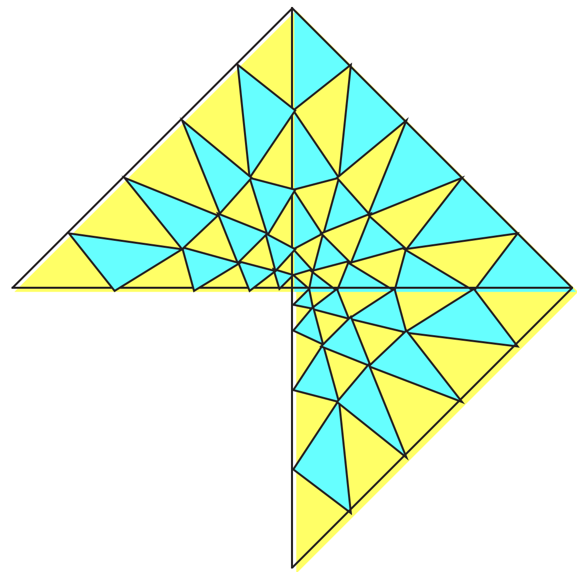
Approximation of Singularities by Finite Element Methods

With a regular and uniform mesh, the convergence rate is bounded by

$$\boxed{h^{\sigma[\Omega,L,B]}}$$

With a sufficiently *refined* mesh, optimal convergence rates are recovered.





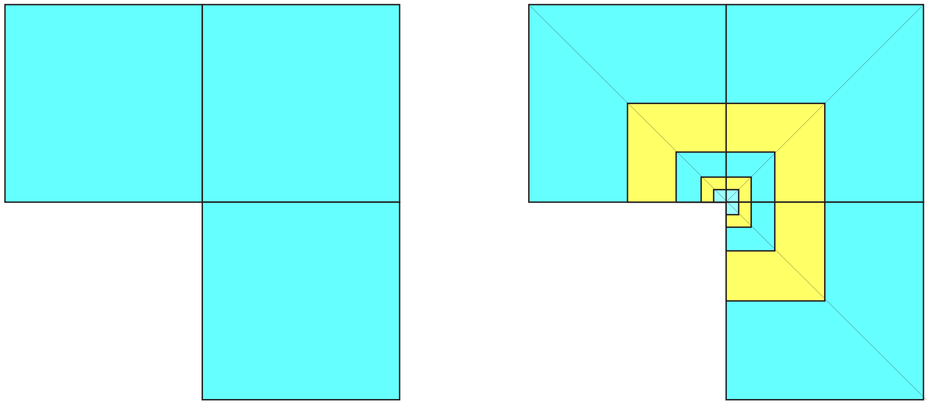
With a Boundary Element Method, refined meshes yield very good results, especially in 2D (1D boundary).

Approximation of Singularities by Higher Order Methods

By an approximation with degree N polynomials (spectral method or p version of finite elements), the convergence rate is *doubled*, compared with the expected rate (*cf* Sobolev regularity):

$$\boxed{N^{-2\cdot\sigma[\Omega,L,B]}}$$

The h-p version (geometrical refinement) yields theoretical exponential rates for analytic data.



With *mortars*, spectral or p version can be coupled with finite elements.

The Maxwell "bug"

The bilinear form associated with the divergence-regularized formulation is

$$a(u,v) \quad = \quad \int_\Omega \big(\operatorname{curl} u \cdot \operatorname{curl} v + \operatorname{div} u \;\; \operatorname{div} v\big)\, \mathrm{d}x.$$

The variational space for the electric field is the space $X_N(\Omega)$:

$$X_N(\Omega) = \{u \in L^2(\Omega)^3 \; ; \; \operatorname{curl} u \in L^2(\Omega)^3, \; \operatorname{div} u \in L^2(\Omega), \; u \times n = 0 \;\text{ on } \partial\Omega\}$$

The related "regular" space is $\boxed{H_N(\Omega) := X_N(\Omega) \cap H^1(\Omega)^3}$

If Ω is a polyhedron with plane faces

- $X_N(\Omega) = H_N(\Omega) \quad \Longleftrightarrow \quad \Omega$ is convex,
- $H_N(\Omega)$ is a closed subspace of $X_N(\Omega)$ for its natural norm.

If Ω is a non-convex polyhedron, the problem has *two distinct* formulations and *two distinct* solutions in each variational space $H_N(\Omega)$ or $X_N(\Omega)$. Moreover any *rot-div conform* method based on $\mathbb{P}_1$ elements converges towards the wrong solution.

Maxwell and Lamé eigenvalues in a polygon

Eigenvalues of the parameter-dependent problems associated with the form $a[s]$

$$s > 0 \ : \quad a[s](u,v) \ = \int_\Omega \big(\operatorname{curl} u \ \operatorname{curl} v + s \ \operatorname{div} u \ \operatorname{div} v\big)\,\mathrm{d}x.$$

Here $\operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$ and $\operatorname{div} u = \partial_1 u_1 + \partial_2 u_2$.

The eigenvalues in X_N are

- **Either *independent* of s and $=$ Maxwell eigenv. They coincide with the Neumann eigenv. of $-\Delta$ and the eigenm. are the vector curls of the Δ Neumann eigenf.**
- **Or *linear dependent* on s. They have the form $s\nu$ with ν the Dirichlet eigenv. of $-\Delta$ and the eigenm. are the gradients of the Δ Dirichlet eigenf.**

The eigenvalues in H_N are the Lamé eigenvalues for the Lamé constants $\mu = 1$ and $\lambda = s - 2$ since, for u and v in H_N

$$\int_\Omega \big(\operatorname{curl} u \operatorname{curl} v + s \operatorname{div} u \operatorname{div} v\big)\,\mathrm{d}x = \int_\Omega \big(2\varepsilon(u) : \varepsilon(v) + (s-2) \operatorname{div} u \operatorname{div} v\big)\,\mathrm{d}x$$

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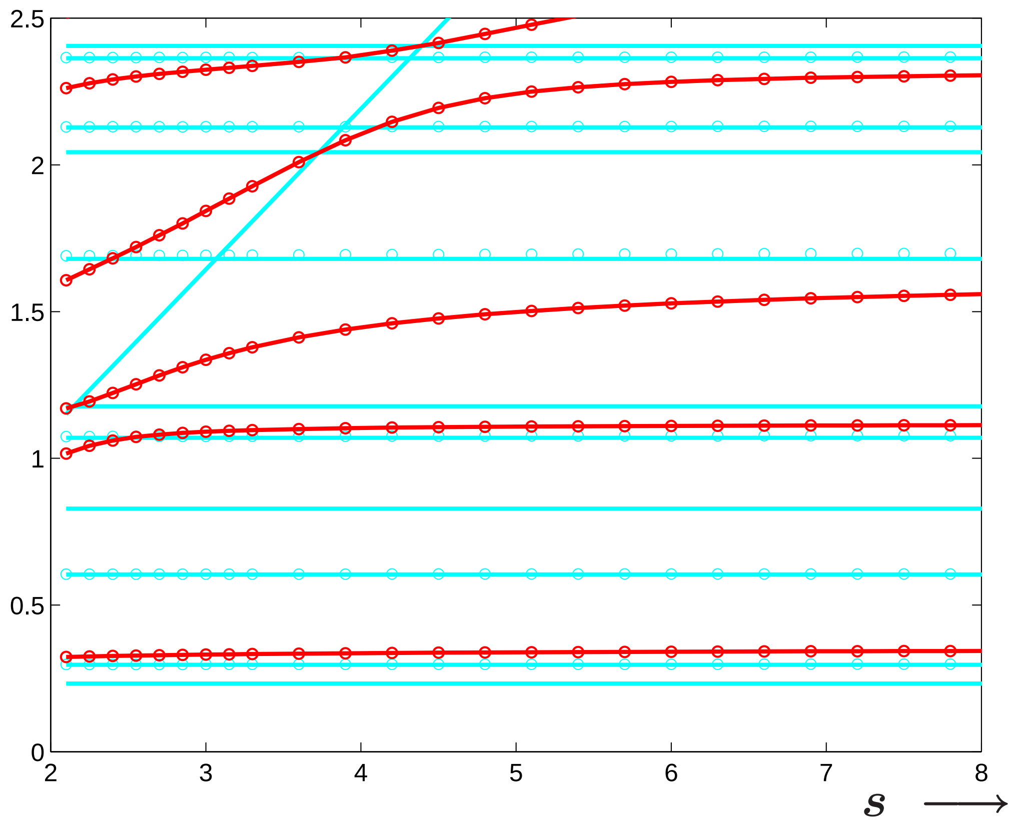
Evidence of the Maxwell “bug”: symmetric domain, ground states

Computation of eigenvalues in X_N and H_N associated with $a[s]$.

Ω is the L – shape polygon: $[0,1] \times [0,1] \setminus [0.75,1] \times [0.75,1]$.



Eigenvalues in X_N (Neumann and $s\times$ Dirichlet) versus s.
Lamé eigenvalues coinciding with Maxwell eigenv.

Eigenvalues in H_N (Lamé eigenv.) $\neq$ eigenv. in X_N, versus s.

One out of two Lamé is a Maxwell eigenv., due to H^2 regularity of even Neumann eigenf.

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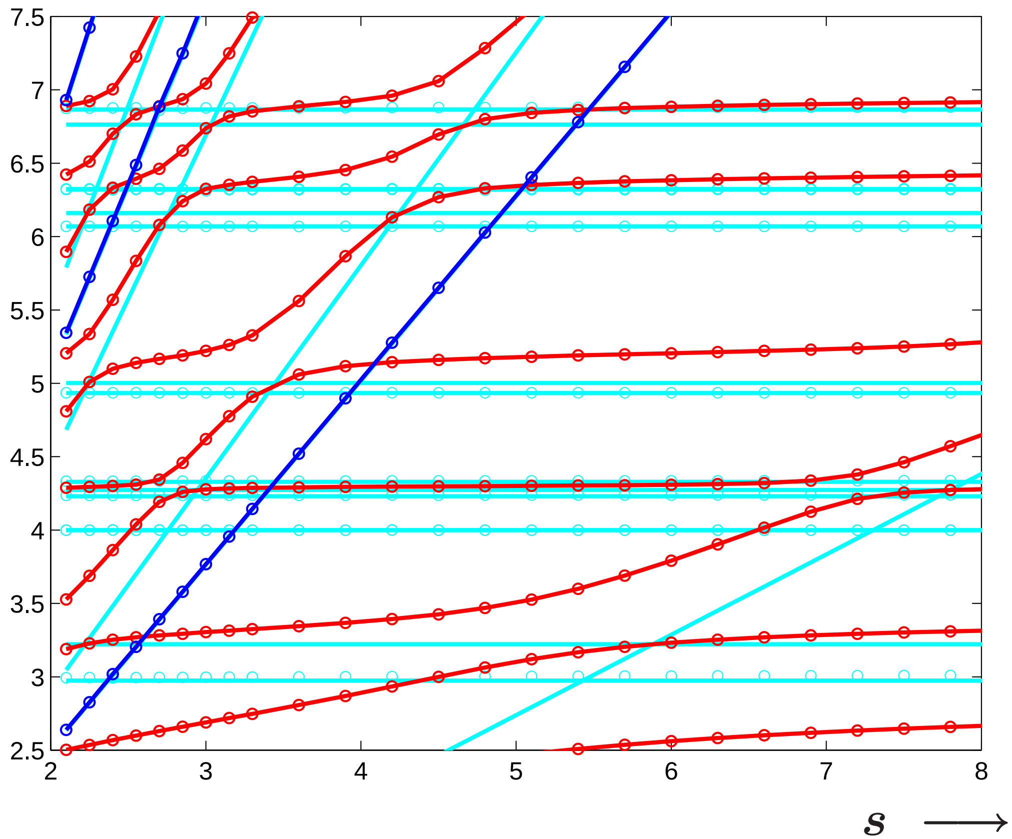
Computed with StressCheck$^{\rm TM}$*. Joint work with Martin Costabel (1998)*

Evidence of the Maxwell “bug”: symmetric domain continued

Ω is the L – shape polygon: $[0,1] \times [0,1] \setminus [0.75,1] \times [0.75,1]$.



Eigenvalues in X_N (Neumann and $s\times$ Dirichlet) versus s.

Lamé eigenvalues coinciding with Maxwell eigenv.

Eigenvalues in H_N (Lamé eigenv.) $\neq$ eigenv. in X_N, versus s.

One out of two Lamé is a Maxwell eigenv., due to H^2 regularity of odd Dirichlet eigenf.

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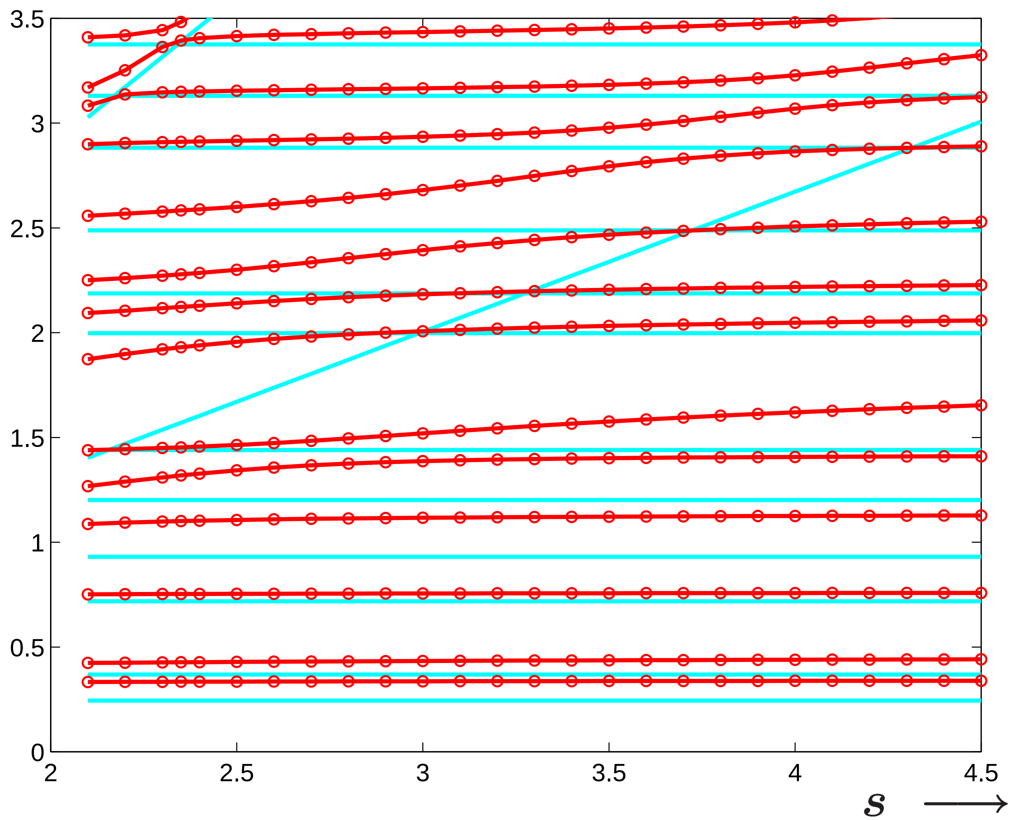
Computed with StressCheck$^{\mathrm{TM}}$. Joint work with Martin Costabel (1998)

Evidence of the Maxwell "bug": non symmetric domain

Computation of eigenvalues in X_N and H_N associated with $a[s]$.

Ω is the L – shaped polygon $[0,1] \times [0,0.87] \setminus [0.72,1] \times [0.61,0.87]$.



Eigenvalues in X_N (Neumann and $s\times$ Dirichlet) versus s.

Eigenvalues in H_N (Lamé eigenv.) $\neq$ eigenv. in X_N versus s.

Note the crossing points between double eigenv. in X_N and eigenv. in H_N.

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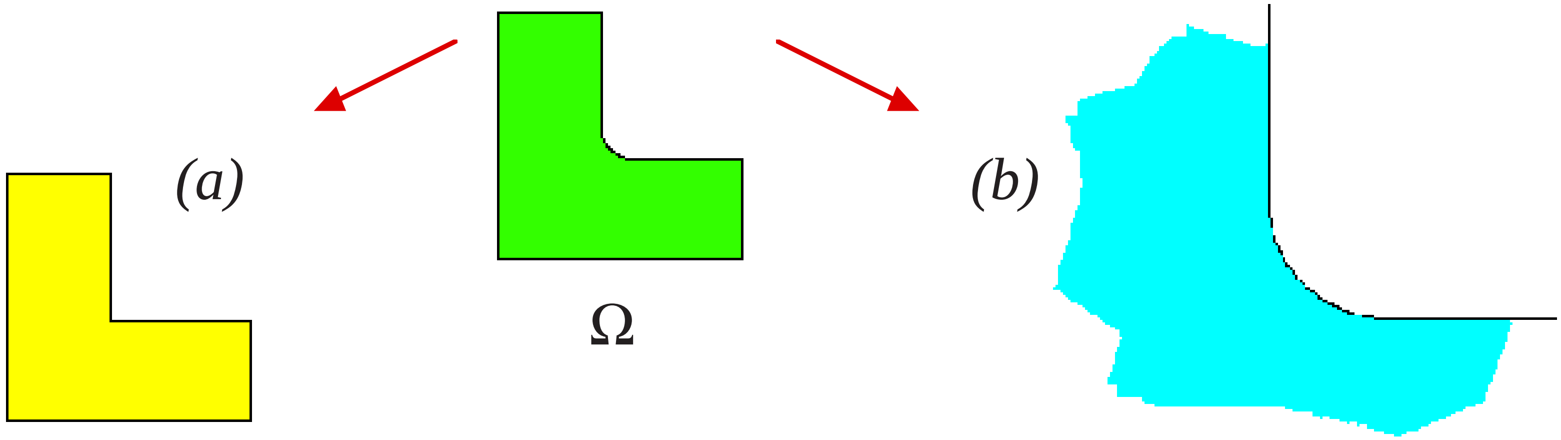
Computed with StressCheck$^{\mathrm{TM}}$. Joint work with Martin Costabel (1998)

If corners are curved...



Small parameter : ε curvature radius. **Opening of the limit angle : ω**

Dirichlet problem for Δ. Solution $u = u_\varepsilon$ in $\Omega = \Omega_\varepsilon$.

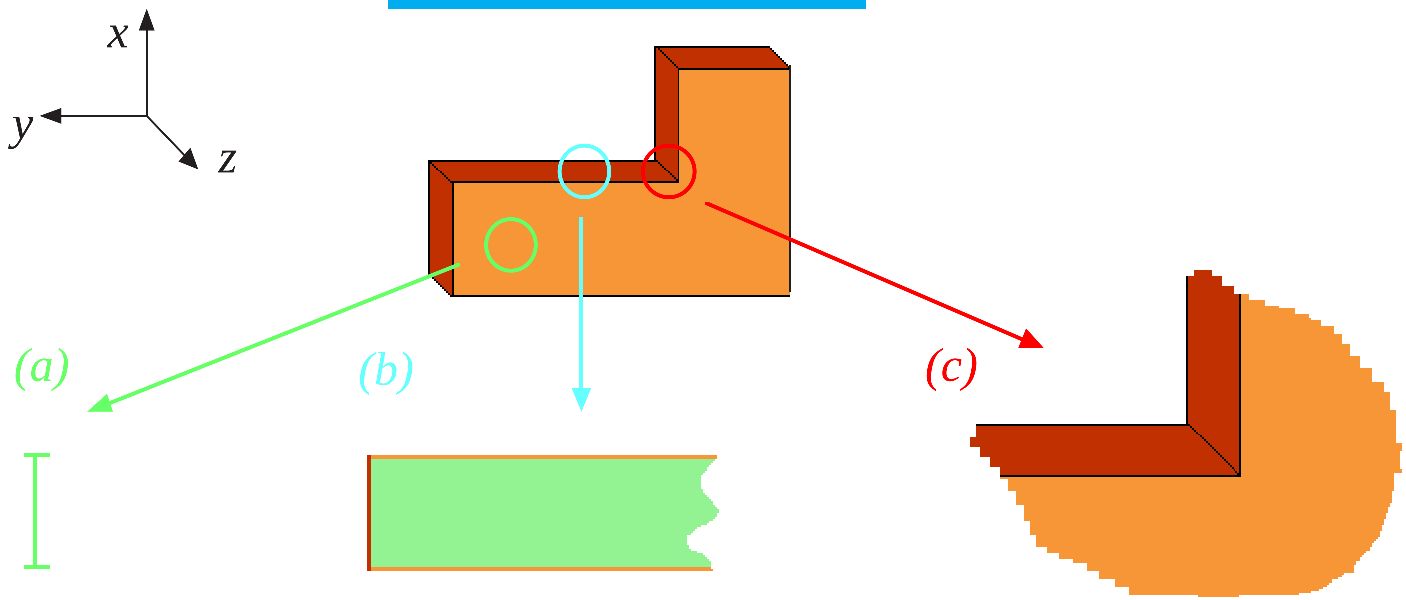
$$u_\varepsilon = \underbrace{v^0(x)}_{(a)} + \sum_{\substack{p,\,q\,\in\,\mathbb{N}\\ p+q\,\geq\,1}} \varepsilon^{\frac{p\pi}{\omega}+q}\, \underbrace{v^{p,q}(x)}_{(a)} + \sum_{\substack{p,\,q\,\in\,\mathbb{N}\\ p+q\,\geq\,1}} \varepsilon^{\frac{p\pi}{\omega}+q}\, \underbrace{w^{p,q}\Big(\frac{x}{\varepsilon}\Big)}_{(b)} .$$

Almost-corners are not better than sharp corners.

Holds true in particular for thin dielectric coating: if the curvature radius is of same order as the thickness of the coating, impedance boundary conditions cannot work.

cf Maz'ya - Nazarov - Plamenevskii 1991

Thin corner plate



(a) **Integration of a Neumann problem**

(b) **Lateral boundary layer profiles**

(c) **Corner boundary layer profiles**

$$\sum_{\substack{p,\,q\,\in\,\mathbb{N}\\ p+q\,\geq\,0}} \varepsilon^{\lambda_p+q}\,\underbrace{v^{p,q}\Big(x,y,\frac{z}{\varepsilon}\Big)}_{(a)} + \sum_{\substack{p,\,q\,\in\,\mathbb{N}\\ p+q\,\geq\,1}} \varepsilon^{\lambda_p+q}\,\underbrace{w^{p,q}\Big(\frac{x}{\varepsilon},y,\frac{z}{\varepsilon}\Big)}_{(b)} + \sum_{\substack{p,\,q\,\in\,\mathbb{N}\\ p+q\,\geq\,1}} \varepsilon^{\lambda_p+q}\,\underbrace{k^{p,q}\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{z}{\varepsilon}\Big)}_{(c)}$$